Asset Modelling

MATH450 - Numerical Integration for Stochastic Differential Equations

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Introduction

In 1900, Louis Bachelier's doctoral thesis, "Theory of Speculation," introduced a model for option pricing using a normal distribution, laying the foundation for quantitative finance. The field gained significant traction in the 1970s with the publication of the Black-Scholes-Merton model. Today, quantitative finance and financial engineering are integral to financial institutions. This paper explores the modeling and analysis of financial assets—such as stocks, options, and bonds—using stochastic differential equations (SDEs) and investigates numerical schemes like the Euler-Maruyama method, the stochastic-θ method, and Euler-type methods based on Lamperti transformations for solving SDEs.

Stochastic setup, framework and theory

Before we can start modelling and solving SDEs, we need to make some assumptions and setups. Therefore, consider the following stochastic basis given by $(\Omega, \mathcal{A}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ and denote the assets price at time t by $S(t)$, the asset's expected rate of return by $I(t)$ and the asset's volatility by $\sigma(t)$. Furthermore, assume that $S(t)$ follows a one-dimensional Geometric Brownian motion (GBM)

$$\mathrm{d}S(t) = I(t)S(t)\,\mathrm{d}t + \sigma(t)S(t)\,\mathrm{d}W(t), \tag{1}$$

where W is a one-dimensional Brownian motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Moreover, we will, at times, assume that I and σ are stochastic and follow their own SDEs described by one of the following equations:

$$\mathrm{d}I(t) = (a - bI(t))\,\mathrm{d}t + c\,\mathrm{d}W_t^{(2)}, \tag{2}$$

$$\mathrm{d}I(t) = (a - bI(t))\,\mathrm{d}t + c\sqrt{I(t)}\,\mathrm{d}W_t^{(2)}, \tag{3}$$

$$\mathrm{d}I(t) = (a - bI(t))\,\mathrm{d}t + c\sqrt{(I(t)\vee 0)}\mathrm{d}W_t^{(2)}, \tag{4}$$

and

$$\mathrm{d}\tilde{\sigma}(t) = -\lambda\tilde{\sigma}(t)\,\mathrm{d}t + f\,\mathrm{d}W_t^{(3)}, \tag{5}$$

where a, b, c, λ and f are non-negative, real constants and $W_t^{(2)}, W_t^{(3)}$ are $(\mathcal{F}_t)$-adapted, one-dimensional Brownian motions, independent from each other and from W described in (1). In addition to the stochastic setup and framework, we back our findings by the following lemmas and theorems covered during the lectures:

Theorem 1 (Law of the Iterated Logarithms [3])**.** *Assume that $(\Omega, \mathcal{A}, \mathbb{P}; (\mathcal{F}_t)_{t\geq 0})$ is a stochastic basis and that W is a one-dimensional Brownian motion adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then, for almost every $\omega \in \Omega$, we have*

$$\lim_{t\to\infty} \inf \frac{W_t(\omega)}{\sqrt{2t\log\log t}} = -1 \quad \textit{and} \quad \lim_{t\to\infty} \sup \frac{W_t(\omega)}{\sqrt{2t\log\log t}} = 1. \tag{6}$$

Lemma 1 (Itô's chain rule [2])**.** *Suppose that $X(\cdot)$ has a stochastic differential*

$$dX = F\,dt + G\,dW,$$

for $F \in \mathbb{L}^1(0,T), G \in \mathbb{L}^2(0,T$. Assume $u : \mathbb{R}\times[0,T] \to \mathbb{R}$, $u = u(x,t)$ is continuous and that its partial derivatives $u_t = \frac{\partial u}{\partial t}$, $u_x = \frac{\partial u}{\partial x}$ and $u_{xx} = \frac{\partial^2 u}{\partial x^2}$ exist and are continuous. Then, $Y(t) := u(X(t),t)$ has the stochastic differential

$$du(X,t) = (u_t + u_x F + \frac{1}{2}u_{xx}G^2)\,dt + u_x G\,dW. \tag{7}$$

Lemma 2 (Itô isometry [2])**.** *We have for all $G, H \in \mathbb{L}^2(0,T)$*

$$\mathbb{E}\left[\int_0^T G\,dW\right] = 0 \quad \textit{and} \quad \mathbb{E}\left[\left(\int_0^T G\,dW\right)^2\right] = \mathbb{E}\left[\int_0^T G^2\,dt\right]. \tag{8}$$

Definition 1 (Asymptotic and mean-square stability)**.** *Suppose $X(t)$ is a solution to the stochastic differential equation*

$$d\,X(t) = F(X(t),t)\,dt + G(X(t),t)\,dW(t).$$

We say that the solution $X(t)$ is exponentially asymptotically stable if

$$\lim_{t\to\infty} \sup \frac{1}{t}\log|X(t)| < 0 \quad a.s \tag{9}$$

and that $X(t)$ is mean-square stable if

$$\lim_{t\to\infty} \mathbb{E}\left[|X(t)|^2\right] = 0. \tag{10}$$

The simple geometric Brownian motion and its properties (Q1)

We start by addressing the problem described in **Q1**. To solve the SDE in (1), we assume that $I(t) \equiv I$ and $\sigma(t) \equiv \sigma$ for some positive, real-valued constants I and σ. Furthermore, assume that $S(0) = S_0$ is known and that $W(0) = 0$ by definition. Let $u(S(t)) = \log S(t)$ such that $\partial_S u = \frac{1}{S(t)}$, $\partial_{SS} u = -\frac{1}{S(t)^2}$ and $\partial_t u = 0$, and denote by $\mu_t = IS(t)$ and $\sigma_t = \sigma S(t)$. Then, by Itô's chain rule,

$$\begin{aligned} \mathrm{d}u &= \left(\partial_t u + \mu_t \partial_S u + \frac{\sigma_t^2}{2}\partial_{SS} u\right)\mathrm{d}t + \sigma_t \partial_S u \,\mathrm{d}W(t) = \left(IS(t)\frac{1}{S(t)} - \frac{\sigma^2 S(t)^2}{2}\frac{1}{S(t)^2}\right)\mathrm{d}t + \sigma S(t)\frac{1}{S(t)}\,\mathrm{d}W(t) \\ &= \left(I - \frac{\sigma^2}{2}\right)\mathrm{d}t + \sigma\,\mathrm{d}W(t) = \mathrm{d}\left(\log S(t)\right). \end{aligned}$$

Integrating on both sides, the solution to the SDE, and thus to problem **Q1-1**, becomes

$$S(t) = S_0 e^{\left(I - \frac{\sigma^2}{2}\right)t + \sigma W(t)}. \tag{11}$$

The solution for $S(t)$ shows some nice properties. For instance, it can be shown that, for a specific value of I, $S(t)$ is a martingale. To do so, we first show that $G(t) = e^{\sigma W(t)}$ is a sub-martingale, or in other words, $\mathbb{E}\left[G(t) \mid \mathcal{F}_s\right] \geq G(s)$ for some positive $s < t$. Let $0 < s < t$ and $\mathcal{F}_s$ be the natural filtration at time s. Then,

$$\mathbb{E}\left[G(t) \mid \mathcal{F}_s\right] = \mathbb{E}\left[e^{\sigma W(t)} \mid \mathcal{F}_s\right] = e^{\sigma W(s)}\mathbb{E}\left[e^{\sigma(W(t)-W(s))} \mid \mathcal{F}_s\right] \tag{12}$$

Denote by $X := W(t) - W(s)$. Its expectation and variance are $\mathbb{E}[X]$ and $\mathrm{Var}[X] = t - s$ so that $X \sim \mathcal{N}(0, t-s)$. Using the law of the unconscious statistician, we have

$$\mathbb{E}\left[e^X \mid \mathcal{F}_s\right] = \int_{-\infty}^{\infty} e^{\sigma x} \cdot \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{x^2}{2(t-s)}}\,\mathrm{d}x = e^{\frac{\sigma^2(t-s)}{2}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{1}{2(t-s)}(x-(t-s)\sigma)^2}\,\mathrm{d}x = e^{\frac{\sigma^2(t-s)}{2}}.$$

Combining the findings from above with (12), results in the solution for problem **Q1-2** given by

$$\mathbb{E}\left[G(t) \mid \mathcal{F}_s\right] = e^{\sigma W(s)}\mathbb{E}\left[e^{\sigma(W(t)-W(s))} \mid \mathcal{F}_s\right] = e^{\sigma W(s) + \frac{\sigma^2(t-s)}{2}} \geq e^{\sigma W(s)} \quad \forall t \geq s,\ \sigma \in \mathbb{R}^+ \tag{13}$$

Hence $\mathbb{E}\left[G(t) \mid \mathcal{F}_s\right]$ is a sub-martingale. Taking the conditional expectation on the solution to the GBM (11) given the filtration $\mathcal{F}_u$ for some non-negative $u \leq t$, and using the results from (13),

$$\begin{aligned} \mathbb{E}\left[S(t) \mid \mathcal{F}_u\right] &= S_0 e^{\left(I - \frac{\sigma^2}{2}\right)t}\,\mathbb{E}\left[e^{\sigma W(t)} \mid \mathcal{F}_u\right] = S_0 e^{\left(I-\frac{\sigma^2}{2}\right)t + \left(I-\frac{\sigma^2}{2}\right)u - \left(I-\frac{\sigma^2}{2}\right)u} e^{\sigma W(u) + \frac{\sigma^2(t-u)}{2}} \\ &= S_0 e^{\left(I-\frac{\sigma^2}{2}\right)u + \sigma W(u)} e^{\left(I-\frac{\sigma^2}{2}\right)(t-u) + \frac{\sigma^2}{2}(t-u)} = S(u) e^{I(t-u)} = S(u), \end{aligned}$$

where the last equation follows by assuming we have a martingale. We can see that the equation holds if and only if $I = 0$ assuming that $t > u$. In finance, this condition implies that we have an arbitrage-free setting such that the expected profit does not change over time as there is no drift component pushing the asset's price up or down over a specific time interval.

Now assume that $I(t)$ and $\sigma(t)$ are time-dependent, and maybe even stochastic, drift and diffusion terms. Using Itô's lemma 7, the solution to (1) with non-constant drift and diffusion terms can be found by integrating the following equation

$$\mathrm{d}\left(\log S(t)\right) = \left(I(t) - \frac{\sigma(t)^2}{2}\right)\mathrm{d}t + \sigma(t)\,\mathrm{d}W(t) \implies S(t) = S_0 \exp\left(\int_0^t \left(I(s) - \frac{\sigma(s)^2}{2}\right)\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W(s)\right) \tag{14}$$

where the latter is a solution to the problem in **Q1-4** and in general to (1).

Stochastic Interest Rates (Q2)

Consider the SDE for $I(t)$ given by (2), assume that $I(0) > 0$ and recall that the parameters a, b, c are non-negative, real constants. We aim to now solve it. As this SDE is reminiscent of the Ornstein-Uhlenbeck type, a standard approach to solve it is to perform a substitution to simplify the drift term $a - bI(t)$. To do so, consider the substitution $Y(t) = e^{bt}I(t)$. By applying Itô's lemma to $Y(t)$, or by directly substituting $I(t) = e^{-bt}Y(t)$, we transform the equation. Substituting $\mathrm{d}I(t)$ and $I(t)$ into the given SDE, and simplifying, we obtain:

$$dY(t) = ae^{bt}\,\mathrm{d}t + ce^{bt}\,\mathrm{d}W_t^{(2)}$$

This is now a simpler SDE that can be integrated directly. Integrating from 0 to t and using $Y(0) = I(0)$, we obtain

$$Y(t) = I(0) + \int_0^t ae^{bs}\,\mathrm{d}s + \int_0^t ce^{bs}\,\mathrm{d}W_s^{(2)} = I(0) + \frac{a}{b}(e^{bt} - 1) + c\int_0^t e^{bs}\,\mathrm{d}W_s^{(2)}$$

Substituting back $I(t) = e^{-bt}Y(t)$, we arrive at

$$I(t) = I(0)e^{-bt} + \frac{a}{b}(1 - e^{-bt}) + c\int_0^t e^{-b(t-s)}\,\mathrm{d}W_s^{(2)}, \tag{15}$$

as the solution for (2), and hence for **Q2-1**.

Consider the previously computed solution to the SDE in (2). Since $I(t)$ is a linear combination of Gaussian increments and deterministic functions, it is normally distributed as well. Its mean and variance are given by

$$\mathbb{E}[I(t)] = I(0)e^{-bt} + \frac{a}{b}(1 - e^{-bt}) \quad \text{and} \quad \mathrm{Var}\,[I(t)] = c^2\int_0^t e^{-2b(t-s)}ds = \frac{c^2}{2b}(1 - e^{-2bt})$$

where Itô isometry given by (8) was used to compute the variance. When $a = b = c = 1$, we have that

$$\mathbb{E}[I(t)] = 1 + (I(0) - 1)e^{-t}, \quad \mathrm{Var}\,[I(t)] = \tfrac{1}{2}(1 - e^{-2t})$$

This means $I(t)$ is a normal random variable with these parameters. The probability that $I(t) < 0$ is then given by evaluating the standard normal CDF at the appropriately standardized value

$$\mathbb{P}\big(I(t) < 0\big) = \Phi\left(\frac{-\mathbb{E}[I(t)]}{\sqrt{\mathrm{Var}\,[I(t)]}}\right).$$

The probabilities that $I(t) < 0$ can be found in Table 1. As we can see, as time grows large, $\mathbb{E}[I(t)]$ approaches 1, and the variance stabilizes, so there remains a nonzero probability of being below zero, however small. From a financial perspective, allowing a process meant to represent quantities like prices or interest rates to become negative may be problematic, as these quantities are often assumed to be non-negative. Put bluntly, having negative interest rates can be considered "un-physical". Thus, the possibility of $I(t) < 0$ can be considered a shortcoming for certain financial modelling purposes.

t	1	5	10	15	20
$\mathbb{P}\big(I(t) < 0\big)$	0.064	0.0786	0.0787	0.0787	0.0787

Table 1: Probabilities of the solution $I(t)$ to the SDE (2) being negative at various times.

We now take a look at the SDE described by (3). Why does this model make more sense financially? Simply because the interest rates are never negative in this model. Let us prove this in detail now:

Proof. Consider first the auxiliary process given by (4), with $I \vee 0 = \max\{I(t), 0\}$. It is evident that the $I \vee 0$ term is always greater than or equal to 0 for all $I(t)$, since 0 is chosen for any $I(t) < 0$. This implies that $\sqrt{I \vee 0}$ is always well-defined in $\mathbb{R}$, and is equivalent to (3) for all $I(t) \geq 0$. Now we take $\tau_\varepsilon = \inf\{t \geq 0 : I(t) = -\varepsilon\}$ for some small enough $\varepsilon = 0$. We want to show rigorously that $\mathbb{P}(\tau_\varepsilon < \infty = 0)$, i.e. there is zero probability that interest rate $I(t)$ will become negative. Let us split this into two cases. In the case of $I(t) > 0$, one ends up with the SDE

$$\mathrm{d}I = (a - bI)\,\mathrm{d}t + c\sqrt{I}\,\mathrm{d}W(t),$$

whereas in the case when $I(t) < 0$, we end up with a deterministic ODE given by

$$\mathrm{d}I = (a - bI)\,\mathrm{d}t.$$

Its solution is thus

$$I(t) = \frac{a}{b} - \frac{1}{b}e^{-bt}$$

when $I(t) < 0$. Notice that this deterministic differential equation has no diffusion term. Given $a, b, c > 0$, for $I(t) \to 0$ we notice that $c\sqrt{I(t) \vee 0} \to 0$. Then the SDE becomes deterministic as in Case 2. Then for the drift term we have that $(a - bI(t)) \overset{I(t)\to 0}{\longrightarrow} a$, i.e. the drift term is positive so that $\mathrm{d}I(t) \to a\,\mathrm{d}t$ and the system has positive gradient $\forall t > 0$. Then since we are given $I(0) > 0$, i.e. interest rates start above 0, it follows that $I(t) > 0 \quad \forall t$ and so never approaches $-\varepsilon \quad \forall \varepsilon > 0$. Therefore we conclude that $\mathbb{P}(\tau_\varepsilon < \infty) = 0$, which implies that $I(t) \geq 0 \quad \forall t$. $\square$

The above model actually represents a mean reverting process. A mean-reverting process is such that when some short rate r is high, it tends towards the long-term average. The drift term moves $I(t)$ towards the mean value and the system is symmetric around the mean. This "restoring" force is proportional to the distance of a state from the mean value. One classical example is the Ornstein-Uhlenbeck process, an exact solution to the Langevin equation. Now for $I(0) = \frac{a}{b}$, we have that

$$\mathbb{E}[I(t)] = \frac{a}{b}e^{-bt} + \frac{a}{b}(1 - e^{-bt}) = \frac{a}{b} = \text{constant}$$

that is, the expected value (mean) of $I(t)$ is time-invariant. On the other hand if $I(0) \neq \frac{a}{b}$, say $I(0) = \gamma \neq \frac{a}{b} \in \mathbb{R}$, then

$$\mathbb{E}[I(t)] = \gamma e^{-bt} + \frac{a}{b}(1 - e^{-bt}) = \frac{a}{b} + \left(\gamma - \frac{a}{b}\right) e^{-bt}$$

with the 2nd term decaying over time and the mean reaching $\frac{a}{b}$ eventually.

Non-negative & Square Volatilities (Q3)

Consider a process $\sigma(t) = \sqrt{v(t)}$, where $v(t) = \tilde{\sigma}^2(t)$ and $\tilde{\sigma}^2(t)$ is described by the SDE (5). We start by solving the SDE for $\tilde{\sigma}^2(t)$. As we can see, the SDE for $\tilde{\sigma}^2(t)$ and $I(t)$ given by (5) and (2) respectively are similar. In fact, they are equivalent if the parameters for $I(t)$ are $a = 0, b = \lambda$ and $c = f$. Therefore, the solution to $\tilde{\sigma}^2(t)$ is as (15) with $a = 0, b = \lambda, c = f$ and naturally $\tilde{\sigma}(t)$ instead of $I(t)$. Hence, the solution to (5) is

$$\tilde{\sigma}(t) = \tilde{\sigma}(0)e^{-\lambda t} + f \int_0^t e^{-\lambda(t-s)} dW_s^{(3)}. \tag{16}$$

Let us now investigate the ergodicity of the process $\tilde{\sigma}(t)$. We can see that it is an Ornstein-Uhlenbeck-type process with mean-reversion level 0 and speed λ. Such a process is Gaussian and ergodic. As $t \to \infty$, its distribution approaches a stationary Gaussian distribution $\mathcal{N}\left(0, \frac{f^2}{2\lambda}\right)$. Thus, the invariant measure μ_∞ is normal with mean 0 and variance $f^2/(2\lambda)$. Its density is given by

$$\rho_\infty(x) = \frac{1}{\sqrt{2\pi(f^2/(2\lambda))}} \exp\left(-\frac{x^2}{2(f^2/(2\lambda))}\right)$$

The Fokker-Planck equation for the density $p(t,x)$ of $\tilde{\sigma}(t)$ is

$$\frac{\partial p}{\partial t}(t,x) = -\frac{\partial}{\partial x}[-\lambda x p(t,x)] + \frac{f^2}{2}\frac{\partial^2 p}{\partial x^2}(t,x)$$

At stationarity, $\partial_t\, p_\infty(x) = 0$. So we substitute $p_\infty(x)$ into the steady-state equation to obtain

$$0 = \lambda p_\infty(x) + \lambda x p_\infty'(x) + \frac{f^2}{2} p_\infty''(x) \implies p_\infty(x) = \frac{\sqrt{\lambda}}{\sqrt{\pi} f} \exp\left(-\frac{\lambda x^2}{f^2}\right) \quad \text{(since } p_\infty(x) \text{ is Gaussian)}$$

we then differentiate w.r.t. x to obtain

$$p_\infty'(x) = -2x \cdot \frac{\lambda}{f^2} \cdot p_\infty(x), \quad p_\infty''(x) = \left(-2\frac{\lambda}{f^2} + 4x^2\frac{\lambda^2}{f^4}\right) \cdot p_\infty(x)$$

Substituting these back in confirms that each term cancels perfectly, leaving the equation satisfied. Hence, p_∞ is indeed the invariant density which satisfies the Fokker-Planck equation.

Now consider the case when $v(t) = \tilde{\sigma}^2(t)$.

Claim 1. *The SDE to the process*

$$v(t) = \tilde{\sigma}^2(t) \tag{17}$$

is

$$dv(t) = k\big(\mu - v(t)\big)\, dt + \nu\sqrt{v(t)}\, dW_t^{(3)}.$$

Proof. We start by applying Itô's lemma 7 to (17) with $\tilde{\sigma}(t)$ given by (16). Then,

$$\mathrm{d}v(t) = 2\tilde{\sigma}(t)\,\mathrm{d}\tilde{\sigma}(t) + (\mathrm{d}\tilde{\sigma}(t))^2.$$

Recalling the SDE for $\tilde{\sigma}(t)$, we find that

$$\mathrm{d}\tilde{\sigma}(t))^2 = f^2\,\mathrm{d}t,$$

and since $v(t) = \tilde{\sigma}^2(t)$, it follows that

$$\mathrm{d}v(t) = 2\tilde{\sigma}(t)(-\lambda\tilde{\sigma}(t)\,\mathrm{d}t + f\,\mathrm{d}W_t^{(3)}) + f^2\,\mathrm{d}t = (-2\lambda v(t) + f^2)\,\mathrm{d}t + 2f\sqrt{v(t)}\,\mathrm{d}W_t^{(3)}.$$

Setting $k = 2\lambda$, $\mu = f^2/(2\lambda)$, and $\eta = 2f$ yields the SDE given in the claim above (17)

$$\mathrm{d}v(t) = k\big(\mu - v(t)\big)\,\mathrm{d}t + \nu\sqrt{v(t)}\,\mathrm{d}W_t^{(3)},$$

which we recognise as a standard Cox-Ingersoll-Ross (CIR)-type model. $\square$

Note that this form, that is the CIR model, ensures $v(t) \geq 0$ *a.s.*, which is a highly desirable property in financial modelling for volatility or variance. From a financial standpoint, such a process is often preferred to linear Gaussian models for volatility, as it avoids the problematic issue of negative values for what should be a non-negative quantity.

Convergence of the Interpolated Euler-Maruyama Method (Q4)

To investigate the convergence rate of the interpolated Euler-Maruyama method, begin with the simplified case for (1), with $I(t) \equiv I$ and $\sigma(t) \equiv \sigma$. Consider a discretised time grid with the partition $P = \{0 = t_0 < t_1 < \ldots < t_N = T\}$ and the Euler-Maruyama method formulated by

$$\hat{S}(t) = S(t_n) + \int_{t_n}^{t} IS(t_n)\,\mathrm{d}s + \int_{t_n}^{t} \sigma S(t_n)\,\mathrm{d}W(s).$$

The one-dimensional GBM can be written as

$$S(t) = S_0 + \int_0^t IS(s)\,\mathrm{d}s + \int_0^t \sigma S(s)\,\mathrm{d}W(s) = S(t_n) + \int_{t_n}^{t} IS(s)\,\mathrm{d}s + \int_{t_n}^{t} \sigma S(s)\,\mathrm{d}W(s).$$

Then, the error at time t is

$$\begin{aligned}
S(t) - \hat{S}(t) &= \int_{t_n}^{t} I\big(S(t) - \hat{S}(t_n)\big)\,\mathrm{d}s + \int_{t_n}^{t} \sigma\big(S(t) - \hat{S}(t_n)\big)\,\mathrm{d}W(s) \\
&= I\int_{t_n}^{t} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}s + \sigma\int_{t_n}^{t} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}W(s) + I\int_{t_n}^{t} \big(S(t) - S(t_n)\big)\mathrm{d}s + \sigma\int_{t_n}^{t} \big(S(t) - S(t_n)\big)\mathrm{d}W(s) \\
&= R_1 + R_2,
\end{aligned}$$

where $R_1 := I\int_{t_n}^{t} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}s + \sigma\int_{t_n}^{t} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}W(s)$ and $R_2 := I\int_{t_n}^{t} \big(S(t) - S(t_n)\big)\mathrm{d}s + \sigma\int_{t_n}^{t} \big(S(t) - S(t_n)\big)\mathrm{d}W(s)$. Taking the expectation of the supremum of the squared error $|S(t) - \hat{S}(t)|^2$ for $0 \leq t \leq T$, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |S(t) - \hat{S}(t)|^2\right] = \mathbb{E}\left[\sup_{0\leq t\leq T} |R_1 + R_2|^2\right] \leq 2\mathbb{E}\left[\sup_{0\leq t\leq T} |R_1|^2\right] + 2\mathbb{E}\left[\sup_{0\leq t\leq T} |R_2|^2\right].$$

For simplicity, we will now compute inequalities for $\mathbb{E}\left[\sup_{0\leq t\leq T} |R_1|^2\right]$ and $\mathbb{E}\left[\sup_{0\leq t\leq T} |R_2|^2\right]$ for themselves.

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |R_1|^2\right] &\leq 2I^2\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\int_{t_n}^{t} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}s\right)^2\right] + 2\sigma^2\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\int_{t_n}^{t} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}W(s)\right)^2\right] \\
&\leq 2I^2T\,\mathbb{E}\left[\int_{t_n}^{T} \big(S(t_n) - \hat{S}(t_n)\big)^2\mathrm{d}s\right] + 8\sigma^2\mathbb{E}\left[\left|\int_{t_n}^{T} \big(S(t_n) - \hat{S}(t_n)\big)\mathrm{d}W(s)\right|^2\right] \\
&\leq 2I^2T\,\mathbb{E}\left[\int_0^{T} \big(S(t_n) - \hat{S}(t_n)\big)^2\mathrm{d}s\right] + 8\sigma^2\mathbb{E}\left[\int_0^{T} \big(S(t_n) - \hat{S}(t_n)\big)^2\mathrm{d}s\right] \leq (2I^2T + 8\sigma^2)\int_0^T \varepsilon(s)\,\mathrm{d}s,
\end{aligned}$$

where we define $\varepsilon(s) := \mathbb{E}\left[\sup_{0\leq\tau\leq s} |S(\tau) - \hat{S}(\tau)|^2\right]$. It remains to find an upper bound for $\mathbb{E}\left[\sup_{0\leq t\leq T} |R_2|^2\right]$.

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T} |R_2|^2\right] &\leq 2I^2\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\int_{t_n}^{t} \big(S(s) - S(t_n)\big)\mathrm{d}s\right)^2\right] + 2\sigma^2\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\int_{t_n}^{t} \big(S(s) - S(t_n)\big)\mathrm{d}W(s)\right)^2\right] \\
&\leq 2I^2T\,\mathbb{E}\left[\int_0^{T} \big(S(s) - S(t_n)\big)^2\mathrm{d}s\right] + 8\sigma^2\mathbb{E}\left[\int_0^{T} \big(S(s) - S(t_n)\big)^2\mathrm{d}s\right] \leq \big(2I^2T + 8\sigma^2\big)\int_0^T \mathbb{E}\left[|S(s) - S(t_n)|^2\right]\,\mathrm{d}s
\end{aligned}$$

Recognising $\mathbb{E}\left[|S(s)-S(t_n)|^2\right]$ as the square of the strong convergence of Euler-Maruyama which is known to be $\mathcal{O}(\Delta t)$, the last inequality becomes

$$\mathbb{E}\left[\sup_{0\le t\le T}|R_2|^2\right]\le(2I^2T+8\sigma^2)C\Delta t.$$

Hence, combining the upper bounds for $\mathbb{E}\left[\sup_{0\le t\le T}|R_1|^2\right]$ and $\mathbb{E}\left[\sup_{0\le t\le T}|R_2|^2\right]$, we have

$$\sqrt{\mathbb{E}\left[\sup_{0\le t\le T}\left(\left|S(t)-\hat{S}(t)\right|^2\right)\right]}\le\sqrt{(2I^2T+8\sigma^2)\int_0^T\varepsilon(s)\,\mathrm{d}s+(2I^2T+8\sigma^2)C\Delta t}=\mathcal{O}\left(\Delta t^{1/2}\right).$$

Thus, the convergence order of the interpolated Euler-Maruyama method is 1/2. As we have computed the order of convergence for the interpolated Euler-Maruyama method for (1) with constant terms for $I(t)$ and $\sigma(t)$, it is important to consider the scenario where $I(t)$ and $\sigma(t)$ follow their own SDEs, as described in (2) and (5), respectively.

Consider now the SDE of the form

$$\mathrm{d}S(t)=\phi_M\left(I(t)\right)S(t)\,\mathrm{d}t+\phi_M\left(\sigma(t)\right)S(t)\,\mathrm{d}W(t),\tag{18}$$

where $\phi_M=\max\{-M,\min\{M,x\}\}$ for some large constant $M>0$. In other words, we assume that both $I(t)$ and $\sigma(t)$ are bounded by a large enough constant M at any positive time t. This is a fair assumption as both $I(t)$ and $\sigma(t)$, given by (15) and (16) respectively, are decreasing functions in t. Hence, there almost surely exists an upper bound M for both solutions of their respective SDEs. Then, the modified one-dimensional GBM can be expressed through the integral form

$$S(t)=S_0+\int_0^t\phi_M\left(I(s)\right)\,\mathrm{d}s+\int_0^t\phi_M\left(\sigma(s)\right)\,\mathrm{d}W(s)=S(t_n)+\int_{t_n}^t\phi_M\left(I(s)\right)\,\mathrm{d}s+\int_{t_n}^t\phi_M\left(\sigma(s)\right)\,\mathrm{d}W(s),$$

and the interpolated Euler-Maruyama scheme as

$$\hat{S}(t)=S(t_n)+\int_{t_n}^t\phi_M\left(I(t_n)\right)\hat{S}(t_n)\,\mathrm{d}s+\int_{t_n}^t\phi_M\left(\sigma(t_n)\right)\hat{S}(t_n)\,\mathrm{d}W(s).$$

Then, the error at time t is

$$\begin{aligned}S(t)-\hat{S}(t)=&\int_{t_n}^t\phi_M\left(I(t_n)\right)\left(S(t_n)-\hat{S}(t_n)\right)\mathrm{d}s+\int_{t_n}^t\phi_M\left(\sigma(t_n)\right)\left(S(t_n)-\hat{S}(t_n)\right)\mathrm{d}W(s)+\\&+\int_{t_n}^t\left[\phi_M\left(I(s)\right)S(s)-\phi_M\left(I(t_n)\right)S(t_n)\right]\mathrm{d}s+\int_{t_n}^t\left[\phi_M\left(\sigma(s)\right)S(s)-\phi_M\left(\sigma(t_n)\right)S(t_n)\right]\mathrm{d}W(s)\le\\&\le M\int_{t_n}^t\left(S(t_n)-\hat{S}(t_n)\right)\mathrm{d}s+M\int_{t_n}^t\left(S(t_n)-\hat{S}(t_n)\right)\mathrm{d}W(s)+\\&+M\int_{t_n}^t\left(S(s)-S(t_n)\right)\mathrm{d}s+M\int_{t_n}^t\left(S(s)-S(t_n)\right)\mathrm{d}W(s),\end{aligned}$$

where we used the fact that $\phi_M(x)\le M\quad\forall\,x\in\mathbb{R}$. As we can see, we end up with the same result as with constant $I(t)$ and $\sigma(t)$. Hence, the order of convergence in the cases with stochastic drift following (2), and stochastic drift and diffusion, following (2) and (16) respectively, are also 1/2. Note that this is in general also the case for (1) with $I(t)$ as $\sigma(t)$ described as previously mentioned. This is because, as previously discussed, $I(t),\sigma(t)$ are decreasing functions in t, such that there will always exist some large enough $M>0$ to bound $I(t)$ and $\sigma(t)$ from above. Therefore, the result above is valid without the function $\phi_M(\cdot)$ as well.

The numerical verification of these findings can be found in Fig. 1, where the numerically computed strong error follows the reference line $\mathcal{O}(\sqrt{\Delta t})$. This suggests that the numerical solution of the interpolated Euler-Maruyama method for the one-dimensional GBM with stochastic drift and diffusion components has indeed a strong convergence rate of 1/2.

Stability of the Geometric Brownian motion and Numerical Stability (Q5)

We now analyse the asymptotic and mean-square stability of (1) with constant drift and diffusion terms, that is $I(t)\equiv I$ and $\sigma(t)\equiv\sigma$ respectively. The solution of the SDE is then given by (11).

Claim 2. *Suppose $S(t)$ is the solution to the SDE (1). Then,*

$$\lim_{t\to\infty}\frac{1}{t}\log S(t)=I-\frac{\sigma^2}{2},\quad a.s.\ \text{if}\ I\neq\frac{\sigma^2}{2}\tag{19}$$

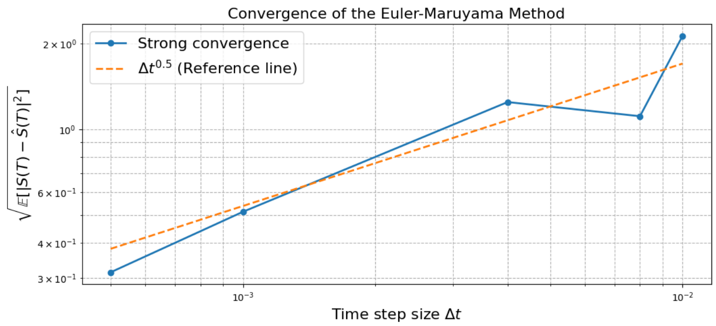


Figure 1: Numerical verification of the strong convergence order of the Euler-Maruyama scheme for (1) with stochastic drift and diffusion as in (2) and (5) respectively, and parameters $a = 0.1, b = c = 0.5, d = \lambda = 1, T = 2, S(0) = 2, I(0) = 0.5$ and $\sigma = 2$. Blue line represents the numerically computed strong error and the dashed orange line is a reference line following $\mathcal{O}\left(\sqrt{\Delta t}\right)$ (**Q4-4**).

Moreover, if $I = \frac{\sigma^2}{2}$, *then the growth of* $\log S(t)$ *is contained by*

$$\lim_{t\to\pm\infty} \frac{\log S(t)}{\sqrt{2t\log\log t}} = \pm\sigma \tag{20}$$

Proof. Using (11), $\log S(t)$ can be expressed by

$$\log S(t) = \log S(0) + \left(I - \frac{\sigma^2}{2}\right)t + \sigma W(t), \quad \text{s.t.} \quad \lim_{t\to\infty}\frac{1}{t}\log S(t) = I - \frac{\sigma^2}{2} + \lim_{t\to\infty}\frac{1}{t}(S(0) + \sigma W(t)).$$

Now recall that the law of iterated logarithms in theorem 1 states that the growth of $W(t)$ is contained by $\pm\sqrt{2t\log\log t}$. As $\sqrt{2t\log\log t} < t$, $\lim_{t\to\infty}\frac{W(t)}{t} = 0$ in the equation above. Hence,

$$\lim_{t\to\infty}\frac{1}{t}\log S(t) = I - \frac{\sigma^2}{2} + \lim_{t\to\infty}\frac{1}{t}(S(0) + \sigma W(t)) = I - \frac{\sigma^2}{2}.$$

Note that $S(t)$ is exponentially asymptotically stable if $I < \frac{\sigma^2}{2}$. For the case when $I = \frac{\sigma^2}{2}$, we have

$$\lim_{t\to+\infty}\sup\frac{\log S(t)}{\sqrt{2t\log\log t}} = \lim_{t\to+\infty}\sup\frac{\log S(0)}{\sqrt{2t\log\log t}} + \lim_{t\to+\infty}\sup\frac{\sigma W(t)}{\sqrt{2t\log\log t}} = \sigma\lim_{t\to+\infty}\sup\frac{W(t)}{\sqrt{2t\log\log t}} \quad \text{and}$$
$$\lim_{t\to-\infty}\inf\frac{\log S(t)}{\sqrt{2t\log\log t}} = \lim_{t\to-\infty}\inf\frac{\log S(0)}{\sqrt{2t\log\log t}} + \lim_{t\to-\infty}\inf\frac{\sigma W(t)}{\sqrt{2t\log\log t}} = \sigma\lim_{t\to-\infty}\inf\frac{W(t)}{\sqrt{2t\log\log t}}.$$

Recall theorem 1. By the law of the iterated logarithms, we can see that the equations above become

$$\lim_{t\to+\infty}\sup\frac{\log S(t)}{\sqrt{2t\log\log t}} = \lim_{t\to+\infty}\sup\frac{\sigma W(t)}{\sqrt{2t\log\log t}} = \sigma \qquad \text{and} \qquad \lim_{t\to-\infty}\inf\frac{\log S(t)}{\sqrt{2t\log\log t}} = \lim_{t\to-\infty}\inf\frac{\sigma W(t)}{\sqrt{2t\log\log t}} = -\sigma.$$

□

Now to deduce a condition for mean square stability, recall that $S(t) = S(0)e^{\left(I-\frac{\sigma^2}{2}\right)t+\sigma W(t)}$ and the mean-square stability condition in (10). Under mean-square stability, we have that

$$\mathbb{E}\left[|S(t)|^2\right] = S(0)^2 e^{2t\left(I-\frac{\sigma^2}{2}\right)}\mathbb{E}\left[e^{2\sigma W(t)}\right] = S(0)^2 e^{2t\left(I-\frac{\sigma^2}{2}\right)}e^{2\sigma^2 t} = S(0)^2 e^{2t\left(I+\frac{\sigma^2}{2}\right)},$$

where $\mathbb{E}\left[e^{2\sigma W(t)}\right]$ was recognised as the moment-generating function for a normal random variable $W(t) \sim \mathcal{N}(0,t)$. Thus, to achieve mean-square stability, we need,

$$\lim_{t\to\infty}\mathbb{E}\left[|S(t)|^2\right] = \lim_{t\to\infty}S(0)^2 e^{2t\left(I+\frac{\sigma^2}{2}\right)} = 0 \iff 2t\left(I + \frac{\sigma^2}{2}\right) < 0 \iff I < -\frac{\sigma^2}{2}, \tag{21}$$

assuming that $S(0) > 0$. Hence, the solution $S(t)$ in (11) is mean-square stable if and only if $I < -\frac{\sigma^2}{2}$ answering **Q5-1**. Figure 2 illustrates the mean-square of $S(t)$ for various interest rates I and time steps Δt, with $S(0) = 2$, $T = 5$, and $\sigma = 0.12$. As observed, smaller values of I tend to keep the mean-square contained, whereas larger values of I exhibit an upward, uncontrolled growth. Notably, none of the tested values of I ensure mean-square stability. To achieve such stability

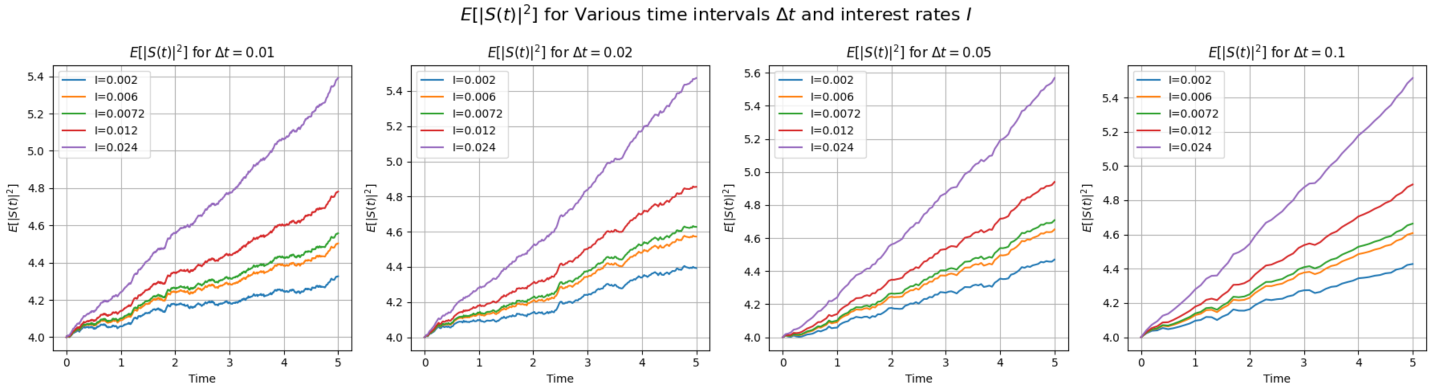


Figure 2: Mean-square of $S(t)$ for time grids $\Delta t = 0.01, 0.02, 0.05, 0.1$ and for $I = 0.002, 0.006, 0.0072, 0.012, 0.024$ (**Q5-2**).

in the SDE, the condition $I < -\frac{\sigma^2}{2} = -0.0072$ must hold. From a financial point of view, this behaviour is undesirable for an asset's price. Recall that the variance of a random variable is defined as

$$\mathrm{Var}[S(t)] = \mathbb{E}[S(t)^2] - \mathbb{E}[S(t)]^2.$$

A rapidly increasing mean-square can result in a high variance, that is, high risk. Assets lacking mean-square stability are thus perceived as risky and may fail to attract investors.

To derive the sufficient conditions for the mean-square stability in **Q5-3**, we define the stochastic-θ method as

$$S_{n+1}^{\theta} = S_n^{\theta} + \theta I S_{n+1}^{\theta} \Delta t + (1-\theta)\Delta t I S_n^{\theta} + \sigma S_n^{\theta} \Delta W_n, \tag{22}$$

where we denote by $(S_n^{\theta})_n$ the numerical solution using the stochastic-θ. Rearranging for S_{n+1}^{θ}, (22) becomes

$$S_{n+1}^{\theta} = \frac{S_n^{\theta}}{1-\theta I \Delta t}\left(1 + (1-\theta)\Delta t I + \sigma \Delta W_n\right).$$

Then, the mean-square stability for the stochastic-θ scheme can be expressed through

$$\mathbb{E}\left[|S_{n+1}^{\theta}|^2\right] = \frac{\mathbb{E}\left[|S_n^{\theta}|^2\right]}{(1-\theta I\Delta t)^2}\mathbb{E}\left[\left(1 + (1-\theta)\Delta t I + \sigma\Delta W_n\right)^2\right] = \mathbb{E}\left[|S_n^{\theta}|^2\right]\frac{\left(1+(1-\theta)^2 I \Delta t\right)^2 + \sigma^2\Delta t}{(1-\theta I \Delta t)^2} = C^n \mathbb{E}\left[|S_0^{\theta}|^2\right],$$

where we defined $C := \frac{\left(1+(1-\theta)^2 I\Delta t\right)^2 + \sigma^2 \Delta t}{(1-\theta I\Delta t)^2}$, assumed independence between S_n^{θ} and ΔW_n and used the fact that $\mathbb{E}\left[\Delta W_n\right] = 0$ and $\mathbb{E}\left[\Delta W_n^2\right] = \Delta t$. Then, for the stochastic-θ method to be mean-square stable, we require that $\lim_{n\to\infty} \mathbb{E}\left[|S_n^{\theta}|^2\right] = 0$, which is possible if and only if $C < 1$. To find sufficient conditions, rearranging for the inequality $C < 1$ for Δt is necessary,

$$C = \frac{\left(1+(1-\theta)^2 I\Delta t\right)^2 + \sigma^2\Delta t}{(1-\theta I \Delta t)^2} < 1 \quad \iff \quad I(2 + (1-2\theta)I\Delta t) < -\sigma^2 \quad \iff \quad \Delta t < -\frac{2\left(I + \frac{\sigma^2}{2}\right)}{(1-2\theta)I^2},$$

such that for $\theta = 0$,

$$\Delta t < -\frac{2\left(I + \frac{\sigma^2}{2}\right)}{I^2}. \tag{23}$$

As the time increments $\Delta t > 0$, it follows that we must have $I < -\frac{\sigma^2}{2}$, that is, the solution to the SDE must be mean-square stable, in order for the sufficient condition for mean-square stability given $\theta = 0$ to hold. In the case of $\theta = 0.5$, the sufficient condition is $I < -\frac{\sigma^2}{2}$ such that the scheme is stable for every $\Delta t > 0$ given that the solution to the SDE is mean-square stable. If the SDE is not mean-square stable, that is $I > -\frac{\sigma^2}{2}$, then the stochastic-θ method is not mean-square stable for any Δt. Lastly, for $\theta = 1$, the stochastic-θ is stable for every $\Delta t > 0$ if the analytical solution to the SDE is mean-square stable. If the mean-square stability condition is not met, we require $\Delta t > 2I + \sigma^2$. The sufficient conditions can be found in Table 2.

For our numerical experiment, we set $S(0) = 2$, $\sigma = 0.12$, $M =$ and $T = 5$ as initial values and use the findings from Table 2 to create both theoretical stable and unstable conditions for the numerical scheme with $\theta = 0, 0.5, 1$ which can be found in Table 3, and using the values in the latter table, we get Figure 3. As shown in the figure, the theoretical stable conditions show mean-square stability, especially for the blue lines where $I = -0.02$ and $\Delta t = 0.001$. The orange line is in theory on

	$\theta = 0$	$\theta = 0.5$	$\theta = 1$
SDE mean-square stable	$\Delta t < -\frac{2(I-\sigma^2/2)}{I^2}$	$\Delta t > 0$	$\Delta t > 0$
SDE mean-square unstable	$\Delta t < -\frac{2(I-\sigma^2/2)}{I^2}$	Unstable $\forall \Delta t \in \mathbb{R}^+$	$\Delta t > 2I + \sigma^2$

Table 2: A summary of sufficient conditions for Δt in the stochastic-θ method applied to the SDE (1) with $I(t) \equiv I$ and $\sigma(t) \equiv \sigma$ in **Q5-3**.

I	Δt	Stability
-0.02	0.001	Stable
-0.01	0.05	Stable
0.024	0.1	Unstable

For $\theta = 0$

I	Δt	Stability
-0.02	0.001	Stable
-0.01	0.05	Stable
0.024	0.01	Unstable

For $\theta = 0.5$

I	Δt	Stability
-0.02	0.001	Stable
0.0	0.01	Unstable
0.024	0.1	Unstable

For $\theta = 1$

Table 3: Stability Conditions for Different θ, Interest Rates I, and Time Steps Δt using the results from Table 2.

the verge of being stable, however, due to computational error, it remains stable at around $S(0)^2 = 4$. The green line is meant to be unstable, and it is visible that it shoots up in a very short span of time.
Now, considering that $\sigma(t)$ is no more constant and follows the SDE described in (5), let $(\sigma_n)_n$ be the Euler Maruyama approximation to the SDE.

Claim 3. *There exists a constant $M > 0$ such that $\mathbb{E}[\sigma_n] \le M$ for all n. Moreover, Δt can be expressed in terms of M, and σ_n is mean square stable iff. $f = 0$, that is, iff. there are no stochastic components describing σ_n.*

Proof. To prove this claim, consider the Euler-Maruyama approximation of the SDE (5). Then,

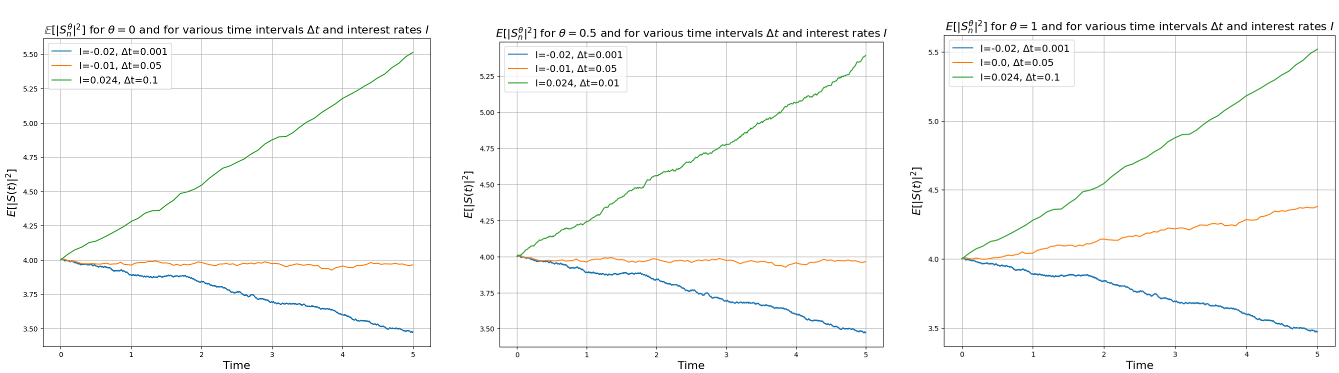
$$\sigma_{n+1} = \sigma_n - \lambda\sigma_n\Delta t + f\Delta W_n = \sigma_n(1-\lambda\Delta t) + f\Delta W_n.$$

Taking the expected value of $|\sigma_{n+1}|^2$ and recalling that $\Delta W_n \sim \mathcal{N}(0,1)$, one gets,

$$\begin{aligned}\mathbb{E}\left[|\sigma_{n+1}|^2\right] &= (1-\lambda\Delta t)^2\mathbb{E}\left[|\sigma_n|^2\right] + 2(1-\lambda\Delta t)\mathbb{E}\left[|\sigma_n|\Delta W_n\right] + f^2\mathbb{E}\left[\Delta W_n^2\right] = \\ &= (1-\lambda\Delta t)^2\mathbb{E}\left[|\sigma_n|^2\right] + f^2\Delta t + \mathcal{O}(\Delta t^2) \equiv (1-2\lambda\Delta t)^2\mathbb{E}\left[|\sigma_n|^2\right] + f^2\Delta t,\end{aligned}$$

where higher order terms for Δt have been dropped. To ensure $\mathbb{E}\left[|\sigma_{n+1}|^2\right]$ does not grow unbounded, impose the inequalities $1 - 2\lambda\Delta t < 1$ and $1 - 2\lambda\Delta t > 0$ such that $\Delta t < \frac{1}{2\lambda}$. The upper bound M is found by

$$\begin{aligned}\mathbb{E}\left[|\sigma_n|^2\right] &= \mathbb{E}\left[|\sigma_0|^2\right](1-2\lambda\Delta t)^n + f^2\Delta t\sum_{k=0}^{n}(1-2\lambda\Delta t)^k = \mathbb{E}\left[|\sigma_0|^2\right](1-2\lambda\Delta t)^n + f^2\Delta t\frac{1-(1-2\lambda\Delta t)^n}{1-(1-2\lambda\Delta t)} \\ &= \left(\mathbb{E}\left[|\sigma_0|^2\right] - \frac{f^2}{2\lambda}\right)\left(1 - \frac{2\lambda T}{n}\right)^n + \frac{f^2}{2\lambda} \le \left(\mathbb{E}\left[|\sigma_0|^2\right] - \frac{f^2}{2\lambda}\right)e^{-2\lambda T} + \frac{f^2}{2\lambda} = M,\end{aligned}$$



(a) $\theta = 0$ **(b)** $\theta = 0.5$ **(c)** $\theta = 1$

Figure 3: Mean-square stability for values for I and Δt found in Table 3 (**Q5-3**).

where we recognise the sum term as a geometric sum which has a known solution. Hence, $M = \left(\mathbb{E}\left[|\sigma_0|^2\right] - \frac{f^2}{2\lambda}\right)e^{-2\lambda T} + \frac{f^2}{2\lambda}$, and we can see that for large enough n or T, $M \to \frac{f^2}{2\lambda}$. Therefore, as $n \to \infty$, $\Delta t < \frac{1}{2\lambda} = \frac{M}{f^2}$. Now the fact that $M \to \frac{f^2}{2\lambda}$ as $n \to \infty$ means that $\lim_{n\to\infty} \mathbb{E}\left[|\sigma_n|^2\right] = \frac{f^2}{2\lambda} = 0 \iff f = 0$, meaning that σ_n is mean-square stable if and only if there is no stochastic component at hand. $\square$

To verify this numerically the conditions on Δt and M, we set $\sigma(0) = 1$ $f = 0.2$, $\lambda = 0.1$, Monte Carlo simulations $= 1000$ and $T = 5$. Then, with the given initial values, we have $M \approx 0.4943$ and $\Delta t \approx 12.3576$ as upper bounds. The fact that the upper bound for Δt is larger than T is handy, as this ensures that the numerical scheme converges to M for every $0 \leq \Delta t \leq T$. Fig. 4 summarises our numerical experiment.

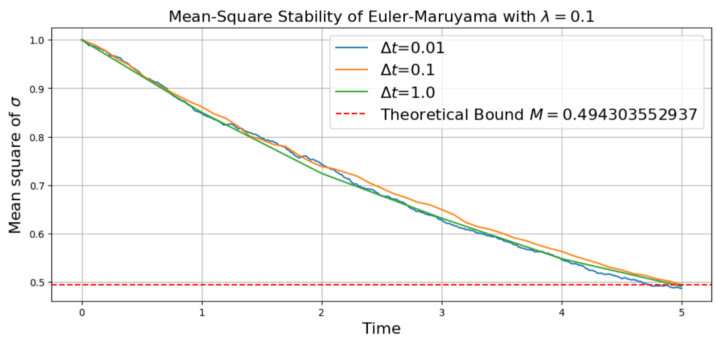


Figure 4: Convergence of $\mathbb{E}\left[|\sigma_n|^2\right]$ for $T = 5$, 1000 Brownian paths, $\Delta t = 0.01, 0.1, 1.0$ respectively.

As we can see, our numerical experiment coincides with the theory, since the mean-square appears to converge to the theoretical bound M for $\Delta t = 0.01, 0.1, 1.0$ (**Q5-4**).

A basket of options (Q6)

Suppose now that we have a basket of options, that is, a collection of multiple financial securities called options, composed of assets whose prices are described by the system of SDEs

$$\mathrm{d}S(t) = IS(t)\mathrm{d}t + \sigma KS(t) \circ \mathrm{d}W(t), \tag{24}$$

where $S(t, \omega) \in \mathbb{R}^n$ for some fixed $\omega \in \Omega$, the matrices $I, K \in \mathbb{R}^{n\times n}$ are symmetric and positive-definite, σ is a non-negative constant, W is an n-dimensional Brownian motion adapted to its natural filtration $(\mathcal{F}_t)_{t\geq 0}$. Then, the solution to the n-dimensional SDE system is given by

$$S(t) = S(0)\exp\left((I - \frac{1}{2}\sigma^2 KK^\top)t + \sigma KW(t)\right) = S(0)\exp\left((I - \frac{1}{2}\sigma^2 K^2)t + \sigma KW(t)\right).$$

This solution can be verified using Itô's lemma 7. Then,

$$G(t) = \mathbb{E}\big[S(t)S(t)^\top\big] = S(0)S(0)^\top \exp\left\{2\left(I - \frac{1}{2}\sigma^2 K^2\right)t\right\}\mathbb{E}\left[\exp\left(2\sigma KW(t)\right)\right] = \tag{25}$$

$$= S(0)S(0)^\top \exp\left\{2\left(I - \frac{1}{2}\sigma^2 K^2\right)t + 2\sigma^2 tK^2\right\} = S(0)S(0)^\top \exp\left\{2t\left(I + \frac{1}{2}\sigma^2 K^2\right)\right\} \tag{26}$$

Approximating the multi-dimensional SDE (24) with the Euler-Maruyama scheme, and computing the error

$$\sqrt{\sum_{n=0}^{N} \|\mathbb{E}\,[\widehat{S_n S_n^\top}] - G(t_n)\|_F^2 \Delta t}, \tag{27}$$

where $(S_n)_n$ is the numerical solution, $t_n = n\Delta t$, $\hat{\mathbb{E}}$ is the Monte-Carlo mean, and $G(t)$ is as in (26). Then, computing the error (27) for $M = 100$ and $M = 2000$ yields the results found in Table 4. When computing the error in (27), there are two main types of error: the discretisation error, which arises from approximating the continuous SDE with finite time steps Δt, and the statistical error, which comes from using a finite number of Monte-Carlo samples M to approximate the expectation. As we can see in Table 4, decreasing Δt and increasing M decreases the error, which coincides with the theoretical analysis.

M	$\Delta t = 0.001$	$\Delta t = 0.01$	$\Delta t = 0.05$	$\Delta t = 0.1$
100	5.7168	7.2158	4.8729	8.7219
2000	1.0912	2.3842	1.3605	1.2399

Table 4: Error values for different M-values and Δt-values with parameters $T = 1, I = 0.01 \cdot I_{d\times d}, \sigma = 10^{-3}, M = 100, d = 5, K = RR^{\top}$, where R is a $d \times d$ Unif(0,1) r.v., and $\Delta t = 0.001, 0.01, 0.05, 0.1$ (**Q6-2**).

An Euler-type Lamperti Transformation (Q7)

The closed-form solution for $\tilde{\sigma}(t)$ given by (16) provides a short expression for the volatility process $\tilde{\sigma}(t)$. However, its use is incompatible with the Euler-type Lamperti transformation approach, wherein $Y_t = \sqrt{v_t}$ with $v_t = \tilde{\sigma}^2(t)$, as developed by Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch [1]. The reasons for this incompatibility arise from both theoretical and practical limitations, which we explore below.

To begin with, the transformation $v_t = \tilde{\sigma}^2(t)$ involves squaring the stochastic integral within $\tilde{\sigma}(t)$. Substituting the closed-form solution into v_t, we obtain

$$v_t = \left(\tilde{\sigma}(0)e^{-\lambda t} + f\int_0^t e^{-\lambda(t-s)}dW_s^{(3)}\right)^2.$$

This expression contains the square of a stochastic integral, which introduces cross-terms and additional "stochasticity" that are not straightforward to handle. The resulting v_t cannot be expressed explicitly in terms of standard Itô calculus, which is crucial in the Lamperti transformation framework. As a result, the closed-form solution becomes difficult to use directly in this context.

Furthermore, the Lamperti transformation approach is fundamentally built on the assumption that the square root of v_t can be updated iteratively in an Euler-type scheme. The stochastic integral within $\tilde{\sigma}(t)$ introduces path-wise dependencies that make $Y_t = \sqrt{v_t}$ challenging to handle. Specifically, the stochastic integral $\int_0^t e^{-\lambda(t-s)}dW_s^{(3)}$ lacks pathwise differentiability, rendering it incompatible with the iterative numerical framework of the Euler-type scheme. This lack of smoothness complicates both theoretical analysis and numerical computation, making the closed-form solution unsuitable for this purpose.

From a computational perspective, the closed-form solution also imposes significant challenges. The Euler-type Lamperti transformation approach relies on iterative updates at each time step. However, using the closed-form solution would require re-evaluating or simulating the stochastic integral $\int_0^t e^{-\lambda(t-s)}dW_s^{(3)}$ at every step. This is computationally expensive and undermines the efficiency of the Euler-type method, which is designed for iterative updates rather than repeated integral evaluations.

Finally, the Lamperti transformation is designed to simplify the SDE for Y_t by converting it into a more manageable form where standard Itô calculus can be directly applied. The closed-form solution for $\tilde{\sigma}(t)$ reintroduces complexity due to the non-explicit nature of v_t and the additional terms that arise from taking the square of the stochastic integral. This complexity conflicts with the simplicity that the Lamperti transformation seeks to achieve.

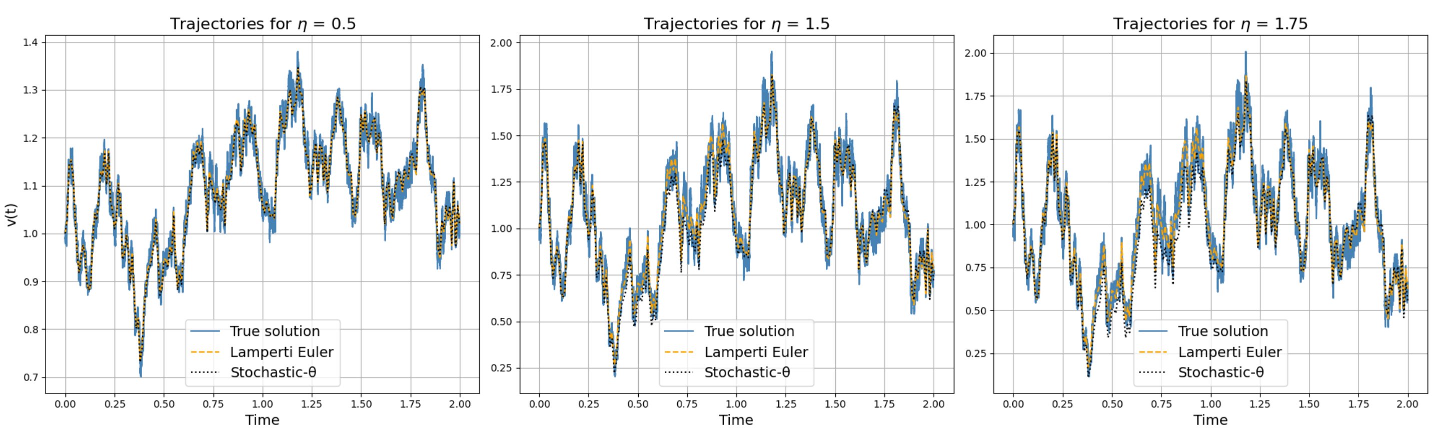


Figure 5: Comparison of numerical approximations with the true solution for $v(t)$ under different values of η. The trajectories are shown for $\eta = 0.5$ (left), $\eta = 1.5$ (middle), and $\eta = 1.75$ (right). The numerical methods compared are the Lamperti Euler scheme (dashed orange) and the stochastic-θ method (dotted black), with the true solution represented as a solid blue line (**Q7-1**).

Fig. 5 shows the numerical simulation of the Euler-Lamperti scheme for different values of η, $\Delta t = 0.01$, $k = \mu = v_0 1, T = 2$ and $M = 500$. As we can see, the numerical methods mostly coincide with the true solution and follow the same pattern.

The Black-Scholes-Merton model (Q8)

The Black-Scholes-Merton model is a popular model for pricing European call or put options under the no-arbitrage assumption. For simplification, consider the case for a call option. The analytical price of the call option is then,

$$C^{\mathrm{BS}}(s,t) = s\mathcal{N}(x_1) - Ke^{-rt}\mathcal{N}(x_2),$$

where $x_1 = \frac{\log(s/K)+(r+\frac{1}{2}\sigma^2 t)}{\sigma\sqrt{T}}$, $x_2 = \frac{\log(s/K)+(r-\frac{1}{2}\sigma^2 t)}{\sigma\sqrt{T}}$, K is the strike price, r is the risk-free interest rate, and s is the asset's price at time $t = 0$. The goal is now to numerically simulate the fair value of the $C(S_0, T)$. To do so, define the payoff function at time T

$$P\big(S(T)\big) = \max\{S(T) - K, 0\},$$

where $S(T)$ is computed numerically using the Euler-Maruyama scheme. Then, under the no-arbitrage assumption, we can express the fair value of the premium by

$$C(S_0, T) = e^{-rT}\mathbb{E}\left[P\big(S(T)\big)\right].$$

The expectation in the equation above is computed using the Monte Carlo method, that is, by generating M paths of Brownian Motions for the SDE (1) and then taking their average. Fig. 6 shows a numerically computed weak convergence of the Euler-Maruyama scheme in blue with a reference line Δt. We can see that the blue line follows the trend of the reference line such that we can conclude that the Euler-Maruyama scheme has a weak convergence rate of 1.

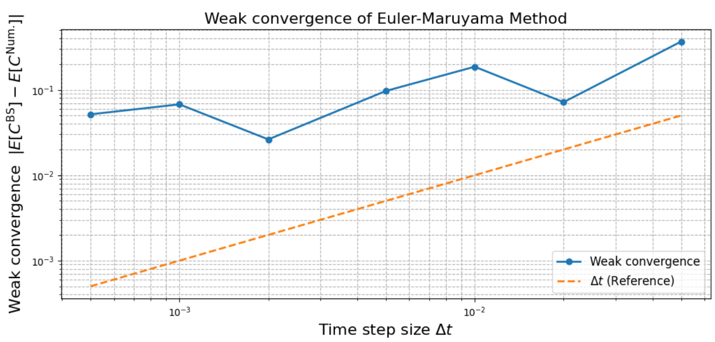


Figure 6: Weak convergence for the Euler-Maruyama method plotted by computing the fair value of the premium using the Black-Scholes-Merton model and an Euler-Maruyama scheme. Computed using $\Delta t = 0.0005, 0.001, 0.002, 0.005, 0.01, 0.02, 0.05$, $S(0) = 8$, $K = 10$, $r = 0.05$, $\sigma = 0.5$, $T = 1$ and $M = 100$ (**Q8**).

Conclusion

In this project, we explored the modelling and numerical analysis of financial assets using stochastic differential equations (SDEs). We started with the assumption of asset prices following a one-dimensional Geometric Brownian Motion (GBM) (1) combining it with both constant and non-constant drift and diffusion terms, examining and solving the SDEs and solving them using numerical schemes such as the Euler-Maruyama scheme, the stochastic-θ method and an Euler-type method based on the Lamperti transformation. Our analysis highlighted the strengths and limitations of various numerical approaches. The Euler-Maruyama method is known to be the most intuitive and simplest to implement, but showed some stability issues for certain parameter values, whereas the stochastic-θ method made stability depend on an exogenous variable θ. Lastly, the Lamperti transformation simplified the SDEs volatility process but showed theoretical and computational challenges, especially with pathwise dependencies and stochastic integrals. To conclude, this project empasised the critical role of numerical methods in financial modelling, and by comparing numerical scheme and analysing stability, we outlined a framework for model selection. In the future, one may even explore higher-order scheme such as the Milstein-Platen scheme to increase accuracy and stability.

References

[1] Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch. "An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process". In: *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences* 468.2140 (2012), pp. 1105–1115.

[2] Lawrence C. Evans. *An Introduction to Stochastic Differential Equations*. Graduate Studies in Mathematics. Providence, Rhode Island: American Mathematical Society, 2013. ISBN: 978-1-4704-1057-4.

[3] Ioannis Karatzas and Steven E. Shreve. *Brownian Motion and Stochastic Calculus*. 2nd. Vol. 113. Graduate Texts in Mathematics. New York: Springer, 1991. ISBN: 978-0-387-97655-6.