2D Finite Differences: Indexing, Vectorization, and Sparsity

6×4 grid, 5-point stencil, interior unknowns, and matrix pattern

Why we number grid points

A 2D grid field $T_{i,j}$ becomes a vector $\mathbf{T}$ so we can write a linear system

$$\mathbf{A}\mathbf{T} = \mathbf{b}.$$

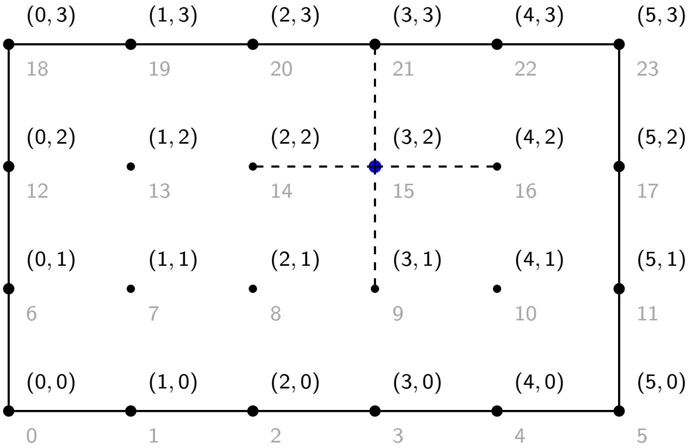
We need:

- **Local indices** (i, j) for neighbors in the stencil.
- **A global index** to store unknowns in a 1D vector.

We will use **row-major ordering** (NumPy-style):

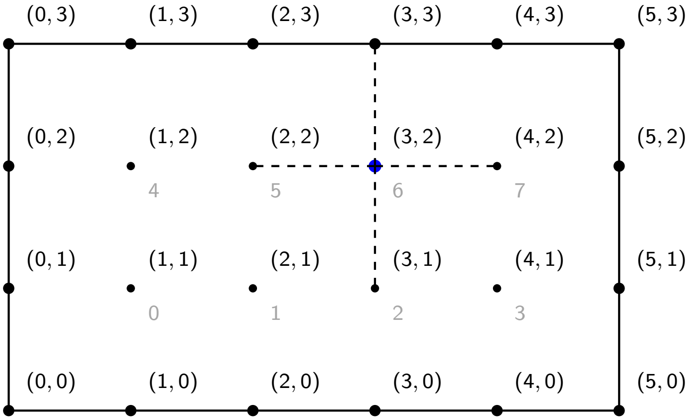
$$\boxed{p = i + j\, N_x.}$$

6×4 grid: (i, j) labels and global index p (interior labeling)



$$p = i + j\,N_x$$

6×4 grid: (i,j) labels and global index k (full labeling)



$$k = (i-1) + (j-1)(N_x - 2)$$

Vector mapping and stencil offsets

With row-major ordering ($N_x = 6$):

$$\boxed{p = i + j\,N_x} \qquad \boxed{i = p \bmod N_x} \qquad \boxed{j = \left\lfloor \frac{p}{N_x} \right\rfloor}$$

For an interior node (i, j) with global index p:

$$\begin{aligned}
(i+1, j) &\Rightarrow p + 1 \\
(i-1, j) &\Rightarrow p - 1 \\
(i, j+1) &\Rightarrow p + N_x \\
(i, j-1) &\Rightarrow p - N_x
\end{aligned}$$

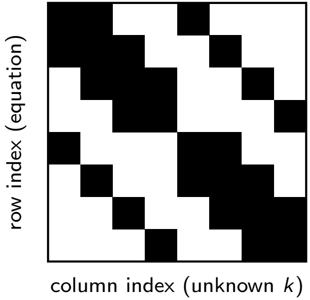
So the 5-point Laplacian stencil becomes the same pattern in vector form:

$$T_{p+1} + T_{p-1} + T_{p+N_x} + T_{p-N_x} - 4\,T_p$$

Sparsity pattern of **A** for interior unknowns (5-point stencil)

- Assume Dirichlet boundary values are known (moved to **b**).
- Unknowns are the **interior** points only: $(N_x - 2) \times (N_y - 2) = 4 \times 2 = 8$.
- Order unknowns in row-major order on the interior grid:

$$k = (i-1) + (j-1)(N_x - 2), \quad i = 1..4,\ j = 1..2.$$



This block-banded structure comes directly from the neighbor offsets ± 1 and $\pm(N_x - 2)$ on the interior grid.

Enforcing Dirichlet boundary conditions using slicing

Suppose the temperature field is stored on a 2D grid

$$T_{i,j}, \qquad i = 0, \dots, N_x - 1,\ j = 0, \dots, N_y - 1,$$

with Dirichlet boundary conditions prescribed on all four edges.

Using NumPy-style slicing, boundary values can be imposed *without loops*:

$$\begin{aligned}
\text{Left boundary:} &\quad T[0,:] = T_L(y), \\
\text{Right boundary:} &\quad T[N_x - 1,:] = T_R(y), \\
\text{Bottom boundary:} &\quad T[:,0] = T_B(x), \\
\text{Top boundary:} &\quad T[:,N_y - 1] = T_T(x).
\end{aligned}$$

Only the interior values

$$T[1 : N_x - 1,\ 1 : N_y - 1]$$

remain unknown and are used to assemble the linear system.

Takeaway

- **Local indices** (i,j) express the PDE stencil naturally.
- A **global index** (row-major) lets us store the unknowns as a vector.
- The 5-point stencil becomes simple vector offsets: ± 1 and $\pm N_x$ (or $\pm(N_x - 2)$ for interior-only).
- Those offsets produce a highly **sparse, structured** matrix $\mathbf{A}$.