Finite Volume Method in 1D

Control Volume Discretization (Chapter 5 figure)

MMAE 450

Internal Energy Equation (with Source)

After substituting the momentum equation into the total energy balance:

$$\rho \frac{De}{Dt} = \nabla \cdot (k \nabla T) + \boldsymbol{\sigma} : \nabla \mathbf{v} + S$$

Assume:
- negligible viscous dissipation
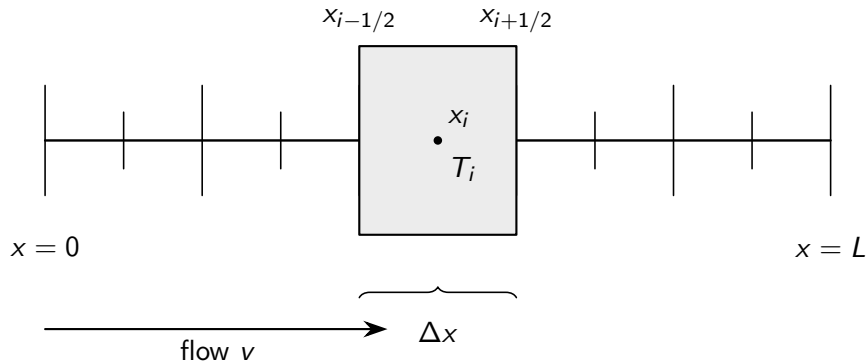- constant ρ, k, c_p
- $e = c_p T$

When we use $e = c_p T$ and apply the definition of the material time derivative,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla,$$

the one-dimensional advection–diffusion equation can be written as

$$\boxed{\frac{\partial T}{\partial t} + v \frac{\partial T}{\partial x} = \alpha \frac{\partial^2 T}{\partial x^2} + \frac{S}{\rho c_p}}$$

1D Finite Volume Discretization (Control Volume View)

Finite Volume Formulation (No Source)

Neglect the source term ($S = 0$).

Write the 1D advection–diffusion equation in conservative form:

$$\frac{\partial T}{\partial t} + \frac{\partial F}{\partial x} = 0, \qquad F \equiv vT - \alpha\frac{\partial T}{\partial x}.$$

Integrate over the control volume $[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}]$:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial T}{\partial t}\, dx + \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial F}{\partial x}\, dx = 0.$$

Applying the Fundamental Theorem of Calculus

Time derivative term:

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial T}{\partial t}\, dx = \frac{d}{dt} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} T\, dx.$$

Flux term (FTC):

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \frac{\partial F}{\partial x}\, dx = F(x_{i+\frac{1}{2}}) - F(x_{i-\frac{1}{2}}).$$

Therefore,

$$\boxed{\frac{d}{dt} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} T\, dx = F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}}$$

$$\text{Storage} = \text{Influx} - \text{Outflux}$$

Cell Average Definition

From the integral balance:

$$\frac{d}{dt} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} T \, dx = F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}.$$

Define the cell average temperature:

$$T_i(t) \equiv \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} T(x, t) \, dx.$$

Therefore,

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} T \, dx = T_i \, \Delta x.$$

Unknown stored at the cell center represents a **cell average**.

Semi-Discrete Finite Volume Equation

Substitute the cell average into the conservation statement:

$$\frac{d}{dt}\left(T_i\,\Delta x\right) = F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}.$$

Since Δx is constant,

$$\boxed{\Delta x \frac{dT_i}{dt} = F_{i-\frac{1}{2}} - F_{i+\frac{1}{2}}}$$

Or equivalently,

$$\frac{dT_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right).$$

Storage = Influx Outflux

Face Flux: Diffusion Term

Recall the total flux:

$$F = vT - \alpha\frac{\partial T}{\partial x}.$$

Diffusive contribution:

$$F^{(d)} = -\alpha\frac{\partial T}{\partial x}.$$

At the face $x_{i+\frac{1}{2}}$, approximate the gradient using a central difference:

$$\left.\frac{\partial T}{\partial x}\right|_{i+\frac{1}{2}} \approx \frac{T_{i+1} - T_i}{\Delta x}.$$

Therefore,

$$\boxed{F^{(d)}_{i+\frac{1}{2}} = -\alpha\frac{T_{i+1} - T_i}{\Delta x}}$$

Face Flux: Convection Term (Central Approximation)

Convective contribution:

$$F^{(c)} = vT.$$

At the face $x_{i+\frac{1}{2}}$:

$$F^{(c)}_{i+\frac{1}{2}} = v\, T_{i+\frac{1}{2}}.$$

Using a central interpolation,

$$T_{i+\frac{1}{2}} = \frac{T_i + T_{i+1}}{2}.$$

Therefore,

$$\boxed{F^{(c)}_{i+\frac{1}{2}} = v\, \frac{T_i + T_{i+1}}{2}}$$

Face Flux at $x_{i-\frac{1}{2}}$ (Central Approximations)

Total flux:

$$F = vT - \alpha\frac{\partial T}{\partial x}.$$

At the left face $x_{i-\frac{1}{2}}$:

$$T_{i-\frac{1}{2}} \approx \frac{T_{i-1} + T_i}{2} \qquad \Rightarrow \qquad F_{i-\frac{1}{2}}^{(c)} = v\,\frac{T_{i-1} + T_i}{2}.$$

$$\left.\frac{\partial T}{\partial x}\right|_{i-\frac{1}{2}} \approx \frac{T_i - T_{i-1}}{\Delta x} \qquad \Rightarrow \qquad F_{i-\frac{1}{2}}^{(d)} = -\alpha\,\frac{T_i - T_{i-1}}{\Delta x}.$$

Therefore,

$$\boxed{F_{i-\frac{1}{2}} = v\,\frac{T_{i-1} + T_i}{2} - \alpha\,\frac{T_i - T_{i-1}}{\Delta x}}$$

Semi-Discrete Update (Central Fluxes at Both Faces)

Finite volume update:

$$\frac{dT_i}{dt} = -\frac{1}{\Delta x}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right), \qquad F = vT - \alpha\frac{\partial T}{\partial x}.$$

Using central face fluxes:

$$F_{i+\frac{1}{2}} = v\,\frac{T_i + T_{i+1}}{2} - \alpha\,\frac{T_{i+1} - T_i}{\Delta x}, \qquad F_{i-\frac{1}{2}} = v\,\frac{T_{i-1} + T_i}{2} - \alpha\,\frac{T_i - T_{i-1}}{\Delta x}.$$

Substitute and simplify:

$$\boxed{\frac{dT_i}{dt} = -v\,\frac{T_{i+1} - T_{i-1}}{2\Delta x} + \alpha\,\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}}$$

Central advection stencil $+$ central diffusion stencil

Time Discretization (Forward Euler)

Semi-discrete equation:

$$\frac{dT_i}{dt} = -v\,\frac{T_{i+1} - T_{i-1}}{2\Delta x} + \alpha\,\frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}.$$

Approximate the time derivative at t^n (Forward Euler):

$$\left.\frac{dT_i}{dt}\right|_{t^n} \approx \frac{T_i^{n+1} - T_i^n}{\Delta t}.$$

Therefore,

$$\boxed{T_i^{n+1} = T_i^n - \frac{v\Delta t}{2\Delta x}\left(T_{i+1}^n - T_{i-1}^n\right) + \frac{\alpha\Delta t}{\Delta x^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right)}$$

Cell Peclet Number and Scheme Behavior

Recall the fully discrete update:

$$T_i^{n+1} = T_i^n - \frac{v\,\Delta t}{2\Delta x}\left(T_{i+1}^n - T_{i-1}^n\right) + \frac{\alpha\,\Delta t}{\Delta x^2}\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right).$$

Define the cell Peclet number:

$$\boxed{\mathrm{Pe}_{\Delta x} = \frac{v\,\Delta x}{\alpha}}$$

Interpretation:

- $\mathrm{Pe}_{\Delta x} \ll 1 \rightarrow$ diffusion-dominated
- $\mathrm{Pe}_{\Delta x} \gg 1 \rightarrow$ advection-dominated
- Central schemes may oscillate for large Peclet numbers

Motivation for upwind schemes

Algorithm: 1D Advection–Diffusion (Central Scheme)

Given:

$$N, \ \Delta x, \ \Delta t, \ v, \ \alpha, \qquad T_i^0 \text{ for } i = 0, \ldots, N$$

Define dimensionless parameters:

$$\lambda_d = \frac{\alpha \Delta t}{\Delta x^2}, \qquad \lambda_c = \frac{v \Delta t}{2 \Delta x}.$$

For each time step $n \to n + 1$:

1. Apply boundary conditions to T_0^n and T_N^n
2. For $i = 1, \ldots, N - 1$:

$$T_i^{n+1} = T_i^n - \lambda_c \left(T_{i+1}^n - T_{i-1}^n \right) + \lambda_d \left(T_{i+1}^n - 2 T_i^n + T_{i-1}^n \right)$$

3. Advance to next time step

Algorithm (IC, Sweep, BC Update)

Initialization:
- ▶ Set initial condition: $T_i^0 = T(x_i, 0)$ for $i = 0, \ldots, N$
- ▶ Enforce boundary conditions on T^0 (set T_0^0, T_N^0)

Time stepping: for $n = 0, 1, 2, \ldots$
1. (BC at current time) Ensure T_0^n and T_N^n satisfy BCs
2. (Interior sweep) For $i = 1, \ldots, N - 1$ compute

$$T_i^{n+1} = T_i^n - \lambda_c \left(T_{i+1}^n - T_{i-1}^n \right) + \lambda_d \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n \right)$$

3. (BC at new time) Set T_0^{n+1} and T_N^{n+1} from BCs