

2D Finite Differences: Indexing, Vectorization, and Sparsity

6×4 grid, 5-point stencil, interior unknowns, and matrix pattern

Why we number grid points

A 2D grid field $T_{i,j}$ becomes a vector \mathbf{T} so we can write a linear system

$$\mathbf{A}\mathbf{T} = \mathbf{b}.$$

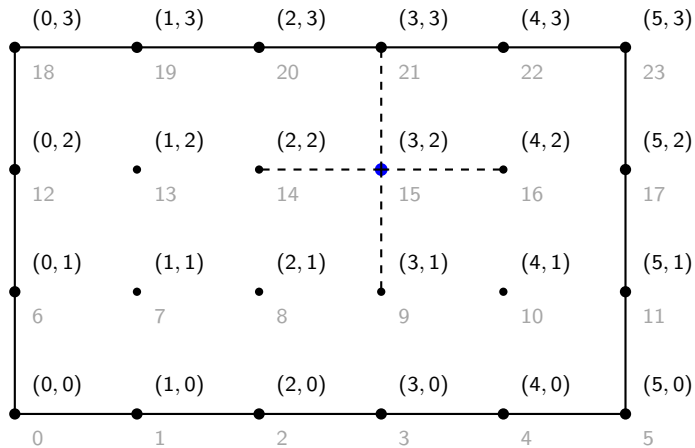
We need:

- ▶ **Local indices** (i, j) for neighbors in the stencil.
- ▶ **A global index** to store unknowns in a 1D vector.

We will use **row-major ordering** (NumPy-style):

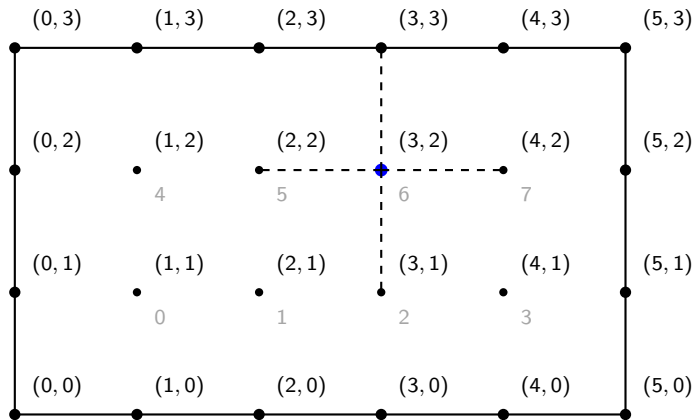
$$p = i + j N_x.$$

6×4 grid: (i,j) labels and global index p (interior labeling)



$$p = i + j N_x$$

6×4 grid: (i,j) labels and global index k (full labeling)



$$k = (i - 1) + (j - 1)(N_x - 2)$$

Vector mapping and stencil offsets

With row-major ordering ($N_x = 6$):

$$p = i + j N_x$$

$$i = p \bmod N_x$$

$$j = \left\lfloor \frac{p}{N_x} \right\rfloor$$

For an interior node (i, j) with global index p :

$$(i + 1, j) \Rightarrow p + 1$$

$$(i - 1, j) \Rightarrow p - 1$$

$$(i, j + 1) \Rightarrow p + N_x$$

$$(i, j - 1) \Rightarrow p - N_x$$

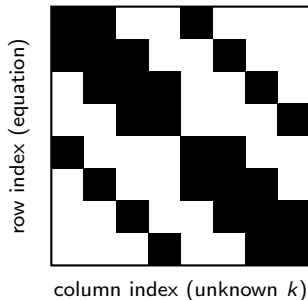
So the 5-point Laplacian stencil becomes the same pattern in vector form:

$$T_{p+1} + T_{p-1} + T_{p+N_x} + T_{p-N_x} - 4T_p$$

Sparsity pattern of \mathbf{A} for interior unknowns (5-point stencil)

- ▶ Assume Dirichlet boundary values are known (moved to \mathbf{b}).
- ▶ Unknowns are the **interior** points only: $(N_x - 2) \times (N_y - 2) = 4 \times 2 = 8$.
- ▶ Order unknowns in row-major order on the interior grid:

$$k = (i - 1) + (j - 1)(N_x - 2), \quad i = 1..4, j = 1..2.$$



This block-banded structure comes directly from the neighbor offsets ± 1 and $\pm(N_x - 2)$ on the interior grid.

Enforcing Dirichlet boundary conditions using slicing

Suppose the temperature field is stored on a 2D grid

$$T_{i,j}, \quad i = 0, \dots, N_x - 1, j = 0, \dots, N_y - 1,$$

with Dirichlet boundary conditions prescribed on all four edges.

Using NumPy-style slicing, boundary values can be imposed *without loops*:

$$\text{Left boundary: } T[0, :] = T_L(y),$$

$$\text{Right boundary: } T[N_x - 1, :] = T_R(y),$$

$$\text{Bottom boundary: } T[:, 0] = T_B(x),$$

$$\text{Top boundary: } T[:, N_y - 1] = T_T(x).$$

Only the interior values

$$T[1 : N_x - 1, 1 : N_y - 1]$$

remain unknown and are used to assemble the linear system.

Takeaway

- ▶ **Local indices** (i, j) express the PDE stencil naturally.
- ▶ A **global index** (row-major) lets us store the unknowns as a vector.
- ▶ The 5-point stencil becomes simple vector offsets: ± 1 and $\pm N_x$ (or $\pm(N_x - 2)$ for interior-only).
- ▶ Those offsets produce a highly **sparse, structured** matrix **A**.