1 TMA4212 - Numerical Solution of Differential Equations by Difference Methods

1.1 Exercise 3

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1a) In this exercise, we want to solve the Black-Scholes equation with artificial B.C's. To do so, we use central differences in space and the Crank-Nicholson method in time. The Black-Scholes equation is given as

$$u_t - \frac{1}{2}\sigma^2 x^2 u_{xx} - rxu_x + cu = 0, \tag{1}$$

where \$$\sigma$, r, c > 0, represent volatility, interest rate, and correlation.

We start by taking the central differences in space. Since we want to end up with an inhomogenous heat-equation, we take the central difference on u_x only. This yields

$$u_x = \frac{1}{2h}\delta_{2h,x}(u_m(t)) = \frac{1}{2h}(u_{m+1}(t) - u_{m-1}(t)) \tag{2}$$

Thus, we get an inhomogenous heat equation and the Crank-Nicolson scheme is applicable

$$u_t = \frac{1}{2}\sigma^2 x_m^2 u_{xx} + \frac{rx_m}{2h}(u_{m+1}(t) - u_{m-1}(t)) - cu_m(t) = \frac{1}{2}\sigma^2 x_m^2 u_{xx} + F_m(t), \tag{3}$$

where $F_m(t) = \frac{rx_m}{2h}(u_{m+1}(t) - u_{m-1}(t)) - cu_m(t)$.

Taking the Crank-Nicolson on the equation above yields us

$$\frac{U_m^{n+1} - U_m^n}{k} = \frac{1}{4h^2}\sigma^2 x_m^2(\delta_x^2 U_m^2 + \delta_x^2 U_m^{n+1}) + \frac{1}{2}(F_m(t^{n+1}) + F_m(t^n))$$
$$\Rightarrow U_m^{n+1} = U_m^n + \frac{1}{4h^2}k\sigma^2 x_m^2(U_{m+1}^{n+1} - 2U_m^{n+1} + U_{m-1}^{n+1} + U_{m+1}^n - 2U_m^n + U_{m-1}^n) + \frac{1}{4}\frac{krx_m}{h}(U_{m+1}^{n+1} - U_{m-1}^{n+1} + U_{m+1}^n - U$$

We denote by $K_1 = \frac{1}{4h^2}k\sigma^2 x_m^2$, $K_2 = \frac{1}{4}\frac{krx_m}{h}$ and $K_3 = \frac{ck}{2}$, and after doing some algebra, we end up with the following solution to the equation

$$(I - A)\vec{U}^{n+1} = (I + A)\vec{U}^n + \vec{g^{n+1}} + \vec{g^n},$$

where

$A = \text{tridiag}\{K_1 - K_2, -2K_1 - K_3, K_1 + K_2\} = \text{tridiag}\left\{\frac{1}{4h^2}k\sigma^2 x_m^2 - \frac{1}{4}\frac{krx_m}{h}, -\frac{1}{2h^2}k\sigma^2 x_m^2 - \frac{ck}{2}, \frac{1}{4h^2}k\sigma^2 x_m^2 + \frac{1}{4}k\right.$

$\vec{U}^{n+1} = [U_0^{n+1}, U_1^{n+1}, \ldots, U_M^{n+1}]^\intercal$

$\vec{U}^n = [U_0^n, U_1^n, \ldots, U_M^n]^\intercal$

$\vec{g^{n+1}} = [(K_1 - K_2)\cdot g_0(x_m, t_{n+1}), 0, \ldots, 0, (K_1 + K_2)\cdot g_1(x_m, t_{n+1})]^\intercal$

$\vec{g^n} = [(K_1 - K_2)\cdot g_0(x_m, t_n), 0, \ldots, 0, (K_1 + K_2)\cdot g_1(x_m, t_n)]^\intercal$

and I is the identity matrix.

One can find the implementation of the scheme above in the `.ipynb`-file that followed the handin.

1b) We now want to show that with Dirichlet's B.C.'s and central differences in space, the Crank-Nicholson (CN) and Forward Euler (FE) methods have both have positive coefficients iff. $\sigma^2 > r$ and a CFL condition holds. Furthermore, we will show that the Backward Euler (BE) method also has positive coefficients for $\sigma^2 > 2$.

In the course, we have seen that a scheme has positive coefficients if, for any point P in the grid, it is of the form

$$-\mathcal{L}_h U_P = \alpha_{PP} U_P - \sum_{\substack{Q \in \mathcal{S}_P \\ Q \neq P}} \alpha_{PQ} U_Q = F_P, \qquad F_P \in \mathbb{R}$$

where we denote $\mathcal{L}_h$ as the linear difference operator $\mathcal{L}_h = a(x)\partial_x^2 + b(x)\partial_x + c(x)$, and the stencil of the scheme corresponding to a point $P = (x_m, t_{n+1})$.

Note that the coefficients α have to satisfy the two following conditions

$$\begin{aligned} &(1) \quad \alpha_{PQ} \geq 0 \quad \forall Q \in \mathcal{S}_P \\ &(2) \quad \alpha_{PP} \geq \sum_{\substack{Q \in \mathcal{S}_P \\ Q \neq P}} \alpha_{PQ}. \end{aligned}$$

1.1.1 CN

We start by checking whether (CN) has positive coefficients.

$$-\mathcal{L}_h U_P = U_m^{n+1}(1 + 2K_1 + K_3) - U_m^{n+1}(K_1 + K_2) - U_{m-1}^{n+1}(K_1 - K_2) - U_{m+1}^{n}(K_1 + K_2) - U_m^{n}(1 - 2K_1 - K_3) - U$$

We can clearly see that

$$\begin{aligned} \alpha_{PP} &= 1 + 2K_1 + K_3 \\ \alpha_{P,Q_1} &= K_1 - K_2 = \alpha_{P,Q_3} \\ \alpha_{P,Q_2} &= K_1 - K_2 = \alpha_{P,Q_5} \\ \alpha_{P,Q_4} &= 1 - 2K_1 - K_3. \end{aligned}$$

Let us now check condition (1).

$$\alpha_{P,Q_1} = K_1 + K_2 = \frac{1}{4h} k x_m \left(\frac{\sigma^2 x_m}{h} + r\right) \geq 0$$

Since all variables are gretaer than zero by defintion, the condition holds for α_{P,Q_1} and α_{P,Q_3}.

$$\alpha_{P,Q_2} = K_1 - K_2 = \frac{1}{4h} k x_m \left(\frac{\sigma^2 x_m}{h} - r\right) \geq 0$$

As we can see, to ensure that the condition holds for α_{P,Q_2} and α_{P,Q_5}, σ^2 has to be greater than r. Thus, $\sigma^2 > r$. Lastly, the condition $\alpha_{P,Q_4}0$ yields a CFL condition since

$$\begin{aligned} &\alpha_{P,Q_4} = 1 - 2K_1 - K_3 = 1 - \frac{1}{2} k \left(\frac{\sigma^2 x_m^2}{h^2} + c\right) 0 \\ &2k\left(\frac{\sigma^2 x_m^2}{h^2} + c\right). \end{aligned}$$

Given that $x_m := mh$, we get for the CFL condition

$$\begin{aligned}
&2k(\frac{\sigma^2 x_m^2}{h^2} + c) = k(\sigma^2 x_m^2 + c) \\
&k \leq \frac{2}{\sigma^2 x_m^2 + c}
\end{aligned}$$

Checking condition (2), we get

$$\begin{aligned}
&\alpha_{PP} \geq \sum_{\substack{Q \in \mathcal{S}_P \\ Q \neq P}} \alpha_{PQ} \\
&1 + 2K_1 + K - 3 \geq 1 + 2K_1 - K_3 \\
&K_3 \geq K_3 \\
&ck/2 \geq -ck/2,
\end{aligned}$$

and since $c, k > 0$, condition (2) holds.

To summarise, 1) we have Dirichlet boundary conditons, allowing us to only apply the scheme on the inner grid points where $x_m \geq h$

2) we found out that $\sigma^2 > r$

3) and that the CFL condition $k \leq \frac{2}{\sigma^2 x_m^2 + c}$ is satisfied since it is not dependent on h.

1.1.2 FE

Before we can check whether the FE for the Black-Scholes equation has positive coefficients, we have to derive it. Using the method of lines (MOL) and taking the central differences, we get

$$\begin{aligned}
&u_x = \frac{1}{2h}(u_{m+1}(t) - u_{m-1}(t)) \\
&u_{xx} = \frac{1}{h^2}(u_{m+1}(t) - 2u_m(t) + u_{m-1}(t)) \\
&\Rightarrow u_t = \frac{1}{2h^2}\sigma^2 x_m^2 h^2 (u_{m+1}(t) - 2u_m(t) + u_{m-1}(t)) + \frac{1}{2h} r x_m (u_{m+1}(t) - u_{m-1}(t)) - c u_m(t)
\end{aligned}$$

Applying FE on the equation above, we get

$$U_m^{n+1} = (K_1 + K_2)U_{m+1}^n + (1 - 2K_1 - K_3)U_m^n + (K_1 - K_2)U_{m-1}^n,$$

where we denote by $K_1 = \frac{1}{2h^2} k \sigma^2 x_m^2$, $K_2 = \frac{1}{2}\frac{k r x_m}{h}$ and $K_3 = ck$.

Taking the linear difference operator, we get

$$-\mathcal{L}_h U_P = U_m^{n+1} - (K_1 + K_2)U_{m+1}^n - (1 - 2K_1 - K_3)U_m^n - (K_1 - K_2)U_{m-1}^n = 0.$$

The corresponding α-coefficients are thus

$\alpha_{P,P} = 1$

s
$\alpha_{P,Q_1} = K_1 + K_2$

s
$\alpha_{P,Q_2} = 1 - 2K_1 - K_3$

s
$\alpha_{P,Q_3} = K_1 - K_2$

Checking condition (1) for positive coefficients, we can clearly see that $\alpha_{P,Q_1} = K_1 + K_2 > 0$, as all coefficients within K_1 and K_2 are positive. The condition $\alpha_{P,Q_2} 0$ yields a CFL condition since

$$\begin{aligned}
\alpha_{P,Q_2} &= 1 - 2K_1 - K_3 = 1 - \frac{1}{h^2} k\sigma^2 x_m^2 - ck 0 \\
&\Rightarrow 1k(\frac{1}{h^2}\sigma^2 x_m^2 + c) = k(\frac{1}{h^2}\sigma^2 (mh)^2 + c) = k(\sigma^2 m^2 + c)
\end{aligned}$$

Thus, a condition for the time step of FE is $k \leq \frac{1}{(\sigma^2 m^2 + c)}$, and since the condition for the time step is not dependent on h, the CFL condition is satisfied.

We now check whether α_{P,Q_3} is positive.

$$\begin{aligned}
\alpha_{P,Q_3} &= K_1 - K_2 = \frac{1}{2h^2} k\sigma^2 x_m^2 - \frac{1}{2}\frac{krx_m}{h} 0 \\
\frac{\sigma^2 x_m}{h} &= \frac{\sigma^2 mh}{h} = \sigma^2 mr
\end{aligned}$$

Thus, $\sigma^2 > r$.

Lastly, we check condition (2).

$$\begin{aligned}
\alpha_{PP} &\geq \sum_{\substack{Q \in \mathcal{S}_P \\ Q \neq P}} \alpha_{PQ} = K_1 + K_2 + 1 - 2K_1 - K_3 + K_1 - K_2 = 1 - K_3 \\
11 - K_3 &= 1 - ck \qquad \Rightarrow 0ck,
\end{aligned}$$

and since $c, k > 0$, condition (2) holds. Thus, FE has positive coefficients iff. $\sigma^2 > r$ with a time step limit of $1k(\sigma^2 m^2 + c)$.

1.1.3 BE

The Backward Euler method has the same scheme as its forward counterpart, except that it uses a forward difference operator. We therefore end up with

$$(1 + 2K_1 + K_3)U_m^{n+1} - (K_1 + K_2)U_{m+1}^{n+1} - (K_1 - K_2)U_{m-1}^{n+1} = U_m^n,$$

with the same constants as in FE.

Taking the linear difference operator on the equation above yields us the following equation

$$-\mathcal{L}_h U_P = (1 + 2K_1 + K_3)U_m^{n+1} - (K_1 + K_2)U_{m+1}^{n+1} - (K_1 - K_2)U_{m-1}^{n+1} - U_m^n = 0,$$

with the correspondant α's being

$\alpha_{P,P} = 1 + 2K_1 + K_3$

s

$\alpha_{P,Q_1} = K_1 + K_2$

s

$\alpha_{P,Q_2} = K_1 - K_2$

s

$\alpha_{P,Q_3} = 1.$

For the same reason as previously mentionned, $\alpha_{P,Q_1}0$, as well as $\alpha_{P,Q_2} = K_1 - K_2 0$ (we discussed this when checking if FE had positive coefficients). Furthemore, $\alpha_{P,Q_3} = 10$. Note that none of the conditions yields a CFL condition that needs to be met. Thus, condition (1) for monotonicity is met iff. $\sigma^2 > r$, and we do not have any constraints on the time step k. It remains now to show that condition (2) holds aswell.

$$\begin{aligned}
&\alpha_{P,P} \geq \sum_{\substack{Q\in\mathcal{S}_P\\ Q\neq P}} \alpha_{PQ} = K_1 + K_2 + K_1 - K_2 + 1\\
&\Rightarrow 1 + K_3 1\\
&\Rightarrow K_3 = ck0,
\end{aligned}$$

and since $c > 0$, the condition abve holds. Thus, to summarise, BE has positive coefficients iff. $\sigma^2 > r$ and there are no CFL conditions to be met.

1c) In this exercise, we want to show that the forward Euler solution of the Black-Scholes equation is stable in L^∞. To do so, we start by recalling the solution to Black-Scholes equation using FE

$$\vec{U}^{n+1} = A\vec{U}^n + k\vec{f}^n + \vec{\tilde{g}}^n,$$

where

$$A = \text{tridiag}\{K_1 - K_2, 1 - 2K_1 - K_3, K_1 + K_2\} = \text{tridiag}\left\{\frac{k\sigma^2 x_m^2}{2h^2} - \frac{krx_m}{2h}, 1 - \frac{k\sigma^2 x_m^2}{h^2} - kc, \frac{k\sigma^2 x_m^2}{2h^2} + \frac{krx_m}{2h}\right\}.$$

We let $g_0 = g_1 = 0$, as well as $\vec{U}^0 = 0$. Thus, using recursion, we get

$$\begin{aligned}
&\vec{U}^{n+1} = A\vec{U}^n + k\vec{f}^n + \vec{\tilde{g}}^n = \vec{U}^{n+1} = A\vec{U}^n + k\vec{f}^n =\\
&= A(A\vec{U}^{n-1} + k\vec{f}^{n-1}) + k\vec{f}^n = \ldots = A^{n+1}\vec{U}^0 + k\sum_{l=0}^{n} A^l \vec{f}^{n-1} = k\sum_{l=0}^{n} A^l \vec{f}^{n-1}.
\end{aligned}$$

Taking the infinity norm, we get

$$\|\vec{U}^{n+1}\|_\infty = \|k\sum_{l=0}^{n} A^l \vec{f}^{n-1}\|_\infty kn \max_{l=0,...n} \|A\|_\infty^l \cdot \|\vec{f}^l\|_\infty kN \max_{l=0,...n} \|A\|_\infty^l \cdot \|\vec{f}^l\|_\infty,$$

with

$\|A\|_\infty = \max\{(1-2K_1-K_3)+(K_1+K_2), (K_1-K_2)+(1-2K_1-K_3)+(K_1+K_2), (K_1-K_2)+(1-2K_1-K$
$= 1-K_3 = 1-kc$
$\Rightarrow \|\vec{U}^{n+1}\|_\infty T \max_{l=0,...,n}(1-kc)^l \max_{l=0,...,n}\|\vec{f}^l\|_\infty T(1-kc)^0 \max_{l=0,...,n}\|\vec{f}^l\|_\infty T\|\vec{f}\|_\infty^n$

Thus, (FE) is stable on RHS.

1d) We now want to show that (FE) is consistent. The truncation error is defined as

$$\tau_{h,m} = \mathcal{L}u_m - \mathcal{L}_h u_m.$$

Thus, the truncation error for the Black-Scholes equation is

$$\begin{aligned}\tau_{h,m} &= u_t - \frac{\sigma^2x^2}{2}u_{xx} - rxu_x + cu - (\frac{1}{k}\Delta_k U_m^n - \frac{\sigma^2x^2}{2h^2}\delta_h^2 U_m^n - \frac{rx}{2h}\delta_h U_m^n + cu) = \\ &= u_t - \frac{\sigma^2x^2}{2}u_{xx} - rxu_x - (\frac{1}{k}\Delta_k U_m^n - \frac{\sigma^2x^2}{2h^2}\delta_h^2 U_m^n - \frac{rx}{2h}\delta_h U_m^n).\end{aligned}$$

We now expand the differences with the help of a Taylor expansion

$$\begin{aligned}\Delta_k U_m^n &= \frac{u(t+k)-u(t)}{k} = u_t(t_0) + \frac{k}{2}u_{tt}(t) + \mathcal{O}(k^2) \\ \delta_h^2 U_m^n &= \frac{u(x+h)-2u(x)+u(x-h)}{h^2} = u_{xx}(x) + \frac{h^2}{12}u^{(4)}(x_0) + \mathcal{O}(h^3) \\ \delta_h U_m^n &= \frac{u(x+h)-u(x-h)}{2h} = u_x(x) + \frac{h^2}{6}u_{xxx}(x) + \mathcal{O}(h^3)\end{aligned}$$

Put together, the truncation error becomes

$$\tau_{h,m} = -\frac{k}{2}u_{tt}(t) + \frac{\sigma^2x^2h^2}{24}u^{(4)}(x_0) + \frac{rx}{6}h^2u_{xxx} + \mathcal{O}(h^3+k^2)$$

Thus, $\tau_{h,m} \to 0$, as $k, h \to 0$, showing that the method is consistent.

1.1.4 L^∞-convergence

The LAx equivalence theorem states that if a method is consistent, stable and linear, which our method is, it is convergent. The error equation is defined as

$\mathcal{L}_h e_m = \tau_m$
$\Rightarrow e_m = -\tau_m = \frac{k}{2}u_{tt}(t) - \frac{\sigma^2x^2h^2}{24}u^{(4)}(x_0) - \frac{rx}{6}h^2u_{xxx} \quad \Rightarrow \|e_m\|_\infty = \|\frac{k}{2}u_{tt}(t) - \frac{\sigma^2x^2h^2}{24}u^{(4)}(x_0) - \frac{rx}{6}h^2u_{xxx}\|_\infty.$

By choosing a point $P(x^*, y^*)$, we can maximise the error term which gives us an upper bound

$$\|e_m\|_\infty C\|\tau\|_\infty CKh^2.$$

Thus, we have a convergence of order 2.

1e) We want to test the convergence in both space and time for the known solution $u(x,t) = cos(x)cos(t)$. We calculate the error with L^∞, and doubele the amount of grid-points for M when checking the convergence for space, and double N when checking the convergence in time. The rate of convergence can be estimated by the formula

$$Rate_i = \frac{\log \frac{e_{h_{i-1}}}{e_{h_i}}}{\log \frac{h_{i-1}}{h_i}}.$$

In the notebook we have plotted a table of errors and the convergence rates and the log-log plots of the errors. For the convergence analasys we can see that the method converges with $O(h^2)$ both is space and time.

1f) In this part we compare the errors $E(t) = max_{x\in[0,2K]|u_{R_{max}}(x,t)-u_R(x,t)|}$. We do the numerical calculations for errors in the range $R = 2K$ to $R = 32K$. When we increase the grid size, we also increase the number of space points, such that the distance between neighbouring points is the same for all grid sizes. In the notebook we have plotted the results in a table, and does not really see any big changes in the error for differnet grid sizes. However we observe the smallest error on the interval $x \in [0, 2K]$ for the grid size $R = 4K$. This is clearly not the result we were supposed to find.