TMA4212 - Numerical solution of differential equations by difference methods: Project 1

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A 1D stationary convection diffusion problem

The one-dimensional stationary convection diffusion problem is an important problem when it for instance comes to transporting, mixing or the decay of a chemical in a fluid moving in a finite tube. The problem can be seen as a boundary problem for a Poisson-like equation, that is,

$$-\partial_x(\alpha(x)\partial_x u) + \partial_x(b(x)u) + c(x)u = f(x) \quad \text{in} \quad \Omega = (0,1), \tag{1}$$

where u is the concentration of the substance, $\alpha(x) > 0$ the diffusion coefficient, $b(x)$ the convective, i.e. fluid velocity, $c(x) \geq 0$ the decay rate of the substance and $f(x)$ the source term.

In this report, we solve the given problem using the finite elements method, as well as by scaling the problem down to a smaller, and more practical domain $\Omega = (0,1)$. Moreover, note that time dependence is negligible for our problem since we look at the stationary time limit.

Theory

To show that any classical solution u of (1) satisfies

$$a(u,v) = F(v) \quad \forall v \in H_0^1(\Omega),\, \Omega = (0,1) \tag{2}$$

where $a(u,v) = \int_0^1 (\alpha u_x v_x - b(x)uv_x + cuv)\,\mathrm{d}x$. Let $H_0^1(\Omega)$ be a space of functions $v(x) \in L^2(\Omega)$ such that a weak derivative of v exists in $L^2(\Omega)$. Assume, for now, that $\alpha(x) = \cos(\frac{\pi}{3}x)$, $c = 5$, $\|b\|_{L^\infty} + \|f\|_{L^2} < \infty$. Moreover, assume that Dirichlet boundary conditions are met, meaning that $u(0) = 0 = u(1)$. With the given assumptions, (1) becomes

$$-\partial_x(\cos(\frac{\pi}{3}x)\partial_x u) + \partial_x(b(x)u) + 5u = f(x) \tag{3}$$

Thereafter, we multiply (3) by $v(x) \in H_0^1(\Omega)$, where $v(0) = 0 = v(1)$ and integrate on both sides. This yields us

$$\int_\Omega -\partial_x(\cos(\frac{\pi}{3}x)\partial_x u)v(x) + \partial_x(b(x)u)v(x) + 5uv(x)\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$$

By integrating by parts the equation above, one ends up with

$$-\cos(\frac{\pi}{3}x)u_x v(x)\Big|_\Omega + \int_\Omega \cos(\frac{\pi}{3}x)u_x v_x\,\mathrm{d}x + b(x)uv|_\Omega - \int_\Omega b(x)uv_x\,\mathrm{d}x + \int_\Omega 5uv\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$$

Now recall that $v(0) = 0 = v(1)$. The equation above can hence be simplified to

$$\int_\Omega \cos(\frac{\pi}{3}x)u_x v_x - b(x)uv_x + cuv\,\mathrm{d}x = \int_\Omega f(x)v(x)\,\mathrm{d}x$$

Define $F(v) := \int_0^1 f(x)v(x)\,\mathrm{d}x$ and $a(u,v) = \int_0^1 \cos(\frac{\pi}{3}x)u_x v_x - b(x)uv_x + cuv\,\mathrm{d}x$. Any classical solution of (1) thus satisfies

$$a(u,v) = F(v).$$

Furthermore, one can show that $a(u,v)$ is a bilinear, continuous as well as a coercive form on $H^1 \times H^1$ and that $F(v)$ is a linear bounded functional on H^1. We will itemize our proof to keep the report more presentable.

Bilinearity on $H^1 \times H^1$: To show that $a(u,v)$ is bilinear, the function must satisfy, for any $c_1, c_2 \in \mathbb{R}$ and $u_1, u_2 \in H^1$,

$$a(c_1u_1 + c_2u_2, v) = c_1a(u_1, v) + c_2a(u_2, v) \tag{4}$$

Applying the equation above to our function $a(u,v)$, we get

$$\begin{aligned} a(c_1u_1 + c_2u_2, v) &= \int_0^1 \cos(\frac{\pi}{3}x)(c_1u_1 + c_2u_2)_x v_x - (c_1u_1 + c_2u_2)bv_x + c(c_1u_1 + c_2u_2)\,\mathrm{d}x = \\ &= \int_0^1 \cos(\frac{\pi}{3}x)c_1(u_1)_x v_x - c_1u_1bv_x + cc_1u_1v\,\mathrm{d}x + \\ &+ \int_0^1 \cos(\frac{\pi}{3}x)c_2(u_2)_x v_x - c_2u_2bv_x + cc_2u_2v\,\mathrm{d}x = c_1a(u_1, v) + c_2a(u_2, v) \end{aligned}$$

As one can see, (4) holds for our case, meaning that a is bilinear.

Continuity on $H^1 \times H^1$: To show that $a(u,v)$ is continuous on $H^1 \times H^1$, we must show that there exist a constant $M \in \mathbb{R}$ such that $a(u,v) \le M\|u\|_{H^1}\|v\|_{H^1}$ $\forall u, v \in H^1$.

$$\begin{aligned} \Rightarrow & a(u,v) \le |a(u,v)| \le \left| \int_0^1 \alpha(x)u_x v_x - b(x)uv_x + cuv\,\mathrm{d}x \right| \\ & \le \|\alpha\|_{L^\infty}\|u_x v_x\|_{L^1} + \|b\|_{L^\infty}\|uv_x\|_{L^1} + \|c\|_{L^\infty}\|uv\|_{L^1} \\ & \overset{\text{Hölder's ineq.}}{\le} \|\alpha\|_{L^\infty}\|u_x\|_{L^2}\|v_x\|_{L^2} + \|b\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} + \|c\|_{L^\infty}\|u\|_{L^2}\|v\|_{L^2} \\ & \le (\|\alpha\|_{L^\infty} + \|b\|_{L^\infty} + \|c\|_{L^\infty})\|u\|_{H^1}\|v\|_{H^1} = M\|u\|_{H^1}\|v\|_{H^1}, \end{aligned}$$

where $M = \|\alpha\|_{L^\infty} + \|b\|_{L^\infty} + \|c\|_{L^\infty}$. Thus, a is continuous on $H^1 \times H^1$.

Linearity of $F(v)$: For $F(v)$ to be linear, we want the following to be satisfied

$$F(c_1v_1 + c_2v_2) = c_1F(v_1) + c_2F(v_2), \tag{5}$$

for any constant $c_1, c_2 \in \mathbb{R}$. Assume therefore that $c_1, c_2 \in \mathbb{R}$ are constants. In our case, we get

$$F(c_1v_1 + c_2v_2) = \int_0^1 (c_1v_1 + c_2v_2)f(x)\,\mathrm{d}x = \int_0^1 c_1v_1f(x)\,\mathrm{d}x + \int_0^1 c_2v_2f(x)\,\mathrm{d}x = c_1F(v_1) + c_2F(v_2)$$

As we can see, (5) holds and hence, $F(v)$ is linear.

Boundedness of $F(v)$: To show that $F(v)$ is bounded, consider the following

$$|F(v)| = \left| \int_0^1 f(x)v(x)\,\mathrm{d}x \right| \le \|f\|_{L^2} \cdot \|v\|_{L^2} \le \|f\|_{L^2} \cdot \|v\|_{H^1}$$

Thus, the norm of $F(v)$ becomes

$$\|F\| = \sup_{v \neq 0} \frac{|F(v)|}{\|v\|_{H^1}} \le \frac{\|f\|_{L^2} \cdot \|v\|_{H^1}}{\|v\|_{H^1}} = \|f\|_{L^2} < \infty,$$

which proves that the functional $F(v)$ is bounded.

Coercivity of $a(u,u)$: Until now, we have shown that the functional $F(v)$ is continuous and linear. Furthermore, we have shown that the form $a(u,v)$ is bilinear and continuous. Another nice property of $a(u,u)$ is its coercive nature. However, before showing that the form is coercive, it can be useful to show that $a(u,u)$ satisfies the Gårding inequality. The Gårding inequality given as

$$a(u,u) \ge (\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty}) \int_0^1 u_x^2\,\mathrm{d}x + (c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty}) \int_0^1 u^2\,\mathrm{d}x \quad \forall \varepsilon > 0, \tag{6}$$

where $\alpha_0 = \min_{x\in[0,1]} \alpha(x)$ and $c_0 = \min_{x\in[0,1]} c(x)$.

$$\begin{aligned}\Rightarrow a(u,u) &= \int_0^1 \alpha u_x^2 - buu_x + cu^2 \,\mathrm{d}x \\ &\geq \min_{x\in[0,1]} \int_0^1 \alpha u_x^2 - \|b\|_{L^\infty} uu_x + cu^2 \,\mathrm{d}x \\ &\geq \int_0^1 \alpha_0 u_x^2 - \|b\|_{L^\infty} uu_x + c_0 u^2 \,\mathrm{d}x,\end{aligned}$$

and using Young's inequality, one can find the following inequality

$$\|b\|_{L^\infty} uu_x \leq \frac{\|b\|_{L^\infty}}{2\varepsilon} u^2 + \frac{\varepsilon \|b\|_{L^\infty}}{2} u_x^2$$

Put together with the inequality for the norm, one gets

$$a(u,u) \geq \int_0^1 (\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty}) u_x^2 \,\mathrm{d}x + (c_0 - \frac{\|b\|_{L^\infty}}{2\varepsilon}) \int_0^1 u^2 \,\mathrm{d}x,$$

which is the Gårding inequality. Thus, the Gårding inequality holds.

Lastly, we want to show that a is coercive, that is, it fulfils the inequality

$$a(u,u) \geq K\|u\|^2_{H^1(0,1)} \quad \forall u \in H^1(0,1),$$

whenever $\|b\|_{L^\infty} < \sqrt{2\alpha_0 c_0} = \sqrt{5}$.

To show coercivity, we start with the Gårding inequality

$$\begin{aligned}a(u,u) &\geq (\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty}) \int_0^1 u^2 \,\mathrm{d}x + (c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty}) \int_0^1 u_x^2 \,\mathrm{d}x \\ &= (\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty})\langle u,u\rangle + (c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty})\langle u_x,u_x\rangle \\ &= (\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty})\|u\|^2_{L^2} + (c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty})\|u_x\|^2_{L^2} \\ &\geq \min(\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty}, c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty})\|u\|_{H^1_0(0,1)}\end{aligned}$$

For a to be coercive, we want the term $K := \min\{\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty}, c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty}\}$ in the equation above to be strictly greater than 0 (or else any function would be coercive if we simply set $K = 0$). This yields two sets of inequalities that we have to solve for

$$\begin{cases}\alpha_0 - \frac{\varepsilon}{2}\|b\|_{L^\infty} > 0 \\ c_0 - \frac{1}{2\varepsilon}\|b\|_{L^\infty} > 0\end{cases}$$

From the first inequality, we get that $\frac{2\alpha_0}{\|b\|_{L^\infty}} > \frac{\alpha_0}{\|b\|_{L^\infty}} > \varepsilon$, whereas the second inequality yields us $2c_0\varepsilon - \|b\|_{L^\infty} > 0$. Hence, $K = \min\{\frac{\alpha_0}{2}, c_0 - \frac{\|b\|^2_{L^\infty}}{2\alpha_0}\}$. Put together, we get

$$2c_0 \frac{\alpha_0}{\|b\|_{L^\infty}} - \|b\|_{L^\infty} > 0 \Rightarrow \|b\|_{L^\infty} < \sqrt{2\alpha_0 c_0} = \sqrt{5}$$

In other words, $a(u,v)$ is coercive only if $\|b\|_{L^\infty} < \sqrt{2\alpha_0 c_0} = \sqrt{5}$.

Now all these proofs that we have shown above are of great importance when it comes to the uniqueness of the solution to our problem. As a matter of fact, the Lax-Milgram theorem states that whenever we work in a Hilbert space $V = H^1$ with a bounded functional F, and that a is a continuous coercive bilinear form, then we can find $u \in H^1_0(0,1)$ such that, for all $v \in H^1_0(0,1)$,

$$a(u,v) = F(v) \tag{7}$$

admits a unique solution.

Since our form a is bilinear, continuous and coercive, as well as F being a bounded, linear functional, we can say, by the Lax-Milgram theorem, that the problem described in (1) admits a unique solution.

Numerical implementation

We are interested in solving (2) with a $\mathbb{P}_1$ FEM on a general grid on $(0,1)$. Let therefore $\alpha, b, c \geq 0$ be nonzero constants. Moreover, define $V_h := X_h^1(\Omega) \cap H_0^1(\Omega)$ where $\Omega = (0,1)$ as before. We want to find $u_h \in V_h$ such that $a(u_h, v_h) = F(v_h) \, \forall v_h \in V_h$. By multiplying with some function $v(x) \in H^1$ and integrating by parts on both sides one gets

$$a(u,v) = \int_0^1 \alpha u_x v_x - b u v_x + c u v \, \mathrm{d}x = \int_0^1 f(x) v \, \mathrm{d}x = F(v). \tag{8}$$

Let $\phi = \{\phi_1, ..., \phi_{M-2}\}$ be the basis of V_h where each basis vector is defined as

$$\phi_i(x) = \begin{cases} \frac{x - x_i}{h_{i-1}}, & x \in [x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h_i}, & x \in [x_i, x_{i+1}], \end{cases} \tag{9}$$

where $h_i = x_{i+1} - x_i$ with $i = 0, \ldots, M-2$. Using this method, the solution to problem (1) can be written as

$$u_h = \sum_{i=0}^{M-2} u_i \phi_i(x) \tag{10}$$

Since (8) has to hold for all $v(x) \in H^1$, the equation

$$a(u_h, \phi_i) = F(\phi_i)$$

has to hold for $i = 0, ..., M-2$. Using (10) and recalling that $a(\cdot\,,\,\cdot)$ is bilinear, we get

$$\sum_j u_j a(\phi_j, \phi_i) = F(\phi_i) \tag{11}$$

Now (11) forms a set of equations that can be written in matrix form

$$A\vec{U} = \vec{F}, \tag{12}$$

where $F_i = F(\phi_i)$, $\vec{U}$ is the approximated solution of the problem we are trying to solve, and $A_{ij} = a(\phi_i, \phi_j)$. To compute $F(\phi_j)$ we use `scipy.integrate.quad`, whereas for A_{ij}, we get

$$A_{ij} = \begin{cases} -\frac{1}{h_{i-1}}\alpha - \frac{1}{2}b + \frac{h_{i-1}}{6}c, & j = i-1, \\ \frac{1}{h_{i-1}}\alpha + \frac{1}{h_i}\alpha + \frac{(h_{i-1}+h_i)}{3}c, & j = i, \\ -\frac{1}{h_i}\alpha + \frac{1}{2}b + \frac{h_i}{6}c, & j = i+1, \\ 0, & \text{otherwise} \end{cases}$$

Testing the scheme on the test function $f(x) = x(1-x)$ yields figure 1 in which one can find a plot of the exact solution u together with the approximated solution u_h. As one can see, the figure includes two plots. On the left plot, we use one hundred equidistant nodes, and on the right plot, we use five nodes which are unevenly spread. The stiffness matrix for the uneaven case can be found in the Jupyter-notebook file. To find the convergence rate of the method, we investigate the behaviour of the error whenever we decrease the step size. This yields figure 2 where we see the loglog plot of the error. The numerically estimated rate of convergence is $p = 1$ in H^1 and $p = 2$ in L^2.

Stability and convergence of Galerkin approximation

To further investigate the behaviour of our numerical solution to the problem described in (1), we want to analyse the stability and convergence of the Galerkin approximation, as well as compare our theoretical findings for the H^1 error bound with our numerical results.

An important part of the convergence proof is Cea's lemma. The lemma helps us find a bound for the approximation error that comes from the Galerkin method and thus, gives us an intuition of how close the approximation can be to the solution. To show that Cea's lemma holds for our problem, we first need to show Galerkin orthogonality for the $\mathbb{P}_1$ FEM of (1).

Galerkin Orthogonality: To show Galerkin orthogonality, let $u \in V = H^1$ and $u_h \in V_h = X_h^1 \cap H_0^1$ be the solutions of the infinite- and finite-dimensional problems respectively. For Galerkin orthogonality to hold we want

$$a(u - u_h, v_h) = 0 \quad \forall v \in V_h \tag{13}$$

to hold. Since the form a of the given problem is a bilinear form, we can write

$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h)$$

Additionally, recall that

$$a(u, v_h) =: F(v_h) \quad \text{and} \quad a(u_h, v_h) =: F(v_h)$$

Thus, the proof for Galerkin orthogonality becomes

$$a(u - u_h, v_h) = a(u, v_h) - a(u_h, v_h) = F(v_h) - F(v_h) = 0 \quad \forall v \in V_h,$$

and (13) holds for our problem.

Cea's lemma: Let $u \in V = H^1$ and $u_h \in V_h = X_h^1 \cap H_0^1$ be the solutions of the infinite and finite dimensional variational problems respectively, and suppose that the hypotheses of Lax-Milgram described previously are satisfied. Notably, assume that a is continuous and coercive with constants $M, K \in \mathbb{R}$. Then,

$$\|u - u_h\|_{H^1} \leq \inf_{v_h \in V_h} \|u - v_h\|_{H^1} \quad \forall v_h \in V_h \tag{14}$$

To prove that the lemma holds, we recall that the Lax-Milgram theorem is satisfied as shown previously, meaning that a is continuous and coercive with constants M and K. By invoking coercivity, we can see that

$$\|u - u_h\|_{H^1} \leq \frac{1}{K} a(u - u_h, u - u_h).$$

Furthermore, by invoking bilinearity and recalling Galerkin orthogonality (13), we expand the right-hand side to

$$a(u - u_h, u - u_h) = a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h).$$

Note that the last term in the equation above is equal to 0 due to Galerkin orthogonality. Combining both equations results in

$$\begin{aligned}
\|u - u_h\|_{H^1}^2 \leq \frac{1}{K} a(u - u_h, u - u_h) &= \frac{1}{K} a(u - u_h, u - v_h) \\
&\frac{M}{K} \|u - u_h\|_{H^1} \cdot \inf_{v_h \in V_h} \|u - v_h\|_{H^1} \\
&\Rightarrow \|u - u_h\|_{H^1} \leq \frac{M}{K} \inf_{v_h \in V_h} \|u - v_h\|_{H^1} \quad \forall v_h \in V_h,
\end{aligned}$$

where $M = (\|a\|_{L^\infty} + \|b\|_{L^\infty} + \|c\|_{L^\infty})$ and $K = \min\{\frac{\alpha_0}{2}, c_0 - \frac{\|b\|_{L^\infty}^2}{2\alpha_0}\}$. Hence, Cea's lemma (14) holds for our problem. This is crucial, as we can now use the lemma to find an H^1 error bound for the problem. To do so, we start by recalling from the course that the interpolation error term in H^1 is described by

$$\|e\|_{H^1} = \|u - I_h u\|_{H^1} \leq 2h\|u_{xx}\|_{L^2},$$

where $I_h u \in V_h$ is the interpolation approximate of u. Starting with Cea's lemma, we can expand it to end up with the error term as follows

$$\|u - u_h\|_{H^1} \leq \frac{M}{K} \inf_{v_h \in V_h} \|u - v_h\|_{H^1} \leq \frac{M}{K} \|u - v_h\|_{H^1} = \frac{M}{K} \|u - I_h u\|_{H^1} \leq \frac{2M}{K} h\|u_{xx}\|_{L^2},$$

where the constants M, K are as described previously. The results from our code are consistent with this result, where the error is of first order in H^1.

Reduced order for non-smooth solutions

We now want to test the scheme for two non-smooth solutions. Consider the following two equations

$$w_1(x) = \begin{cases} \frac{x}{\frac{\sqrt{2}}{2}}, & x \in [0, \frac{\sqrt{2}}{2}], \\ \frac{1-x}{1-\frac{\sqrt{2}}{2}}, & x \in (\frac{\sqrt{2}}{2}, 1], \end{cases} \quad \text{and} \quad w_2(x) = x - x^{\frac{3}{4}} \tag{15}$$

As one can see, both w_1 and w_2 are continuous, but w_1 is not differentiable at $x = \frac{\sqrt{2}}{2}$ and w_2 is not differentiable at $x = 0$. We find a weak derivative of w_1;

$$D(w_1)(x) = \begin{cases} \sqrt{2}, & x \in [0, \frac{\sqrt{2}}{2}), \\ \frac{1}{\frac{\sqrt{2}}{2}-1}, & x \in [\frac{\sqrt{2}}{2}, 1], \end{cases} = \sqrt{2} - \frac{2}{1-\sqrt{2}} \cdot \theta(x - \frac{\sqrt{2}}{2}), x \in [0, 1], \tag{16}$$

where we denote by $\theta(x)$ the Heaviside function. We get

$$\int_0^1 |w_1|^2 \,\mathrm{d}x < \infty \quad \text{and} \quad \int_0^1 |D(w_1)|^2 \,\mathrm{d}x < \infty$$

meaning that $w_1, D(w_1) \in L^2$ and $w_1 \in H^1$. If we want to find the weak derivative of $D(w_1)$, then we need to find the weak derivative of $\theta(x)$. Observe that the weak derivative of the Heaviside step function is the Dirac delta function, which is not an element of L^2. Therefore $w_1 \notin H^2(\Omega)$.

Now consider w_2. Its derivative is given by

$$(w_2)_x(x) = 1 - \frac{3}{4} x^{-\frac{1}{4}} \tag{17}$$

We check if $w_2, (w_2)_x \in L^2$,

$$\begin{aligned} &\int_0^1 |w_2|^2 \,\mathrm{d}x = \int_0^1 (x - x^{\frac{3}{4}})^2 \,\mathrm{d}x < \infty \\ &\int_0^1 |(w_2)_x|^2 \,\mathrm{d}x = \int_0^1 (1 - \frac{3}{4} x^{-\frac{1}{4}})^2 \,\mathrm{d}x = \int_0^1 x - 2x^{\frac{3}{4}} + \frac{9}{8} x^{\frac{1}{2}} \,\mathrm{d}x < \infty \\ &\implies w_2, (w_2)_{xx} \in L^2 \implies w_2 \in H^1. \end{aligned}$$

To check if $w_2 \in H^2$ we need to calculate for the second derivative,

$$\begin{aligned} &(w_2)_{xx}(x) = \frac{3}{16} x^{-\frac{5}{4}}. \\ &\lim_{K \to 0^+} \int_K^1 |(w_2)_{xx}|^2 dx = \lim_{K \to 0^+} \int_K^1 \left(\frac{3}{16}\right)^2 x^{-\frac{5}{2}} dx = \lim_{K \to 0^+} -\frac{6}{16^2} \left[1 - K^{-\frac{3}{2}}\right] = -\frac{6}{16^2} + \frac{6}{16^2} \lim_{K \to 0^+} K^{-\frac{3}{2}} \not< \infty. \\ &\implies (w_2)_{xx} \notin L^2 \implies w_2 \notin H^2. \end{aligned}$$

When finding F for w_1 we get a problem with u_{xx} since it is not in H^2. This can be fixed with the help of integration by parts

$$\begin{aligned} \int_0^x f\phi \,\mathrm{d}x &= \int_0^x -\alpha u_{xx}\phi + bu_x\phi + cu\phi \,\mathrm{d}x \\ &= \alpha \int_0^x u_x \phi_x \,\mathrm{d}x + b \int_0^x u_x \phi \,\mathrm{d}x + c \int_0^x u\phi \,\mathrm{d}x = g(\phi). \end{aligned}$$

For w_2, also using integration by parts

$$F(\phi) = \int_0^x f\phi \,\mathrm{d}x = -\int_0^x g\phi_x \,\mathrm{d}x, \tag{18}$$

where $g = \int_0^x f \,\mathrm{d}x$.

Since $w_1, w_2 \in H^1$, but not in H^2 we can no longer expect the convergence rate of $p = 1$ in H^1 and $p = 2$ in L^2, which is guaranteed for functions in H^2. After doing numerical testing, the numerically estimated convergence for w_1 is $p = 0,53$ in H^1 and $p = 1,54$ in L^2. For w_2 the numerically estimated convergence is $p = 0,27$ in H^1 and $p = 1,26$ in L^2, see Figure 3. The convergence for both functions is lower than the guaranteed convergence of functions in H^2.

Refinement near singularities at $x = 0$

In the following section, we are looking at solutions for functions that have singularities at $x = 0$. Consider the functions $f_1(x) = x^{-\frac{2}{5}}$ and $f_2(x) = x^{-\frac{7}{5}}$ where $x \in (0, 1)$. We check if $f_1, f_2 \in L^2$.

$$
\begin{aligned}
&\int_0^1 |f_1|^2 \,\mathrm{d}x = \int_0^1 x^{-\frac{4}{5}} \,\mathrm{d}x = 5 \left[x^{\frac{1}{5}} \right]_0^1 = 5 < \infty \\
&\int_0^1 |f_2|^2 \,\mathrm{d}x = \lim_{K \to 0^+} \int_K^1 x^{-\frac{14}{5}} \,\mathrm{d}x = -\frac{5}{9} \lim_{K \to 0^+} \left[x^{-\frac{9}{5}} \right]_K^1 = -\frac{5}{9} \lim_{K \to 0^+} \left[1 - K^{-\frac{9}{5}} \right] = -\frac{5}{9} + \frac{5}{9} \lim_{K \to 0^+} K^{-\frac{9}{5}} \not< \infty \\
&\implies f_1 \in L^2 \text{ and } f_2 \notin L^2.
\end{aligned}
$$

When one uses f_1 as the right-hand side, one can use (8) as before since $f_1 \in L^2$. Moreover, notice that

$$
-\frac{5}{2}(f_1)_x = -\frac{5}{2}\left(-\frac{2}{5} x^{-\frac{7}{2}}\right) = x^{-\frac{7}{2}} = f_2.
$$

Lastly, solving with f_2 as the right-hand side, we can use (18) to solve for $F(\phi)$, where $g = -\frac{5}{2} f_1$. We now want to check whether placing the nodes closer to the singularity yields a smaller error than with an equidistant grid. Let $x_0 = 0$ and $x_i = r^{M-i}$, $i = 1, \ldots, M$, for some $r \in (0, 1)$, define a new refined grid.

In the following example we used $M = 50$ nodes and $a = 1$, $b = -70$, $c = 1$. We found that $r = 0.8$ is a suitable value for both f_1 and f_2 by calculating the errors for different values of r. Since we do not have the exact solutions to the problems, we utilise an approximation from a very fine grid when computing the errors. In figure 4, we compare the solutions for both equidistant and refined grid on f_1. In figure 5, we do the same for f_2. Looking at the plots, one can see that using the refined grid gives a more accurate approximation near the singularity. To further investigate, we tested with $M = 10$, 20, 50, 100 and suitable r-values for each case. In table 1, one can find errors for f_1 in H^1 and L^2, whereas table 2 shows errors for f_2 in H^1 and L^2.

Norm	Parameters	Coarse	Refined
H^1	$M = 10$, $r = 0.5$	0.144858	0.019281
L^2	$M = 10$, $r = 0.5$	0.003366	0.000222
H^1	$M = 20$, $r = 0.7$	0.108834	0.009657
L^2	$M = 20$, $r = 0.7$	0.001301	0.000064
H^1	$M = 50$, $r = 0.8$	0.055006	0.005851
L^2	$M = 50$, $r = 0.8$	0.000268	0.000025
H^1	$M = 100$, $r = 0.9$	0.028822	0.002557
L^2	$M = 100$, $r = 0.9$	0.000071	0.000006

Table 1: Error table for f_1 in H^1 and L^2.

Norm	Parameters	Coarse	Refined
H^1	$M = 10$, $r = 0.4$	2.647061	0.564115
L^2	$M = 10$, $r = 0.4$	0.034751	0.003799
H^1	$M = 20$, $r = 0.6$	2.602543	0.177010
L^2	$M = 20$, $r = 0.6$	0.021170	0.000681
H^1	$M = 50$, $r = 0.8$	2.298950	0.066916
L^2	$M = 50$, $r = 0.8$	0.008674	0.000123
H^1	$M = 100$, $r = 0.9$	1.978961	0.025310
L^2	$M = 100$, $r = 0.9$	0.004003	0.000028

Table 2: Error table for f_2 in H^1 and L^2.

The tables shows that we get a smaller error in both H^1 and L^2 with more nodes closer to the singularity at $x = 0$ compared to equidistant nodes. This is the case for both f_1 and f_2.

Conclusion

In this report, we studied a one-dimensional stationary convection diffusion problem using the finite elements method (FEM). We showed the method's validity by proving essential mathematical properties such as the Lax-Milgram theorem, the Galerkin approximation and Cea's lemma, and by applying the method to both smooth and non-smooth solutions. Our results coincided with the theoretical convergence rates and highlighted the effectiveness of a graded grid approach near singularities.

A Appendix

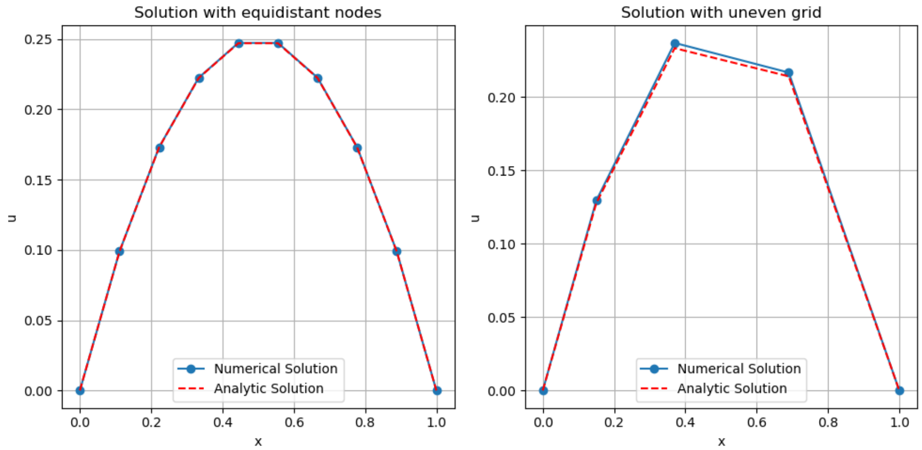

Figure 1: Solutions for $f(x) = x(1-x)$. Left plot: uneven grid with five nodes. Right plot: Grid with 100 equidistant nodes. Bottom: Stiffness matrix of the numerical solution on the left.

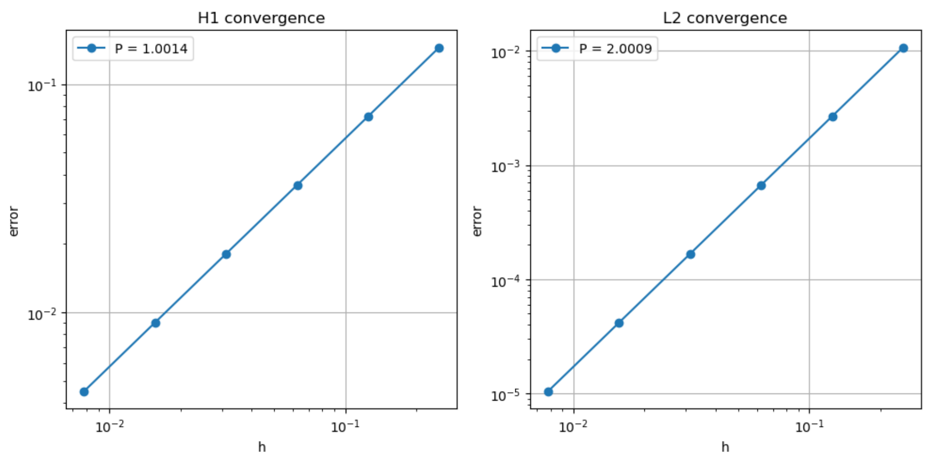

Figure 2: Loglog-plots for $f(x) = x(1-x)$. Left plot: Error in H^1. Right plot: Error in L^2.

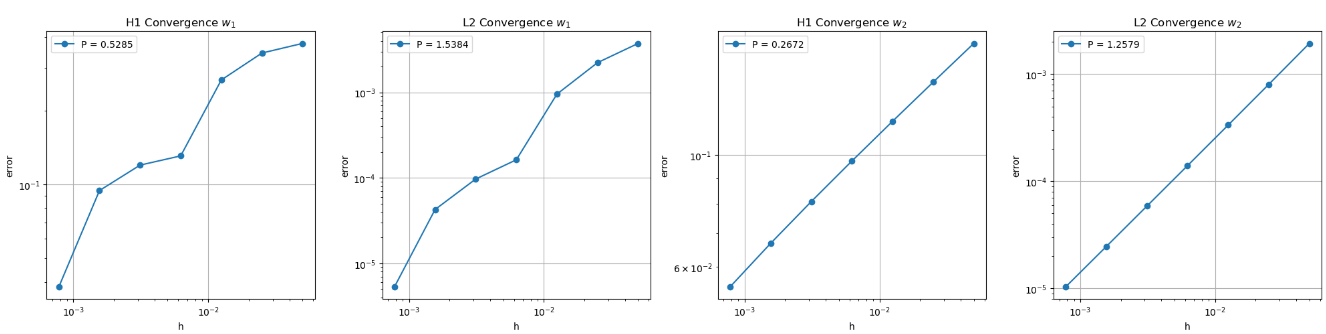

Figure 3: Loglog-plots for $w_1(x)$ and $w_2(x)$. Left plot: Error of w_1 in H^1. Left centre plot: Error of w_1 in L^2. Right centre plot: Error of w_2 in H^1. Right plot: Error of w_2 in L^2.

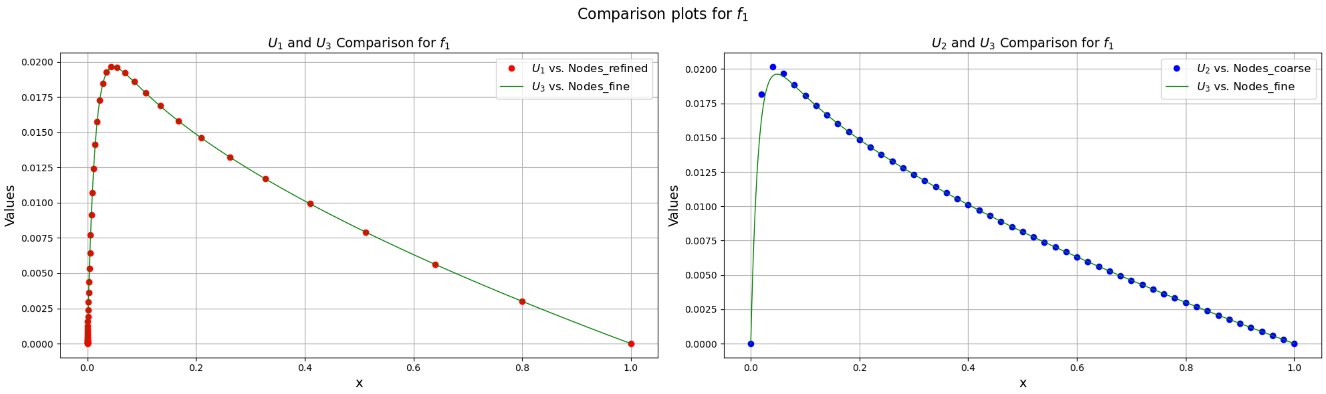

Figure 4: Loglog-plots for $w_1(x)$ and $w_2(x)$. Left plot: Error of w_1 in H^1. Left centre plot: Error of w_1 in L^2. Right centre plot: Error of w_2 in H^1. Right plot: Error of w_2 in L^2.

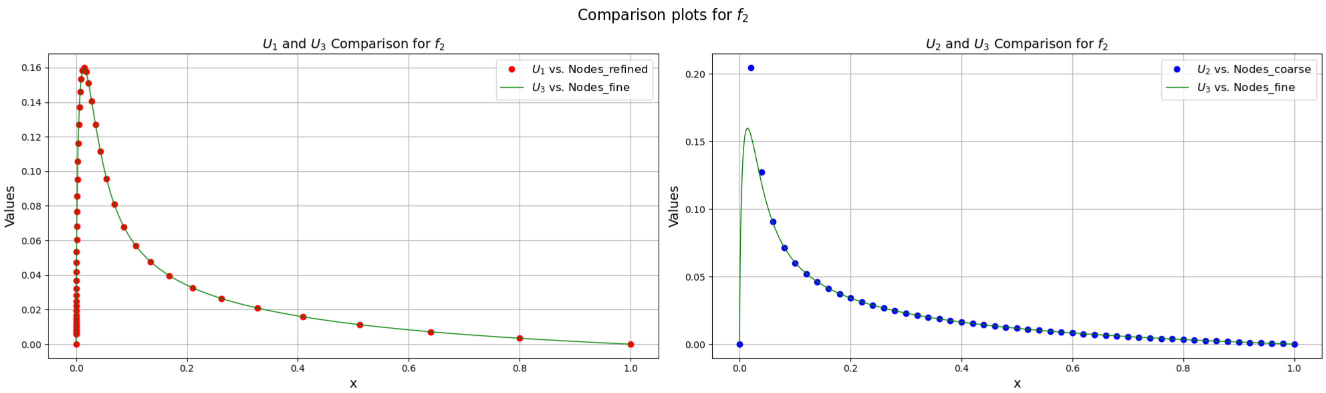

Figure 5: Loglog-plots for $w_1(x)$ and $w_2(x)$. Left plot: Error of w_1 in H^1. Left centre plot: Error of w_1 in L^2. Right centre plot: Error of w_2 in H^1. Right plot: Error of w_2 in L^2.