NISHARG GOSAI SML Assignment 1

1. Let $\vec{u} = [3,6,1]^T, \vec{v} = [2,0,3]^T, \vec{w} = [1,5,8]^T$. Find,
   a. $3\vec{u} + 4\vec{v}$
   b. $2\vec{u} + 4\vec{v} - 5\vec{w}$

a) $$3\begin{bmatrix}3\\6\\1\end{bmatrix} + 4\begin{bmatrix}2\\0\\3\end{bmatrix} = \begin{bmatrix}9\\18\\3\end{bmatrix} + \begin{bmatrix}8\\0\\12\end{bmatrix} = \begin{bmatrix}17\\18\\15\end{bmatrix}$$

b) $$2\begin{bmatrix}3\\6\\1\end{bmatrix} + 4\begin{bmatrix}2\\0\\3\end{bmatrix} - 5\begin{bmatrix}1\\5\\8\end{bmatrix} = \begin{bmatrix}6\\12\\2\end{bmatrix} + \begin{bmatrix}8\\0\\12\end{bmatrix} - \begin{bmatrix}5\\25\\40\end{bmatrix}$$

$$= \begin{bmatrix}9\\-13\\-26\end{bmatrix}$$

$$\boxed{\begin{array}{l} a)\ [17,18,15]^T \\ b)\ [9,-13,-26]^T \end{array}}$$

2. Find the coefficients c_1, c_2, c_3 such that $\sum_{i=1}^{3} c_i u_i = [0,3,5]^T$. Here $u_1 = [-1,1,-1]^T, u_2 = [2,1,2]^T, u_3 = [1,0,2]^T$

$$\begin{bmatrix} -c_1 \\ c_1 \\ -c_1 \end{bmatrix} + \begin{bmatrix} 2c_2 \\ c_2 \\ 2c_2 \end{bmatrix} + \begin{bmatrix} c_3 \\ 0 \\ 2c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$-c_1 + 2c_2 + c_3 = 0 \quad - ①$

$c_1 + c_2 = 3 \quad - ②$

$-c_1 + 2c_2 + 2c_3 = 5 \quad - ③$

From ① & ③

$-c_3 + 2c_3 = 5$

$\boxed{c_3 = 5}$

From ① & ②

$-c_1 + 2(3 - c_1) + c_3 = 0$

$-c_1 + 6 - 2c_1 + 5 = 0$

$-3c_1 = -11$

$\boxed{c_1 = \frac{11}{3}}$

Putting c_1 in ②

$\frac{11}{3} + c_2 = 3$

$c_2 = 3 - \frac{11}{3}$

$\boxed{c_2 = \frac{-2}{3}}$

$\boxed{c_1 = \frac{11}{3}, \; c_2 = -\frac{2}{3}, \; c_3 = 5}$

3. Let $\vec{u} = [2,3,1]^T, \vec{v} = [4,-3,1]^T, \vec{w} = [1,2,3]^T$

   a. Which pair of distinct vectors are most colinear?

   b. Which pair is most orthogonal?

   c. Are $\vec{u}, \vec{v}$ and $\vec{w}$ linearly independent?

$u = [2,3,1]^T$
$v = [4,-3,1]^T$
$w = [1,2,3]^T$

$\vec{u}.\vec{v} = 2.4 + 3.(-3) + 1.1 = 0$
$\vec{u}.\vec{w} = 2.1 + 3.2 + 1.3 = 11$
$\vec{v}.\vec{w} = 4.1 + (-3).2 + 1.3 = 4 - 6 + 3 = 1$

$\|u\| = \sqrt{2^2+3^2+1^2} = \sqrt{4+9+1} = \sqrt{14}$
$\|v\| = \sqrt{4^2+(-3)^2+1^2} = \sqrt{16+9+1} = \sqrt{26}$
$\|w\| = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

Comparing cosine angles

$\cos(\theta_{uv}) = \frac{0}{\sqrt{14}.\sqrt{26}} = 0$ b) (Since 0, $\vec{u}$ & $\vec{v}$ are orthogonal)

$\cos(\theta_{uw}) = \frac{11}{\sqrt{14}.\sqrt{14}} = \frac{11}{14}$

$\cos(\theta_{vw}) = \frac{1}{\sqrt{26}\sqrt{14}}$ a) ($\vec{v}$ & $\vec{w}$ are most collinear)

Linear independence

$$\text{Matrix} = \begin{bmatrix} 2 & 4 & 1 \\ 3 & -3 & 2 \\ 1 & 1 & 3 \end{bmatrix}$$

Determinant of matrix $\neq 0$, then linearly independent

$= 2(-3)(3) + 4(2)(1) + 1(3)(1) - 1(-3)(1) - 2(2)(2) - 3(4)(3)$
$= -18 + 8 + 3 - (-3) - 8 - 36$
$= \underline{\underline{-44}}$

$\therefore$ vectors are linearly independent (c)

4. Let $\vec{u} = [0,4,3]^T$ and $\vec{v} = [2,4,4]^T$. Find
   a. the projection of $\vec{u}$ onto $\vec{v}$
   b. the L_2 distance between $\vec{u}$ and $\vec{v}$
   c. the L_1 distance between $\vec{u}$ and $\vec{v}$

a) Projection of $\vec{u}$ onto $\vec{v}$ $= \dfrac{(\vec{u}\cdot\vec{v})}{\|v\|^2}\vec{v}$

$$= \frac{0(2)+4(4)+3(4)}{\left(\sqrt{2^2+4^2+4^2}\right)^2}[2,4,4]$$

$$= \frac{16+12}{36}[2,4,4]$$

$$= \frac{28}{36}[2,4,4]$$

$$= \frac{7}{9}[2,4,4]$$

$$= \left[\frac{14}{9}, \frac{28}{9}, \frac{28}{9}\right]^T$$

b) L_2 distance $= \sqrt{(2-0)^2+(4-4)^2+(4-3)^2}$

$$= \sqrt{4+1}$$

$$= \sqrt{5}$$

c) L_1 distance $= (2-0)+(4-4)+(4-3)$

$$= 2+1$$

$$= 3$$

5. Write the vector $\vec{z} = [1,0,5]^T$ as a linear combination of $\vec{p_1} = [1,0,1]^T, \vec{p_2} = [1,1,3]^T$, $\vec{p_3} = [2,0,-1]^T$

$$c_1\vec{p_1} + c_2\vec{p_2} + c_3\vec{p_3} = \vec{z}$$

$$c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\3\end{bmatrix} + c_3\begin{bmatrix}2\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

$$\begin{bmatrix}c_1\\0\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\c_2\\3c_2\end{bmatrix} + \begin{bmatrix}2c_3\\0\\-c_3\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

$c_1 + c_2 + 2c_3 = 1$ — ①

$\boxed{c_2 = 0}$ — ②

$c_1 + 3c_2 - c_3 = 5$ — ③

$\therefore\ c_1 + 2c_3 = 1$

$c_1 - c_3 = 5$

$c_1 = 5 + c_3$

$5 + c_3 + 2c_3 = 1$

$5 + 3c_3 = 1$

$\boxed{c_3 = -\frac{4}{3}}$

$c_1 - \left(-\frac{4}{3}\right) = 5$ (From ③)

$c_1 + \frac{4}{3} = 5$

$c_1 = 5 - \frac{4}{3}$

$\boxed{c_1 = \frac{11}{3}}$

$$\boxed{\therefore\ \vec{z} = \frac{11}{3}\vec{p_1} + 0\vec{p_2} + \left(\frac{-4}{3}\right)\vec{p_3}}$$

6. Find all t, k so that $\vec{u} = [t, k]^T$ and $\vec{v} = [2,3]^T$ are orthogonal and $\vec{u}$ has unit length.

1) $\vec{u}$ ~~&~~ $\vec{v}$ are orthagonal then $\vec{u}.\vec{v} = 0$

2) $\vec{u}$ has unit length then $||u|| = 1$

Eq^n: $2t + 3k = 0$ — (1)

$t^2 + k^2 = 1$ — (2)

From (1)

$$t = \frac{-3k}{2}$$

Putting above in (2)

$$\left(\frac{-3}{2}k\right)^2 + k^2 = 1$$

$$\frac{9k^2}{4} + k^2 = 1$$

$$\frac{9k^2 + 4k^2}{4} = 1$$

$$9k^2 + 4k^2 = 4$$

$$13k^2 = 4$$

$$\boxed{k = \sqrt{\frac{4}{13}}}$$

Substitute k in (2)

$$t^2 + \left(\sqrt{\frac{4}{13}}\right)^2 = 1$$

$$t^2 + \frac{4}{13} = 1$$

$$t^2 = 1 - \frac{4}{13}$$

$$t = \sqrt{\frac{9}{13}}$$

$$\boxed{t = -\frac{3}{\sqrt{13}}}$$

7. Let $\mathcal{L}$ be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$, with $\mathcal{L}([0,1]^T) = [-1,0]^T$ and $\mathcal{L}([1,0]^T) = [3,5]^T$
   d. Find a matrix representation for $\mathcal{L}$ in the standard ordered basis
   e. Find $\mathcal{L}([2,2]^T)$ and $\mathcal{L}([1,-1]^T)$
   f. Prove or Disprove: $[2,3]^T$ is an eigenvector of $\mathcal{L}$

d) Matrix representation in standard ordered basis

$$A = \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix}$$

e) $$L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 10 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

f) For $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ to be eigenvector of L, λ exists such that

$$L\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = A \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

$$L\left(\begin{bmatrix} 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 10 \end{bmatrix} \neq \lambda \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$\therefore$ Not eigenvector

8. Consider the linear operator from $\mathbb{R}^2$ to $\mathbb{R}^2$ defined by matrix $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $V = \{[1,1]^T, [1,-1]^T\}$, both in the standard ordered basis

g. Show that V is a basis for $\mathbb{R}^2$

h. Find matrix K to express the linear operator in the basis V

g) To show V is a basis for R^2, we show that

i) vectors are linearly independent

ii) span R^2

$$\begin{bmatrix} c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} c_1 + c_2 = 0 \\ c_1 - c_2 = 0 \end{matrix}$$

$\therefore$ $c_1 = c_2 = 0$

$\therefore$ Vectors are linearly independent vectors in R^2

$\therefore$ They forms a basis for R^2

h) The matrix K can be found using

$$K = V^{-1} B V$$

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} V_{22} & -V_{12} \\ -V_{21} & V_{11} \end{bmatrix}$$

$$K = V^{-1} B V$$

$$= \frac{1}{-2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\boxed{K = \begin{bmatrix} 5 & -1 \\ -2 & 0 \end{bmatrix}}$$

9. Let X denote a Gaussian column vector with mean vector $m_x = [2,3]^T$, and covariance matrix $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then random vector $Y = AX$, where $A = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$
   i. Determine $E[Y]$
   j. Find and simplify the covariance matrix of Y
   k. Completely specify the probability density function of vector Y

i) $E[Y] = A.m_x = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ 13 \end{bmatrix}$

j) $Cov(Y) = A. Cov(X). A^T$

$= A . I . A^T$

$= \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & -8 \\ -8 & 13 \end{bmatrix}$

k) $f_y(y) = \frac{1}{2\pi \sqrt{\|C_y\|}} \exp\left(\frac{-1}{2} (y - m_y)^T C_y^{-1} (y - m_y)\right)$

$= \frac{1}{2\pi \sqrt{5(13)-(-8)(-8)}} \exp\left(\frac{-1}{2} \left(y - \begin{bmatrix} -8 \\ 13 \end{bmatrix}\right)^T \begin{bmatrix} 5 & -8 \\ -8 & 13 \end{bmatrix}^{-1} \left(y - \begin{bmatrix} -8 \\ 13 \end{bmatrix}\right)\right)$

$= \frac{1}{2\pi \sqrt{65}} \exp\left(\frac{-1}{2} \left(\frac{1}{65}\right) \begin{bmatrix} y_1 + 8 \\ y_2 - 13 \end{bmatrix}^T \begin{bmatrix} 13 & 8 \\ 8 & 5 \end{bmatrix} \begin{bmatrix} y_1 + 8 \\ y_2 - 13 \end{bmatrix}\right)$

10. Let X and Y be random variables with joint pdf

$$f_{xy}(x,y) = \begin{cases} \frac{K}{16}, x \in (-4,4) \ \ and \ \ y \in (2,4) \\ 0, \qquad\qquad other \end{cases}$$

l. Find constant K
m. Prove or Disprove: X and Y are orthogonal
n. Prove or Disprove: X and Y are independent
o. Find $P[Y \leq 3 | X \geq 0]$

l) For any PDF, the total prob. must be 1

$$\Rightarrow \int_{-4}^{4}\int_{2}^{4} \frac{K}{16}\, dx dy = 1$$

$$\frac{K}{16}\int_{-4}^{4} (4-2) dy = 1$$

$$\frac{K}{16}\left[2y\right]_{-4}^{4} = 1$$

$$\frac{K}{16}(8-(-8)) = 1$$

$$\underline{\underline{K=1}}$$

m) For X & Y to be orthogonal, $Cov(x,y)=0$

$$Cov(x,y) = E[XY] - E[X]E[Y] \quad - \text{①}$$

$$E[XY] = \frac{K}{16}\int_{-4}^{4}\int_{2}^{4} xy\, dy dx$$

$$= \frac{1}{16}\int_{-4}^{4} [16x - 4x]\, dx$$

$$= \frac{1}{16}\left[\frac{1}{12}\cdot 12x^2\right]_{-4}^{4}$$

$$= \frac{1}{16}\left[96-96\right] = \underline{\underline{0}}$$

$$E[X] = \frac{K}{16}\int_{-4}^{4}\int_{2}^{4} x\, dy dx$$

$$= \frac{1}{16}\int_{-4}^{4} (4x-2x)\, dx$$

$$= \frac{1}{16}[x^2]_{-4}^{4} = \underline{\underline{0}}$$

$$E[Y] = \frac{K}{16}\int_{-4}^{4}\int_{2}^{4} y\, dy dx$$

$$= \frac{1}{16}\int_{-4}^{4} 6\, dx$$

$$= \frac{1}{2}$$

$$\therefore Cov(x,y) = 0 - 0\left(\frac{1}{2}\right) = 0$$

$\therefore$ X & Y are orthogonal

n) X & Y independent if $f_{xy}(xy) = f_x(x) \cdot f_y(y)$

$$f_{xy}(x,y) = \frac{1}{16}$$

$$f_x(x) = \int_2^4 \frac{1}{16}\, dy = \frac{1}{8}$$

$$f_y(y) = \int_{-4}^4 \frac{1}{16}\, dx = \frac{1}{2}$$

$$f_{xy}(x,y) = f_x(x) \cdot f_y(y) = \frac{1}{16}$$

$\therefore$ X & Y are independent

o) $P[Y \le 3 \mid X \geq 0] = \dfrac{P(X \geq 0, Y \le 3)}{P(X \geq 0)}$

Finding $P[X \geq 0 \mid Y \le 3] = \int_0^4 \int_2^3 \frac{1}{16}\, dy\, dx = \frac{1}{16} \int_0^4 dx = \frac{1}{4}$

Finding $P[X \geq 0] = \int_0^4 f_x(x)\, dx = \frac{1}{8} \int_0^4 dx = \frac{1}{2}$

$$P[Y \le 3 \mid X \geq 0] = \frac{1/4}{1/2} = \underline{\underline{\frac{1}{2}}}$$

11. Coin C_1 is selected with probability $\frac{1}{3}$; otherwise, coin C_2 is chosen for a single flip. C_1 produces heads with probability $\frac{1}{7}$ and C_2 produces heads with probability $\frac{7}{8}$

p. A head is produced. Find the probability that C_2 was selected.

q. Prove or Disprove: the probability of heads is $\frac{1}{2}$

p.) C_2 selected given heads

Using Bayes,

$$P(C_2 | Head) = \frac{P(Head | C_2) . P(C_2)}{P(Head)}$$

- $P(Head | C_2) = \frac{7}{8}$
- $P(C_2) = \frac{2}{3}$
- $P(Head) = P(Head | C_1) . P(C_1) + P(Head | C_2) . P(C_2)$

$$= \frac{1}{7} \cdot \frac{1}{3} + \frac{7}{8} \cdot \frac{2}{3}$$

$$= \frac{1}{21} + \frac{14}{24} = \frac{8}{168} + \frac{98}{168} = \frac{106}{168} = \frac{53}{84}$$

$$P(C_2 | Head) = \frac{\frac{7}{8} \cdot \frac{2}{3}}{\frac{53}{84}} = \frac{14}{24} \cdot \frac{84}{53} = \frac{1176}{1272} = \boxed{0.924}$$

q) $P(Head) = \frac{53}{84} \simeq 0.631 \neq \frac{1}{2}$

$\therefore P(Heads) = \frac{1}{2}$ is disproven