Nonlinear Systems in Two Dimensions

Two-Term Linearization in x and y

MMAE 350

Nonlinear Systems of Equations

Many engineering problems lead to systems of nonlinear equations:

$$\mathbf{F}(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Examples include:

- nonlinear material behavior,
- geometric nonlinearity,
- nonlinear boundary conditions.

To solve these systems, we rely on *local linearization*.

A Two-Dimensional Nonlinear System

Consider a system of two nonlinear equations:

$$
\begin{aligned}
F_1(x, y) &= 0, \\
F_2(x, y) &= 0.
\end{aligned}
$$

We seek a solution (x, y) that satisfies both equations simultaneously.

In general, these equations cannot be solved analytically.

Local Linearization About a Point

Let (x_0, y_0) be a current approximation to the solution.

We approximate the nonlinear system near this point using a first-order Taylor expansion.

This is a direct generalization of the 1D Taylor expansion.

Two-Term Taylor Expansion (2D)

Each component of **F** is expanded about (x_0, y_0):

$$F_1(x,y) \approx F_1(x_0,y_0) + \frac{\partial F_1}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F_1}{\partial y}(x_0,y_0)(y-y_0),$$

$$F_2(x,y) \approx F_2(x_0,y_0) + \frac{\partial F_2}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial F_2}{\partial y}(x_0,y_0)(y-y_0).$$

Higher-order terms are neglected.

Linearized System in Matrix Form

The linearized system can be written compactly as:

$$\mathbf{F}(x, y) \approx \mathbf{F}(x_0, y_0) + \mathbf{J}(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix},$$

where $\mathbf{J}$ is the Jacobian matrix:

$$\mathbf{J} = \begin{bmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{bmatrix}.$$

What Does the Jacobian Represent?

- The Jacobian generalizes the concept of a derivative to multiple dimensions
- It describes how the system changes with respect to small changes in (x, y)
- It provides the best linear approximation near the current point

Key Idea

Nonlinear problems are solved by repeatedly solving linearized systems.

Why Linearization Matters

- Enables iterative solution methods (Newton-type methods)
- Converts nonlinear problems into a sequence of linear solves
- Forms the foundation of nonlinear finite element analysis

Looking Ahead

Next, we will use this linearization to construct Newton's method for solving nonlinear systems.