MMAE 350 – Homework 3

Direct and Iterative Solvers for Linear Systems

Due: **Friday, February 7, 2026 at 11:59 PM**

Overview

In this homework, you will explore two fundamental approaches for solving linear systems of equations:

- the *Thomas Algorithm*, a direct solver specialized for tridiagonal systems, and
- the *Gauss–Seidel method*, an iterative solver applicable to more general sparse systems.

You will solve each problem analytically by hand and then verify your results using a Jupyter notebook. The goal is to understand both the *algorithmic structure* and the *computational behavior* of each method.

Instructions

- Show all steps clearly in your handwritten or typed solution.
- A companion Jupyter notebook is provided to *check* your results.
- The notebook should not be used to replace the required analytical work.

Submit the following to Canvas:

- A single PDF containing your written solutions.
- Your completed Jupyter notebook (`.ipynb`).

Problem 1: Thomas Algorithm (Direct Solver)

Consider the tridiagonal linear system

$$\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},$$

where

$$\boldsymbol{A} = \begin{bmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 15 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$

(a) Hand solution using the Thomas Algorithm

(a) Write the forward sweep equations for the Thomas Algorithm.

(b) Compute the modified coefficients step by step.

(c) Perform back substitution to obtain the solution vector $\boldsymbol{x}$.

Clearly show your forward-sweep table and all intermediate values.

(b) Verification using a Jupyter notebook

Using the provided notebook:

- Enter the matrix $\boldsymbol{A}$ and vector $\boldsymbol{b}$.
- Solve the system using a Thomas Algorithm implementation.
- Confirm that the numerical solution matches your hand calculation.

Problem 2: Gauss–Seidel Method (Iterative Solver)

In this problem, you will solve the *same linear system* using the Gauss–Seidel method.

(a) One Gauss–Seidel sweep by hand

Assume the initial guess

$$\boldsymbol{x}^{(0)} = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T .$$

Perform one full Gauss–Seidel sweep to compute $\boldsymbol{x}^{(1)}$. Clearly indicate which values are:

- newly updated within the current sweep, and
- carried over from the previous iteration.

(b) Second sweep

Perform a second Gauss–Seidel sweep to compute $\boldsymbol{x}^{(2)}$. You may compute all components explicitly or compute one component in detail and explain the pattern.

(c) Convergence discussion

In 2–3 sentences, answer the following:

- Why does the Gauss–Seidel method converge for this system?
- How does the converged solution compare to the Thomas Algorithm solution?

(d) Verification using a Jupyter notebook

Using the provided notebook:

- Implement the Gauss–Seidel method.
- Track the solution vector and residual norm versus iteration.
- Verify convergence to the same solution obtained in Problem 1.

Submission Checklist

Before submitting, make sure that:

- All analytical steps are clearly shown.
- Numerical values are reported with reasonable precision.
- Your notebook runs without errors.
- Your notebook output confirms your hand calculations.

Learning Takeaways

After completing this assignment, you should be able to:

- Apply a direct solver to a structured linear system.
- Perform Gauss–Seidel iterations by hand.
- Explain the difference between direct and iterative solvers.
- Use computational tools to verify analytical work.