MMAE 450 — Homework 2

Newton's Method for Nonlinear Systems

Due: January 27, 2026

Objectives

The objectives of this assignment are to:

- Introduce Newton's method for solving systems of nonlinear equations.
- Reinforce residual-based formulations of governing equations.
- Practice computing Jacobian matrices analytically or symbolically.
- Implement a Newton solver using numerical linear algebra.

This homework emphasizes algorithmic structure and physical interpretation rather than algebraic complexity.

Problem: Steady-State Nonlinear Heat Balance

Consider two thermal nodes with unknown temperatures T_1 and T_2. The nodes exchange heat by linear conduction, and node 1 loses heat to the environment by thermal radiation.

The steady-state governing equations are given by

$$R_1(T_1, T_2) = k\,(T_2 - T_1) + \sigma\,(T_1^4 - T_\infty^4), \tag{1}$$

$$R_2(T_1, T_2) = k\,(T_1 - T_2). \tag{2}$$

Here:

- $k > 0$ is a conduction coefficient,
- $\sigma > 0$ is a radiation coefficient,
- T_∞ is the ambient temperature.

(a) Residual Formulation

Write the system (1)–(2) in vector residual form,

$$\boldsymbol{R}(\boldsymbol{T}) = \begin{bmatrix} R_1(T_1, T_2) \\ R_2(T_1, T_2) \end{bmatrix} = \mathbf{0}, \qquad \boldsymbol{T} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}.$$

Clearly identify the unknown vector and the residual components.

(b) Jacobian Matrix

Compute the Jacobian matrix

$$\boldsymbol{J}(\boldsymbol{T}) = \frac{\partial \boldsymbol{R}}{\partial \boldsymbol{T}}.$$

You may compute the Jacobian:

- analytically, or
- using a symbolic tool such as `SymPy`.

(c) Newton Iteration

Write the Newton linear system solved at iteration k and the update equation for $\boldsymbol{T}^{(k+1)}$.

(d) Numerical Solution

Implement Newton's method in Python using **NumPy only** (no symbolic operations inside the solver).

Your implementation must:

- use a relative residual convergence criterion

  $$\frac{\|\boldsymbol{R}(\boldsymbol{T}^{(k)})\|}{\|\boldsymbol{R}(\boldsymbol{T}^{(0)})\|} < \varepsilon,$$

  where ε is a user-prescribed tolerance,
- report the converged solution (T_1, T_2),
- report the number of Newton iterations required.

You may choose reasonable numerical values for k, σ, and T_∞.

(e) Convergence Behavior

Plot the relative residual norm versus Newton iteration number and comment briefly on the observed convergence behavior.

Submission Instructions

Submit a single Jupyter notebook (`.ipynb`) containing:

- a clear problem setup,
- residual and Jacobian definitions,
- the Newton solver implementation,
- results and discussion.

Ensure that your notebook is well-organized and clearly documented.