Koopman-Based Control: Bilinearization, Controllability and Optimal Control of Control-Affine Nonlinear Systems

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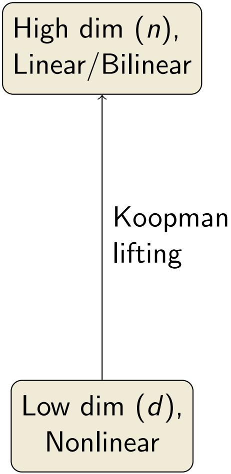
Koopman operator: A way out from nonlinearity

- Nonlinear system: difficult to analyze controllability and control.
- Operator theoretic approach: **Linear** operator operating on a infinite-dimensional Hilbert Space.
- Evolution of **observable** functions: Koopman operator [B.O. Koopman, 1931].
- Curse of dimensionality: **infinite-dimensional**



Koopman operator: An overview

- Autonomous dynamical system: $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{f} : \mathbb{X} \to \mathbb{X}$, $\mathbb{X} \subset \mathbb{R}^d$ compact manifold.
- The flow map of the system:
  $\mathbf{\Phi} : \mathbb{R} \times \mathbb{X} \to \mathbb{X}$, $\mathbf{\Phi}(0, \mathbf{x}) = \mathbf{x}$ and $\mathbf{\Phi}(s, \mathbf{\Phi}(t, \mathbf{x})) = \mathbf{\Phi}(s + t, \mathbf{x})$.
- $\mathcal{F}$ is a Banach space of complex-valued observables $\varphi : \mathbb{X} \to \mathbb{C}$.

Definition (Koopman) [Mezić, 2015]

The *Koopman* semigroup of operators $\mathcal{U}^t : \mathcal{F} \to \mathcal{F},\ t > 0$ associated with flow $\mathbf{\Phi}$ is defined by

$$(\mathcal{U}^t \varphi)(\cdot) = \varphi \circ \mathbf{\Phi}(t, \cdot), \quad \varphi \in \mathcal{F}. \tag{1}$$

Eigenvalue, eigenfunction, modes and infinitesimal generator

- Being linear operator $\mathcal{U}^t$ is characterized by eigenvalue and eigenfunctions.
- $(\lambda, \phi(\cdot))$ is an eigenvalue-eigenfunction pair: $(\mathcal{U}^t\phi)(\cdot) = e^{\lambda t}\phi(\cdot)$.
- Infinitesimal generator of $\mathcal{U}^t$: $\lim_{t\to 0} \dfrac{\mathcal{U}^t - I}{t} = \mathbf{f}\cdot\nabla = L_{\mathbf{f}}$.
- The infinitesimal generator satisfies the eigenvalue equation $L_{\mathbf{f}}\phi = \lambda\phi$.
- vector valued observable: $\mathbf{g}(\cdot) = \sum_{i=1}^{\infty} \phi_i(\cdot) \underbrace{\mathbf{v}_i^{\mathbf{g}}}_{\text{eigenmodes}}$.
- Time-varying observable $\psi(t,\mathbf{x}) \triangleq \mathcal{U}^t\varphi(\mathbf{x})$ solves the PDE

$$\begin{aligned} \frac{\partial\psi}{\partial t} &= L_{\mathbf{f}}\psi, \\ \psi(0,\mathbf{x}) &= \varphi(\mathbf{x}). \end{aligned} \tag{2}$$

Prior works and contributions

Existing Literature

- Koopman spectral properties of dynamical systems [Mezíc *et al*, 2005].
- Connection to Dynamic Mode Decomposition [Rowley *et al*, 2009].
- Koopman Canonical Transform (KCT) [Surana, 2016].

Contributions

- Sufficient conditions for bilinearizability using KCT.[1]
- Convergence of Koopman Bilinear Form
- Reachability in Koopman Bilinear Form (KBF).[1]
- Approximate bilinearization and optimal control.

[1] D. Goswami and D. A. Paley, "Global Bilinearization and Controllability of Control-Affine Nonlinear Systems: A Koopman Spectral Approach", IEEE CDC 2017.

Plan

System for bilinearization

Control-affine system

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m} \mathbf{g}_i(\mathbf{x}) u_i \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}), \end{aligned} \tag{3}$$

where $\mathbf{x} \in \mathbb{X} \subseteq \mathbb{R}^d$, $u_i \in \mathbb{R}$, for $i = 1, \ldots, m$ and $\mathbf{y} \in \mathbb{R}^p$.

- Applying Eq. (2) (Observation PDE) for system (3), we get an infinite dimensional bilinear form:

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \underbrace{L_{\mathbf{f}} \psi + \sum_{i=1}^{m} u_i L_{\mathbf{g}_i} \psi,}_{\text{like } A\mathbf{x} + \sum_{i=1}^{m} u_i B_i \mathbf{x}} \\ \psi(0, \mathbf{x}) &= \varphi(\mathbf{x}). \end{aligned} \tag{4}$$

Koopman Canonical Transform

- Goal: Project the infinite-dimensional operators $L_{\mathbf{g}_i}$ on the finite dimensional basis.
- Natural choice of basis functions: Koopman eigenfunctions.
- Let $\lambda_i, \phi_i(\cdot)$ are the eigenvalue-function pairs for the unactuated Koopman operator, i.e., $L_{\mathbf{f}}$.

Assumption 1 [Surana 2016]

$\exists\ \phi_i, i = 1, 2, \ldots, n$ such that

$$\mathbf{x} = \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{x}}, \quad \mathbf{h}(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x})\mathbf{v}_i^{\mathbf{h}},$$

where $\mathbf{v}_i^{\mathbf{x}} \in \mathbb{C}^d$ and $\mathbf{v}_i^{\mathbf{h}} \in \mathbb{C}^p$.

Koopman Canonical Transform (contd.)

The Transform [Surana 2016]

$$
\begin{aligned}
T(\mathbf{x}) &= [\tilde{\phi}_1(\mathbf{x}), \ldots, \tilde{\phi}_n(\mathbf{x})]^T \\
\tilde{\phi}_i(\mathbf{x}) &= \phi_i(\mathbf{x}), \text{ if } \phi_i : \mathbb{X} \to \mathbb{R} \\
(\tilde{\phi}_i(\mathbf{x}), \tilde{\phi}_{i+1}(\mathbf{x}))^T &= (2Re(\phi_i(\mathbf{x})), -2Im(\phi_i(\mathbf{x})))^T, \\
& \quad \text{if } \phi_i : \mathbb{X} \to \mathbb{C} \\
& \quad \text{and assuming } \phi_{i+1} = \overline{\phi}_i.
\end{aligned} \tag{5}
$$

Define the following:

- $\tilde{\mathbf{v}}_i^{\mathbf{x}} \triangleq \mathbf{v}_i^{\mathbf{x}}$ if ϕ_i is real-valued, and $[\tilde{\mathbf{v}}_i^{\mathbf{x}}, \tilde{\mathbf{v}}_{i+1}^{\mathbf{x}}] \triangleq [\mathrm{Re}\, \mathbf{v}_i^{\mathbf{x}}, \mathrm{Im}\, \mathbf{v}_i^{\mathbf{x}}]$ if ϕ_i is complex-valued.
- $C^{\mathbf{x}} \triangleq [\tilde{\mathbf{v}}_1^{\mathbf{x}} | \ldots | \tilde{\mathbf{v}}_n^{\mathbf{x}}]$.
- $\tilde{\mathbf{v}}_i^{\mathbf{h}}$ and $C^{\mathbf{h}}$ are defined similarly.

Towards bilinearization

Transformed System

$$\begin{aligned} \mathbf{x} &= C^{\mathbf{x}}\mathbf{z}, \\ \dot{\mathbf{z}} &= D\mathbf{z} + \sum_{i=1}^{m} L_{\mathbf{g}_i} T(\mathbf{x}) u_i|_{\mathbf{x}=C^{\mathbf{x}}\mathbf{z}}, \\ \mathbf{y} &= C^{\mathbf{h}}\mathbf{z}, \end{aligned} \tag{6}$$

Here $D \in \mathbb{R}^{n\times n}$ is a block diagonal matrix with diagonal entry $D_{i,i} = \lambda_i$ if ϕ_i is a real-valued eigenfunction, or $\begin{bmatrix} D_{i,i} & D_{i,i+1} \\ D_{i+1,i} & D_{i+1,i+1} \end{bmatrix} = |\lambda_i| \begin{bmatrix} \cos(\angle\lambda_i) & \sin(\angle\lambda_i) \\ -\sin(\angle\lambda_i) & \cos(\angle\lambda_i) \end{bmatrix}$ if ϕ_i is complex.

- Note: the eigenfunctions can be calculated from the time series data with Extended Dynamic Mode Decomposition (EDMD) with a suitable choice of dictionary.

Bilinearizability Results: Sufficient Conditions

- To bilinearize the transformed system (6), we need to impose conditions on the eigenspace of $L_{\mathbf{g}_i}$ as described in the Theorem 1.

Theorem 1: Bilinearization in countable basis [Goswami 2017]

The systems (3) and (6) are bilinearizable in a countable basis if the **eigenspace of** $L_{\mathbf{f}}$, i.e., the Koopman operator corresponding to the drift vector field, is an **invariant subspace of** $L_{\mathbf{g}_i}, i = 1, \ldots, m$, i.e., the Koopman operators related to the control vector fields.

Corollary 2.1: The systems (3) and (6) are bilinearizable if the drift vector field $\mathbf{f} \equiv 0$, i.e., it is a pure control-affine system.

Bilinearizability results: Sufficient conditions (cont.)

Towards finite dimensional bilinearization: stronger conditions

Theorem 2: Bilinearizability in finite dimension [Goswami 2017]

Suppose $\exists \{\phi_j : j = 1, \ldots, n\}$, $n \in \mathbb{N}$, $n < \infty$ such that ϕ_j, $j = 1, \ldots, n$ are the Koopman eigenfunctions of the unactuated system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ and $\text{span}\{\phi_1, \ldots, \phi_n\}$ forms an **invariant subspace of** $L_{\mathbf{g}_i}$, $i = 1, \ldots, m$. Then the system (3) and, in turn system (6), are bilinearizable with an n dimensional state space.

Sufficiency Fails: Approximate Bilinearization

The condition of Theorem 2 essentially states that $L_{\mathbf{g}_i} T(\mathbf{x}) = B_i T(\mathbf{x})$ for some B_i, $i = 1 \ldots, m$,.
If the condition is not satisfied, we approximate B_i with the following convex quadratic optimization problem:

Approximate Bilinearization [Goswami preprint]

$$\begin{aligned} &\underset{B_i}{\text{minimize}} \, \| L_{\mathbf{g}_i} T(\mathbf{x}) - B_i T(\mathbf{x}) \|^2_{\mathcal{L}^2_m(\mathbf{X})} \\ &\text{for } i = 1, \ldots, m, \end{aligned} \tag{7}$$

where $\mathbf{X}$ is a truncation of the state-space $\mathbb{X}$ such that $m(\mathbf{X}) < \infty$.

The solution is given by $Q{B_i^*}^T = R_i$ where $Q_{kl} = \langle \tilde{\phi}_k, \tilde{\phi}_l \rangle_{\mathcal{L}^2_m(\mathbf{X})}$ and $(R_i)_{kl} = \left\langle \frac{\partial \tilde{\phi}_l}{\partial \mathbf{x}} \mathbf{g}_i, \tilde{\phi}_k \right\rangle_{\mathcal{L}^2_m(\mathbf{X})} = \left\langle L_{\mathbf{g}_i} \tilde{\phi}_l, \tilde{\phi}_k \right\rangle_{\mathcal{L}^2_m(\mathbf{X})}$.

Bilinear Form

Koopman Bilinear Form (KBF) [Goswami 2017]

Now with coordinates $\mathbf{z} = T(\mathbf{x})$, the transformed system is

$$\dot{\mathbf{z}} = D\mathbf{z} + \sum_{i=1}^{m} B_i \mathbf{z} u_i, \ \mathbf{z} \in \mathbb{R}^n, \ n < \infty. \tag{8}$$

We are going to use this form for reachability analysis and optimal control.

Data-driven construction of Koopman Bilinear Form

- Take snapshot pairs $\overline{X} = [\mathbf{x}_1, \dots, \mathbf{x}_N] \xrightarrow{\Delta t} \overline{Y} = [\mathbf{y}_1, \dots, \mathbf{y}_N]$.
- Dictionary of functions: $\mathcal{D} = [h_1, \dots, h_K] \subset \mathcal{F}$.
- Define $\mathbf{H}(\mathbf{x}) = [h_1(\mathbf{x}), \dots, h_n(\mathbf{x})]$, $\mathbf{G} = \dfrac{1}{N} \sum_{i=1}^{N} \mathbf{H}(\mathbf{x}_i)^* \mathbf{H}(\mathbf{x}_i)$,

  and $\mathbf{A} = \dfrac{1}{N} \sum_{i=1}^{N} \mathbf{H}(\mathbf{x}_i)^* \mathbf{H}(\mathbf{y}_i)$.
- Solve the optimization problem $\underset{\mathbf{K}}{\text{minimize}} \|\mathbf{G}\mathbf{K} - \mathbf{A}\|_F^2$ by $\mathbf{K} \triangleq \mathbf{G}^\dagger A$ to get the approximate Koopman operator.
- Let μ_j, ξ_j be the j^{th} eigenvalue-vector pair of $\mathbf{K}$. $\phi_j = \mathbf{H}\xi_j$ is an approximate eigenfunction of $\mathcal{K}$ with eigenvalue $\lambda_j = \dfrac{\ln \mu_j}{\Delta t}$.
- Approximate $C^{\mathbf{x}} \approx \overline{X} T_{\mathbf{x}}^\dagger$, where $T_{\mathbf{x}} \triangleq [T(\mathbf{x}_1), \dots, T(\mathbf{x}_N)]$.

Plan

Non-recurrent surface and richness of the eigenspace

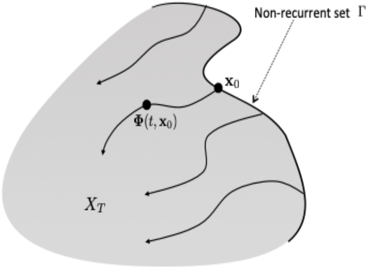
Question: Is the span of the Koopman eigenfunctions rich enough to approximate continuous functions of $\mathbf{x}$?

Non-recurrent set

$\mathbf{x} \in \Gamma \implies \mathbf{\Phi}(t, \mathbf{x}) \notin \Gamma$,
$\forall\, t \in (0, T],\ T > 0.$

Eigenfunction construction

Choose $\lambda \in \mathbb{C}$, $g \in \mathcal{C}(\Gamma)$.
$\phi_{\lambda,g}(\mathbf{x}) = e^{-\lambda\tau(\mathbf{x})} g(\mathbf{\Phi}(\tau(\mathbf{x}), \mathbf{x})),\ \forall\, \mathbf{x} \in X_T,$
where $\tau(\mathbf{x}) = \inf_{t \in \mathbb{R}} \{t | \mathbf{\Phi}(\mathbf{t}, \mathbf{x}) \in \Gamma\}$
and
$X_T = \bigcup_{t \in [0,T]} \{\mathbf{\Phi}(t, \mathbf{x}_0) | \mathbf{x}_0 \in \Gamma\}$



Denseness of the eigenfunctions

It turns out that with the proper choice of boundary functions g and with a non-recurrent set for $T > 0$, the set of eigenfunction is dense in $\mathcal{C}(X_T)$.

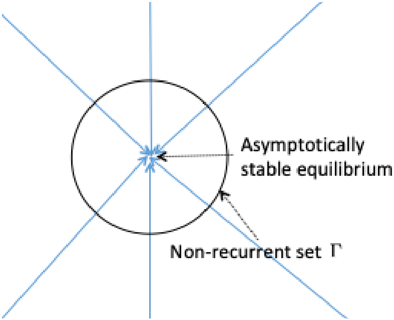
Theorem 3 (Denseness)

$\Lambda \subset \mathbb{C}$ nontrivial, $L(\Lambda) \triangleq \{\sum_{k=1}^{N} \alpha_k \lambda_k | \lambda_k \in \Lambda, \alpha_k \in \mathbb{N}_0, N \in \mathbb{N}\}$,
$\Gamma \in \mathbb{X}$: **non-recurrent set closed** in $\mathbb{R}^d$, $G \subset \mathcal{C}(\Gamma)$: **a dense unital subalgebra** (i.e., G is closed under multiplication and contains a multiplicative identity) of $\mathcal{C}(\Gamma)$, no finite escape time in $[0, T]$.
Then $\mathbf{\Phi}_{\Lambda,G} \triangleq \{\phi_{\lambda,g} | \lambda \in L(\Lambda), g \in G\}$ is dense in $\mathcal{C}(X_T)$, i.e., for any $\epsilon > 0$ and $\xi \in \mathcal{C}(X)$, $\exists\ \phi_1, \ldots, \phi_n \in \mathbf{\Phi}_{\Lambda,G}$ such that

$$\sup_{\mathbf{x} \in X_T} \left| \xi(\mathbf{x}) - \sum_{i=1}^{n} v_i \phi_i(\mathbf{x}) \right| < \epsilon.$$

Existence of non-recurrent surface

- **Rectifiable dynamics**: If $\exists$ a diffeomorphism $h : Y' \subset \mathbb{R}^d \longrightarrow D \subset \mathbb{X}$ through which $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ is conjugate to $\dot{\mathbf{y}} = (0, \dots, 0, 1)^T$, then a non-recurrent surface can be constructed [Korda 2018].
- **Asymptotically stable equilibrium point**: Non rectifiable, but level sets of Lyapunov function comes handy. Same type of constructions for divergent (no stable manifold)



Plan

Myhill semigroup and matrix differential equation

- Myhill semigroup is the semigroup of flow maps that maps the initial condition to a final state under a piecewise control signal.
- For bilinear system

$$\dot{\mathbf{z}} = A\mathbf{z} + \sum_{i=1}^{m} B_i \mathbf{z} u_i,\ \mathbf{z} \in \mathbb{R}^n,\ \mathbf{u} \in \mathbb{R}^m, \tag{9}$$

  the Myhill semigroup is the semigroup of matrices $Z \in \mathbb{R}^{n \times n}$ given by the matrix differential equation

$$\dot{Z}(t) = AZ(t) + \sum_{i=1}^{m} B_i Z(t) u_i,\ Z(0) = I, \tag{10}$$

  with $\mathbf{z}(t) = Z(t)\mathbf{z}(0)$ for any $\mathbf{z}(0) \in \mathbb{R}^n$.
- Hence the reachability analysis of the MDE (10) is equivalent to the reachability analysis of the bilinear ODE (9).

Definitions concerning Lie algebra

- Denote $\{X_i : i = 1, \ldots, n\}_A$ as the **smallest Lie algebra** containing $\{X_i : i = 1, \ldots, n\}$.
- Matrix exponentials of the elements of a Lie algebra generates Lie group with normal matrix multiplication.
- Let $\{\exp\{X_i\} : i = 1, \ldots, n\}_G$ be the **smallest Lie group** containing $\{\exp\{X_i\} : i = 1, \ldots, n\}$.
- $\forall\, A, B \in \mathbb{R}^{n\times n}$ and $k = 0, 1, \ldots$ we define $\mathrm{ad}_A^{k+1} B \triangleq [A, \mathrm{ad}_A^k B]$ with $\mathrm{ad}_A^0 B \triangleq B$.

Reachability results: Wei-Norman Lemma

- Wei-Norman lemma provides a way to express the solutions of a bilinear differential equations in terms of the product of matrix exponentials.
- The matrix exponentials pave the way of using Lie algebraic structure.

Lemma 1 [Wei-Norman 1964]

$$\dot{Z}(t) = \sum_{i=1}^{m} B_i Z(t) u_i(t), \; Z(0) = I, \quad \Rightarrow \quad Z(t) = \prod_{i=1}^{l} \exp(h_i(t) B_i)$$

$h_i(t)$ piecewise continuous and $Z(t) \in \mathbb{R}^{n \times n}$, $\{B_i : i = 1, \ldots, l\}$ is the extension of $\{B_i : i, \ldots, m\}$ to a basis of $\{B_i : i = 1, \ldots, m\}_A$.

Brockett's results on reachability

Reachability results for drift-free Matrix Differential Equation

Lemma 2 [Brockett 1973]

$$\dot{Z}(t) = \sum_{i=1}^{m} B_i Z(t) u_i(t),\ Z(0) = I \tag{11}$$

corresponds to the KBF(8) with $\mathbf{f}_0 \equiv 0$. $Z_1 \in \mathbb{R}^{n\times n}$ is in the reachable space of (11) if and only if $Z_1 \in \{\exp\{\{B_i : i = 1, \dots, m\}_A\}\}_G$.

Brockett's results on reachability (Cont.)

Lemma 3 [Brockett 1973]

$$\dot{Z}(t) = DZ(t) + \sum_{i=1}^{m} B_i Z(t) u_i(t), \;\; Z(0) = I \tag{12}$$

Assume $[\mathrm{ad}_D^k B_i, B_j] = 0$ for $i, j = 1, \ldots, m$ and $k = 0, 1, \ldots, n^2 - 1$. Let $\mathcal{L} = \mathrm{span}\{\mathrm{ad}_D^k B_i : \; i = 1, \ldots, m, \; k = 0, 1, \ldots, n^2 - 1\}$. Then Z_1 is reachable at time t_1 through continuous controls if and only if $\exists\, L \in \mathcal{L}$ such that

$$Z_1 = \exp(t_1 D) \exp(L).$$

The condition seems restrictive, but usually holds with sparse B_i.

Reachability of Koopman Bilinear Form

The reachability of the resultant bilinear system

$$\dot{\mathbf{z}} = D\mathbf{z} + \sum_{i=1}^{m} B_i \mathbf{z} u_i, \tag{13}$$

can be expressed with help of Lemma 2 and 3 as the following.

Theorem 4 (Reachability of Koopman Bilinear Form) [Goswami 2017]

- Given a transformed state $\mathbf{z}_1$, if $\exists\ Z_1 \in \mathbb{R}^{n\times n}$ such that $\mathbf{z}_1 = Z_1\mathbf{z}_0$, then $\mathbf{z}_1$ is reachable from $\mathbf{z}_0$ in KBF (13) if Z_1 is reachable from $Z(0) = I$ in the MDE (12).
- Conversely if $\mathbf{z}_1$ is reachable from $\mathbf{z}_0$ in KBF (13), then $\exists\ Z_1$ in the reachable set of the MDE (12) from $Z(0) = I$ such that $\mathbf{z}_1 = Z_1\mathbf{z}_0$.

Plan

Optimal Control of Koopman Bilinear Form

$$\begin{aligned}
&\underset{\mathbf{u}(t)}{\text{minimize}} && \frac{1}{2}\int_{t_0}^{t_f}\left(\mathbf{x}^T Q\mathbf{x} + \mathbf{u}^T\mathbf{u}\right)dt \\
&\text{subject to} && \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \sum_{i=1}^{m}\mathbf{g}_i(\mathbf{x})u_i, \\
& && \mathbf{x}(t_0) = \mathbf{x}_0,\ \mathbf{x}(t_f) = \mathbf{x}_f, \\
& && \mathbf{u} \in \mathcal{U},
\end{aligned}$$

$$\begin{aligned}
&\underset{\mathbf{u}(t)}{\text{minimize}} && \frac{1}{2}\int_{t_0}^{t_f}\left(\mathbf{z}^T C^{\mathbf{x}^T} QC^{\mathbf{x}}\mathbf{z} + \mathbf{u}^T\mathbf{u}\right)dt \\
&\text{subject to} && \dot{\mathbf{z}} = D\mathbf{z} + \sum_{i=1}^{m}B_i\mathbf{z}u_i, \\
& && \mathbf{z}(t_0) = T(\mathbf{x}_0),\ C^{\mathbf{x}}\mathbf{z}(t_f) = \mathbf{x}_f, \\
& && \mathbf{u} \in \mathcal{U},
\end{aligned}$$

- We use Pontryagin's principle and shooting method to solve it.
- Pre-Hamiltonian: $H(t, \mathbf{z}, \mathbf{p}, \mathbf{v}) = \mathbf{p}^T\left(D\mathbf{z} + \sum_{i=1}^{m}B_i\mathbf{z}v_i\right) - \mathcal{L}(\mathbf{z}, \mathbf{v})$
- Costate equation: $\dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{z}}^T = -\left(D + \sum_{i=1}^{m}B_iu_i\right)^T\mathbf{p} + C^{\mathbf{x}^T}QC^{\mathbf{x}}\mathbf{z}.$
- Optimal control $\mathbf{u}_i^*(t) = \underset{\mathbf{v}}{\operatorname{argmax}}\, H(t, \mathbf{z}, \mathbf{p}, \mathbf{v}) = \mathbf{p}^T(t)B_i\mathbf{z}(t)$ with transversality $\mathbf{p}(t_f) \perp \ker C^{\mathbf{x}}$.

Plan

Numerical Simulation

Example System [Rowley 2006]

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2, \tag{14}$$

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \lambda x_1 \\ \mu x_2 + (2\lambda - \mu)cx_1^2. \end{pmatrix}$$

Koopman eigenvalue-eigenfunction pairs for $L_{\mathbf{f}}$ are as follows:

- $\phi_1(\mathbf{x}) = x_1 = z_1$ with eigenvalue λ,
- $\phi_2(\mathbf{x}) = x_2 - cx_1^2 = z_2$ with eigenvalue μ,
- $\phi_3(\mathbf{x}) = x_1^2 = z_3$ with eigenvalue 2λ, and
- $\phi_4(\mathbf{x}) = 1 = z_4$ with eigenvalue 0.

Applied controls: $u_1 = \cos(2\pi t)$, a sinusoidal excitation, and $u_2 = -x_2 = -(z_2 + cz_1^2) = -(z_2 + cz_3)$, a state feedback.

- Choosing $\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix}1 & x_1^2\end{bmatrix}^T$ and $\mathbf{g}_2(\mathbf{x}) = \begin{bmatrix}0 & 1\end{bmatrix}^T$ makes the system completely bilinearizable according to Theorem 2.
- Choosing $\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix}1 & \cos x_1\end{bmatrix}^T$ does not satisfy the conditions of the Theorem 2, but we use the approximate B_i with $\mathbf{X} = [0, 30] \times [-10, 0]$.

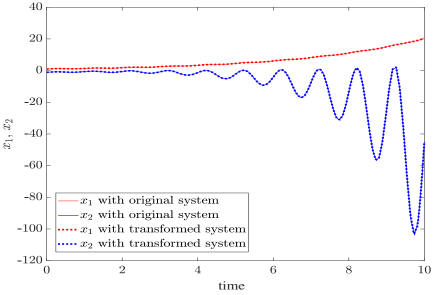


Figure: Exact bilinearization

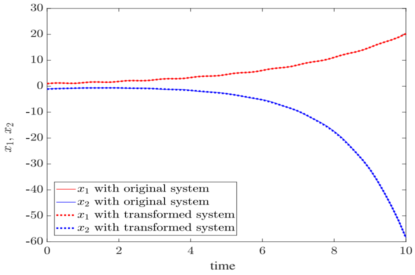
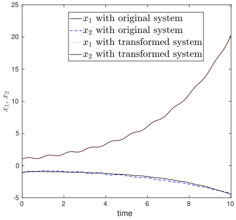


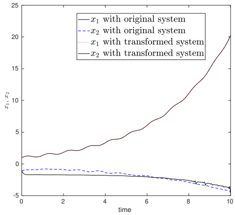
Figure: Approximate bilinearization

Effect of truncation

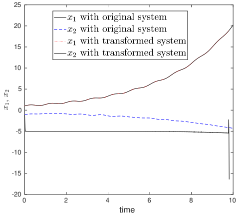
- $\mathbf{g}_1(\mathbf{x}) = \begin{bmatrix} 1 & \cos x_1 \end{bmatrix}^T$ and $\mathbf{g}_2(\mathbf{x}) = \begin{bmatrix} 0 & x_2^2 \end{bmatrix}^T$



(a)



(b)



(c)

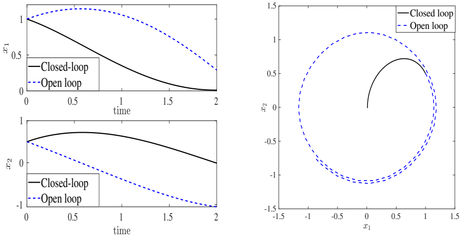
Figure: Comparison of the approximate bilinearization with different trucation: (a) $\mathbf{X} = [0, 30] \times [-5, 0]$, (b) $\mathbf{X} = [0, 30] \times [-10, 0]$, (c) $\mathbf{X} = [0, 30] \times [-30, 0]$

Optimal Control Simulation: Example 1

Controlled Pendulum

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0.01x_2 - \sin x_1 + u, \end{aligned} \tag{15}$$

EDMD with monomials up to 5^{th} degree, $\mathbf{X} = [-1, 1] \times [-1, 1]$.



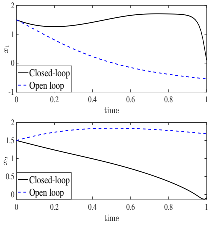
(a) (b)

Optimal Control Simulation: Example 2

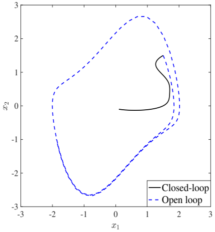
Van der Pol equation

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= (1 - x_1^2)x_2 - x_1 + u
\end{aligned}
\tag{16}
$$

EDMD with monomials up to 5^{th} degree, $\mathbf{X} = [-3, 3] \times [-3, 3]$.



(a)



(b)

Conclusion

- Nonlinear systems, in general, are difficult to analyze and control.
- (Linear) Koopman operator and Koopman Canonical Transform present an effective way to globally bilinearize a nonlinear system.
- Sufficient conditions for bilinearizability have been derived.
- When the conditions are failed to be satisfied, a framework is designed for an approximate bilinearization using quadratic programming on $\mathcal{L}^2$ norm.
- The resulting Koopman Bilinear Form (KBF) is subjected to controllability analysis using Lie algebraic structure and optimal control design using Pontryagin's principle.

Future Challenges

- Performance quantization of KBF and the optimal control thereof.
- Alternative definition of Koopman operator when an arbitrary control is present.
- Relation with the nonlinear controllability and Koopman Bilinear Form.

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