Hitting Probabilities and Hitting Times for Stochastic Fluid Flows

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Abstract

Recently there has been considerable interest in Markovian stochastic fluid flow models. A number of authors have used different methods to calculate quantities of interest. In this paper, we consider a fluid flow model, formulated so that time is preserved, and derive expressions for return probabilities to the initial level, the Laplace-Stieltjes transforms (for arguments with nonnegative real part only) and moments of the time taken to return to the initial level, excursion probabilities to high/low levels, and the Laplace-Stieltjes transforms of sojourn times in specified sets. An important feature of our results is their physical interpretation within the stochastic fluid flow environment, which is given. This allows for further implementation of our expressions in the calculation of other quantities of interest.

Novel aspects of our treatment include the calculation of probability densities with respect to level and an argument under which we condition on the infimum of the levels at which a "down-up period" occurs.

Significantly, these results are achieved with techniques applied directly within the fluid flow model, without the need for discretization or transformation to other equivalent models.

Keywords Markovian fluid model, hitting probabilities, hitting times, sojourn times.

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1 Introduction

Stochastic fluid flow models have been studied using a variety of techniques by, for example, Anick, Mitra and Sondhi [2], Rogers [23], Asmussen [3], Ramaswami [22], Da Silva Soares and Latouche [9] and, more recently, Ahn and Ramaswami [1]. In such models, the rate of increase or decrease of the level of a fluid is governed by the state of an underlying continuous-time Markov chain. Specifically, when the underlying Markov chain is in state i, the fluid increases at rate c_i, which can be positive, zero or negative. Of interest have been the questions of when such a model is stable and, if so, what is its stationary distribution. To address these questions, Anick, Mitra and Sondhi used spectral methods, Asmussen conditioned on the maximum level of the fluid in a busy period, Ramaswami conditioned on the last epoch of increase in a busy period and Da Silva Soares and Latouche used conditioning on both the first and the last epochs of increase in a busy period.

Each of Asmussen [3], Ramaswami [22] and Da Silva Soares and Latouche [9] considered a simplified model in which c_i is constrained to be 1 or -1. The reason for doing this is that more general models can be transformed into such a model, while preserving the hitting probabilities on the initial level. Moreover, the stationary distribution of the original model can be shown to be equivalent to that of the transformed model.

This transformation, however, does not preserve the times taken to traverse sample paths. Thus, to calculate moments of elapsed times, it is necessary to work within the general model. We do this here. Like Ramaswami [22], Da Silva Soares and Latouche [9] and Ahn and Ramaswami [1], we apply matrix-analytic methods [16, 18, 19, 21]. However, unlike them, we work with the original fluid flow model, rather than an equivalent discrete-level model. Because of this, our expressions have useful physical interpretations within the stochastic fluid flow environment and can be applied in further analysis of the model.

Ahn and Ramaswami [1] also considered a general fluid model. They derived several interesting results for the transient analysis of this model. They achieved these results by the method of stochastic coupling to a queue. First, they constructed an appropriate queueing model, then derived several results for this model, and finally, by taking limits, derived the results for the original fluid flow model. Our methodology is significantly different. We apply our analysis directly within the fluid flow model. The technique that we introduce here in order to do so is the calculation of probability densities *with respect to level*, combined with an application of the semi-group property. The performance measures analyzed here are different from those studied by Ahn and Ramaswami [1], with one exception (see Remark 1 in Section 3). Our contribution is not only the new methodology, but also important new results.

We achieve our results by conditioning on the infimum of all levels at which a "down-up period" occurs. A "down-up period" occurs when the fluid level has been decreasing, possibly remained constant for some time, and then begins to increase. Via this conditioning, we derive expressions for return probabilities, and moments of the elapsed times taken for return sample paths. We calculate probability densities with respect to level, and thus treat level like 'time'. The expressions determined here can be used to compute return probabilities to the initial level, expected elapsed times and other moments of these return journeys, excursion probabilities to high/low levels, and expected sojourn times in specified sets.

We introduce the fluid flow model in Section 2, and derive the expressions for the return probabilities to the initial level, the Laplace-Stieltjes transforms and moments of return times in Section 3. In Section 4 we give formulae for the excursion probabilities to high/low levels. Here we extend the results, developed by Latouche and Taylor [17] for a quasi-birth-and-death process with a countable number of levels, to the fluid model which can be considered to have an uncountable number of levels. In Section 5 we establish the results for the expected sojourn times in specified sets. In view of the existing methods, our approach has important advantages. These are discussed in Section 6.

In [5], we discuss the implementation of algorithms to compute the expressions given here. Further, in [6] we shall consider a model in which the fluid level is bounded, both from below and above, and derive expressions for the Laplace-Stieltjes transforms of several time-related performance measures.

2 Fluid Flow Model

Our model is of a class discussed before by a number of authors. Notable amongst these are Anick, Mitra and Sondhi [2], Rogers [23], Asmussen [3], Ramaswami [22], Da Silva Soares and Latouche [9], and Ahn and Ramaswami [1]. Let $\widehat{M}(t) \in \mathbb{R}^+$ denote the level of fluid in a container at time t. The rate of increase of $\widehat{M}(t)$ is determined by the state $\varphi(t)$ of a continuous-time Markov chain with finite state space $\mathcal{S}$ and generator $\mathcal{T} = [\mathcal{T}_{ij}]$. Specifically, when $\widehat{M}(t) > 0$, the rate of increase of $\widehat{M}(t)$ is equal to $c_{\varphi(t)}$, which may be positive, negative or zero. If $\widehat{M}(t) = 0$, then $\widehat{M}(t)$ can change only if $c_{\varphi(t)} > 0$, in which case the rate of increase is $c_{\varphi(t)}$. Let $\mathcal{S}_0$, $\mathcal{S}_1$ and $\mathcal{S}_2$ denote the set of states, or *phases*, $i \in \mathcal{S}$ for which $c_i = 0$, $c_i > 0$ and $c_i < 0$ respectively.

We shall need to consider the "doubly-infinite" fluid process $\{(M(t), \varphi(t)) : t \in \mathbb{R}^+\}$, constructed from $(\widehat{M}(t), \varphi(t))$ by allowing negative fluid values. This process has

- fluid level $M(t) \in \mathbb{R}$,
- phase $\varphi(t) \in \mathcal{S}$, which moves according to an irreducible Markov chain with infinitesimal generator $\mathcal{T}$, and
- net rate c_i of input to the infinite fluid buffer for all $M(t) \in \mathbb{R}$, which is zero when $i \in \mathcal{S}_0$, positive if $i \in \mathcal{S}_1$ and negative if $i \in \mathcal{S}_2$.

Let $|\mathcal{S}_0| = s_0$, $|\mathcal{S}_1| = s_1$, $|\mathcal{S}_2| = s_2$ and partition the generator $\mathcal{T}$ according to $\mathcal{S} = \mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2$ so that

$$\mathcal{T} = \begin{bmatrix} T_{00} & T_{01} & T_{02} \\ T_{10} & T_{11} & T_{12} \\ T_{20} & T_{21} & T_{22} \end{bmatrix}.$$

Further, let C_1 be a diagonal $s_1 \times s_1$ matrix such that $[C_1]_{ii} = c_i$ for all $i \in \mathcal{S}_1$ and C_2 be a diagonal $s_2 \times s_2$ matrix such that $[C_2]_{ii} = -c_i$ for all $i \in \mathcal{S}_2$. Let $\lambda_i = -[\mathcal{T}]_{ii}$ and Λ be the diagonal $s_1 \times s_1$ matrix such that $\Lambda_{ii} = \lambda_i$ for all $i \in \mathcal{S}_1$.

Observe that the transition structure of the process $(M(t), \varphi(t))$ is independent of the fluid level, that is the process $(M(t), \varphi(t))$ is fluid-level homogenous. The transition structure of the process $(\widehat{M}(t), \varphi(t))$ is independent of the fluid level away from level zero. The process $(\widehat{M}(t), \varphi(t))$ is upward homogenous.

If the process $(\widehat{M}(t), \varphi(t))$ is positive recurrent, then the process $(M(t), \varphi(t))$ must be transient. In physical terms, if there is a downward drift (to level zero) in the process $(\widehat{M}(t), \varphi(t))$, then the process $(M(t), \varphi(t))$ will also have a downward drift (to $-\infty$). By the same argument, if $(\widehat{M}(t), \varphi(t))$ is transient (with drift to $+\infty$), then the process $(M(t), \varphi(t))$ is transient (with drift to $+\infty$). Finally, if $(\widehat{M}(t), \varphi(t))$ is null recurrent, then $(M(t), \varphi(t))$ is null recurrent.

3 Return Times to the Initial Level

Let $\theta(x) = \inf\{t > 0 : M(t) = x\}$ be the first passage time to level x in the process $(M(t), \varphi(t))$.

For all $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, let

$$\begin{aligned} \Psi_{ij} &= P[\theta(z) < \infty \ , \ \varphi(\theta(z)) = j \mid M(0) = z, \varphi(0) = i], \\ \Xi_{ji} &= P[\theta(z) < \infty \ , \ \varphi(\theta(z)) = i \mid M(0) = z, \varphi(0) = j] \end{aligned}$$

and let $\Psi = [\Psi_{ij}]$ and $\Xi = [\Xi_{ij}]$. The entry Ψ_{ij} is the probability that, starting from level z in phase $i \in \mathcal{S}_1$, the process $(M(t), \varphi(t))$ first returns to level z in finite time and does so in phase $j \in \mathcal{S}_2$, while avoiding levels below z. The entry Ξ_{ji} is the probability that, starting from level z in phase $j \in \mathcal{S}_2$, the process $(M(t), \varphi(t))$ first returns to level z in finite time, and does so in phase $i \in \mathcal{S}_1$, while avoiding levels above z. Note that, since these probabilities do not depend on the starting level, we have dropped the subscript z from the notation.

We can define a level-independent matrix analogous to Ψ for the process $(\widehat{M}(t), \varphi(t))$. In fact, because of the upward homogeneity of both processes, this matrix is the same as Ψ. However, we cannot define a level-independent matrix analogous to Ξ for the process $(\widehat{M}(t), \varphi(t))$, because the influence of the boundary level zero ensures that the probability analogous to Ξ_{ij} depends on z.

Note the symmetry of the matrices Ψ and Ξ. First, observe that

- the set of paths starting from level z in some phase in $\mathcal{S}_1$, first reaching level z in finite time, and doing so in some phase in $\mathcal{S}_2$, while avoiding levels below z,

is symmetrical to

- the set of paths starting from level z in some phase in $\mathcal{S}_2$, first reaching level z in finite time, and doing so in some phase in $\mathcal{S}_1$, while avoiding levels above z.

Consequently, an expression for Ξ can be derived from an expression for Ψ by swapping $\mathcal{S}_1$ and $\mathcal{S}_2$ and any relevant notation.

For $i \in \mathcal{S}_1$, let

$$\psi_i(j,u) = P[\theta(z) \leq u \ , \varphi(\theta(z)) = j \mid M(0) = z, \varphi(0) = i]. \tag{1}$$

The function $\psi_i(j,u)$ is the joint probability mass/distribution function that, starting from level z in phase $i \in \mathcal{S}_1$, the process $(M(t), \varphi(t))$ first returns to level z at time less than or equal to u, and does so in phase $j \in \mathcal{S}_2$, while avoiding levels below z. Define the matrix $\psi(u)$ such that $[\psi(u)]_{ij} = \psi_i(j,u)$. It follows that

$$\Psi = \int_0^\infty d\psi(u). \tag{2}$$

The matrices recording the n-th moments of the return times are

$$\Upsilon^{(n)} = \int_0^\infty u^n d\psi(u). \tag{3}$$

So, for example, $\Upsilon^{(1)}_{ij}$ is the expected elapsed time of a return journey to the initial level, which starts in phase $i \in \mathcal{S}_1$ and ends in phase $j \in \mathcal{S}_2$.

Below we develop a method for calculating the return probabilities Ψ_{ij}, and the moments $\Upsilon^{(n)}_{ij}$ of the return times. We do this by calculating the Laplace-Stieltjes transforms of the return times. Expressions for Ξ_{ij} and corresponding expected elapsed times and other moments follow by symmetry. First, we introduce the important matrix $Q(s)$ and establish two supporting lemmas, which contain the physical interpretation of the matrices in $Q(s)$. The main result of this section is contained in Theorem 1, which gives two expressions for the Laplace-Stieltjes transform $\mathcal{L}([\psi(u)]_{ij})$ of $[\psi(u)]_{ij}$. As corollaries, we subsequently give the results for Ψ_{ij}, $\Upsilon^{(1)}_{ij}$ and other moments. In the proof of Theorem 1, we introduce a new technique: conditioning on the infimum of all levels at which a "down-up period" occurs.

Let

$$Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

where

$$\begin{aligned} Q_{11} &= C_1^{-1}[T_{11} - T_{10}(T_{00})^{-1}T_{01}], \\ Q_{22} &= C_2^{-1}[T_{22} - T_{20}(T_{00})^{-1}T_{02}], \\ Q_{12} &= C_1^{-1}[T_{12} - T_{10}(T_{00})^{-1}T_{02}], \\ Q_{21} &= C_2^{-1}[T_{21} - T_{20}(T_{00})^{-1}T_{01}]. \end{aligned}$$

Asmussen [3, Lemma 3.1] and Rogers [23] suggested "pre-processing" a fluid model identical to ours into a process with net input rates restricted to 1 and -1 only and transition rates between phases given by Q. This mapping does not change the matrices Ψ and Ξ. Consequently, by the expressions for the model with net input rates equal to 1 or -1 in [9], we have

$$\begin{aligned}\Psi &= \int_0^\infty e^{Q_{11}x} Q_{12} e^{(Q_{22}+Q_{21}\Psi)x} dx \\ &= \int_0^\infty e^{(Q_{11}+\Psi Q_{21})x} Q_{12} e^{Q_{22}x} dx.\end{aligned}$$

However, the above-mentioned pre-processing does not preserve the elapsed times of sample paths that return to the initial level, and so cannot be used if we are interested in the Laplace-Stieltjes transforms and moments $\Upsilon_{ij}^{(n)}$ of the return times. Therefore, we introduce the following important matrix, which is useful for calculating these quantities. Let

$$Q(s) = \begin{bmatrix} Q_{11}(s) & Q_{12}(s) \\ Q_{21}(s) & Q_{22}(s) \end{bmatrix},$$

where, for s with $Re(s) \geq 0$,

$$\begin{aligned}Q_{11}(s) &= C_1^{-1}[(T_{11} - sI) - T_{10}(T_{00} - sI)^{-1}T_{01}], \\ Q_{22}(s) &= C_2^{-1}[(T_{22} - sI) - T_{20}(T_{00} - sI)^{-1}T_{02}], \\ Q_{12}(s) &= C_1^{-1}[T_{12} - T_{10}(T_{00} - sI)^{-1}T_{02}], \\ Q_{21}(s) &= C_2^{-1}[T_{21} - T_{20}(T_{00} - sI)^{-1}T_{01}]. \end{aligned} \tag{4}$$

Note that $Q \equiv Q(0)$. The condition $Re(s) \geq 0$ guarantees the existence of the inverse $(T_{00} - sI)^{-1}$. Lemmas 1 and 2 below contain the physical interpretation of the matrices in $Q(s)$.

Let $f(t) = \int_0^t |c_{\varphi(t)}|dt$. $f(t) \geq 0$ can be interpreted as the total amount of fluid that has flowed into or out of the buffer during the time interval $(0, t]$. Let $\omega(x) = \inf\{t > 0 : f(t) = x\}$. For $i, j \in \mathcal{S}_1 \cup \mathcal{S}_2$, let

$$\begin{aligned}&\delta_i^y(j,t) = \\ &\qquad P[\omega(y) \leq t, \varphi(\omega(y)) = j | M(0) = 0, \varphi(0) = i].\end{aligned} \tag{5}$$

The function $\delta_i^y(j,t)$ is the joint probability mass/distribution function that, starting from level zero in phase i, the the total amount of fluid that has flown into or out of the buffer first reaches y at time less than or equal to t, and does so in phase j. Let $\hat{\Delta}^y(s)$ be the matrix of the Laplace-Stieltjes transforms whose (i,j)-th entry is $\int_0^\infty e^{-st} d\delta_i^y(j,t)$. Also, let $\delta^y(u)$ be such that $[\delta^y(u)]_{ij} = \delta_i^y(j,u)$.

Below we show that the matrix $\hat{\Delta}^y(s)$ defines a semigroup and that the matrix $Q(s)$ is the generator of this semigroup.

Lemma 1 *For $Re(s) \geq 0$, the matrix $\hat{\Delta}^y(s)$ is given by*

$$\hat{\Delta}^y(s) = e^{Q(s)y}.$$

Proof

For all $y > 0$, $z > 0$, we have

$$\delta_i^{y+z}(j,t) = \sum_{k \in \mathcal{S}} \int_0^t \delta_i^y(k, t-u) d\delta_k^z(j,u),$$

and so

$$\hat{\Delta}^{y+z}(s) = \hat{\Delta}^y(s)\hat{\Delta}^z(s). \tag{6}$$

Also,

$$\hat{\Delta}^0(s) = I. \tag{7}$$

By (6) and (7), $\{\hat{\Delta}^y(s), y > 0\}$ is a continuous semi-group and so, by [8], $\hat{\Delta}^y(s)$ must be of the form $e^{G(s)y}$, where the generator $G(s)$ is defined by

$$G(s) = \frac{d}{dh}\hat{\Delta}^h(s)\bigg|_{h=0}. \tag{8}$$

Assume that the process starts from level zero in phase i and is observed until the total amount of fluid that has flown into or out of the buffer reaches $h > 0$ and does so in phase j.

We first assume $i, j \in \mathcal{S}_1$ and show that

$$[G(s)]_{ij} = [Q_{11}(s)]_{ij}.$$

Essentially there are only three cases to consider for small h, as all other events happen with probability $o(h)$.

1. The phase process remains in phase i until the total amount of fluid that has flown into or out of the buffer reaches h. Conditioning on this event happening, this happens at time h/c_i and so $\delta_i^h(i,t)$ has a jump at the point $t = h/c_i$ and its Laplace-Stieltjes transform is $e^{-s(h/c_i)}$. Multiply this by the probability density $e^{-\lambda_i(h/c_i)}$ of this event and store the result in a diagonal $s_1 \times s_1$ matrix $\hat{\Delta}_1^h(s)$ with (i,i)-th entry given by
$$[\hat{\Delta}_1^h(s)]_{ii} = e^{-s(\frac{h}{c_i})}e^{-\lambda_i(\frac{h}{c_i})}.$$
It follows that,
$$\frac{d}{dh}[\hat{\Delta}_1^h(s)]_{ii}\bigg|_{h=0} = -\frac{s+\lambda_i}{c_i},$$
and so
$$\frac{d}{dh}\hat{\Delta}_1^h(s)\bigg|_{h=0} = -C_1^{-1}(sI+\Lambda). \tag{9}$$

2. The phase process makes a single transition from i to some phase $j \neq i \in \mathcal{S}_1$ when the total amount of fluid that has flown into or out of the buffer reaches some u in $(0,h]$.

   - First, the probability that $\varphi(t)$ leaves phase i when the total amount of fluid that has flown into or out of the buffer is less than or equal to u, and so at time less than or equal to u/c_i, is given by $1 - e^{-\lambda_i(u/c_i)}$. Differentiating this with respect to the level u, we see that the probability density that the phase process leaves state i when the total amount of fluid that has flown into or out of the buffer is qual to $u \in (0,h]$ is $(\lambda_i/c_i)e^{-\lambda_i(u/c_i)}$.
   - On leaving the phase i, the probability that $\varphi(t)$ moves to phase j is given by $\mathcal{T}_{ij}/\lambda_i$.
   - The probability that the process remains in phase j until the total amount of fluid that has flown into or out of the buffer reaches h, that is for the remaining time $(h-u)/c_j$, is given by $e^{-\lambda_j((h-u)/c_j)}$.

   Consequently, the probability density that the process makes a single transition from i to $j \neq i \in \mathcal{S}_1$ when the total amount of fluid that has flown into or out of the buffer is equal to $u \in (0,h]$ is
$$\frac{1}{c_i}e^{-\lambda_i(\frac{u}{c_i})}\mathcal{T}_{ij}e^{-\lambda_j(\frac{h-u}{c_j})}. \tag{10}$$

When this event occurs, the time at which the total amount of fluid that has flown into or out of the buffer reaches h is $u/c_i + (h-u)/c_j$. Thus $\delta_i^h(j,t)$ has a jump at the point $t = u/c_i + (h-u)/c_j$ with probability density (10). Multiplying $e^{-s(u/c_i+(h-u)/c_j)}$ by the probability density (10) of occurrence and integrating, results in an $s_1 \times s_1$ matrix $\hat{\Delta}_2^h(s)$ with(i,i)-th entry 0, and (i,j)-th entry, for $i \neq j$, given by

$$[\hat{\Delta}_2^h(s)]_{ij} = \int_0^h e^{-s(\frac{u}{c_i}+\frac{h-u}{c_j})} \frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} \mathcal{T}_{ij} e^{-\lambda_j(\frac{h-u}{c_j})} du.$$

We have,

$$\begin{aligned}
\frac{d}{dh}[\hat{\Delta}_2^h(s)]_{ij}\Big|_{h=0} &= \frac{d}{dh}\left[\frac{\mathcal{T}_{ij}}{c_i} e^{-h(\frac{s+\lambda_j}{c_j})} \int_0^h e^{-u(\frac{\lambda_i+s}{c_i}-\frac{\lambda_j+s}{c_j})} du\right]_{h=0} \\
&= \frac{\mathcal{T}_{ij}}{c_i}\frac{d}{dh}\left[\int_0^h e^{-u(\frac{\lambda_i+s}{c_i}-\frac{\lambda_j+s}{c_j})} du\right]_{h=0} \\
&= \frac{\mathcal{T}_{ij}}{c_i}, \quad \text{for } i \neq j,
\end{aligned}$$

and so

$$\frac{d}{dh}\hat{\Delta}_2^h(s)\Big|_{h=0} = C_1^{-1}(T_{11} + \Lambda). \tag{11}$$

3. The phase process makes a transition from i to some phase in $\mathcal{S}_0$ when the total amount of fluid that has flown into or out of the buffer is in $(0,h]$, spends some time in the set $\mathcal{S}_0$, and then makes a transition from the set $\mathcal{S}_0$ to some phase $j \in \mathcal{S}_1$, and remains in phase j until the the total amount of fluid that has flown into or out of the buffer reaches h.

   - By the argument in step 2, the probability density that the phase process leaves state i when the total amount of fluid that has flown into or out of the buffer is equal to $u \in (0,h]$, and so at time u/c_i, is $(\lambda_i/c_i)e^{-\lambda_i(u/c_i)}$.
   - The probability that, on leaving the phase i, $\varphi(t)$ moves to $k \in \mathcal{S}_0$ is $\mathcal{T}_{ik}/\lambda_i$.
   - The probability density that the phase process spends time t in the set $\mathcal{S}_0$ and then makes a transition from the set $\mathcal{S}_0$ to phase $j \in \mathcal{S}_1$ is given by $[e^{T_{00}t}T_{01}]_{kj}$.

- The probability that the phase process remains in phase j as the total amount of fluid that has flown into or out of the buffer moves from u to h is given by $e^{-\lambda_j((h-u)/c_j)}$.

Consequently, the probability density that the phase process makes a transition from i to some phase in $\mathcal{S}_0$ when the total amount of fluid that has flown into or out of the buffer is in $u \in (0, h]$, spends time t in the set $\mathcal{S}_0$, and then makes a transition from the set $\mathcal{S}_0$ to phase $j \in \mathcal{S}_1$, and remains in phase j until the total amount of fluid that has flown into or out of the buffer reaches h, is given by

$$\frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} [T_{10} e^{T_{00}t} T_{01}]_{ij} e^{-\lambda_j(\frac{h-u}{c_j})}. \tag{12}$$

When this event occurs, the time taken for the total amount of fluid that has flown into or out of the buffer to reach h is $u/c_i + t + (h - u)/c_j$. Multiplying $e^{-s(u/c_i+t+(h-u)/c_j)}$ by the probability density (12) and integrating with respect to both u and t, results in an $s_1 \times s_1$ matrix $\hat{\Delta}_3^h(s)$ with the (i,j)-th entry, $[\hat{\Delta}_3^h(s)]_{ij}$ given by

$$\int_0^h e^{-s(\frac{u}{c_i}+\frac{h-u}{c_j})} \frac{1}{c_i} e^{-\lambda_i(\frac{u}{c_i})} [T_{10} \int_0^\infty e^{-st} e^{T_{00}t} T_{01} dt]_{ij} e^{-\lambda_j(\frac{h-u}{c_j})} du.$$

In a manner similar to step 2, we obtain

$$\frac{d}{dh}[\hat{\Delta}_3^h(s)]_{ij}\bigg|_{h=0} = \frac{1}{c_i}[T_{10} \int_0^\infty e^{-st} e^{T_{00}t} T_{01} dt]_{ij},$$

and so

$$\frac{d}{dh}\hat{\Delta}_3^h(s)\bigg|_{h=0} = -C_1^{-1} T_{10}(T_{00} - sI)^{-1} T_{01}. \tag{13}$$

Now, by (8), (9), (11) and (13), we have, for $i, j \in \mathcal{S}_1$,

$$\begin{aligned}
[G(s)]_{ij} &= [\frac{d}{dh}\hat{\Delta}^h(s)\bigg|_{h=0}]_{ij} \\
&= [\frac{d}{dh}\Big(\hat{\Delta}_1^h(s) + \hat{\Delta}_2^h(s) + \hat{\Delta}_3^h(s)\Big)\bigg|_{h=0}]_{ij} \\
&= [C_1^{-1}\{(T_{11} - sI) - T_{10}(T_{00} - sI)^{-1} T_{01}\}]_{ij} \\
&= [Q_{11}(s)]_{ij}.
\end{aligned}$$

The proof for the matrix $Q_{12}(s)$ follows by an argument analogous to steps 2 to 3 of the proof above (Simply, replace $j \in \mathcal{S}_1$, T_{11}, T_{01} and $Q_{11}(s)$ with $j \in \mathcal{S}_2$, T_{12}, T_{02} and $Q_{12}(s)$, respectively). The results for the matrices $Q_{22}(s)$ and $Q_{21}(s)$ follow by symmetry. ■

We now introduce the concepts of an "up-down period" and a "down-up period", which we later use in our conditioning in the proof of Theorem 1. Assume that the process $(M(t), \varphi(t))$ starts from some level z in some phase $i \in \mathcal{S}_1 \cup \mathcal{S}_2$. If $i \in \mathcal{S}_1$, we let $\chi_0 = 0$, and, for $n \geq 0$,

$$\begin{aligned}
\tau_{n+1} &= \inf\{t \in (\chi_n, \infty) : \varphi(t) \in S_2\}, \\
\xi_{n+1} &= \inf\{t \in (\chi_n, \tau_{n+1}) : M(t) = M(\tau_{n+1})\}, \\
\chi_{n+1} &= \inf\{t \in (\tau_{n+1}, \infty) : \varphi(t) \in S_1\}, \\
\eta_{n+1} &= \inf\{t \in (\tau_{n+1}, \chi_{n+1}) : M(t) = M(\chi_{n+1})\}.
\end{aligned}$$

Similarly, if $i \in \mathcal{S}_2$, we let $\tau_0 = 0$, and, for $n \geq 0$,

$$\begin{aligned}
\chi_{n+1} &= \inf\{t \in (\tau_n, \infty) : \varphi(t) \in S_1\}, \\
\eta_{n+1} &= \inf\{t \in (\tau_n, \chi_{n+1}) : M(t) = M(\chi_{n+1})\}, \\
\tau_{n+1} &= \inf\{t \in (\chi_{n+1}, \infty) : \varphi(t) \in S_2\}, \\
\xi_{n+1} &= \inf\{t \in (\chi_{n+1}, \tau_{n+1}) : M(t) = M(\tau_{n+1})\}.
\end{aligned}$$

If a transition from $\mathcal{S}_1$ to $\mathcal{S}_2$ occurs at τ_n, then $\xi_n = \tau_n$. However, it might also happen that a transition from $\mathcal{S}_0$ to $\mathcal{S}_2$ occurs at τ_n, in which case ξ_n is the time that $\varphi(t)$ enters $\mathcal{S}_0$, which is strictly less than τ_n. A portion of the sample path that starts at time ξ_i and ends at time τ_i is referred to as an "up-down period". Similarly, a portion of the sample path that starts at time η_i and ends at time χ_i is referred to as a "down-up period". In Figure 1 below, an "up-down" period is indicated with a thick-solid line and two "down-up" periods are indicated with a thick-dashed line and a star, respectively.

Next, we define matrices $\hat{\mathcal{A}}^y(s)$, $\hat{\mathcal{B}}^y(s)$, $\hat{\mathcal{G}}^y(s)$ and $\hat{\mathcal{D}}^y(s)$. In Lemma 2 we give the expressions for these matrices in terms of the matrices $Q_{11}(s)$, $Q_{22}(s)$, $Q_{12}(s)$ and $Q_{21}(s)$, respectively.

Assume that the process $(M(t), \varphi(t))$ starts in phase $i \in \mathcal{S}_1$ at level zero and stays in the set $\mathcal{S}_1 \cup \mathcal{S}_0$ at least until reaching level $y > 0$. For $i, j \in \mathcal{S}_1$, let

$$\alpha_i^y(j,t) = P[\theta(y) \leq t, \varphi(\theta(y)) = j, M(\tau_1) \geq y | M(0) = 0, \varphi(0) = i]. \quad (14)$$

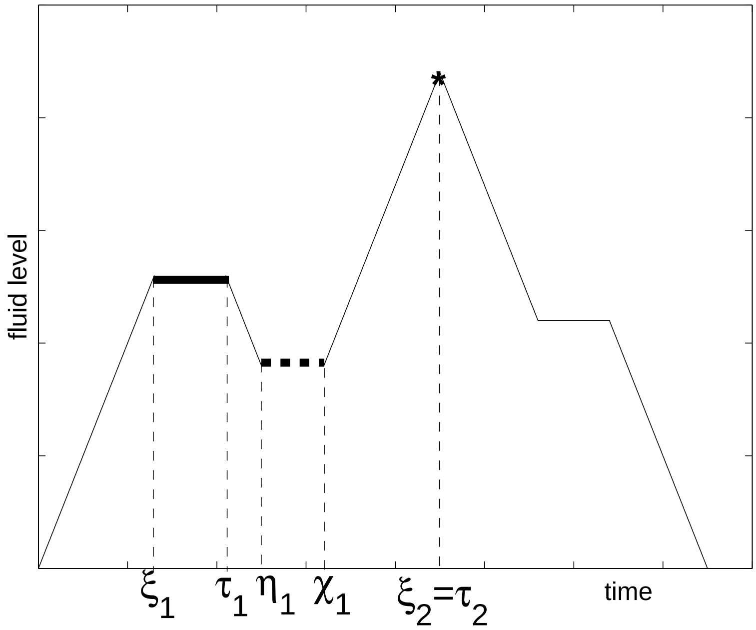

Figure 1: A sample path and its “up-down” and “down-up” periods.

$\alpha_i^y(j,t)$ is the joint probability mass/distribution function that, starting from level zero in phase i, the process first reaches level y at time less than or equal to t, and does so in phase j, and the level at time τ_1 is greater than or equal to y. Let $\hat{\mathcal{A}}^y(s)$ be the matrix of the Laplace-Stieltjes transforms whose (i,j)-th entry is $\int_0^\infty e^{-st} d\alpha_i^y(j,t)$. Also, let $\alpha^y(u)$ be such that $[\alpha^y(u)]_{ij} = \alpha_i^y(j,u)$.

Similarly, assume that the process $(M(t),\varphi(t))$ starts in phase $i \in \mathcal{S}_2$ at level $y > 0$ and stays in the set $\mathcal{S}_2 \cup \mathcal{S}_0$ at least until reaching level zero. For $i,j \in \mathcal{S}_2$, let

$$\beta_i^y(j,t) = P[\theta(0) \le t, \varphi(\theta(0)) = j, M(\chi_1) \le 0 | M(0) = y, \varphi(y) = i]. \tag{15}$$

$\beta_i^y(j,t)$ is the joint probability mass/distribution function that, starting from level y in phase i, the process first reaches level zero at time less than or equal to t, and does so in phase j, and the level at time χ_1 is less than or equal to zero. Let $\hat{\mathcal{B}}^y(s)$ be the matrix of the Laplace-Stieltjes transforms whose (i,j)-th entry is $\int_0^\infty e^{-st} d\beta_i^y(j,t)$. Also, let $\beta^y(u)$ be such that $[\beta^y(u)]_{ij} = \beta_i^y(j,u)$.

Assume that the process $(M(t),\varphi(t))$ starts in phase $i \in \mathcal{S}_1$ at level zero. For $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$, $y > 0$, let

$$\gamma_i^y(j,t) = P[\tau_1 \le t, M(\tau_1) \le y, \varphi(\tau_1) = j | M(0) = 0, \varphi(0) = i]. \tag{16}$$

$\gamma_i^y(j,t)$ is the joint probability mass/distribution function that, starting from level zero in phase $i \in \mathcal{S}_1$, the process first enters the set $\mathcal{S}_2$ at time less than or equal to t and at level less than or equal to y, and does so in phase j. Let $\hat{\mathcal{G}}^y(s)$ be the matrix of the Laplace-Stieltjes transforms whose (i,j)-th entry is $\int_0^\infty e^{-st} d\gamma_i^y(j,t)$, and let $\gamma^y(u)$ be such that $[\gamma^y(u)]_{ij} = \gamma_i^y(j,u)$.

Similarly, assume that the process $(M(t), \varphi(t))$ starts in phase $i \in \mathcal{S}_2$ at some level $y > 0$. For $i \in \mathcal{S}_2$, $j \in \mathcal{S}_1$, let

$$\delta_i^y(j,t) = P[\chi_1 \leq t, M(\chi_1) \geq 0, \varphi(\chi_1) = j | M(0) = y, \varphi(0) = i]. \tag{17}$$

$\delta_i^y(j,t)$ is the joint probability mass/distribution function that, starting from level y in phase $i \in \mathcal{S}_2$, the process first enters the set $\mathcal{S}_1$ at time less than or equal to t and at level greater than or equal to zero, and does so in phase j. Let $\hat{\mathcal{D}}^y(s)$ be the matrix of the Laplace-Stieltjes transforms whose (i,j)-th entry is $\int_0^\infty e^{-st} d\delta_i^y(j,t)$, and let $\delta^y(u)$ be such that $[\delta^y(u)]_{ij} = \delta_i^y(j,u)$.

The result below states that the families of matrices $\hat{\mathcal{A}}^y(s)$ and $\hat{\mathcal{B}}^y(s)$ define semigroups and that the matrices $Q_{11}(s)$ and $Q_{22}(s)$ are generators of these semigroups. Further, it contains a physical interpretation for the matrices $Q_{12}(s)$ and $Q_{21}(s)$ in terms of matrices $\hat{\mathcal{G}}^y(s)$ and $\hat{\mathcal{D}}^y(s)$, respectively. Note that $Q_{11}(0) \equiv Q_{11}$, $Q_{22}(0) \equiv Q_{22}$, $Q_{12}(0) \equiv Q_{12}$ and $Q_{21}(0) \equiv Q_{21}$.

Lemma 2 *Let $Re(s) \geq 0$. The matrices $\hat{\mathcal{A}}^y(s)$ and $\hat{\mathcal{B}}^y(s)$ are given by*

$$\hat{\mathcal{A}}^y(s) = e^{Q_{11}(s)y} \quad \text{and} \quad \hat{\mathcal{B}}^y(s) = e^{Q_{22}(s)y}.$$

The matrices $\hat{\mathcal{G}}^y(s)$ and $\hat{\mathcal{D}}^y(s)$ satisfy the equations

$$\frac{d}{dh}\hat{\mathcal{G}}^h(s)\bigg|_{h=0} = Q_{12}(s) \quad \text{and} \quad \frac{d}{dh}\hat{\mathcal{D}}^h(s)\bigg|_{h=0} = Q_{21}(s).$$

Proof

The result for $\hat{\mathcal{A}}^y(s)$ follows by an argument analogous to the proof of Lemma 1. The result for $\hat{\mathcal{G}}^y(s)$ follows by an argument analogous to steps 2 and 3 in the proof of Lemma 1 (Simply, replace $j \in \mathcal{S}_1$, T_{11}, T_{01} and $Q_{11}(s)$ with $j \in \mathcal{S}_2$, T_{12}, T_{02} and $\hat{\mathcal{G}}^y(s)$, respectively).

The results for $\hat{\mathcal{B}}^y(s)$ and $\hat{\mathcal{D}}^y(s)$ follow then by symmetry. ■

Corollary 1 *Suppose that the process $(M(t), \varphi(t))$ starts from level zero in phase $i \in \mathcal{S}_1$. The probability that it will first reach level y at some finite* *time afterwards, and do so in phase $j \in \mathcal{S}_1$ while the phase process stays in the set $\mathcal{S}_1 \cup \mathcal{S}_0$ is given by the (i,j)-th entry of the matrix*

$$e^{Q_{11}y}. \tag{18}$$

The expected elapsed time of this journey is given by the (i,j)-th entry of the matrix

$$\sum_{n=1}^{\infty} \frac{y^n}{n!} \sum_{i=0}^{n-1} (Q_{11})^i C_1^{-1}[I + T_{10}(T_{00}^{-1})^2 T_{01}](Q_{11})^{n-1-i}. \tag{19}$$

Similarly, suppose that the process $(M(t), \varphi(t))$ starts from level y in phase $i \in \mathcal{S}_2$. The probability that it will first reach level zero at some finite time afterwards, and do so in phase $j \in \mathcal{S}_2$, while the phase process stays in the set $\mathcal{S}_2 \cup \mathcal{S}_0$ is given by the (i,j)-th entry of the matrix

$$e^{Q_{22}y}. \tag{20}$$

The expected elapsed time of this journey is given by the (i,j)-th entry of the matrix

$$\sum_{n=1}^{\infty} \frac{y^n}{n!} \sum_{i=0}^{n-1} (Q_{22})^i C_2^{-1}[I + T_{20}(T_{00}^{-1})^2 T_{02}](Q_{22})^{n-1-i}. \tag{21}$$

Proof

We prove (18) and (19) for the case with $i, j \in \mathcal{S}_1$. The expressions (20) and (21) follow by symmetry. Let $s \geq 0$. Note that $Q_{11}(s)\mathbf{e} \leq Q_{11}\mathbf{e}$, and so $\hat{\mathcal{A}}^y(s)$ exists. Therefore, the change of order of limit and infinite summation, and the change of order of differentiation and infinite summation, in the proof below, are justified.

To obtain (18), take the limit as $s \to 0^+$ in $\hat{\mathcal{A}}^y(s)$. The matrix (19) recording the expected elapsed times is given by $-\lim_{s\to 0^+} \frac{d}{ds}\hat{\mathcal{A}}^y(s)$. Differentiating $\hat{\mathcal{A}}^y(s)$ with respect to s and taking the limit as $s \to 0^+$ gives

$$\begin{aligned}
\lim_{s\to 0^+} \frac{d}{ds}\hat{\mathcal{A}}^y(s) &= \lim_{s\to 0^+} \sum_{n=1}^{\infty} \frac{\left(\frac{d}{ds}[Q_{11}(s)]^n\right) y^n}{n!} \\
&= \sum_{n=1}^{\infty} \frac{y^n}{n!} \sum_{i=0}^{n-1} (Q_{11})^i \Big(\lim_{s\to 0^+} \frac{d}{ds} Q_{11}(s) \Big) Q_{11}^{n-i-1}.
\end{aligned}$$

Note that

$$\begin{aligned}\lim_{s\to 0^+}\frac{d}{ds}Q_{11}(s) &= \lim_{s\to 0^+}\frac{d}{ds}C_1^{-1}[(T_{11}-sI)+T_{10}\Big(\int_0^\infty e^{-su}e^{T_{00}u}du\Big)T_{01}] \\ &= -C_1^{-1}[I+T_{10}\Big(\int_0^\infty ue^{T_{00}u}du\Big)T_{01}] \\ &= -C_1^{-1}[I+T_{10}(T_{00}^{-1})^2T_{01}]\end{aligned}$$

and so the result follows. ■

Lemma 3 *For s with $Re(s) \geq 0$, the spectra of $Q_{11}(s)$ and $Q_{22}(s)$ are contained in the open left half plane.*

Proof
Note that $e^{Q_{11}(s)}$ is a matrix recording Laplace-Transforms of a *probability* function. From the physical interpretation of the matrix $e^{Q_{11}(s)}$ given in Lemma 1, for $Re(s) \geq 0$ and $y > 0$ we have $||e^{Q_{11}(s)y}|| < 1$. Consequently, by [10, Theorem 3.14], the spectrum of $Q_{11}(s)$ is contained in the open left half plane. An identical argument holds for $Q_{22}(s)$. ■

Let $\hat{\Psi}(s)$ be the matrix such that $[\hat{\Psi}(s)]_{ij} = \mathcal{L}([\psi(u)]_{ij})$. Thus, for $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, $\hat{\Psi}(s)$ records the Laplace-Stieltjes transforms of the times taken by sample paths that start in phase i at level z and first return to level z in phase j, while avoiding levels below zero.

Theorem 1 *The matrix $\hat{\Psi}(s)$ satisfies*

$$\hat{\Psi}(s) = \int_0^\infty e^{Q_{11}(s)y}\Big(Q_{12}(s)+\hat{\Psi}(s)Q_{21}(s)\hat{\Psi}(s)\Big)e^{Q_{22}(s)y}dy. \tag{22}$$

For $Re(s) \geq 0$, this integral exists and satisfies the equation

$$Q_{12}(s)+\hat{\Psi}(s)Q_{21}(s)\hat{\Psi}(s)+Q_{11}(s)\hat{\Psi}(s)+\hat{\Psi}(s)Q_{22}(s) = 0. \tag{23}$$

Furthermore, if s is real then $\hat{\Psi}(s)$ is the minimal nonnegative solution of this equation.

Remark 1 *Ahn and Ramaswami [1] also studied the matrix $\hat{\Psi}(s)$, but their methodology was different. They studied a sequence of discrete level processes that converges to the fluid flow process in the limit. The analysis here is applied directly within the fluid flow model. Equations (31) to (33) in Theorem 12 in [1] reduce to (23). As pointed out in [1], the solutions to this equation for a fixed complex s are not unique. Our observation that, for real s, $\hat{\Psi}(s)$ is the minimal nonnegative solution is new.*

Proof
Assuming that the process $(M(t), \varphi(t))$ starts from level zero in some phase in $\mathcal{S}_1$, let

$$\begin{aligned}(\eta^*, \chi^*) &= ((\eta_r, \chi_r) : M(\chi_r) = \inf_i M(\chi_i), \chi_i \in (0, \theta(0)), i, r \geq 1),\\ \sigma_1 &= \inf\{t \in (0, \theta(0)) : M(t) = M(\chi^*)\},\\ \sigma_2 &= \sup\{t \in (0, \theta(0)) : M(t) = M(\chi^*)\}.\end{aligned}$$

Thus η^* and χ^* are the times marking the beginning and the end of the "down-up period" which occurs at the minimum level reached during the time interval $(0, \theta(0))$. These are unique with probability one. The times σ_1 and σ_2 mark the first and the last time this minimum level is reached, respectively.

Also, let

$$(\xi^*, \tau^*) = ((\xi_r, \tau_r) : M(\tau_r) = \sup_i M(\tau_i), \tau_i \in (0, \theta(0)), i, r \geq 1).$$

ξ^* and τ^* are times marking the beginning and the end of the "up-down period" which occurs at the maximum level reached during the time interval $(0, \theta(0))$.

Assume that the process $(M(t), \varphi(t))$ starts from level zero in phase $i \in \mathcal{S}_1$ and reaches level zero again at some finite time afterwards, and does so in phase $j \in \mathcal{S}_2$, while avoiding levels below zero. During this journey, either

- there are no "down-up periods", or
- there is a positive number of "down-up periods".

Denote the matrix recording the Laplace-Stieltjes transforms of elapsed times for sample paths of the first type by $\hat{\mathcal{J}}_1(s)$. By conditioning on $y = M(\tau^*)$, the maximum level reached, we partition the sample path starting from fluid level zero in phase $i \in \mathcal{S}_1$ and returning to level 0 in phase $j \in \mathcal{S}_2$ into the three following stages (see Figure 2):

- The phase process remains in $\mathcal{S}_1 \cup \mathcal{S}_0$ for the duration of time $v = \xi^*$, until it reaches level y and does so in some phase $k \in \mathcal{S}_1$.
- An "up-down period" occurs at the end of which, at time τ^*, the phase process enters the set $\mathcal{S}_2$, and does so in some phase ℓ whereupon the fluid level begins to decrease.

- Starting from level y and phase ℓ, the phase process remains in $\mathcal{S}_2 \cup \mathcal{S}_0$ for the duration of time $z = \theta(0) - \tau^*$, until it first returns to level zero, and does so in phase $j \in \mathcal{S}_2$.

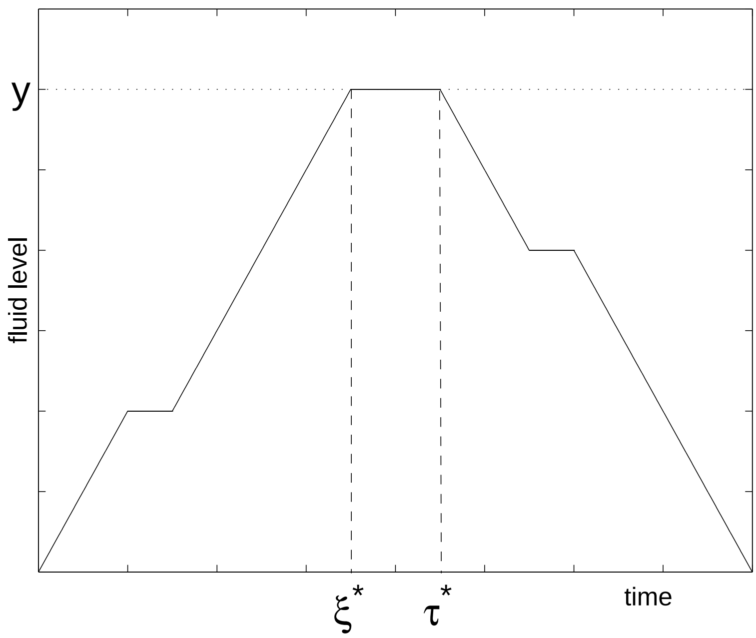

Figure 2: A sample path and its level y.

Conditioning on y, by Lemma 2, we can express $\hat{\mathcal{J}}_1(s)$ in the form

$$\begin{aligned}\hat{\mathcal{J}}_1(s) &= \int_{y=0}^{\infty}\int_{v=0}^{\infty}\int_{z=0}^{\infty} e^{-s(v+z)} d\alpha^y(v) Q_{12}(s) d\beta^y(z) dy \\ &= \int_0^{\infty} \hat{\mathcal{A}}^y(s) Q_{12}(s) \hat{\mathcal{B}}^y(s) dy.\end{aligned}$$

In order to calculate the matrix recording the Laplace-Stieltjes transforms of elapsed times for sample paths of the second type, denoted by $\hat{\mathcal{J}}_2(s)$, we condition on the level $w = M(\chi^*)$, the infimum of all levels in a sample path, at which a "down-up period" occurs. We illustrate this in Figure 3. Note that this conditioning is analogous to the conditioning used for the simple random walk by Kennedy [15].

A sample path starting from fluid level zero in phase $i \in \mathcal{S}_1$ and returning to level 0 in phase $j \in \mathcal{S}_2$ can be broken up into five stages. These are:

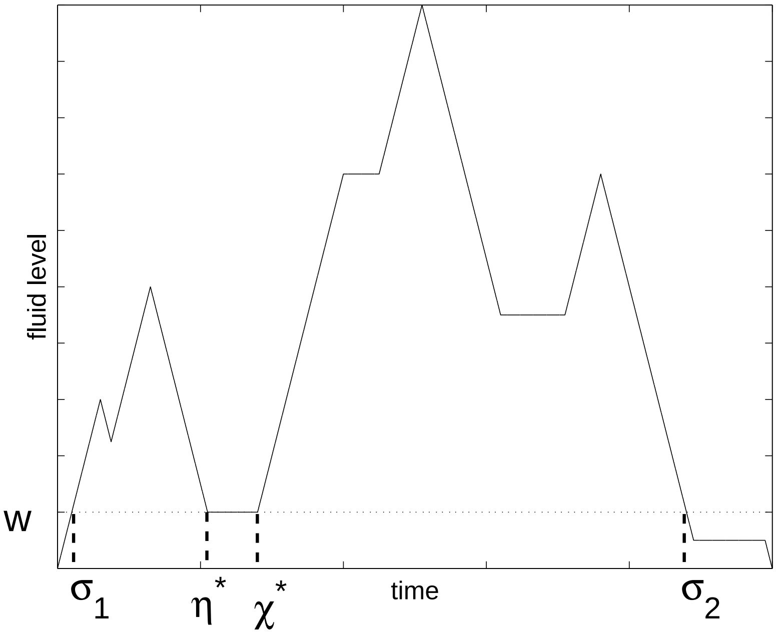

Figure 3: A sample path and its level w.

- The phase process remains in $\mathcal{S}_1 \cup \mathcal{S}_0$ for the duration of time $v = \sigma_1$, until it reaches level w and does so in some phase $k \in \mathcal{S}_1$.

- Starting from level w and phase $k \in \mathcal{S}_1$, the process remains in levels above w for the duration of time at most ν, until it first returns to level w in some phase $k^{'} \in \mathcal{S}_2$, at time η^*. By the spatial homogeneity of the process, the probability distribution for this path is $\psi(\nu)_{kk'}$.

- A "down-up period" occurs at the end of which, at time χ^*, the phase process enters the set $\mathcal{S}_1$, and does so in some phase $\ell^{'} \in \mathcal{S}_1$ whereupon the fluid level begins to increase.

- Starting from level w and phase $\ell^{'} \in \mathcal{S}_1$, the process remains in levels above w for the duration of time at most u, until it first returns to level w, and does so in some phase $\ell \in \mathcal{S}_2$, at time σ_2. The probability distribution of this path is $\psi(u)_{\ell'\ell}$.

- Starting from level w and phase $\ell \in \mathcal{S}_2$,the phase process remains in $\mathcal{S}_2 \cup \mathcal{S}_0$ for the duration of time $z = \theta(0) - \sigma_2$, until it first returns to level zero, and does so in phase $j \in \mathcal{S}_2$.

Conditioning on w, by Lemma 2, we can express $\hat{\mathcal{J}}_2(s)$ in the form

$$\int_{w=0}^{\infty}\int_{v=0}^{\infty}\int_{\nu=0}^{\infty}\int_{u=0}^{\infty}\int_{z=0}^{\infty} e^{-s(v+\nu+u+z)} d\alpha^w(v) d\psi(\nu) Q_{21}(s) d\psi(u) d\beta^w(z) dw$$
$$= \int_{w=0}^{\infty} \hat{\mathcal{A}}^w(s)\hat{\Psi}(s)Q_{21}(s)\hat{\Psi}(s)\hat{\mathcal{B}}^w(s)dw.$$

Since $\hat{\Psi}(s) = \hat{\mathcal{J}}_1(s) + \hat{\mathcal{J}}_2(s)$, (22) follows by Lemma 2. By Lemma 3 and Theorem 9.2 in [7], the integral

$$Y = \int_0^{\infty} e^{Q_{11}(s)y}\Big(Q_{12}(s) + \hat{\Psi}(s)Q_{21}(s)\hat{\Psi}(s)\Big)e^{Q_{22}(s)y}dy$$

exists and is a solution to the equation

$$Q_{11}(s)Y + YQ_{22}(s) = -\Big(Q_{12}(s) + \hat{\Psi}(s)Q_{21}(s)\hat{\Psi}(s)\Big),$$

and so, by (22), (23) follows. It only remains to prove that for $s \geq 0$ $\hat{\Psi}(s)$ is the minimal nonnegative solution. This can be done by following standard techniques [16, 20]. Let $\hat{\Psi}_n(s)$ be the Laplace-Stieltjes transform $\hat{\Psi}(s)$ restricted to paths with at most n "down-up periods" so that

$$\hat{\Psi}(s) = \lim_{n\to\infty} \hat{\Psi}_n(s).$$

We can show by mathematical induction that for $s \geq 0$ the sequence $\{\hat{\Psi}_n(s)\}$ is increasing and bounded from above by any nonnegative solution to (23). Consequently, its limit $\hat{\Psi}(s)$ is the minimal nonnegative solution of (23). ■

As corollaries of Theorem 1, we establish equations for Ψ_{ij} and other moments. The result below includes two such equations. As far as the authors know, the first is new. For the model with nonzero net input rates, an equation analogous to the second was earlier established by Rogers [23] using Wiener-Hopf factorization. This is a nonsymmetric Riccati equation, the algorithmic solution of which was discussed by Guo [13]. We discuss the physical interpretations of these algorithms and compare their performance in [5].

Corollary 2 *The matrix Ψ satisfies:*

$$\Psi = \int_0^{\infty} e^{Q_{11}y}[Q_{12} + \Psi Q_{21}\Psi]e^{Q_{22}y}dy. \tag{24}$$

Furthermore, Ψ is the minimal nonegative solution of

$$Q_{12} + \Psi Q_{21} \Psi + Q_{11} \Psi + \Psi Q_{22} = 0. \tag{25}$$

Proof

Let $s \geq 0$. The result follows from Theorem 1 by taking limits as $s \to 0^+$. ■

For notational convenience we write $\Upsilon^{(0)} = \Psi$. The next result gives a recursive formula, which allows for the calculation of the matrix $\Upsilon^{(n)}$, once the matrices $\Upsilon^{(0)}$, $\Upsilon^{(1)}$,...,$\Upsilon^{(n-1)}$ are known. To solve equations such as (27) below, we recommend the Bartels-Stewart algorithm [4]. Here, and throughout, we define $\sum_k^{\ell}(\ldots) \equiv 0$ if $k > \ell$.

Corollary 3 *If the matrix Π, defined as the Kronecker sum*

$$\Pi = (Q_{22} + Q_{21}\Psi)^t \oplus (Q_{11} + \Psi Q_{21}) \tag{26}$$

is nonsingular, then, for $n \geq 1$, the matrix $\Upsilon^{(n)}$ is finite and is the unique solution of the equation,

$$\begin{aligned}
&(Q_{11} + \Psi Q_{21})\Upsilon^{(n)} + \Upsilon^{(n)}(Q_{22} + Q_{21}\Psi) \\
&\quad = (-1)^{n+1} Q_{12}^{(n)} - \Psi \sum_{j=1}^{n} (-1)^j \binom{n}{j} Q_{21}^{(j)} \Upsilon^{(n-j)} \\
&\qquad - \sum_{i=1}^{n-1} \binom{n}{i} \Upsilon^{(i)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} Q_{21}^{(j)} \Upsilon^{(n-i-j)} \\
&\qquad - \sum_{i=1}^{n} (-1)^i \binom{n}{i} Q_{11}^{(i)} \Upsilon^{(n-i)} - \sum_{i=0}^{n-1} (-1)^{n-i} \binom{n}{i} \Upsilon^{(i)} Q_{22}^{(n-i)},
\end{aligned} \tag{27}$$

where

$$\begin{aligned}
Q_{21}^{(0)} &= Q_{21}, \\
Q_{11}^{(1)} &= -C_1^{-1}[I + T_{10}(T_{00}^{-1})^2 T_{01}], \\
Q_{22}^{(1)} &= -C_2^{-1}[I + T_{20}(T_{00}^{-1})^2 T_{02}], \\
Q_{11}^{(i)} &= -i! C_1^{-1} T_{10} (T_{00}^{-1})^{i+1} T_{01}, \ i > 1, \\
Q_{22}^{(i)} &= -i! C_2^{-1} T_{20} (T_{00}^{-1})^{i+1} T_{02}, \ i > 1, \\
Q_{12}^{(i)} &= -i! C_1^{-1} T_{10} (T_{00}^{-1})^{i+1} T_{02}, \ i \geq 1, \\
Q_{21}^{(i)} &= -i! C_2^{-1} T_{20} (T_{00}^{-1})^{i+1} T_{01}, \ i \geq 1.
\end{aligned}$$

If the matrix Π is singular, then $\Upsilon^{(n)}$ is infinite.

Proof
Let $s \geq 0$. The matrices $Q_{11}^{(i)}$, $Q_{22}^{(i)}$, $Q_{12}^{(i)}$ and $Q_{21}^{(i)}$ as defined above are the limits as $s \to 0^+$ of the i-th derivatives of $Q_{11}(s)$, $Q_{22}(s)$, $Q_{12}(s)$ and $Q_{21}(s)$ with respect to s. Differentiating (23) n times with respect to s and taking limits as $s \to 0^+$ gives, by Leibnitz's rule [11],

$$\begin{aligned} 0 \;=\; & Q_{12}^{(n)} + \sum_{i=0}^{n}(-1)^i \binom{n}{i} \Upsilon^{(i)} \sum_{j=0}^{n-i}(-1)^{n-i-j}\binom{n-i}{j} Q_{21}^{(j)}\Upsilon^{(n-i-j)} \\ & + \sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i} Q_{11}^{(i)}\Upsilon^{(n-i)} + \sum_{i=0}^{n}(-1)^i\binom{n}{i}\Upsilon^{(i)}Q_{22}^{(n-i)}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} & \sum_{i=0}^{n}(-1)^i \binom{n}{i} \Upsilon^{(i)} \sum_{j=0}^{n-i}(-1)^{n-i-j}\binom{n-i}{j} Q_{21}^{(j)}\Upsilon^{(n-i-j)} \\ & = \sum_{i=1}^{n-1}\binom{n}{i}\Upsilon^{(i)}\sum_{j=0}^{n-i}(-1)^{n-j}\binom{n-i}{j}Q_{21}^{(j)}\Upsilon^{(n-i-j)} + (-1)^n\Upsilon^{(n)}Q_{21}\Psi \\ & + \Psi\sum_{j=1}^{n}(-1)^{n-j}\binom{n}{j}Q_{21}^{(j)}\Upsilon^{(n-j)} + (-1)^n\Psi Q_{21}\Upsilon^{(n)}, \end{aligned} \tag{29}$$

$$\sum_{i=0}^{n}(-1)^{n-i}\binom{n}{i}Q_{11}^{(i)}\Upsilon^{(n-i)} = \sum_{i=1}^{n}(-1)^{n-i}\binom{n}{i}Q_{11}^{(i)}\Upsilon^{(n-i)} + (-1)^n Q_{11}\Upsilon^{(n)}, \tag{30}$$

and

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\Upsilon^{(i)}Q_{22}^{(n-i)} = \sum_{i=0}^{n-1}(-1)^i\binom{n}{i}\Upsilon^{(i)}Q_{22}^{(n-i)} + (-1)^n\Upsilon^{(n)}Q_{22}. \tag{31}$$

Equation (27) follows by (28), (29), (30) and (31). By [12] and mathematical induction, if Π is nonsingular, then for all $n \geq 1$ (27) has a unique solution and $\Upsilon^{(n)}$ is finite. Otherwise, $\Upsilon^{(n)}$ is infinite. ■

By putting $n = 1$ in Corollary 3 we have a method for calculating the expected elapsed times $\Upsilon_{ij}^{(1)}$. Once Ψ is known, the matrix $\Upsilon^{(1)}$ can be computed using the Bartels-Stewart algorithm [4].

Corollary 4 *If the matrix Π is nonsingular, then the matrix $\Upsilon^{(1)}$, which records the expected elapsed times $\Upsilon^{(1)}_{ij}$, is finite and is the unique solution of the equation*

$$\begin{aligned}&(Q_{11}+\Psi Q_{21})\Upsilon^{(1)}+\Upsilon^{(1)}(Q_{22}+Q_{21}\Psi)\\&\quad=-C_1^{-1}T_{10}(T_{00}^{-1})^2T_{02}-\Psi C_2^{-1}T_{20}(T_{00}^{-1})^2T_{01}\Psi\\&\qquad-C_1^{-1}[I+T_{10}(T_{00}^{-1})^2T_{01}]\Psi-\Psi C_2^{-1}[I+T_{20}(T_{00}^{-1})^2T_{02}].\end{aligned}$$

If Π is singular, then $\Upsilon^{(1)}$ is infinite.

Note that if the matrix Π is nonsingular, then by Corollary 4 the matrix $\Upsilon^{(1)}$ is finite, which implies that the process is either positive recurrent or transient. Conversely, if the matrix Π is singular, then the process is null recurrent.

4 Excursion Probabilities

In the previous section we considered probabilities and elapsed times for sample paths that return to the initial starting level. In this section, we study sample paths that move to a level different from their initial starting point. We specifically are interested in sample paths over which the level $M(t)$ increases or decreases by some amount x.

For all i, $j \in \mathcal{S}_1 \cup \mathcal{S}_2$, $z \in \mathbb{R}$ and $x > 0$, let

$$G_{ij}(x) = P[\theta(z-x) < \infty\ ,\ \varphi(\theta(z-x)) = j \mid M(0) = z, \varphi(0) = i], \quad (32)$$

and

$$H_{ij}(x) = P[\theta(z+x) < \infty, \varphi(\theta(z+x)) = j \mid M(0) = z, \varphi(0) = i]. \quad (33)$$

Also, let $G(0) = \lim_{x\to 0^+} G(x)$ and $H(0) = \lim_{x\to 0^+} H(x)$. $G_{ij}(x)$ is the probability that, starting from level z in phase $i \in \mathcal{S}_1 \cup \mathcal{S}_2$, the process $(M(t), \varphi(t))$ will first reach level $(z-x)$ in finite time and do so in phase $j \in \mathcal{S}_1 \cup \mathcal{S}_2$. $H_{ij}(x)$ is the probability that, starting from level z in phase $i \in \mathcal{S}_1 \cup \mathcal{S}_2$, the process $(M(t), \varphi(t))$ will first reach level $(z+x)$ in finite time and do so in phase $j \in \mathcal{S}_1 \cup \mathcal{S}_2$. Since $(M(t), \varphi(t))$ is a homogenous process, the matrices $G(x)$ and $H(x)$ are level-independent. Consequently, we have dropped the subscript z from the notation.

We can define a level-independent matrix $\widehat{G}(x)$ analogous to $G(x)$ for the process $(\widehat{M}(t), \varphi(t))$. The matrix $\widehat{G}(x)$ is, in fact, the same as $G(x)$, because of the upward homogeneity of both processes. We cannot, however, define a level-independent matrix analogous to $H(x)$ for the process $(\widehat{M}(t), \varphi(t))$, because $(\widehat{M}(t), \varphi(t))$ is not downward homogeneous: for the process $(\widehat{M}(t), \varphi(t))$, the probabilities of fluid level first reaching level $(z+x)$ in finite time, starting from level z, depend on the choice of the initial level z.

Also, for all i, $j \in \mathcal{S}_1 \cup \mathcal{S}_2$ and $0 \leq x < y$, let

$$G_{ij}(x,y) = P[\theta(0) < \infty \ , \ \theta(0) < \theta(y) \ , \ \varphi(\theta(0)) = j \mid M(0) = x, \varphi(0) = i], \tag{34}$$

and

$$H_{ij}(x,y) = P[\theta(y) < \infty \ , \ \theta(y) < \theta(0) \ , \ \varphi(\theta(y)) = j \mid M(0) = x, \varphi(0) = i], \tag{35}$$

with $G(x,x) = \lim_{y \to x^+} G(x,y)$ and $H(x,x) = \lim_{y \to x^+} H(x,y)$. $G_{ij}(x,y)$ is the probability that, starting from level x in phase $i \in \mathcal{S}_1 \cup \mathcal{S}_2$, the process $(M(t), \varphi(t))$ will first reach level zero in finite time, and do so in phase $j \in \mathcal{S}_1 \cup \mathcal{S}_2$, while avoiding level y. $H_{ij}(x,y)$ is the probability that, starting from level x in phase $i \in \mathcal{S}_1 \cup \mathcal{S}_2$, the process $(M(t), \varphi(t))$ will first reach level y in finite time, and do so in phase $j \in \mathcal{S}_1 \cup \mathcal{S}_2$, while avoiding level zero.

We can define level-independent matrices $\widehat{G}(x,y)$ and $\widehat{H}(x,y)$ analogous to $G(x,y)$ and $H(x,y)$ for the process $(\widehat{M}(t), \varphi(t))$. In fact $\widehat{G}(x,y) = G(x,y)$ and $\widehat{H}(x,y) = H(x,y)$. Further, observe the symmetry of formulae for the matrices $G(x,y)$ and $H(x,y)$ and note that $G(x) = \lim_{y \to \infty} G(x,y)$ and $H(x) = \lim_{y \to \infty} H(y, x+y)$.

Da Silva Soares and Latouche [9] noted that $\Psi_{ij} = \lim_{x \to 0^+} [G(x)]_{ij}$ and $\Xi_{ji} = \lim_{x \to 0^+} [H(x)]_{ji}$. The matrices Ψ and Ξ can be written in terms of $G(x,y)$ and $H(x,y)$ via the relation $\Psi_{ij} = \lim_{y \to \infty} [G(0,y)]_{ij}$ and $\Xi_{ji} = \lim_{x \to \infty} [H(x,x)]_{ji}$.

Below we derive expressions for the matrices $G(x)$ and $H(x)$ in terms of the matrices Ψ and Ξ and expressions for the matrices $G(x,y)$ and $H(x,y)$ in terms of the matrices $G(x)$ and $H(x)$.

By [9] we have

$$G(x) = \begin{bmatrix} 0 & G_{12}(x) \\ 0 & G_{22}(x) \end{bmatrix}. \tag{36}$$

where the decomposition is made according to $\mathcal{S}_1 \cup \mathcal{S}_2$. Similarly, as $c_i < 0$ for all $i \in \mathcal{S}_2$, we also have

$$H(x) = \begin{bmatrix} H_{11}(x) & 0 \\ H_{21}(x) & 0 \end{bmatrix}. \tag{37}$$

Theorem 2 *The non-zero blocks of the matrices $G(x)$ and $H(x)$ in (36) and (37) are given by*

$$\begin{aligned} G_{22}(x) &= e^{(Q_{22}+Q_{21}\Psi)x}, \\ G_{12}(x) &= \Psi e^{(Q_{22}+Q_{21}\Psi)x}, \\ H_{11}(x) &= e^{(Q_{11}+Q_{12}\Xi)x}, \\ H_{21}(x) &= \Xi e^{(Q_{11}+Q_{12}\Xi)x}. \end{aligned}$$

Proof
The mapping into a process with unit rates of increase or decrease, discussed in Section 3 does not change the matrices $G(x)$ and $H(x)$. Therefore, as noted in da Silva Soares and Latouche [9], the expressions for $G_{12}(x)$ and $G_{22}(x)$ follow by a straightforward adaptation of Theorem 3.2 in Ramaswami [22]. The expressions for the matrices $H_{11}(x)$ and $H_{21}(x)$ follow by symmetry. ■

Corollary 5 *The matrices $G_{12}(x)$, $G_{22}(x)$, $H_{11}(x)$ and $H_{21}(x)$ are positive matrices for $x > 0$.*

Proof
In the Introduction, we have assumed that $\mathcal{T}$ is irreducible. Therefore, by [14], Ψ is a positive matrix and, by symmetry, Ξ is a positive matrix. The result then follows from Theorem 2. ■

Now partition $G(x, y)$ and $H(x, y)$ according to $\mathcal{S}_1 \cup \mathcal{S}_2$. Note that we have

$$G(x, y) = \begin{bmatrix} 0 & G_{12}(x, y) \\ 0 & G_{22}(x, y) \end{bmatrix},$$

as $c_i > 0$ for all $i \in \mathcal{S}_1$. Similarly,

$$H(x, y) = \begin{bmatrix} H_{11}(x, y) & 0 \\ H_{21}(x, y) & 0 \end{bmatrix},$$

as $c_i < 0$ for all $i \in \mathcal{S}_2$.

Latouche and Taylor [17, Theorem 4.2] established a relationship between matrices similar to $G(x)$, $G(x,y)$, $H(x)$, $H(x,y)$ for a countable-level model. An analogous relationship holds for the uncountable-level model considered here. This is given in the theorem below.

Theorem 3 *For $0 \leq x \leq y$, the matrix*

$$\begin{bmatrix} G(x,y) & H(x,y) \end{bmatrix}$$

is a solution of the equation

$$\begin{bmatrix} G(x,y) & H(x,y) \end{bmatrix} \begin{bmatrix} I & H(y) \\ G(y) & I \end{bmatrix} = \begin{bmatrix} G(x) & H(y-x) \end{bmatrix}. \tag{38}$$

Also,

$$\begin{bmatrix} G(x,y) & H(x,y) \end{bmatrix} \mathbf{e} = \mathbf{e}, \tag{39}$$

where $\mathbf{e}$ *is a column of ones of the appropriate dimension.*

Proof
The argument here follows that presented in [17]. We assume $x < y$. The case $x = y$ follows by taking limits as $y \to x^+$.

Note that

$$G(x) = G(x,y) + H(x,y)G(y), \tag{40}$$

because, starting from level x, the first visit to level zero can occur either by

- reaching level zero while avoiding level y, or
- reaching level y while avoiding level zero, and then reaching level zero.

By symmetry, we also have

$$H(y-x) = H(x,y) + G(x,y)H(y). \tag{41}$$

Hence the matrices $G(x,y)$ and $H(x,y)$ satisfy equation (38).

If the process $(\widehat{M}(t), \varphi(t))$ is positive recurrent then, by their physical interpretation, the matrix $G(x)$ is stochastic while $H(x)$ is substochastic, but not stochastic. If the process $(\widehat{M}(t), \varphi(t))$ is null recurrent, then their physical interpretation tells us that both $G(x)$ and $H(x)$ are stochastic. If the process $(\widehat{M}(t), \varphi(t))$ is transient, then the matrix $H(x)$ is stochastic, while $G(x)$ is substochastic, but not stochastic.

Suppose that the matrix $G(x)$ is stochastic. Then by (40) we have

$$\begin{aligned} \begin{bmatrix} G(x,y) & H(x,y) \end{bmatrix} \mathbf{e} &= G(x,y)\mathbf{e} + H(x,y)\mathbf{e} \\ &= (G(x) - H(x,y)G(y))\mathbf{e} + H(x,y)\mathbf{e} \\ &= \mathbf{e} - H(x,y)\mathbf{e} + H(x,y)\mathbf{e} \\ &= \mathbf{e}, \end{aligned}$$

giving (39). The proof for the case with $H(x)$ stochastic is analogous. The physical interpetation of (39) is that, assuming that the process starts from level x, after some finite time and with probability one, it will either reach level zero or reach level y. ■

Corollary 6 *If $(\widehat{M}(t), \varphi(t))$ is either a positive recurrent or transient process, then for $0 \leq x \leq y$ we have*

$$\begin{bmatrix} G(x,y) & H(x,y) \end{bmatrix} = \begin{bmatrix} G(x) & H(y-x) \end{bmatrix} \begin{bmatrix} I & H(y) \\ G(y) & I \end{bmatrix}^{-1}.$$

Proof

Since, we can directly verify that

$$\begin{bmatrix} I & H(y) \\ G(y) & I \end{bmatrix}^{-1} =$$
$$\begin{bmatrix} I & -H(y) \\ -G(y) & I \end{bmatrix} \begin{bmatrix} (I - H(y)G(y))^{-1} & 0 \\ 0 & (I - G(y)H(y))^{-1} \end{bmatrix},$$

the corollary follows immediately from Theorem 3, provided we can show that $(I - H(y)G(y))^{-1}$ and $(I - G(y)H(y))^{-1}$ exist.

Let $E_i(y)$ be the expected total number of visits to level y, assuming that the process starts from level y in phase i. Since $(\widehat{M}(t), \varphi(t))$ is transient or positive recurrent, the process $(M(t), \varphi(t))$ is transient by the observations in Section 3. Hence, $E_i(y)$ is finite. Note that $\sum_j [(G(y)H(y))^k]_{ij}$ is the probability that starting from (i, y) the process $(M(t), \varphi(t))$ alternately visits level zero and y at least k times, before never visiting y again. Consequently, $\sum_j (I - G(y)H(y))^{-1}_{ij} \leq E_i(y)$. Hence, $(I - G(y)H(y))^{-1}$ is finite. Similarly we can show that $(I - H(y)G(y))^{-1}_{ij}$ is finite. The result follows. ■

5 Sojourn Times in Specified Sets

Consider the following question. Suppose that the process $(M(t), \varphi(t))$ starts from level zero in phase $i \in \mathcal{S}_1$ and then first returns to level zero at some finite time afterwards, while avoiding levels below zero, and does so in phase $j \in \mathcal{S}_2$. What is the expected time Υ^y_{ij} spent in levels above the (possibly large) level $y > 0$ during this journey? For all $i \in \mathcal{S}_1$, $j \in \mathcal{S}_2$, let $[\Upsilon^y]_{ij} = \Upsilon^y_{ij}$. Let U^y be the time spent above level $y + z$ on a sample path that starts with $M(0) = z$ and finishes at time $\theta(z)$. For $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, let

$$\psi^y_i(j, u) = P[U^y \leq u, \varphi(\theta(z)) = j \mid M(0) = z, \varphi(0) = i]. \tag{42}$$

Thus, $\psi^y_i(j, u)$ is the probability that, starting from level z in phase $i \in \mathcal{S}_1$ at time zero, the process $(M(t), \varphi(t))$ spends time less than or equal to u above level $(y + z)$ and first returns to level z in phase $j \in \mathcal{S}_2$, while avoiding levels below z.

Define the matrix $\psi^y(u)$ such that $[\psi^y(u)]_{ij} = \psi^y_i(j, u)$. Then,

$$\Upsilon^y = \int_0^\infty u d\psi^y(u).$$

For all $i \in \mathcal{S}_1$ and $j \in \mathcal{S}_2$, let $\hat{\Psi}^y(s)$ be the matrix recording the Laplace-Stieltjes transforms $[\hat{\Psi}^y(s)]_{ij} = \mathcal{L}([\psi^y(u)]_{ij}$. We have $\Upsilon^y = -\lim_{s \to 0^+} \frac{d}{ds} \hat{\Psi}^y(s)$.

The result below gives an expression for the matrix $\hat{\Psi}^y(s)$. As a corollary we give the expression for the matrix Υ^y. Also, we give the expression for the matrix recording the probabilities that during the return journey to the initial level z, the process hits level $(y + z)$. Similarly, the n-th moments of U^y can be obtained from $\lim_{s \to 0^+} (-1)^n \left(\hat{\Psi}^y(s)\right)^{(n)}$.

Theorem 4 *The matrix $\hat{\Psi}^y(s)$ is given by*

$$\hat{\Psi}^y(s) = G_{12}(0, y) + H_{11}(0, y)\hat{\Psi}(s)\left(\sum_{N=0}^{\infty} (H_{21}(y, y)\hat{\Psi}(s))^N\right) G_{22}(y, y).$$

Proof
Assume that the process $(M(t), \varphi(t))$ starts in level zero in phase $i \in \mathcal{S}_1$,

first returns to level zero at some finite time afterwards, and does so in phase $j \in \mathcal{S}_2$. Then either the process spends no time above level y or some time above level y on this sample path. The probability of the first event is $G_{12}(0, y)$. Assume the second event happens, that is a sample path in Ψ^y occurs, and that the time spent in levels above y is less than or equal to u. Then the following three stages must occur.

- Starting from level zero in phase $i \in \mathcal{S}_1$, the process must reach level y in some phase $k \in \mathcal{S}_1$ while avoiding level zero. The probability of this is $[H_{11}(0, y)]_{ik}$.

- Starting from level y in phase $k \in \mathcal{S}_1$ the process must return to level y, and do so in some phase $k' \in \mathcal{S}_2$, while avoiding levels below y. Then, starting from level y in phase $k' \in \mathcal{S}_2$ the process returns to level y in some phase in $\mathcal{S}_1$, while avoiding level zero and then returns to level y in some phase in $\mathcal{S}_2$, while avoiding levels below y, It can do this any number of times (including none). Let the joint probability mass/distribution function that $(M(t), \varphi(t))$ starts the second stage in phase k' and finishes this stage in phase $\ell \in \mathcal{S}_2$ and with total time spent in levels above $y > 0$ less than or equal to u be given by $\phi_{k'}(\ell, u)$, and let $\phi(u)$ be such that $[\phi(u)]_{k'\ell} = \phi_{k'}(\ell, u)$.

- Starting from level y in phase $\ell \in \mathcal{S}_2$, the process must first return to level zero, while avoiding level y, and do so in phase $j \in \mathcal{S}_2$. The probability of this is $[G_{22}(y, y)]_{\ell j}$.

As a result of this decomposition of the sample path, $\hat{\Psi}^y(s)$ must satisfy the equation

$$\begin{aligned}\hat{\Psi}^y(s) &= G_{12}(0, y) + \int_0^\infty e^{-su} H_{11}(0, y) d\phi(u) G_{22}(y, y) \\ &= G_{12}(0, y) + H_{11}(0, y)\Big(\int_0^\infty e^{-su} d\phi(u)\Big) G_{22}(y, y). \qquad (43)\end{aligned}$$

We now show that

$$\int_0^\infty e^{-su} d\phi(u) = \sum_{N=1}^{\infty} \hat{\Psi}(s)(H_{21}(y, y)\hat{\Psi}(s))^{N-1}. \qquad (44)$$

Let $\phi^{(N)}(u)$ be the matrix recording the same probability distribution as $\phi(u)$, with the extra condition that $(M(t), \varphi(t))$ visits level y in some phase in $\mathcal{S}_1$ exactly N times. Thus

$$\phi(u) = \sum_{N=1}^{\infty} \phi^{(N)}(u). \tag{45}$$

When $N = 1$ we have

$$\begin{aligned}\int_0^{\infty} e^{-su} d\phi^{(1)}(u) &= \int_0^{\infty} e^{-su} d\psi(u) \\ &= \hat{\Psi}(s).\end{aligned}$$

Assume that

$$\int_0^{\infty} e^{-su} d\phi^{(N)}(u) = \hat{\Psi}(s)(H_{21}(y,y)\hat{\Psi}(s))^{N-1}. \tag{46}$$

Then for $(N+1)$ we have

$$\begin{aligned}\int_0^{\infty} e^{-su} d\phi^{(N+1)}(u) &= \int_{u=0}^{\infty} e^{-su} \Big(\int_{\nu=0}^{u} d\phi^{N}(\nu) H_{21}(y,y) d\psi(u-\nu) \Big) \\ &= \int_{\nu=0}^{\infty} \int_{u=\nu}^{\infty} e^{-su} d\phi^{N}(\nu) H_{21}(y,y) d\psi(u-\nu) \\ &= \int_{\nu=0}^{\infty} \int_{u=0}^{\infty} e^{-s(u+\nu)} d\phi^{N}(\nu) H_{21}(y,y) d\psi(u) \\ &= \Big(\int_{\nu=0}^{\infty} e^{-s\nu} d\phi^{N}(\nu) \Big) H_{21}(y,y) \Big(\int_{u=0}^{\infty} e^{-su} d\psi(u) \Big) \\ &= \hat{\Psi}(s)(H_{21}(y,y)\hat{\Psi}(s))^{N-1} H_{21}(y,y)\hat{\Psi}(s) \\ &= \hat{\Psi}(s)(H_{21}(y,y)\hat{\Psi}(s))^{N}.\end{aligned}$$

Therefore the result is established by mathematical induction. Consequently, by (45) we have

$$\begin{aligned}\int_0^{\infty} e^{-su} d\phi(u) &= \int_0^{\infty} e^{-su} \sum_{N=1}^{\infty} d\phi^{(N)}(u) \\ &= \sum_{N=1}^{\infty} \hat{\Psi}(s)(H_{21}(y,y)\hat{\Psi}(s))^{N-1}.\end{aligned}$$

Hence, by (43), the result follows. ■

Corollary 7 *The probability that, starting from level z in phase $i \in \mathcal{S}_1$, the process $(M(t), \varphi(t))$ visits level $(y+z)$, first returns to level z in finite time and does so in phase $j \in \mathcal{S}_2$, while avoiding levels below z, is the (i,j)-th entry in the matrix Ψ^y given by*

$$\Psi^y = H_{11}(0,y)\Psi(I - H_{21}(y,y)\Psi)^{-1}G_{22}(y,y). \tag{47}$$

Further, if the process is positive recurrent or transient, then the matrix Υ^y is finite and given by

$$\begin{aligned}\Upsilon^y &= H_{11}(0,y)\Big[I + \Psi(I - H_{21}(y,y)\Psi)^{-1}H_{21}(y,y)\Big] \times \\ &\quad \Upsilon^{(1)}(I - H_{21}(y,y)\Psi)^{-1}G_{22}(y,y). \end{aligned} \tag{48}$$

If the process is null recurrent, then the matrix Υ^y is infinite.

Proof

Let $s \geq 0$. The proof of (47) and (48) will follow easily if we can justify a change in the order of limit and infinite summation, and in the order of derivative and infinite summation. For all $y > 0$ and $i \in \mathcal{S}_2$, there is a positive probability that, starting from level y in phase i, $M(t)$ will reach level zero before level y. For example, by the proof of Lemma 1, the probability that the process remains in phase i until level zero is first reached is equal to $(\lambda_i/c_i)e^{-\lambda_i y/c_i)}$, which is positive. Because of this, all row sums of $H_{21}(y,y)$ are strictly less than one, that is $H_{21}(y,y)\mathbf{e} < \mathbf{e}$. We have $H_{21}(y,y)\hat{\Psi}(s)\mathbf{e} \leq H_{21}(y,y)\mathbf{e} < \mathbf{e}$ and so $(I - H_{21}(y,y)\hat{\Psi}(s))^{-1}$ exists.

Observe that, by the decomposition of the sample path in the proof of Theorem 4, we have

$$\Psi^y = \lim_{s\to 0}[\hat{\Psi}^y(s) - G_{12}(0,y)].$$

Hence, (47) follows by calculating limits as $s \to 0^+$ in Theorem 4.

Let

$$W = \lim_{s\to 0^+}\frac{d}{ds}(I - H_{21}(y,y)\hat{\Psi}(s))^{-1}.$$

We have

$$W = \lim_{s\to 0^+}\frac{d}{ds}\sum_{N=0}^{\infty}(H_{21}(y,y)\hat{\Psi}(s))^N$$

$$
\begin{aligned}
&= \lim_{s\to 0^+} \frac{d}{ds}\left(H_{21}(y,y)\hat{\Psi}(s) + H_{21}(y,y)\hat{\Psi}(s)\sum_{N=0}^{\infty}(H_{21}(y,y)\hat{\Psi}(s))^N \right) \\
&= H_{21}\Upsilon^{(1)} + \Upsilon^{(1)}(I - H_{21}(y,y)\Psi)^{-1} + H_{21}(y,y)\Psi W, \qquad (49)
\end{aligned}
$$

and so

$$
W = (I - H_{21}(y,y)\Psi)^{-1}\left(H_{21}\Upsilon^{(1)} + H_{21}\Upsilon^{(1)}(I - H_{21}(y,y)\Psi)^{-1} \right). \qquad (50)
$$

Consequently, (48) follows by differentiating with respect to s and then taking limits as $s \to 0^+$ in Theorem 4. ■

6 Conclusion

We have established theoretical results for several performance measures for a fluid-flow model with general rates of increase or decrease. An important feature of these results is that they are suitable for the calculation of expected times and other moments. All the measures considered here can be calculated using efficient algorithms. For example, conditioning on the level w introduced in the proof of Theorem 1, leads to the physical interpretation of an algorithm in [13], which can be used to calculate Ψ from (25). In a forthcoming paper [5], we consider this and several other algorithms, including some new ones, and give their physical interpretations. Moreover, we compare their performance, which depends on the physical properties of the process.

Besides the immediate practical consequences of our results, this paper provides the theoretical basis for future research, by introducing a simple and efficient method that allows for the performance analysis to remain within the fluid flow environment. Two main components of this method are

- the application of the semi-group property, and
- the calculation of probability densities with respect to the fluid level.

The interpretation of expressions in terms of physical properties of fluid-flow processes is an important part of the research, as understanding the formulae in physical terms is very valuable in their applications. Therefore, it is an advantage of our results that the physical interpretation is both an essential part of every expression as well as a useful tool used in the proofs.

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