Heat Conduction: From Balance Law to FTCS

(building directly on the divergence theorem)

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Where we are headed (10 minutes)

- Start with a **global energy balance** on a body Ω with boundary Γ.
- Use the **divergence theorem** to convert boundary fluxes to a volume statement.
- Close the model with **Fourier's law** to obtain the heat equation.
- Specialize to 1D and build the **FTCS** explicit update.

Global energy balance (integral form)

Let $T(\boldsymbol{x}, t)$ be temperature, ρ density, c specific heat, $\boldsymbol{q}$ conductive heat flux, and S volumetric heat generation (power per unit volume).

Total thermal energy:

$$E(t) = \int_\Omega \rho c \, T(\boldsymbol{x}, t) \, \mathrm{d}V.$$

Balance law (rate of change = net inflow through boundary + internal generation):

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \rho c \, T \, \mathrm{d}V = -\int_\Gamma \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_\Omega S \, \mathrm{d}V.$$

Divergence theorem ⇒ local balance

Assuming sufficient smoothness, move the time derivative inside:

$$\int_\Omega \rho c \, \frac{\partial T}{\partial t} \, \mathrm{d}V = - \int_\Gamma \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S + \int_\Omega S \, \mathrm{d}V.$$

Use the divergence theorem on the boundary flux term:

$$\int_\Gamma \boldsymbol{q} \cdot \boldsymbol{n} \, \mathrm{d}S = \int_\Omega \nabla \cdot \boldsymbol{q} \, \mathrm{d}V.$$

Therefore,

$$\int_\Omega \left(\rho c \, \frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{q} - S \right) \mathrm{d}V = 0 \quad \Rightarrow \quad \rho c \, \frac{\partial T}{\partial t} + \nabla \cdot \boldsymbol{q} - S = 0 \ \text{ in } \Omega.$$

Constitutive law: Fourier's law

For isotropic conduction,

$$\boldsymbol{q} = -k\,\nabla T,$$

where k is thermal conductivity.
Substitute into the local balance:

$$\rho c\,\frac{\partial T}{\partial t} = \nabla\cdot(k\,\nabla T) + S.$$

If k is constant:

$$\rho c\,\frac{\partial T}{\partial t} = k\,\nabla^2 T + S, \qquad \frac{\partial T}{\partial t} = \alpha\nabla^2 T + \frac{S}{\rho c}, \qquad \alpha = \frac{k}{\rho c}.$$

1D specialization and boundary conditions

On $0 \leq x \leq L$:

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial x^2}(x,t) + \frac{S(x,t)}{\rho c}.$$

Typical boundary conditions:

- Dirichlet: $T(0,t) = T_L(t), \quad T(L,t) = T_R(t).$
- Neumann (prescribed flux): $-k \dfrac{\partial T}{\partial x}(0,t) = q_n^*(t).$
- Insulated end is the special case $q_n^*(t) = 0$.

Key idea: the PDE says "rate of temperature change" = "net diffusion in/out" + "sources."

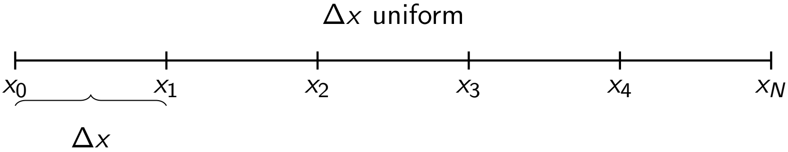
Grid and notation (space–time lattice)

Uniform grid:

$$x_i = i\,\Delta x, \quad i = 0, 1, \ldots, N, \qquad t^n = n\,\Delta t, \quad n = 0, 1, 2, \ldots$$

Store nodal values:

$$T_i^n \approx T(x_i, t^n).$$



FTCS: Forward Time, Centered Space

Approximate derivatives at (x_i, t^n):

$$\frac{\partial T}{\partial t}\bigg|_{i,n} \approx \frac{T_i^{n+1} - T_i^n}{\Delta t}, \qquad \frac{\partial^2 T}{\partial x^2}\bigg|_{i,n} \approx \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}.$$

Insert into the 1D heat equation:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} + \frac{S_i^n}{\rho c}.$$

Define the diffusion number

$$\lambda = \frac{\alpha \, \Delta t}{(\Delta x)^2}.$$

Explicit update (interior nodes):

$$T_i^{n+1} = T_i^n + \lambda \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n \right) + \Delta t \, \frac{S_i^n}{\rho c}.$$

Interpretation: stencil = local energy balance

Rewrite the update to highlight "in minus out":

$$T_i^{n+1} = (1 - 2\lambda) T_i^n + \lambda T_{i-1}^n + \lambda T_{i+1}^n + \Delta t \, \frac{S_i^n}{\rho c}.$$

- $\lambda T_{i\pm1}^n$ are contributions from neighbors (*heat diffuses in*).
- $-2\lambda T_i^n$ is the net *outflow* contribution.
- $\Delta t \, S/(\rho c)$ raises temperature when $S > 0$.

If $S = 0$, the update is a weighted average (plus the old value) *only if* λ is not too large.

Stability: why FTCS needs a time-step restriction

For the 1D diffusion equation with $S = 0$, von Neumann analysis gives

$$0 \leq \lambda \leq \frac{1}{2} \qquad \Longleftrightarrow \qquad \Delta t \leq \frac{(\Delta x)^2}{2\alpha}.$$

- Physically: you cannot let diffusion move information “too far” in one explicit step.
- Numerically: if λ is too large, high-frequency modes grow $\Rightarrow$ oscillation/blow-up.

Rule of thumb for class demos: pick $\lambda \approx 0.25$.

Algorithm slide (what students will code)

Given α, Δx, Δt, N and initial data T_i^0:

1. Compute $\lambda = \alpha \Delta t / (\Delta x)^2$ and ensure $\lambda \leq 1/2$.
2. For $n = 0, 1, 2, \ldots$:
    2.1 Enforce boundary conditions to obtain T_0^n and T_N^n.
    2.2 For $i = 1, \ldots, N-1$ update

$$T_i^{n+1} = T_i^n + \lambda \left(T_{i+1}^n - 2T_i^n + T_{i-1}^n \right) + \Delta t \, \frac{S_i^n}{\rho c}.$$

Next lecture: implicit methods $\Rightarrow$ linear systems (tridiagonal) and the Thomas algorithm.

One-slide recap

Balance law: $$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho c T \,\mathrm{d}V = -\int_\Gamma \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}S + \int_\Omega S\,\mathrm{d}V$$

Divergence theorem: $$\int_\Gamma \boldsymbol{q}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_\Omega \nabla\cdot\boldsymbol{q}\,\mathrm{d}V \;\Rightarrow\; \rho c\, T_{,t} + \nabla\cdot\boldsymbol{q} - S = 0$$

Fourier law: $$\boldsymbol{q} = -k\nabla T \;\Rightarrow\; T_{,t} = \alpha\nabla^2 T + \frac{S}{\rho c}$$

FTCS (1D): $$T_i^{n+1} = T_i^n + \lambda\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right) + \Delta t\,\frac{S_i^n}{\rho c}, \quad \lambda \le \tfrac{1}{2}$$