Midterm Exam 1

MMAE 350 — Computational Mechanics

Time: 75 minutes

Instructions

- Answer all questions.
- Show all work for calculation problems.
- Write clearly and concisely.
- One handwritten formula sheet (front/back) is allowed.
- No computers, tablets, or calculators with symbolic capability.

Part I: Conceptual Understanding (30 points)

Question 1: Direct vs. Iterative Solvers (10 points)

(a) In your own words, explain the difference between a *direct solver* and an *iterative solver* for linear systems.

(b) Give one advantage and one disadvantage of each type of solver.

(c) Why is `numpy.linalg.solve` typically faster than a loop-based Gauss–Seidel implementation written in Python?

Question 2: Matrix Structure (10 points)

A linear system arises from a one-dimensional finite difference discretization of a differential equation.

(a) What special structure does the resulting matrix usually have?

(b) How does this structure affect storage requirements and computational cost?

(c) Which solver introduced in this course is specifically designed to exploit this structure?

Question 3: Residuals and Convergence (10 points)

(a) Define the residual vector $\mathbf{r} = \mathbf{b} - A\mathbf{x}$.

(b) What does the residual measure, mathematically or physically?

(c) Why is the *relative residual norm* often preferred over the absolute residual norm?

Part II: Hand Calculations (30 points)

Question 4: One Gauss–Seidel Iteration (15 points)

Consider the linear system

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 15 \\ 10 \\ 10 \end{bmatrix}.$$

(a) Starting from the initial guess $\mathbf{x}^{(0)} = (0, 0, 0)^T$, perform **one Gauss–Seidel iteration**.

(b) Clearly indicate which values are updated and reused during the sweep.

Question 5: Thomas Algorithm (15 points)

Solve the following tridiagonal system using the Thomas algorithm:

$$\begin{aligned} 2x_1 - x_2 &= 1, \\ -x_1 + 2x_2 - x_3 &= 0, \\ - x_2 + 2x_3 &= 1. \end{aligned}$$

(a) Write down the modified coefficients after the forward sweep.

(b) Perform the backward substitution to compute x_1, x_2, x_3.

Part III: Nonlinear Systems and Interpretation (30 points)

Question 6: Newton's Method (1D) (15 points)

Consider the nonlinear equation

$$f(x) = x^2 - 2.$$

(a) Write down the Newton update formula for this problem.

(b) Starting from $x^{(0)} = 1$, perform **one Newton iteration**.

(c) Briefly explain why Newton's method converges faster than the bisection method when it converges.

Question 7: Interpreting Computational Results (15 points)

A student applies three methods to solve a linear system of increasing size n:

Method	**Observed Behavior**
`numpy.linalg.solve`	Fast for small–medium n, slows for large n
Gauss–Seidel (Python loop)	Slow, many iterations
Thomas algorithm	Fast, scales approximately linearly

(a) Which method scales best and why?

(b) Why does Gauss–Seidel perform poorly in pure Python even though the algorithm is simple?

(c) For very large n, which method would you choose and why?

Question 7: Interpreting Newton's Method (15 points)

A student applies Newton's method to solve a nonlinear equation $f(x) = 0$ using different initial guesses. The following behavior is observed:

- Initial guess $x^{(0)} = 0.1$: method converges slowly
- Initial guess $x^{(0)} = 1.0$: method converges rapidly
- Initial guess $x^{(0)} = 2.5$: method fails to converge

(a) Explain why Newton's method can converge at very different rates depending on the initial guess.

(b) What properties of the function $f(x)$ and its derivative $f'(x)$ influence convergence?

(c) Give one practical strategy an engineer might use to improve the robustness of Newton's method in practice.

End of Exam