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The exponential degree distribution in complex networks: Non-equilibrium network theory, numerical simulation and empirical data

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ABSTRACT

The exponential degree distribution has been found in many real world complex networks, based on which, the random growing process has been introduced to analyze the formation principle of such kinds of networks. Inspired from the non-equilibrium network theory, we construct the network according to two mechanisms: growing and adjacent random attachment. By using the Kolmogorov–Smirnov Test (KST), for the same number of nodes and edges, we find the simulation results are remarkably consistent with the predictions of the non-equilibrium network theory, and also surprisingly match the empirical databases, such as the Worldwide Marine Transportation Network (WMTN), the Email Network of University at Rovira i Virgili (ENURV) in Spain and the North American Power Grid Network (NAPGN). Our work may shed light on interpreting the exponential degree distribution and the evolution mechanism of the complex networks.

1. Introduction

Through empirical analysis of many real world complex networks, the scale-free property [1–3] has been detected extensively, and representative examples consist of the world wide web [4], the collaboration network [5], the public transportation networks [6–10] and the graph of human language [11,12], etc. Another subsequent achievement was made by the research team of Barabasi and Albert, namely as the BA model [13], which mainly refers to two mechanisms in the formation process of the network: growing and preferential linking. And it has successfully interpreted such commonly shared characteristics in complex networks [14].

However, other than the scale-free degree distribution [15], the exponential format has also been discovered in many real world networks, such as the Worldwide Marine Transportation Network (WMTN) [16], the Email Network of University at Rovira i Virgili (ENURV) in Spain [17] and the North American Power Grid Network (NAPGN) [18], etc.

This new class of network gave birth to several questions. What has led such kinds of networks to this point? How can we design a better network? To answer these questions, it is foremost to characterize the mechanisms, which are responsible for their evolution [19]. Therefore, to understand the basic principles of the structural organization regarding such kinds of networks will be of vital importance, which is the main focus of this paper.

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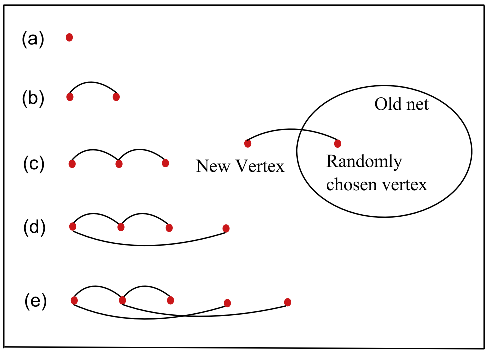

Fig. 1. An example of a non-equilibrium random network. At each time step, a new vertex is added to the graph and randomly connects to some existing vertexes via an edge.

As a rule, these networks are not static, but evolving ones, the state of which is far from equilibrium, and the formation process of which has been influenced by varieties of factors, covering the political, economical, cultural, historical and geographical aspects; one cannot fully understand the evolution of such networks without a holistic perspective [19].

Take the WMTN [16] as an example. The structure of the WMTN is mostly determined by the actions of the marine transportation companies; when they arrange their sailing schedules, they will be concerned about the destination of the cargo, the weather, the ocean currents, the impost, and the politics, etc. Moreover, the structure of the network is also the outcome of numerous historical accidents arising from geographical, political, and economic factors. These many factors, which influence the topological structure, may make them different from the scale-free networks.

In this paper, we firstly introduce the non-equilibrium network theory [20], which indeed generates the exponential degree distributions. Based on the ideology of the theory, we try to construct the network. For the same number of nodes and edges, the degree distributions and the degree distribution exponents of the simulated networks are compared with those of the theory and the empirical data. It is demonstrated that the simulation results conform with the non-equilibrium network theory and the empirical data curiously [16–18].

The whole text is organized as follows: we show the non-equilibrium network theory in Section 2. Section 3 presents the construction of the network according to the mechanisms of the non-equilibrium network theory. In Section 4, we analyze the simulated networks, and also compare them with the theory and the empirical data. Conclusions and discussions are given in the last part, Section 5.

2. The non-equilibrium network theory

For the non-equilibrium network [20], if new edges in a growing network become attached to vertices at random, without any preference, the degree distribution will be exponential, which may be derived as follows:

As is depicted in Fig. 1, at each time step, we add one new vertex and attach it to a randomly selected old vertex. In the definition, the variable $s = 1, 2, 3, \ldots$ marks the vertices and $p(k, s, t)$ represents the probability that a vertex s has degree k at time t. We start the growth of the network from two doubly connected vertices, $s = 1$ and $s = 2$, at time $t = 2$. Thus the master equation, which images the evolution of the probability, can be described as

$$p(k, s, t) = \frac{1}{t} p(k-1, s, t) + \left(1 - \frac{1}{t}\right) p(k, s, t), \tag{1}$$

where the initial and the boundary conditions of the probability are $p(k, s = 1, 2, t = 2) = \delta_{k,2}$ and $p(k, s = t, t > 2) = \delta_{k,1}$, respectively.

The total degree distribution $P(k, t)$ of the entire network follows from the above probability for individual vertices:

$$P(k, t) = \frac{1}{t} \sum_{s=1}^{t} p(k, s, t). \tag{2}$$

Using this definition and applying $\sum_{s=1}^{t}$ to both sides of Eq. (1), we can obtain the following master equation of the total degree distribution:

$$(t+1)P(k, t+1) - tP(k, t) = P(k-1, t) - P(k, t) + \delta_{k,1}. \tag{3}$$

The continuum limit of the above equation is

$$\frac{\partial P(k, t)}{\partial t} + P(k, t) = \frac{\partial [tP(k, t)]}{\partial t} = P(k-1, t) - P(k, t) + \delta_{k,1}. \tag{4}$$

Table 1
The empirical and simulation results of the three networks, where $\langle k_e \rangle$ and $\langle k_s \rangle$ represent the average degree, while $P_e(k)$ and $P_s(k)$ depict the degree distributions.

Network	Size	$\langle k_e \rangle$	$P_e(k)$	$\langle k_s \rangle$	$P_s(k)$
NAPGN	4941	2.67	$e^{-k/2.0}$	2.58	$e^{-k/2.26}$
ENURV	1133	9.62	$e^{-k/9.2}$	9.68	$e^{-k/9.65}$
WMTN	676	7.6	$e^{-k/7.2}$	7.62	$e^{-k/7.5}$

Eq. (4) for the stationary degree distribution $P(k) \equiv P(k, t \to \infty)$ is

$$2P(k) - P(k-1) = \delta_{k,1}. \tag{5}$$

Passing the continuum degree limit in Eq. (5) for $P(k)$, we can get

$$\frac{\mathrm{d}P(k)}{\mathrm{d}k} = -P(k). \tag{6}$$

Even for the infinite networks, all the moments of the degree distribution are finite, $M_m \equiv \sum_{k=0}^{\infty} k^m P(k) < \infty$, therefore the degree distribution has a natural scale of the order of the average degree, and the solution of the above master equation for the degree distribution $P(k)$ is

$$P(k) \propto \mathrm{e}^{-k/\bar{k}}, \tag{7}$$

where $\bar{k}$ represents the average degree of the network.

3. Constructions of the networks: numerical simulation

From the above analysis, we could find the exponential degree distribution might be indeed generated from the non-equilibrium network theory. However, the question is, how can we construct such kinds of networks from numerical simulation?

Considering the formation process of these three real world complex networks, for example, in the Worldwide Marine Transportation Network, the seaports firstly like to establish links with their neighborhood; in the Email Network, people tend to send emails to their acquaintances; while in the Power Grid Network, the distributing substations and transmission substations are apt to build connections with the nearest generating substations.

Inspired from this commonly shared characteristics and the above theory, we carried out the simulation of the networks according to two mechanisms: growing and adjacent random attachment.

The simulation approaches are as follows:

(1) Initialization: we start the construction of the networks with n vertices, $n = 6$, 8 or 10, and several edges have been assigned among them, with the probability $p = 0.6$.

(2) At each time step, a new vertex is added to the network. Firstly, only one edge is randomly distributed from this vertex to the vertices already present in the network, with probability $1/N$, N is the number of the old vertices.

(3) Then, $m - 1$ edges are randomly assigned to the first order neighbors of the firstly chosen vertex in Step (2). If $m - 1$ is greater than the number of first order neighbors of the firstly chosen vertex, the rest of the connections will be randomly established among the second order or even the third order neighbors of the first chosen vertex.

(4) When the number of vertices and edges approximately equals those of the real world complex networks, the construction of the network will be accomplished.

4. Comparisons about the results of non-equilibrium network theory, numerical simulation and empirical data

Based on the above mechanisms, for the same number of vertices and edges of the WMTN, the NAPGN and the ENURV, we carried out the simulations of the networks, and the cumulative degree distributions of the simulated networks are presented in Fig. 2.

The Kolmogorov–Smirnov test [21] has been employed to test the three cumulative degree distributions in the software SPSS, which suggests the P values equal 0.56, 0.53 and 0.55, being larger than 0.05. Therefore we can conclude the cumulative degree distributions of the three networks all follow the exponential law.

By using the maximum likelihood estimation (MLE) [21] in Matlab, the exponential exponents of the cumulative degree distributions are calculated respectively. The average degree of the simulated networks is also calculated accordingly, which have been shown in Table 1.

From Table 1, on the one hand, we can find the average degree and the degree distributions of the simulated and empirical networks [16–18] are so close to each other, which suggests the results of the simulated networks match the empirical data very well.

On the other hand, for both the simulated and the real world networks, the average degree approximately equal the natural scale of the exponential degree distributions. Therefore, we can draw that the empirical and simulated results are remarkably consistent with the predictions of the non-equilibrium network theory.

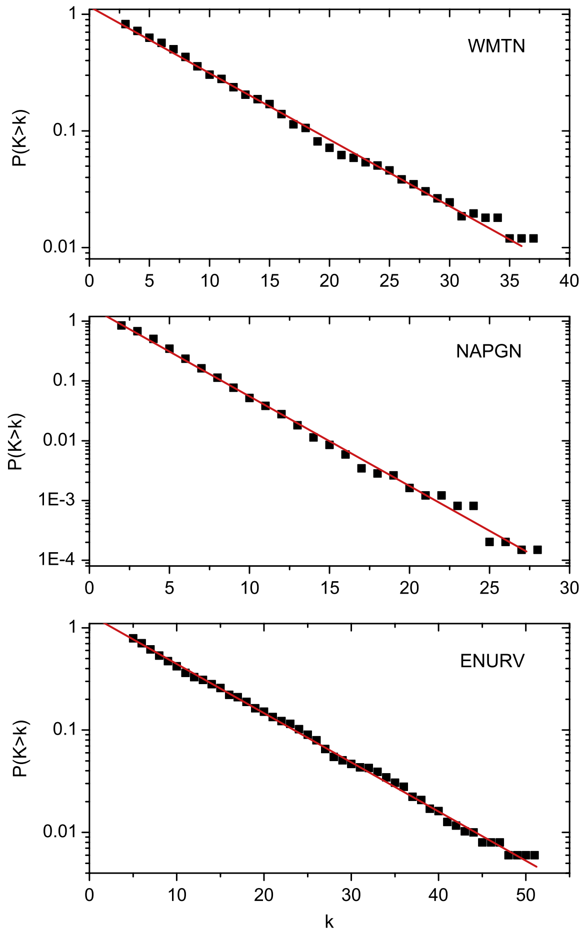

Fig. 2. Simulation results about the degree distributions of the North American Power Grid Network (NAPGN), the Email Network of University at Rovira i Virgili (ENURV) in Spain and the Worldwide Marine Transportation Network (WMTN), which follow the exponential format, $P(k) \propto e^{-k/\bar{k}}$, with $\bar{k} = 2.26$, 9.65 and 7.5.

5. Conclusions and discussions

Based on two mechanisms: growing and adjacent random attachment, we have introduced a new class of complex networks. The simulation results indicate this class of networks show the exponential degree distributions, which are analogous to the results of the empirical data. Moreover, the exponential exponents of the numerical simulation and the empirical data are surprisingly coincident with the predictions of the non-equilibrium network theory.

Our work may shed light on interpreting the exponential degree distribution in complex networks, and provide a way to understand their evolution mechanisms. However, there are still so many questions. Whether there exist some other mechanisms during the evolution process, which are more close to the real networks? How do the parameters referred to in the numerical simulation affect the topology of the network, such as the clustering coefficient, the shortest path length, or the degree correlations, etc? Therefore, future work might pay attention to such problems.

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