Qiskit Fall Fest at KU workshop2

목차

1. 행렬 (matrix)

1) 정의: 수 또는 변수 등의 일련의 개체들을 행(row)과 열(column)에 맞추어 직사각형 모양으로 순서있게 배열하여 괄호로 묶은 것.

ex)

$$\begin{array}{cc} & \text{열, column} \\ \text{행, row} & \begin{bmatrix} 1 & 9 & -13 \\ 20 & -6 & 5 \end{bmatrix} \end{array} \Rightarrow 2\times 3 \text{ 행렬}$$

$$A = \begin{bmatrix} 1 & 9 & -13 \\ 20 & -6 & 5 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

$m \times n$ 행렬

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad \begin{array}{l} a_{ij} \in \mathbb{R} \\ a_{i,j} \in \mathbb{C} \end{array}$$

2) 행렬 곱셈

$(m \times n) \cdot (n \times k) \Rightarrow m \times k$

$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ $B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nk} \end{bmatrix}$

$C = AB$

$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ \vdots & \vdots & & \vdots \\ c_{m1} & c_{m2} & \cdots & c_{mk} \end{bmatrix}$ 이 때, $i = 1, \ldots, m$, $j = 1, \ldots, k$ 에 대하여

$c_{ij} = \sum_{t=1}^{n} a_{it} b_{tj}$

ex1) $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

ex2) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$

ex3) $\begin{bmatrix} 1 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 11 \end{bmatrix} = 40$

ex4) $\begin{bmatrix} 7 \\ 11 \end{bmatrix} \begin{bmatrix} 1 & 3 \end{bmatrix} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

3) 전치 행렬 (Transposed matrix)

Transposed matrix : 행과 열을 교환한 행렬

ex1) $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

ex2) $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $B^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$

ex3) $C = \begin{bmatrix} 1 & 0 & 2 \\ 7 & 3 & 8 \end{bmatrix}$ $C^T = \begin{bmatrix} 1 & 7 \\ 0 & 3 \\ 2 & 8 \end{bmatrix}$

4) Conjugate transpose (a.k.a Hermitian transpose)

└ complex conjugate + transpose

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ $A^\dagger = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 1+i & 3 \\ -1 & 6-7i \end{bmatrix}$ $B^\dagger = \begin{bmatrix} 1-i & -1 \\ 3 & 6+7i \end{bmatrix}$

$A^\dagger = A^H = (A^*)^T = (A^T)^*$ $(A^\dagger)_{ij} = \overline{A_{ji}}$

5) 항등행렬 (Identity matrix)

$$\mathbb{1}_n = \underbrace{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}_{n} \Big\} n$$

$m \times n$ matrix A : $\mathbb{1}_m A = A \mathbb{1}_n = A$

ex) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

6) 역행렬 (Inverse matrix)

square matrix $n \times n$ 행렬 A에 대해 $AB = BA = \mathbb{1}_n$ 을 만족하는 행렬이 있다면, 행렬 B는 행렬 A의 역행렬 이라고 한다.

역행렬 표기: A^{-1} $\Rightarrow$ $A^{-1}A = AA^{-1} = \mathbb{1}$

2x2 행렬에서의 역행렬.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad-bc \neq 0 \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* 판별식(determinant)

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad-bc \begin{cases} \text{if } 0 & \cancel{A^{-1}} \\ \text{else} & A^{-1} \end{cases}$

2. 벡터(vector)

1) 정의: 크기와 방향을 갖는 값

2) 표기: $\vec{a}$ $\mathbf{a}$ $|a\rangle$ $\vec{a} = (a_1, a_2)$

3) 열벡터, 행벡터

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$ $\begin{bmatrix} 9 & 8 \end{bmatrix}, \begin{bmatrix} 7 & 13 & 17 & 5 \end{bmatrix}$

4) Dirac notation (bra-ket 표기)

$|\psi\rangle$: ket vector (column)

$\langle\psi|$: bra vector (row)

$|\psi\rangle = \begin{bmatrix} 1 \\ 2+3i \end{bmatrix}$ $\langle\psi| = (|\psi\rangle)^{\dagger} = \begin{bmatrix} 1 & 2-3i \end{bmatrix}$

$|\phi\rangle = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ $\langle\phi| = \begin{bmatrix} 3 & 5 \end{bmatrix}$

5) 내적

i) 실수

$$|a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad |b\rangle = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Longrightarrow a_1b_1 + a_2b_2 + a_3b_3$$

$$= \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \langle a| \, |b\rangle$$

$$= \langle a|b\rangle$$

ii) 복소수

$$|a\rangle = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad |b\rangle = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$\langle a|b\rangle = \begin{bmatrix} a_1^* & a_2^* & a_3^* \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1^*b_1 + a_2^*b_2 + a_3^*b_3$$

$$\langle b|a\rangle = \begin{bmatrix} b_1^* & b_2^* & b_3^* \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = a_1b_1^* + a_2b_2^* + a_3b_3^* = \langle a|b\rangle^*$$

6) 크기

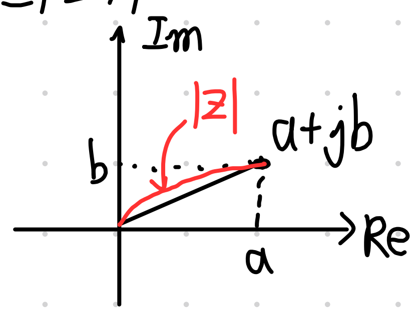
$$\left| |\psi\rangle \right| = \sqrt{\langle\psi|\psi\rangle}$$

cf) 복소수 $z = a + jb$ (단, $a, b \in \mathbb{R}$ and $j = \sqrt{-1}$)

$$z^* = \bar{z} = a - jb$$

$$z^* z = z z^* = a^2 + b^2$$

복소수의 크기



$$|z| = \sqrt{a^2 + b^2} = \sqrt{z^* z}$$

$$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix} \quad a, b \in \mathbb{C}$$

$$\langle\psi|\psi\rangle = \begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = a^* a + b^* b$$

$$= |a|^2 + |b|^2$$

$$\left| |\psi\rangle \right| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{|a|^2 + |b|^2}$$

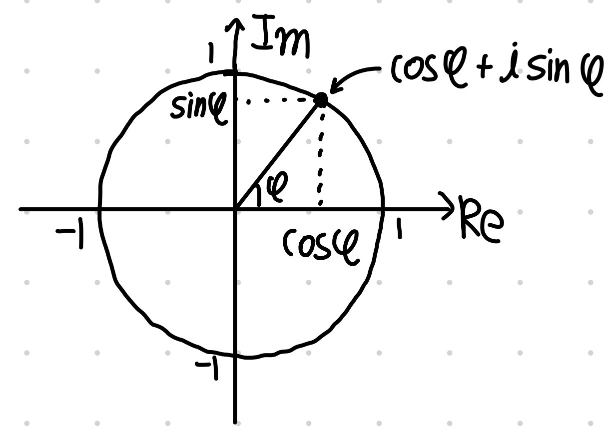
3. Single qubit & Bloch sphere

1)

Classical computer	Quantum computer
bit/register	qubit
0 or 1	superposition : $\|\psi\rangle = \cos\frac{\theta}{2}\|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\|1\rangle$

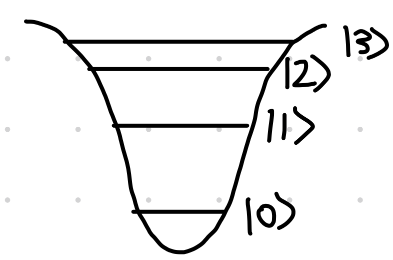
cf)

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$
$$e^{i0} = \cos 0 + i\sin 0 = 1$$
$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$$
$$e^{i\pi} = \cos\pi + i\sin\pi = -1$$
$$e^{i\frac{3}{2}\pi} = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i$$



2) Qubit, Qutrit, Qudit

Qubit : 2-level system, basis : $\{|0\rangle, |1\rangle\}$
Qutrit : 3-level system, basis : $\{|0\rangle, |1\rangle, |2\rangle\}$
Qudit : d-level system, basis : $\{|0\rangle, |1\rangle, \cdots, |d-1\rangle\}$



3) Famous 6 single qubit state

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad e^{i0} = 1$$

$$|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \qquad e^{i\pi} = -1$$

$$|+i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \qquad e^{i\frac{\pi}{2}} = i$$

$$|-i\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) \qquad e^{i\frac{3\pi}{2}} = -i$$

4) Inner product of 6 states.

$\langle 0|0\rangle$

$\langle 1|1\rangle$

$\langle +|+\rangle$

$\langle -|-\rangle$

$\langle +i|+i\rangle$

$\langle -i|-i\rangle$

$\langle 0|1\rangle$

$\langle +|-\rangle$

$\langle +i|-i\rangle$

$\Downarrow$

$\langle \psi|\psi\rangle = 1$: The sum of all probabilities is 1.

$|\psi\rangle = \begin{bmatrix} a \\ b \end{bmatrix}$

단, $|a|^2+|b|^2=1$

$\Downarrow$

$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}|1\rangle$

$\begin{cases} \theta \in [0, \pi] \\ \varphi \in [0, 2\pi) \end{cases}$

$\Downarrow$

Bloch sphere

5) Bloch sphere

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

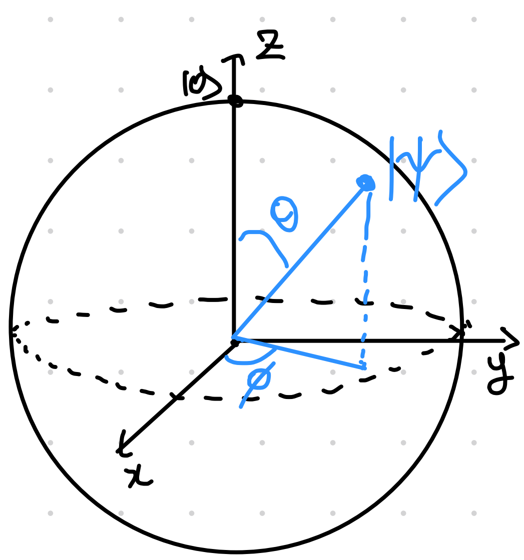
phase

$e^{i\phi} = \cos\phi + i\sin\phi \implies$ ① $\phi=0 : e^{i0} = 1$

② $\phi=\frac{\pi}{2} : e^{i\frac{\pi}{2}} = i$

③ $\phi=\pi : e^{i\pi} = -1$

④ $\phi=\frac{3}{2}\pi : e^{i\frac{3}{2}\pi} = -i$



Bloch Sphere

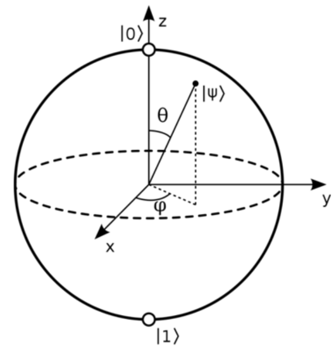
$$|\psi>= \cos\frac{\theta}{2}|0> + e^{i\phi}\sin\frac{\theta}{2}|1>$$

$$|+>= \frac{1}{\sqrt{2}}|0> + \frac{1}{\sqrt{2}}|1>$$

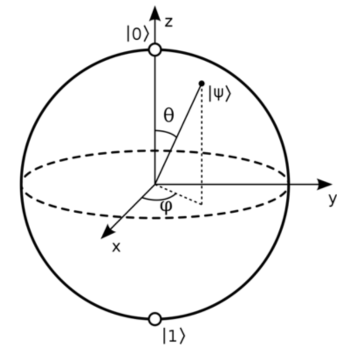
$$|->= \frac{1}{\sqrt{2}}|0> - \frac{1}{\sqrt{2}}|1>$$

$$|+i>= \frac{1}{\sqrt{2}}|0> + \frac{i}{\sqrt{2}}|1>$$

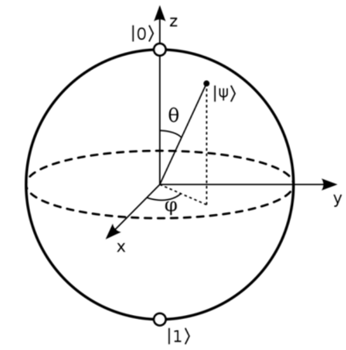
$$|-i>= \frac{1}{\sqrt{2}}|0> - \frac{i}{\sqrt{2}}|1>$$



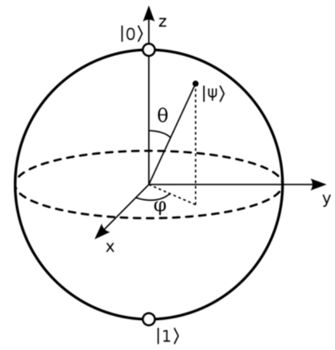
$$|0>= \begin{bmatrix}1\\0\end{bmatrix}$$



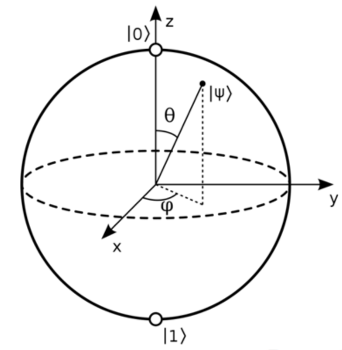
$$|+>= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$



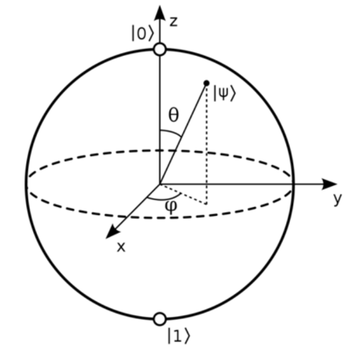
$$|+i>= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$$



$$|1>= \begin{bmatrix}0\\1\end{bmatrix}$$



$$|->= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$



$$|-i>= \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$$

6) Single qubit gates

① Pauli X

$-\boxed{X}-$ $\quad \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

$X|0\rangle =$

$X|1\rangle =$

② Pauli Y

$-\boxed{Y}-$ $\quad \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$

③ Pauli Z

$-\boxed{Z}-$ $\quad \begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$

$Z|+\rangle =$

$Z|-\rangle =$

④ Hadamard gate

$-\boxed{H}-$ $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H|0\rangle =$ $\quad H|+\rangle =$

$H|1\rangle =$ $\quad H|-\rangle =$

⇒ make superposition!

$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

X basis ⟷ Z basis

⑤ S gate

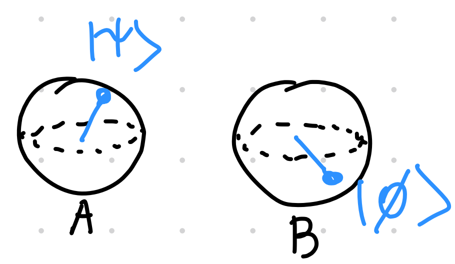
$-\boxed{S}-$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$

⑥ T gate

$-\boxed{T}-$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{bmatrix}$

4. Multiple qubits

1) Tensor product



① $|\psi_A\rangle$ and $|\phi_B\rangle$ ⇒ $|\psi_A\rangle \otimes |\phi_B\rangle$

ex) $|0\rangle_A \otimes |0\rangle_B$

② vector representation

$$|0\rangle_A \otimes |0\rangle_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ 0\begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 1 \times 1 \\ 1 \times 0 \\ 0 \times 1 \\ 0 \times 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} \otimes \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} c \\ d \end{bmatrix} \\ b\begin{bmatrix} c \\ d \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ac \\ ad \\ bc \\ bd \end{bmatrix}, \quad |1\rangle \otimes |0\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

③ simplification

$$|0\rangle_A \otimes |0\rangle_B = |0\rangle_A |0\rangle_B = |00\rangle_{AB} = |00\rangle$$

④ tensor product of operator

$$X_A \otimes I_B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 0\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$(X_A \otimes I_B)\, |0\rangle_A \otimes |0\rangle_B = X_A|0\rangle \otimes I_B|0\rangle$$

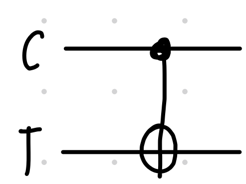
$$= |1\rangle_A \otimes |0\rangle_B$$

$$\Downarrow$$

$$\begin{bmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & 1 & & \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

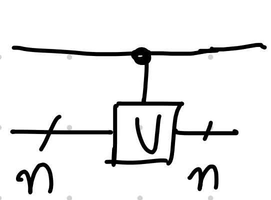
2) C-NOT gate

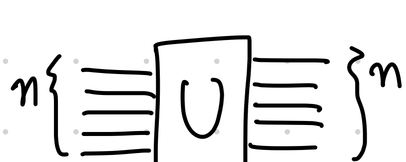
$$CNOT = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & & 1 \\ & & 1 & \end{bmatrix}$$



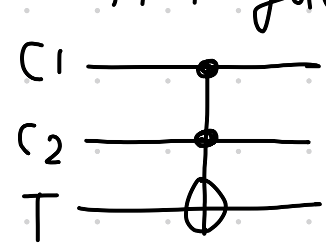
Before		After	
Control	Target	Control	Target
$|0\rangle$	$|0\rangle$	$|0\rangle$	$|0\rangle$
$|0\rangle$	$|1\rangle$	$|0\rangle$	$|1\rangle$
$|1\rangle$	$|0\rangle$	$|1\rangle$	$|1\rangle$
$|1\rangle$	$|1\rangle$	$|1\rangle$	$|0\rangle$

cf) controlled-U gate





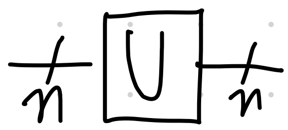
cf) Toffoli gate

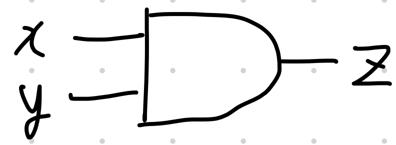


- Gates are Unitary operator!

  $U^{\dagger} = U^{-1}$: unitary operator is invertible

  $\Rightarrow$ input n qubits $\Rightarrow$ output also n qubits





- Entanglement $\longleftrightarrow$ Separable state

$$|\Phi^{+}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B\right)$$