Resolving Inconsistencies in Simple Temporal Problems

A Parameterized Approach

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Overview

- *Simple Temporal Problem* (STP) is an influential formalism for encoding and reasoning about temporal relations.
- STP constraints: $a \leq x_i - x_j \leq b$, where x_i, x_j represent points in time and a, b are rational or infinite values.
- STP consistency can be checked in polynomial time.
- But what if STP constraints are inconsistent?
- We study ALMOST STP: the problem of resolving few inconsistencies using tools from *parameterized complexity*.
- For two large classes of STP constraints (one-sided and equation constraints), we find fpt algorithms.
- We determine complexity of all classes of STP constraints.

Simple Temporal Problem (STP)

Introduced by Dechter, Meiri, and Pearl in 1989.

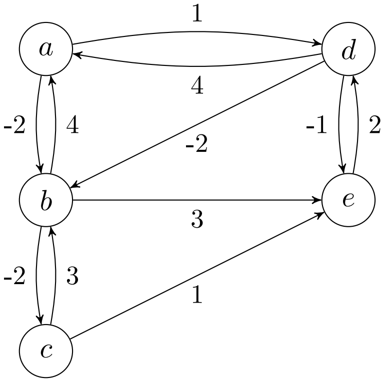
Objects: points in time $x_1, x_2, \ldots, x_n$.

Constraints: $a \leq x_i - x_j \leq b$, where $a, b \in \mathbb{Q} \cup \{-\infty, \infty\}$.

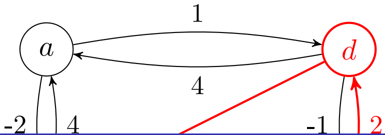
Examples of constraints:

$$
\begin{aligned}
1 &\leq x_i - x_j \leq 2, & & \\
-\infty &\leq x_i - x_j \leq -2 & & \text{(one-sided)}, \\
1 &\leq x_i - x_j \leq \infty & & \text{(one-sided)}, \\
1 &\leq x_i - x_j \leq 1 & \equiv x_i - x_j = 1 \qquad & \text{(equation)}.
\end{aligned}
$$

Checking consistency requires polynomial time.



$$
\begin{aligned}
-1 \le d - a &\le 4 \\
2 \le b - a &\le 4 \\
2 \le c - b &\le 3 \\
1 \le e - d &\le 2 \\
2 \le b - d &\le \infty \\
-\infty \le c - e &\le 1
\end{aligned}
$$

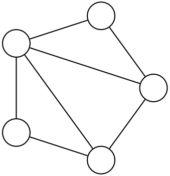


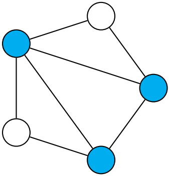
Almost STP

- How to deal with with inconsistent instances?
- *Remove some constraints to achieve consistency.*
- Call this problem Almost STP.
- Almost STP is NP-hard.
- *Restrict the set of allowed constraints.*
- Almost STP is in P only when restricted to trivial constraints ($a \leq x_i - x_j \leq b$, where $a \leq 0 \leq b$) and NP-hard otherwise.
- *Assume that removing few constraints is enough.*
- Study complexity of Almost STP parameterized by k – number of constraints to be removed.

Parameterized Complexity

k-Vertex Cover



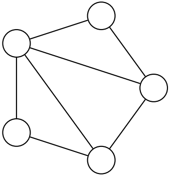


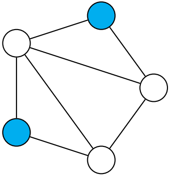
Cover all edges with k vertices.

Solvable in $f(k) \cdot \text{poly}(n)$ time.

In FPT

k-Independent Set



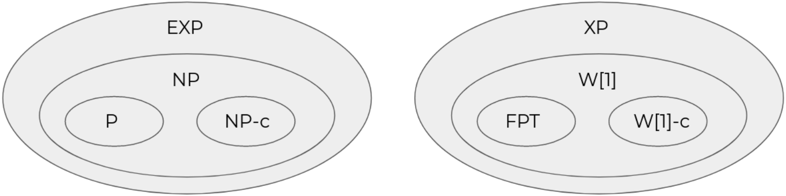


Find k non-adjacent vertices.

Solvable in $n^{O(k)}$ time.

W[1]-hard

Parameterized Complexity Classes



Back to ALMOST STP

- Let $\mathcal{S}$ contain $a \leq x_i - x_j \leq b$ for all $a, b \in \mathbb{Q} \cup \{-\infty, \infty\}$.
- For every subset $\mathcal{A}$ of $\mathcal{S}$, what is the parameterized complexity of ALMOST STP restricted to $\mathcal{A}$?
- Some subsets of $\mathcal{S}$:
  - Trivial constraints: $a \leq x_i - x_j \leq b$, where $a \leq 0 \leq b$.
  - One-sided constraints: $a \leq x_i - x_j$, where $a \geq 0$.
  - Equation constraints: $a \leq x_i - x_j \leq a \;\equiv\; x_i - x_j = a$.
- $1 \leq x_i - x_j \leq 2$ is not trivial, one-sided or equation.

Classification Theorem

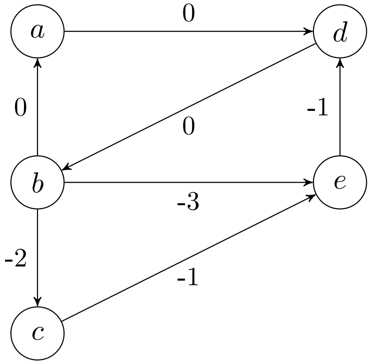
Theorem

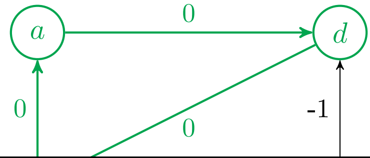
Almost STP *restricted to* $\mathcal{A} \subseteq \mathcal{S}$ *is*

1. *in constant time if* $\mathcal{A}$ *only contains trivial constraints*,
2. *in FPT if* $\mathcal{A}$ *only contains one-sided constraints*,
3. *in FPT if* $\mathcal{A}$ *only contains equation constraints, and*
4. *W[1]-hard otherwise.*

One-sided constraints

Examples: $0 \le d - a$, $1 \le d - e$, $2 \le c - b$, ...





- At most one arc for every pair.
- Labels either zero or negative.
- Negative cycles are bad.

Classification Theorem

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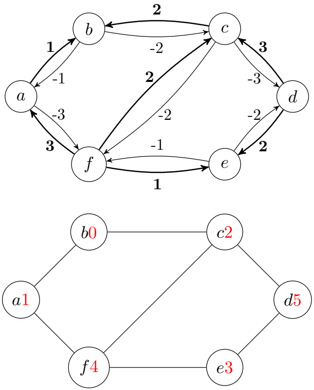
Equations



- $a - b = 1$: $a \xrightarrow{1} b$, $b \xrightarrow{-1} a$.
- Values propagate. E.g., set $a = 1$.
- Nonzero cycles are bad.
- **Goal:** find k arcs that intersect every nonzero cycle.
- High level idea: use iterative compression and multicut to separate "conflicting" variables.

Classification Theorem

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4. *W[1]-hard otherwise.*

W[1]-hard Cases (1/3)

Theorem (Göke et al.)

If $\mathcal{A}$ contains $x_i - x_j \leq 1$ and $x_i - x_j \geq 1$, then ALMOSTSTP *restricted to $\mathcal{A}$ is W[1]-hard.*

- $x_i - x_j \leq 2$ and $x_i - x_j \geq 2$ imply W[1]-hardness.
- What about $x_i - x_j \leq 2$ and $x_i - x_j \geq 3$?
- $x_i - x_j \leq 2$ implements $x_i - x_j \leq 6$:
  $x_i - y \leq 2$, $y - y' \leq 2$, $y' - x_j \leq 2$.
- $x_i - x_j \geq 3$ implements $x_i - x_j \geq 6$.
- $x_i - x_j \leq 6$ and $x_i - x_j \geq 6$ imply W[1]-hardness.

W[1]-hard Cases (2/3)

Lemma

If $\mathcal{A}$ contains $x_i - x_j \leq a$ and $x_i - x_j \geq b$ for any $a, b \in \mathbb{Q}_{>0}$, then ALMOSTSTP *restricted to $\mathcal{A}$ is W[1]-hard.*

- What about $1 \leq x_i - x_j \leq 2$?
- We can express $x_i - x_j = 2$:
  $1 \leq x_i - x_j \leq 2$, $2 \leq x_i - x_j \leq 4$.
- $1 \leq x_i - x_j \leq 2$ implements $2 \leq x_i - x_j \leq 2n + 2$ $\forall n \in \mathbb{N}$:
  $y - x_i = 2n - 2$, $2n \leq y - x_j \leq 4n$.
- For large enough n (in $O(\#\text{variables})$), $2n + 2 \approx \infty$ in STP.
- $1 \leq x_i - x_j \leq 2$ expresses $x_i - x_j \leq 2$ and $x_i - x_j \geq 2$.

W[1]-hard Cases (3/3)

Lemma

If $\mathcal{A}$ contains
(a) $x_i - x_j \leq a$ and $x_i - x_j \geq b$ for any $a, b \in \mathbb{Q}_{>0}$, or
(b) $a \leq x_i - x_j \leq b$ for some $0 < a < b < \infty$,
then ALMOSTSTP *restricted to $\mathcal{A}$ is W[1]-hard.*

Finally, we prove that if $\mathcal{A}$ is not trivial, one-sided, or equation, then it either implements two constraints from (a) or the constraint from (b).

Questions for Future

- What if we allow unary constraints, e.g. $1 \leq x_i \leq 3$?
- What if we allow strict constraints, e.g. $1 < x_i - x_j \leq 2$?
- For which other problems X is ALMOST X interesting?
- ALMOST STP assumes that the *additive* error is small. What about the *multiplicative* error? Can we check if $(1 - \epsilon)$ fraction of STP constraints are consistent? This question is asking about *robust approximation*.

Thank you!