Heat Conduction: From Balance Law to FTCS

(Boundary Conditions)

Mike Gosz

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Boundary Conditions for the Cooling Sphere HW3

We solve the radially symmetric heat equation in a sphere:

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \right], \qquad 0 < r < R.$$

Two boundary conditions are required:

- **Center symmetry (Neumann):**

$$\left. \frac{\partial T}{\partial r} \right|_{r=0} = 0 \quad \Longleftrightarrow \quad q_r(0, t) = 0.$$

- **Surface convection (Robin):**

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=R} = h \left(T(R, t) - T_\infty \right).$$

Neumann BC at $r = 0$ via a Balance Law (Center Control Volume)

Consider a small spherical control volume around the origin, with radius $\Delta r/2$.

$$\rho c_p V \frac{\mathrm{d}T_0}{\mathrm{d}t} = -qA,$$

where q is the outward heat flux at $r = \Delta r/2$.

Geometry:

$$V = \frac{4}{3}\pi \left(\frac{\Delta r}{2}\right)^3, \qquad A = 4\pi \left(\frac{\Delta r}{2}\right)^2.$$

Approximate the conductive flux across the boundary:

$$q \approx -k \frac{T_1 - T_0}{\Delta r}.$$

Neuman BC at $r = 0$: Conduction Balances Convection

Substitute and simplify:

$$\frac{\mathrm{d}\,T_0}{\mathrm{d}t} = 6\alpha\,\frac{T_1 - T_0}{\Delta r^2}.$$

Forward Euler in time gives the **center update**:

$$T_0^{n+1} = T_0^n + 6\lambda\left(T_1^n - T_0^n\right), \qquad \lambda = \frac{\alpha\Delta t}{\Delta r^2}.$$

Robin BC at $r = R$: Conduction Balances Convection

At the surface, the boundary condition is

$$-k \left.\frac{\partial T}{\partial r}\right|_{r=R} = h\left(T(R,t) - T_\infty\right).$$

Interpretation:

- $-k\,\partial T/\partial r$ is the **conductive heat flux leaving the solid**
- $h(T - T_\infty)$ is the **convective heat flux into the air**

This is a **Robin (mixed) boundary condition** because it couples:

temperature at the boundary   and   temperature gradient at the boundary.

In a finite difference method, we enforce this condition at each time step to determine the surface behavior.

Robin BC Implementation: Ghost Node at the Surface

Approximate the surface derivative using a centered difference:

$$\left.\frac{\partial T}{\partial r}\right|_{r=R} \approx \frac{T^n_{N+1} - T^n_{N-1}}{2\Delta r}.$$

Insert into the Robin condition:

$$-k\,\frac{T^n_{N+1} - T^n_{N-1}}{2\Delta r} = h\left(T^n_N - T_\infty\right).$$

Solve for the ghost value T^n_{N+1}:

$$\boxed{T^n_{N+1} = T^n_{N-1} - 2\Delta r\,\frac{h}{k}\left(T^n_N - T_\infty\right).}$$

Then the surface node update uses the same stencil as interior nodes, with T^n_{N+1} substituted where T_{N+1} appears.

Summary: How We Enforce Boundary Conditions

- **Center ($r = 0$), Neumann / symmetry:**

$$T_0^{n+1} = T_0^n + 6\lambda\,(T_1^n - T_0^n)\,.$$

- **Surface ($r = R$), Robin / convection:**

$$-k\,\frac{\partial T}{\partial r}\bigg|_{r=R} = h\,(T_N - T_\infty)\,, \qquad T_{N+1}^n = T_{N-1}^n - 2\Delta r\,\frac{h}{k}\,(T_N^n - T_\infty)\,.$$

- **Interior ($i = 1, \ldots, N-1$):** use the standard spherical FTCS stencil.

Takeaway: boundary conditions are enforced *at every time step*, and they determine the updates at nodes where the standard interior stencil does not apply.