MMAE 350 — Homework #4
Nonlinear Equations and Newton's Method

Problem: Axial Deformation of a Nonlinear Bar

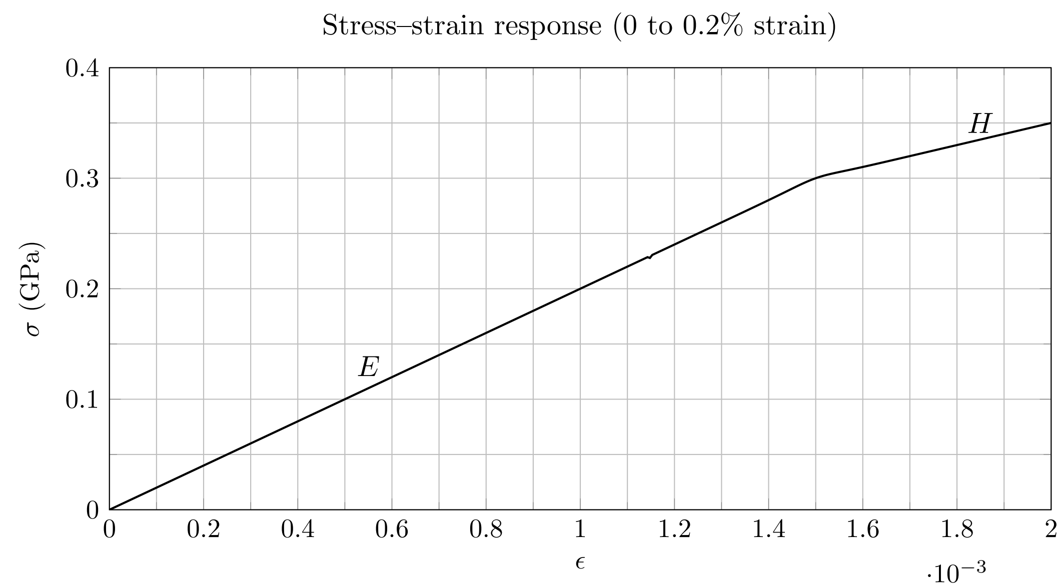

Figure 1: Stress–strain curve for a steel-like material, showing the elastic region and post-yield hardening over the strain range $0 \leq \epsilon \leq 0.002$.

Consider a uniform bar of length L and cross-sectional area A, subjected to a constant axial force P. The bar material exhibits a smooth nonlinear stress–strain relationship intended to mimic yielding and post-yield hardening.

Constitutive law

Let ϵ denote the axial strain and $\sigma(\epsilon)$ the axial stress. Define the yield strain

$$\epsilon_y = \frac{\sigma_y}{E}, \tag{1}$$

and the smooth transition function

$$s(\epsilon) = \frac{1}{2}\left(1 + \tanh\left(\frac{\epsilon - \epsilon_y}{\delta}\right)\right), \tag{2}$$

where $\delta > 0$ controls the sharpness of the transition (smaller δ gives a sharper knee).

The stress–strain law is

$$\sigma(\epsilon) = (1 - s(\epsilon))\, E\epsilon \;+\; s(\epsilon)\left[\sigma_y + H(\epsilon - \epsilon_y)\right]. \tag{3}$$

Kinematics and equilibrium

The axial strain is related to end displacement u by

$$\epsilon = \frac{u}{L}. \tag{4}$$

Force equilibrium requires that the internal axial force equals the applied load:

$$P = A\,\sigma(\epsilon). \tag{5}$$

Hence the displacement u satisfies the nonlinear equation

$$f(u) = A\,\sigma\left(\frac{u}{L}\right) - P = 0. \tag{6}$$

where the stress σ is given by Equation (3).

(b) Newton's method formulation

1. Compute $f'(u)$. (*Hint: use the chain rule and SymPy*)
2. Write the Newton iteration
$$u^{(k+1)} = u^{(k)} - \frac{f\left(u^{(k)}\right)}{f'\left(u^{(k)}\right)}. \tag{7}$$
3. Briefly explain the physical meaning of linearizing $f(u)$ about the current iterate.

(c) Numerical solution

Use Newton's method to solve for the displacement u using the following data:

Table 1: Material properties, geometry, and loading parameters used in the analysis.

Symbol	**Description**	**Value**
E	Young's modulus	200 GPa
σ_y	Yield stress	300 MPa
H	Post-yield hardening modulus	1×10^{11} Pa
δ	Transition parameter	5×10^{-5}
L	Bar length	1 m
A	Cross-sectional area	$1 \times 10^{-4}\,\mathrm{m}^2$
P	Applied axial load	150 kN

1. Use the linear-elastic solution as your initial guess:
$$u^{(0)} = \frac{PL}{AE}. \tag{8}$$
2. Iterate until $|f(u)| < 10^{-8}$.
3. Report the converged displacement and the number of Newton iterations required.
4. Make a plot of $f(u)$ versus u/L over $0 \leq u/L \leq 0.002$, and indicate the root on the plot.

(d) Interpretation

1. Compute the displacement predicted by linear elasticity.
2. Compare the linear and nonlinear solutions (numerically and in words).
3. Over the strain range considered (up to 0.2%), does the nonlinear response make the bar effectively stiffer or softer than linear elasticity?

Submission Instructions

Submit a single Jupyter notebook (`.ipynb`) containing:

- Your derivations (typeset using Markdown and LaTeX or cut and paste a screenshot or photo of handwritten work into markdown cell),
- Your Newton implementation,
- Numerical results,
- Interpretation, including your plot.