MMAE 450 — Homework 04: 2D Transient Heat Conduction in a Plate

Due: Wednesday, February 4, 2026 (11:59 pm)

Problem: Transient Cooling of a Heated Patch in a Rectangular Plate (Dirichlet BCs)

A thin rectangular plate occupies the domain

$$0 \le x \le L_x, \qquad 0 \le y \le L_y,$$

with constant thermal diffusivity α. Neglecting through-thickness gradients, the temperature field $T(x, y, t)$ satisfies the two-dimensional heat equation

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) \qquad \text{in } \Omega,\ t > 0. \tag{1}$$

Given data

Use the following values:

- Plate dimensions: $L_x = 1.0\,\mathrm{m}$, $L_y = 0.5\,\mathrm{m}$
- Thermal diffusivity: $\alpha = 1.0 \times 10^{-4}\,\mathrm{m^2/s}$
- Uniform grid including boundaries: $N_x = 51$, $N_y = 26$

Thus,

$$\Delta x = \frac{L_x}{N_x - 1}, \qquad \Delta y = \frac{L_y}{N_y - 1}.$$

Boundary conditions (Dirichlet on all edges)

All four edges are held at ambient temperature (take $0\,^\circ\mathrm{C}$):

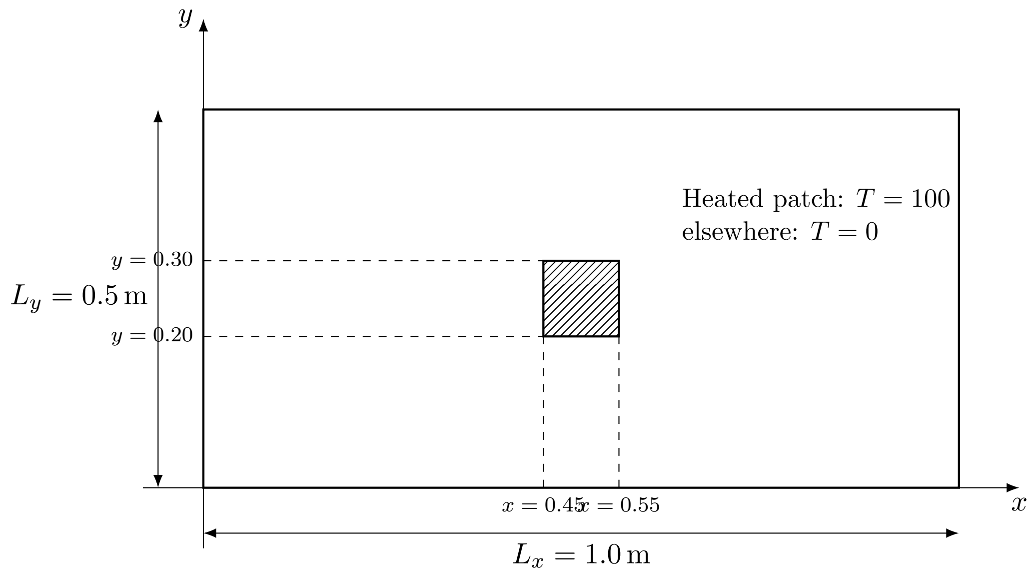
$$T(0, y, t) = 0, \quad T(L_x, y, t) = 0, \quad T(x, 0, t) = 0, \quad T(x, L_y, t) = 0.$$

Initial condition: heated patch

At $t = 0$, the plate is at ambient temperature everywhere *except* for a heated rectangular patch:

$$T(x, y, 0) = \begin{cases} 100, & 0.45 \le x \le 0.55 \text{ and } 0.20 \le y \le 0.30, \\ 0, & \text{otherwise.} \end{cases}$$

Geometry and initial condition sketch



Numerical method requirement: FTCS

Use an FTCS discretization of (**??**) on the uniform grid:

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \alpha\,\Delta t \left[\frac{T_{i+1,j}^{n} - 2T_{i,j}^{n} + T_{i-1,j}^{n}}{\Delta x^2} + \frac{T_{i,j+1}^{n} - 2T_{i,j}^{n} + T_{i,j-1}^{n}}{\Delta y^2} \right], \tag{2}$$

for interior nodes $i = 1, \ldots, N_x - 2$ and $j = 1, \ldots, N_y - 2$. Enforce the Dirichlet boundary values at every time step.

Stability requirement

Your choice of Δt must satisfy the 2D FTCS stability condition

$$\alpha\,\Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \leq \frac{1}{2}. \tag{3}$$

Choose a "safe" Δt (e.g. some fraction of the maximum stable value) and clearly document it.

Deliverables

Submit **(i)** a short PDF summary (1–2 pages) and **(ii)** a Jupyter notebook (`.ipynb`) that reproduces all results.

1. **Grid and time step.** Report Δx, Δy, your chosen Δt, and verify (**??**) numerically.
2. **Contour plots.** Produce filled contour plots of $T(x, y, t)$ at the following times:

   $$t \in \{0,\ 10,\ 50,\ 200\}\ \mathrm{s}.$$

   Use consistent contour levels (or a consistent color scale) so the decay is visually clear.

3. **Center-point history.** Let $(x_c, y_c) = (0.5, 0.25)$. Plot $T(x_c, y_c, t)$ versus time over the same simulation window.

4. **Maximum temperature decay.** Compute and plot

$$T_{\max}(t) = \max_{(x,y)\in\Omega} T(x, y, t)$$

versus time.

5. **Cooling time metric.** Report the first time $t_{10\%}$ at which $T_{\max}(t)$ drops below $10\,^{\circ}\mathrm{C}$.

Notes and good practice

- Use clear variable names and consistent indexing (i for x, j for y).
- Boundary nodes are prescribed values (not unknowns). Enforce them directly.
- Include at least one brief code cell that prints a quick stability check value:

$$\eta = \alpha\,\Delta t\left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right) \quad \text{and confirm} \quad \eta \le \frac{1}{2}.$$

Optional extension (extra credit)

Implement a fully implicit method (Backward Euler or Crank–Nicolson) in 2D and solve the linear system at each time step using Gauss–Seidel (or SOR). Compare:

- allowable time steps,
- runtime,
- accuracy of the temperature histories.