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Note on the Runge-Kutta Method[1]

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A comparison is made between the standard Runge-Kutta method of solving the differential equation $y'=f(x,y)$ and a method of numerical quadrature. By examples it is shown that the Runge-Kutta method may be unfavorable even for simple functions f.

One of the most celebrated methods for the numerical solution of differential equations is the one originated by Runge[2] and elaborated by Heun,[3] Kutta,[4] Nyström,[5] and others. This method is usually given considerable prominence in texts where numerical methods are discussed.

In contrast to step-by-step procedures based on formulas for numerical quadrature the Runge-Kutta method (as it is usually called) enjoys two conspicuous advantages:

(1) No special devices are required for starting the computation.

(2) The length of the step can be modified at any time in the course of the computation without additional labor.

On the other hand it is open to two major objections:

(1) The process does not contain in itself any simple means for estimating the error or for detecting computation mistakes. It is true that Bieberbach[6] has found an expression which provides an upper bound for the error at a given step of the Runge-Kutta process (or more accurately, the Kutta process). However this estimate depends on quantities which do not appear directly in the computation, and therefore requires some additional separate calculation.

(2) Each step requires four substitutions into the differential equation. For the case of complicated equations this may demand an excessive amount of labor per step.

By accident or design it happens that examples usually chosen in textbooks to illustrate the Runge-Kutta method are such that the method appears in a very favorable light.

It is the purpose of this note to exhibit examples where the Runge-Kutta method does not make a very good showing when compared with a method based on numerical quadrature. For the comparison we employ the commonly used form of the Runge-Kutta method (actually due to Kutta). If the differential equation is

$$\frac{dy}{dx}=f(x,y),$$

the step from x_n to $x_{n+1}=x_n+h$ is made by the formulas

$$y_{n+1}=y_n+(k_1+2k_2+2k_3+k_4)/6,$$

where

$$\begin{aligned}
k_1&=hf(x_n, y_n),\\
k_2&=hf(x_n+\tfrac{1}{2}h, y_n+\tfrac{1}{2}k_1),\\
k_3&=hf(x_n+\tfrac{1}{2}h, y_n+\tfrac{1}{2}k_2),\\
k_4&=hf(x_n+h, y_n+k_3).
\end{aligned}$$

These formulas yield an approximation of the fourth degree in h, that is, the expansion in powers of h of y_{n+1} defined by these formulas agrees through terms of the fourth degree with the expansion of y_{n+1} obtained directly from the differential equation.

[1] The preparation of this paper was sponsored (in part) by the Office of Naval Research.

[2] C. Runge, Ueber die Numerische Auflösung von Differentialgleichungen, Math. Ann. **46**, 167 (1895).

[3] K. Heun, Zeitschrift Math. Phys. **45**, 23 (1900).

[4] W. Kutta, Zeitschrift Math. Phys. **46**, 435 (1901).

[5] E. J. Nyström, Acta Societatis Scientiarum Fennicae **50**, No. 13, 1 to 55, (1925).

[6] L. Bieberbach, Theorie der Differentialgleichungen, Berlin (1946) (now Dover Publications, New York, N. Y.), page 54.

This is to be compared with the method[7] that makes the step from x_n to x_{n+1} by using

$$\bar{y}_{n+1}=y_{n-3}+4hy'_{n-1}+\frac{8h}{3}\,\delta^2 y'_{n-1}$$

to obtain a trial value $\bar{y}_{n+1}$ of y_{n+1}, then calculating y'_{n+1} approximately from the differential equation, and finally computing a corrected value of y_{n+1} by

$$y_{n+1}=y_{n-1}+2hy'_n+\frac{h}{3}\,\delta^2 y'_n.$$

This method is also of fourth degree in h in the sense that the error terms are proportional to h^5. The difference

$$\bar{y}_{n+1}-y_{n+1}=E_{n+1}$$

serves as a control, since it may be shown that the error of y_{n+1} (ignoring rounding errors and errors accumulated from previous steps) is approximately $E_{n+1}/29$. In actual calculation it is customary to tabulate E_n and to correct the trial value by estimating E_{n+1} from the trend of the E_n already known. When this is done and when the number of places retained in y is kept such that $E_{n+1}/29$ is not significant, it will not be necessary to recompute y'_{n+1} after y_{n+1} has been corrected. Thus after the computation is well under way we need only *one* substitution per step.

Thus this method has two advantages:

1. By internal evidence the computation shows how many places in y can be accepted as reliable (of course accumulation of rounding errors, etc., excepted).
2. Except at the beginning, only one substitution is required per step.

Two disadvantages may be cited:

1. Special devices are necessary to get started.
2. Change of the interval h in the course of a computation is somewhat troublesome.

Now let us consider a particular example

$$\frac{dy}{dx}=\frac{5y}{1+x},\ y=1 \text{ when } x=0.$$

In each case take $h=0.1$ and carry the computation to $x=1$.

[7] Milne, W. E., Numerical integration of ordinary differential equations, Am. Math. Monthly **33**, 455 (1926). (The formulas of the present paper are expressed in terms of central differences but are actually identical with those of the above reference.)

In Kutta's method, without making some independent investigation, we have no idea how many places in y to retain. In the comparison below, we arbitrarily retained four decimal places.

The results obtained by the two methods are given in table 1 and are compared with those obtained from the actual solution $y=(1+x)^5$.

TABLE 1. *Solution of* $dy/dx=5y/(1+x)$, $y=1$, *at* $x=0$.

x	Kutta	Error	Milne	Error	$(1+x)^5$
0	1.0000	--------	1.0000	--------	1.0000
.1	1.6103	0.0002	1.6105	--------	1.6105
.2	2.4878	.0005	2.4883	--------	2.4883
.3	3.7119	.0010	3.7129	--------	3.7129
.4	5.3765	.0017	5.3782	--------	5.3782
.5	7.5911	.0027	7.5937	0.0001	7.5938
.6	10.4819	.0039	10.4857	.0001	10.4858
.7	14.1931	.0055	14.1985	.0001	14.1986
.8	18.8882	.0075	18.8956	.0001	18.8957
.9	24.7509	.0101	24.7609	.0001	24.7610
1.0	31.9867	.0133	31.9999	.0001	32.0000

For this simple differential equation a comparison of the times required by the two methods is not very significant. However, since in this example Kutta requires 40 substitutions while the second method requires 15 or so (depending on exactly how the start is made), the difference would obviously be significant for more difficult equations.

It is of some interest to make the comparison for the equations

$$\frac{dy}{dx}=\frac{4y}{1+x}$$

and

$$\frac{dy}{dx}=\frac{2y}{1+x},$$

with $y=1$ at $x=0$, and $h=0.1$.

For the first of these Kutta is in error at $x=1$ by 0.00242, and for the second by 0.0000205. In both of these cases the second method gives *exact* values.

The foregoing examples serve to show that even for very innocent looking differential equations the Runge-Kutta method may give very bad results. Since the accuracy is in any case difficult to ascertain, the possibility of such errors occurring is a serious indictment of the Runge-Kutta method.

LOS ANGELES, June 16, 1949.