SIGNAL PROCESSING & FILTER DESIGN

B3 Option – 8 lectures
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Recommended texts

Analogue filters

- Paul Horowitz, Winfield Hill. *The art of electronics*. 2nd Ed. Cambridge University Press. Especially useful for getting a feel for the issues in analogue design.
- G.B Clayton. *Linear integrated circuit applications.*, MacMillan 1975.

Digital signal processing

- Lynn, PA. *An Introduction to the analysis and processing of signals.* A useful introductory text.
- Oppenheim, Willsky & Nawab. *Signals and Systems*, 2nd Ed. Prentice Hall.
- Oppenhem & Schafer. *Digital Signal Processing.*, Prentice Hall. A good book covering basic concepts and advanced topics.
- Matlab manuals (for signal processing & system identification toolboxes)

Lecture 1 - Introduction

1.1 Introduction

Signal processing is the treatment of signals (information-bearing waveforms or data) so as to extract the wanted information, removing unwanted signals and noise. The two applications mentioned below are familiar to me, but I could have equally well chosen applications as diverse as the analysis of seismic waveforms or the recording of moving rotor blade pressures and heat transfer rates.

1.1.1 Examples of Signal Processing Applications

Processing of speech signals

Up to now, human–machine communication has been almost entirely by means of keyboards and screens, but there are substantial disadvantages in this approach for many applications. The human speech perception and production processes are so complex that very complicated signal processing algorithms are required even to solve simple problems such as the recognition of single words spoken by one speaker. The list below includes some of the main applications of signal processing to speech:

- Speech storage and transmission (in order to minimise bandwidth, a set of parameters describing the speech production process is sent rather than the speech waveform itself.
- Speech enhancement (for example, improving the intelligibility of speech in a very noisy environment).
- Speech synthesis (e.g. voice output from a computer).
- Speech recognition (e.g. voice input into a computer).

- Speaker verification and identification.

Processing of Biomedical Signals

The body generates a multitude of bio-electric signals. Well known (and well analysed) signals are, for example, the electrical activity of the brain (the *electroencephalogram* or EEG) and the electrical activity of the heart (the *electrocardiogram* or ECG). Both of these signals are small (the EEG is of order μV and the ECG or order mV) and are often heavily corrupted by artifacts from the subject's breathing and body movements as well as electrical artifacts such as 50Hz noise. Signal processing is a valuable asset in clinical and research medicine for detection and analysis as well as for the removal of artifacts. Figure 1.1 shows an example of a section of EEG.

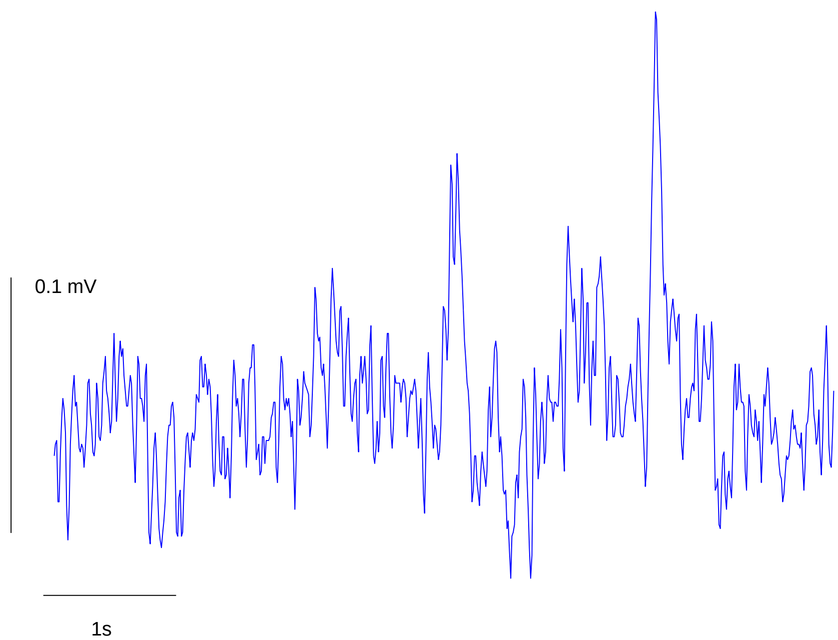

Figure 1.1: *A typical section of EEG signal. The large positive spikes are artifacts caused by eye movements.*

1.1.2 Analogue vs. Digital Signal Processing

Most of the signal processing techniques mentioned above could be used to process the original analogue (continuous-time) signals or their digital version (the signals are sampled in order to convert them to sequences of numbers). For example, the earliest type of "voice coder" developed was the *channel vocoder* which consists mainly of a bank of band-pass filters. More recent versions of this vocoder use digital (rather than analogue) filtering, although this does not result directly in any improvement in performance; however, one advantage of the digital implementation is that it can be "re-configured" as another type of vocoder (for example, the *linear prediction vocoder*); it is also much easier to encrypt digital rather than analogue data, for applications where communication must remain secure. The trend is, therefore, towards digital signal processing systems; even the well-established radio receiver has come under threat. The other great advantage of digital signal processing lies in the ease with which *non-linear* processing may be performed. Almost all recent developments in modern signal processing are in the digital domain. This lecture course concentrates on the basics though.

It is, however, important not to neglect analogue signal processing and some of the reasons for this should become clear during the course. On a more practical point, you will have noted that filters are the main topic of this course, and a thorough grounding in the design of analogue filters is a pre-requisite to understanding much of the underlying theory of digital filtering.

1.1.3 Summary/Revision of basic definitions

1.1.4 Linear Systems

A linear system may be defined as one which obeys the **Principle of Superposition**. If $x_1(t)$ and $x_2(t)$ are inputs to a linear system which gives rise to outputs $y_1(t)$ and $y_2(t)$ respectively, then the combined input $ax_1(t) + bx_2(t)$ will give rise to an output $ay_1(t) + by_2(t)$, where a and b are arbitrary constants.

Notes

- If we represent an input signal by some support in a frequency domain, $\mathcal{F}_{in}$ (i.e. the set of frequencies present in the input) then no new frequency support will be required to model the output, i.e.

$$\mathcal{F}_{out} \subseteq \mathcal{F}_{in}$$

- Linear systems can be broken down into simpler sub-systems which can be re-arranged in any order, i.e.

$$x \longrightarrow g_1 \longrightarrow g_2 \equiv x \longrightarrow g_2 \longrightarrow g_1 \equiv x \longrightarrow g_{1,2}$$

1.1.5 Time Invariance

A time-invariant system is one whose properties do not vary with time (i.e. the input signals are treated the same way regardless of their time of arrival); for example, with discrete systems, if an input sequence $x(n)$ produces an output sequence $y(n)$, then the input sequence $x(n-n_o)$ will produce the output sequence $y(n-n_o)$ for all n_o.

1.1.6 Linear Time-Invariant (LTI) Systems

Most of the lecture course will focus on the design and analysis of systems which are both linear and time-invariant. The basic linear time-invariant operation is "filtering" in its widest sense.

1.1.7 Causality

In a causal (or realisable) system, the present output signal depends only upon present and previous values of the input. (Although all practical engineering systems are necessarily causal, there are several important systems which are non-causal (non-realisable), e.g. the ideal digital differentiator.)

1.1.8 Stability

A stable system (over a finite interval T) is one which produces a bounded output in response to a bounded input (over T).

1.2 Linear Processes

Some of the common signal processing functions are amplification (or attenuation), mixing (the addition of two or more signal waveforms) or un-mixing[1] and

[1]This linear *unmixing* turns out to be one of the most interesting current topics in signal processing.

filtering. Each of these can be represented by a linear time-invariant "block" with an input-output characteristic which can be defined by:

- The *impulse response* $g(t)$ in the *time domain*.
- The *transfer function* in a *frequency domain*. We will see that the choice of frequency basis may be subtly different from time to time.

As we will see, there is (for the systems we examine in this course) an *invertable* mapping between the time and frequency domain representations.

1.3 Time-Domain Analysis – convolution

Convolution allows the evaluation of the output signal from a LTI system, given its *impulse response* and input signal.

The input signal can be considered as being composed of a succession of impulse functions, each of which generates a weighted version of the impulse response at the output, as shown in 1.2.

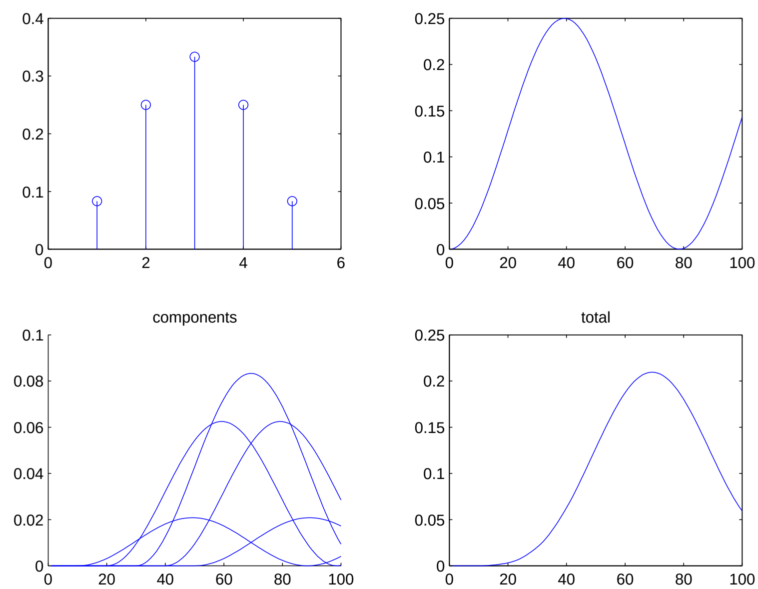

Figure 1.2: *Convolution as a summation over shifted impulse responses.*

The output at time t, $y(t)$, is obtained simply by adding the effect of each separate impulse function – this gives rise to the *convolution integral*:

$$y(t) = \sum_{\tau} \{x(t-\tau)d\tau\} g(\tau) \xrightarrow{d\tau \to 0} \int_0^\infty x(t-\tau) g(\tau) d\tau$$

τ is a dummy variable which represents time measured "back into the past" from the instant t at which the output $y(t)$ is to be calculated.

1.3.1 Notes

- Convolution is commutative. Thus:

$$y(t) \text{ is also } \int_0^\infty x(\tau) g(t-\tau) d\tau$$

- For discrete systems convolution is a summation operation:

$$y[n] = \sum_{k=0}^{\infty} x[k] g[n-k] = \sum_{k=0}^{\infty} x[n-k] g[k]$$

- **Relationship between convolution and correlation** The general form of the convolution integral

$$f(t) = \int_{-\infty}^{\infty} x(\tau) g(t-\tau) d\tau$$

  is very similar[2] to that of the cross-correlation function relating 2 variables $x(t)$ and $y(t)$

$$R_{xy}(\tau) = \int_{-\infty}^{\infty} x(t) \cdot y(t-\tau) dt$$

  Convolution is hence an integral over *lags* at a fixed *time* whereas *correlation* is the integral over *time* for a fixed *lag*.

[2]Note that the lower limit of the integral can be $-\infty$ or 0. Why?

- **Step response** The step function is the time integral of an impulse. As integration (and differentiation) are *linear* operations, so the order of application in a LTI system does not matter:

$$\delta(t) \longrightarrow \int dt \longrightarrow g(t) \longrightarrow \text{step response}$$

$$\delta(t) \longrightarrow g(t) \longrightarrow \int dt \longrightarrow \text{step response}$$

1.4 Frequency-Domain Analysis

LTI systems, by definition, may be represented (in the continuous case) by linear differential equations (in the discrete case by linear *difference* equations). Consider the application of the linear differential operator, $\mathcal{D}$, to the function $f(t) = e^{st}$:

$$\mathcal{D}f(t) = sf(t)$$

An equation of this form means that $f(t)$ is the *eigenfunction* of $\mathcal{D}$. Just like the eigen analysis you know from matrix theory, this means that $f(t)$ and any linear operation on $f(t)$ may be represented using a set of functions of exponential form, and that this function may be chosen to be *orthogonal*. This naturally gives rise to the use of the *Laplace* and *Fourier* representations.

- **The Laplace transform**:

  $$X(s) \longrightarrow \text{Transfer function } G(s) \longrightarrow Y(s)$$

  where,

  $$X(s) = \int_0^\infty x(t)e^{-st}dt \qquad \text{Laplace transform of } x(t)$$

  $$Y(s) = G(s)X(s)$$

  where $G(s)$ can be expressed as a *pole-zero* representation of the form:

  $$G(s) = \frac{A(s-z_1)\dots(s-z_m)}{(s-p_1)(s-p_2)\dots(s-p_n)}$$

  (NB: The inverse transformation, ie. obtaining $y(t)$ from $Y(s)$, is not a straightforward mathematical operation.)

- **The Fourier transform**:

$$X(j\omega) \longrightarrow \text{Frequency response } G(j\omega)Y(j\omega)$$

where,

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \quad \text{Fourier transform of } x(t)$$

and

$$Y(j\omega) = G(j\omega)X(j\omega)$$

The output time function can be obtained by taking the inverse Fourier transform:

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(j\omega)e^{j\omega t}d\omega$$

1.4.1 Relationship between time & frequency domains

Theorem
If $g(t)$ is the *impulse response* of an LTI system, $G(j\omega)$, the Fourier transform of $g(t)$, is the *frequency response* of the system.

Proof
Consider an input $x(t) = A\cos\omega t$ to an LTI system. Let $g(t)$ be the impulse response, with a Fourier transform $G(j\omega)$.

Using convolution, the output $y(t)$ is given by:

$$y(t) = \int_{0}^{\infty} A\cos\omega(t-\tau)g(\tau)d\tau$$

$$= \frac{A}{2}\int_{0}^{\infty} e^{j\omega(t-\tau)}g(\tau)d\tau + \frac{A}{2}\int_{0}^{\infty} e^{-j\omega(t-\tau)}g(\tau)d\tau$$

$$= \frac{A}{2}e^{j\omega t}\int_{-\infty}^{\infty} g(\tau)e^{-j\omega\tau}d\tau + \frac{A}{2}e^{-j\omega t}\int_{-\infty}^{\infty} g(\tau)e^{j\omega\tau d\tau}$$

(lower limit of integration can be changed from 0 to $-\infty$ since $g(\tau) = 0$ for $t < 0$)

$$= \frac{A}{2}\{e^{j\omega t}G(j\omega) + e^{-j\omega t}G(-j\omega)\}$$

Let $G(j\omega) = Ce^{j\phi}$ ie. $C = |G(j\omega)|, \quad \phi = arg\{G(j\omega)\}$

$$\text{Then } y(t) = \frac{AC}{2}\{e^{j(\omega t+\phi)} + e^{-j(\omega t+\phi)}\} = CA\cos(\omega t + \phi)$$

i.e. an input sinusoid has its amplitude scaled by $|G(j\omega)|$ and its phase changed by arg$\{G(j\omega)\}$, where $G(j\omega)$ is the Fourier transform of the impulse response $g(t)$.

Theorem
Convolution in the time domain is equivalent to multiplication in the frequency domain i.e.

$$y(t) = g(t) * x(t) \equiv \mathcal{F}^{-1}\{Y(j\omega) = G(j\omega)X(j\omega)\}$$

and

$$y(t) = g(t) * x(t) \equiv \mathcal{L}^{-1}\{Y(s) = G(s)X(s)\}$$

Proof
Consider the general integral (Laplace) transform of a shifted function:

$$\begin{aligned} \mathcal{L}\{f(t-\tau)\} &= \int_t f(t-\tau)e^{-st}dt \\ &= e^{-s\tau}\mathcal{L}\{f(t)\} \end{aligned}$$

Now consider the Laplace transform of the convolution integral

$$\begin{aligned} \mathcal{L}\{f(t) * g(t)\} &= \int_t\int_\tau f(t-\tau)g(\tau)d\tau e^{-st}dt \\ &= \int_\tau g(\tau)e^{-s\tau}d\tau\mathcal{L}\{f(t)\} \\ &= \mathcal{L}\{g(t)\}\mathcal{L}\{f(t)\} \end{aligned}$$

By allowing $s \to j\omega$ we prove the result for the Fourier transform as well.