Local Polynomial Approximation and Taylor Series

Constructing Numerical Methods from First Principles

MMAE 350

Local Polynomial Approximation

In numerical analysis, we approximate functions locally using polynomials.

The strategy is:

- Assume a local polynomial model
- Determine coefficients by enforcing agreement at a point
- Use the result to construct numerical methods

This approach underlies finite differences and finite elements.

Assumed Local Approximation

We assume that near a point x_0, the function $f(x)$ can be approximated by

$$f(x) \approx d + a(x - x_0) + b(x - x_0)^2 + c(x - x_0)^3.$$

The coefficients a, b, c, and d are unknown and will be determined by matching derivatives at x_0.

Matching the Function Value

Evaluate the approximation at $x = x_0$:

$$f(x_0) = d.$$

$$\boxed{d = f(x_0)}$$

The constant term ensures the approximation is exact at the expansion point.

Matching the First Derivative

Differentiate the approximation:

$$\frac{d}{dx} = a + 2b(x - x_0) + 3c(x - x_0)^2.$$

Evaluating at $x = x_0$:

$$f'(x_0) = a.$$

$$\boxed{a = f'(x_0)}$$

Matching the Second Derivative

Differentiate again:

$$\frac{d^2}{dx^2} = 2b + 6c(x - x_0).$$

Evaluating at $x = x_0$:

$$f''(x_0) = 2b.$$

$$\boxed{b = \frac{1}{2} f''(x_0)}$$

Matching the Third Derivative

Differentiate a third time:

$$\frac{d^3}{dx^3} = 6c.$$

Evaluating at $x = x_0$:

$$f^{(3)}(x_0) = 6c.$$

$$\boxed{c = \frac{1}{6} f^{(3)}(x_0)}$$

Recovered Taylor Expansion

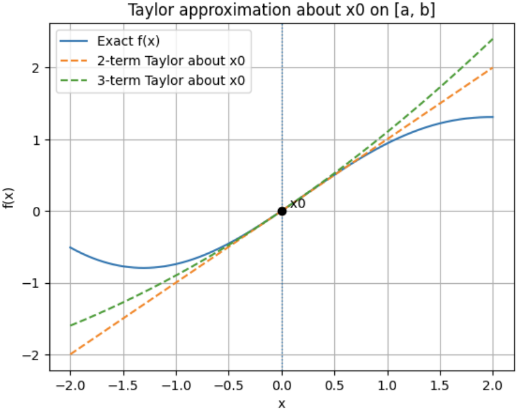
Substituting the coefficients into the approximation:

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{6}(x - x_0)^3.$$

This is the Taylor expansion of $f(x)$ about x_0, truncated after the cubic term.

Taylor Expansion About the Center of the Interval

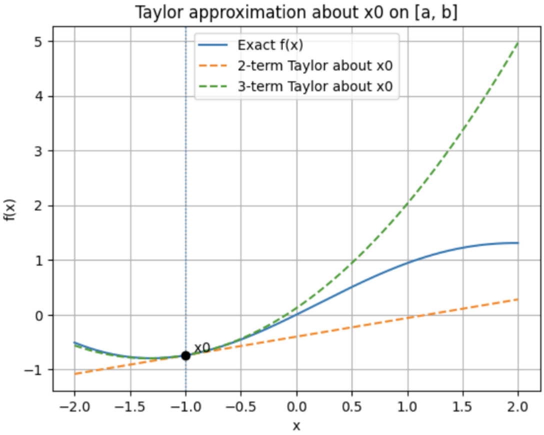
$$f(x) = \sin x + \frac{1}{10}x^2$$



Key Observation

Taylor Expansion About an Off-Center Point

$$f(x) = \sin x + \frac{1}{10}x^2$$



Key Observation

Key Takeaways

- Taylor series are constructed by matching derivatives at a point
- Each additional term enforces agreement of one higher derivative
- The constant term ensures exact agreement at the expansion point
- This construction will be reused to derive finite difference formulas