

Matrix Algebra for Engineering Systems

Solving Linear Systems

MMAE 350

Tuesday, January 20

Where We Are in the Course

In the previous lecture, we:

- ▶ Introduced matrices and vectors
- ▶ Defined the identity matrix and matrix inverse
- ▶ Wrote linear systems in the form $A\mathbf{x} = \mathbf{b}$
- ▶ Explored numerical issues using the Hilbert matrix

Today we focus on how engineers *actually solve* linear systems.

Why Linear Systems Matter in Engineering

Linear systems appear everywhere in engineering:

- ▶ Structural analysis (forces and displacements)
- ▶ Heat conduction (nodal temperatures)
- ▶ Electrical circuits (currents and voltages)
- ▶ Fluid flow and transport problems

In most applications:

$$A\mathbf{x} = \mathbf{b} \quad \text{with} \quad A \in \mathbb{R}^{n \times n}, \quad n \gg 1.$$

The Linear System $A\mathbf{x} = \mathbf{b}$

We seek a vector \mathbf{x} such that:

$$A\mathbf{x} = \mathbf{b}.$$

- ▶ A : system matrix (geometry, material properties, connectivity)
- ▶ \mathbf{x} : unknowns (displacements, temperatures, potentials)
- ▶ \mathbf{b} : known loads, sources, or boundary data

This is the core mathematical structure of many engineering models.

Why Not Use the Matrix Inverse?

From algebra, we know:

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

However, in practice:

- ▶ Computing A^{-1} is expensive
- ▶ It amplifies numerical errors
- ▶ It is unnecessary for solving systems

Engineers almost never compute matrix inverses explicitly.

What We Learned from the Hilbert Matrix

The Hilbert matrix example showed that:

- ▶ Some matrices are *ill-conditioned*
- ▶ Small changes in \mathbf{b} can cause large changes in \mathbf{x}
- ▶ Numerical accuracy depends on the matrix, not just the algorithm

This motivates solving linear systems carefully.

Solving Linear Systems: The Right Way

Instead of computing A^{-1} , we:

Solve $A\mathbf{x} = \mathbf{b}$ directly

- ▶ Gaussian elimination
- ▶ LU factorization
- ▶ Iterative methods (later in the course)

These approaches are:

- ▶ More efficient
- ▶ More stable
- ▶ Scalable to large systems

Gaussian Elimination (Conceptual View)

Gaussian elimination:

- ▶ Transforms the system into an equivalent upper-triangular system
- ▶ Uses row operations
- ▶ Preserves the solution

After elimination:

$$U\mathbf{x} = \mathbf{c},$$

which can be solved by back-substitution.

LU Factorization

Many solvers factor the matrix:

$$A = LU$$

- ▶ L : lower triangular
- ▶ U : upper triangular

Then solve:

$$L\mathbf{y} = \mathbf{b},$$

$$U\mathbf{x} = \mathbf{y}.$$

This is the foundation of most numerical linear algebra software.

What Software Actually Does

When you write in Python:

```
x = np.linalg.solve(A, b)
```

Behind the scenes:

- ▶ No inverse is computed
- ▶ The system is factorized
- ▶ Forward and backward substitution are used

This is exactly what professional engineering codes do.

Conditioning and Accuracy

The accuracy of the solution depends on:

- ▶ The conditioning of A
- ▶ The numerical precision
- ▶ The solver algorithm

Even a *perfect* algorithm cannot fix a badly conditioned problem.

Key Takeaways

- ▶ Linear systems are central to engineering analysis
- ▶ The form $A\mathbf{x} = \mathbf{b}$ appears everywhere
- ▶ Matrix inverses are almost never used in practice
- ▶ Conditioning matters as much as algorithms
- ▶ Modern solvers rely on matrix factorization