

## 2D Finite Differences: Indexing, Vectorization, and Sparsity

$6 \times 4$  grid, 5-point stencil, interior unknowns, and matrix pattern

## Why we number grid points

A 2D grid field  $T_{i,j}$  becomes a vector  $\mathbf{T}$  so we can write a linear system

$$\mathbf{AT} = \mathbf{b}.$$

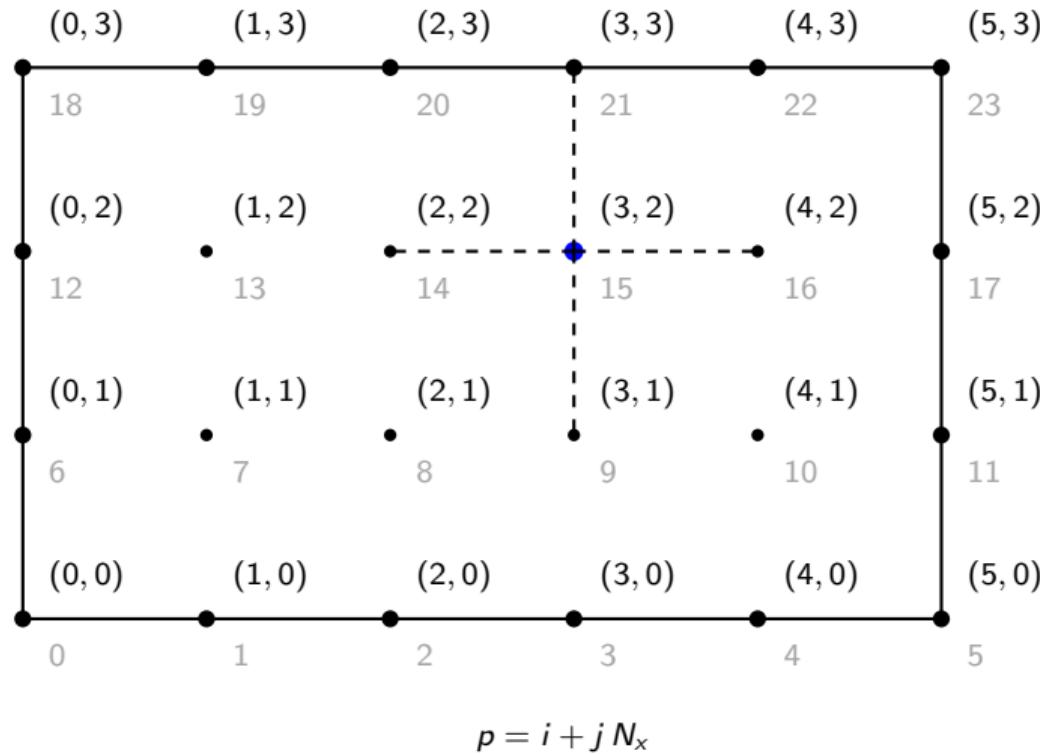
We need:

- ▶ **Local indices**  $(i,j)$  for neighbors in the stencil.
- ▶ **A global index** to store unknowns in a 1D vector.

We will use **row-major ordering** (NumPy-style):

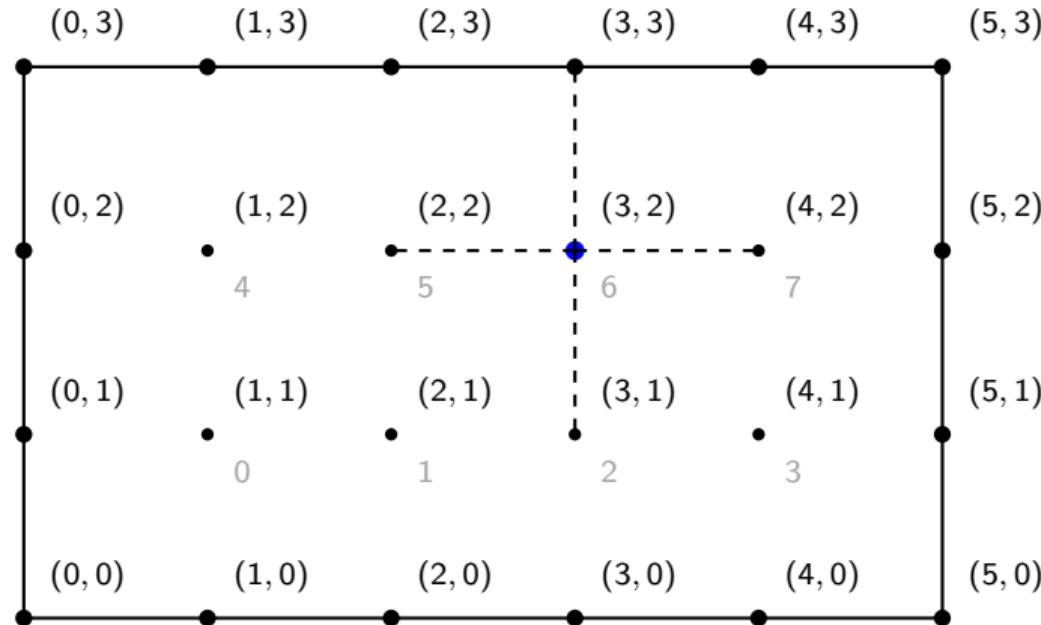
$$p = i + j N_x.$$

$6 \times 4$  grid:  $(i, j)$  labels and global index  $p$  (interior labeling)



$$p = i + j N_x$$

## $6 \times 4$ grid: $(i, j)$ labels and global index $k$ (full labeling)



## Vector mapping and stencil offsets

With row-major ordering ( $N_x = 6$ ):

$$p = i + j N_x$$

$$i = p \bmod N_x$$

$$j = \left\lfloor \frac{p}{N_x} \right\rfloor$$

For an interior node  $(i, j)$  with global index  $p$ :

$$(i+1, j) \Rightarrow p + 1$$

$$(i-1, j) \Rightarrow p - 1$$

$$(i, j+1) \Rightarrow p + N_x$$

$$(i, j-1) \Rightarrow p - N_x$$

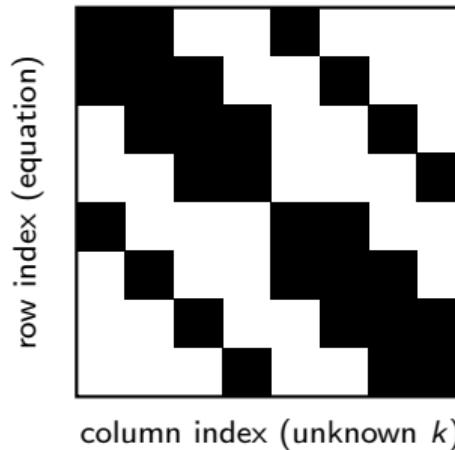
So the 5-point Laplacian stencil becomes the same pattern in vector form:

$$T_{p+1} + T_{p-1} + T_{p+N_x} + T_{p-N_x} - 4T_p$$

## Sparsity pattern of $\mathbf{A}$ for interior unknowns (5-point stencil)

- ▶ Assume Dirichlet boundary values are known (moved to  $\mathbf{b}$ ).
- ▶ Unknowns are the **interior** points only:  $(N_x - 2) \times (N_y - 2) = 4 \times 2 = 8$ .
- ▶ Order unknowns in row-major order on the interior grid:

$$k = (i - 1) + (j - 1)(N_x - 2), \quad i = 1..4, j = 1..2.$$



This block-banded structure comes directly from the neighbor offsets  $\pm 1$  and  $\pm(N_x - 2)$  on the interior grid.

## Enforcing Dirichlet boundary conditions using slicing

Suppose the temperature field is stored on a 2D grid

$$T_{i,j}, \quad i = 0, \dots, N_x - 1, j = 0, \dots, N_y - 1,$$

with Dirichlet boundary conditions prescribed on all four edges.

Using NumPy-style slicing, boundary values can be imposed *without loops*:

$$\text{Left boundary: } T[0, :] = T_L(y),$$

$$\text{Right boundary: } T[N_x - 1, :] = T_R(y),$$

$$\text{Bottom boundary: } T[:, 0] = T_B(x),$$

$$\text{Top boundary: } T[:, N_y - 1] = T_T(x).$$

Only the interior values

$$T[1 : N_x - 1, 1 : N_y - 1]$$

remain unknown and are used to assemble the linear system.

## Takeaway

- ▶ **Local indices**  $(i, j)$  express the PDE stencil naturally.
- ▶ A **global index** (row-major) lets us store the unknowns as a vector.
- ▶ The 5-point stencil becomes simple vector offsets:  $\pm 1$  and  $\pm N_x$  (or  $\pm(N_x - 2)$  for interior-only).
- ▶ Those offsets produce a highly **sparse, structured** matrix **A**.