Indicial Notation, Material Derivative, and Balance Laws

Outline (if time allows)

- Why indicial notation?
- Common operators in component form
- Material time derivative
- Balance laws: mass, momentum, energy
- Advection–diffusion equation
- Reduction to 1D

Why Indicial Notation?

- Compact representation of vector/tensor equations
- Eliminates ambiguity in differentiation
- Essential for deriving and implementing balance laws
- Natural bridge to component-based numerical methods

Basic Conventions

Index notation rules

- Indices: $i, j, k = 1, 2, 3$
- Repeated indices imply summation (Einstein convention)
- Free indices must match on both sides of an equation

Examples

$$\mathbf{v} = v_i \, \mathbf{e}_i, \qquad \mathbf{a} \cdot \mathbf{b} = a_i b_i$$

Vectors in a Cartesian Basis

Write vectors in terms of components and Cartesian base vectors:

$$\mathbf{a} = a_i \, \mathbf{e}_i, \qquad \mathbf{b} = b_j \, \mathbf{e}_j$$

- $\mathbf{e}_i$ are orthonormal Cartesian base vectors
- Components a_i, b_j are scalars

Dot Product in Index Form

Start from the basis expansion:

$$\mathbf{a} \cdot \mathbf{b} = (a_i \mathbf{e}_i) \cdot (b_j \mathbf{e}_j) = a_i b_j \, (\mathbf{e}_i \cdot \mathbf{e}_j)$$

For Cartesian base vectors,

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}$$

so

$$\mathbf{a} \cdot \mathbf{b} = a_i b_j \delta_{ij} = a_i b_i .$$

Second-Order Tensors in a Cartesian Basis

Represent a second-order tensor using dyads:

$$\boldsymbol{\sigma} = \sigma_{ij} \left(\mathbf{e}_i \otimes \mathbf{e}_j \right)$$

- σ_{ij} are the tensor components
- $(\mathbf{e}_i \otimes \mathbf{e}_j)\mathbf{v} = \mathbf{e}_i(\mathbf{e}_j \cdot \mathbf{v})$

Traction: $\boldsymbol{\sigma} \cdot \mathbf{n}$ in Components

Let

$$\mathbf{n} = n_k \, \mathbf{e}_k, \qquad \boldsymbol{\sigma} = \sigma_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j).$$

Then the traction vector is

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \sigma_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) \cdot (n_k \mathbf{e}_k) = \sigma_{ij} n_k \, \mathbf{e}_i (\mathbf{e}_j \cdot \mathbf{e}_k).$$

Using $\mathbf{e}_j \cdot \mathbf{e}_k = \delta_{jk}$,

$$\boldsymbol{\sigma} \cdot \mathbf{n} = \sigma_{ij} n_k \, \delta_{jk} \, \mathbf{e}_i = (\sigma_{ij} n_j) \, \mathbf{e}_i.$$

So, in components:

$$(\boldsymbol{\sigma} \cdot \mathbf{n})_i = \sigma_{ij} n_j.$$

$\nabla \cdot \boldsymbol{\sigma}$ in a Cartesian Basis

Write the gradient operator and stress tensor in a Cartesian basis:

$$\nabla = \mathbf{e}_j \frac{\partial}{\partial x_j}, \qquad \boldsymbol{\sigma} = \sigma_{ij} \left(\mathbf{e}_i \otimes \mathbf{e}_j \right).$$

Then

$$\nabla \cdot \boldsymbol{\sigma} = \left(\mathbf{e}_j \frac{\partial}{\partial x_j} \right) \cdot \left(\sigma_{ik} \, \mathbf{e}_i \otimes \mathbf{e}_k \right).$$

Use the identity $\mathbf{e}_j \cdot \left(\mathbf{e}_i \otimes \mathbf{e}_k \right) = \delta_{ji} \, \mathbf{e}_k$ to obtain

$$\nabla \cdot \boldsymbol{\sigma} = \frac{\partial \sigma_{ik}}{\partial x_j} \delta_{ji} \, \mathbf{e}_k = \left(\frac{\partial \sigma_{ij}}{\partial x_j} \right) \mathbf{e}_i.$$

So, in components:

$$(\nabla \cdot \boldsymbol{\sigma})_i = \frac{\partial \sigma_{ij}}{\partial x_j}.$$

$\boldsymbol{\sigma} \cdot \mathbf{n}$

$$(\boldsymbol{\sigma} \cdot \mathbf{n})_i = \sigma_{ij} n_j$$

- Traction vector on a surface
- Boundary term in momentum and energy balances

$\nabla \mathbf{v}$

$$(\nabla \mathbf{v})_{ij} = \frac{\partial v_i}{\partial x_j}$$

- Velocity gradient tensor
- Encodes strain rates and rotation

$\nabla^2 T$

$$\nabla^2 T = \frac{\partial^2 T}{\partial x_j \partial x_j}$$

- Scalar Laplacian
- Appears in heat conduction and diffusion

$\nabla^2 \mathbf{v}$

$$(\nabla^2 \mathbf{v})_i = \frac{\partial^2 v_i}{\partial x_j \partial x_j}$$

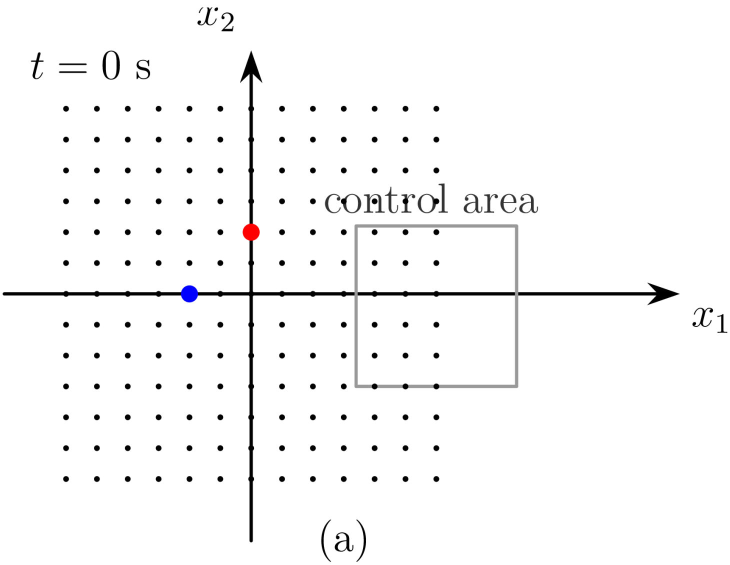
- Vector Laplacian
- Appears in viscous momentum transport

$\mathbf{v} \cdot \mathbf{b}$

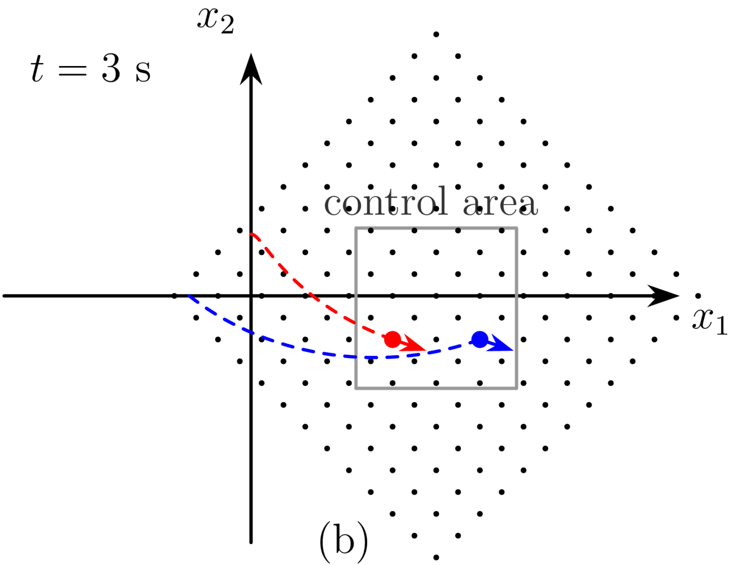
$$\mathbf{v} \cdot \mathbf{b} = v_i b_i$$

- Power density of body forces
- Appears in the energy balance

Eulerian and Lagrangian Descriptions



Eulerian and Lagrangian Descriptions



Material Time Derivative

For a scalar field $f(\mathbf{x}, t)$:

$$\frac{Df}{Dt} = \frac{\partial f}{\partial t} + v_j \frac{\partial f}{\partial x_j}$$

- Time rate of change following a particle
- Combines local and convective effects

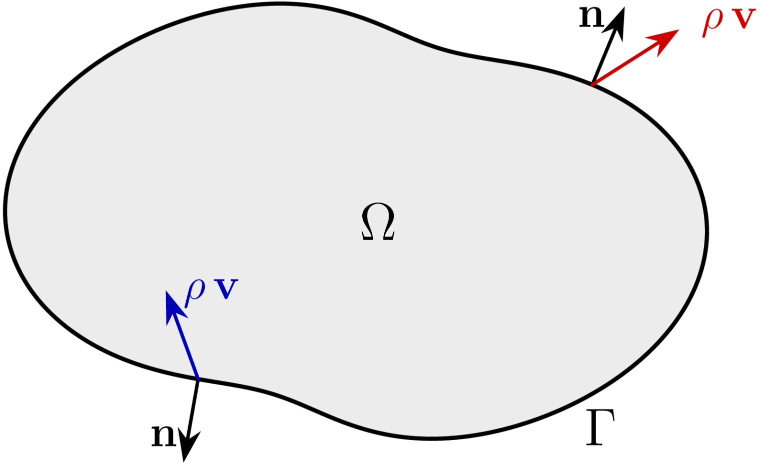
Material Derivative of Velocity

$$\frac{Dv_i}{Dt} = \frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}$$

Vector form

$$\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$

Control Volume for Mass Balance

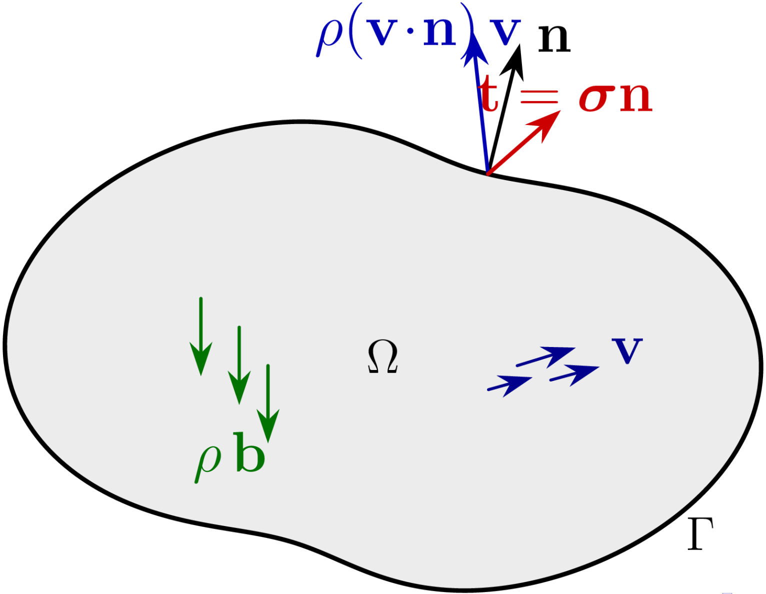


Mass Balance

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho v_j)}{\partial x_j} = 0$$

- Continuity equation
- Expresses conservation of mass

Control Volume for Momentum Balance

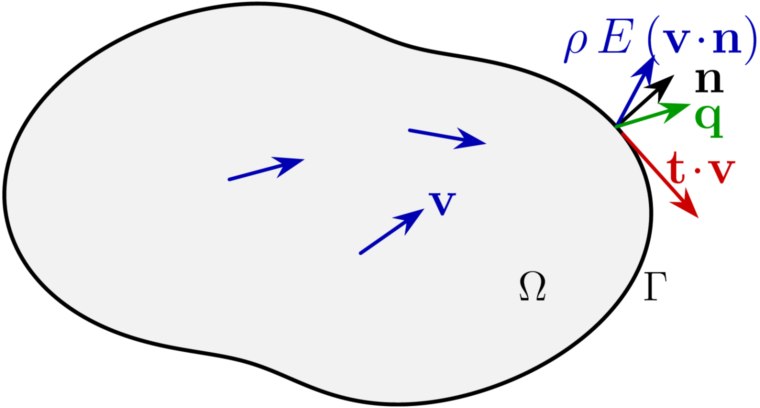


Momentum Balance

$$\rho \frac{Dv_i}{Dt} = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i$$

- Newton's second law per unit volume
- Stress divergence + body forces

Control Volume for Energy Balance



Energy Balance

$$\rho \frac{De}{Dt} = \sigma_{ij} \frac{\partial v_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} + \rho S$$

- Internal energy form
- Mechanical power, heat flux, and sources

Thermal Energy Balance

Assuming:

- Constant density and heat capacity
- Fourier's law: $q_i = -k\,\partial T/\partial x_i$

$$\rho c \frac{DT}{Dt} = k\nabla^2 T + \rho S$$

Expanded Form

$$\rho c \left(\frac{\partial T}{\partial t} + v_j \frac{\partial T}{\partial x_j} \right) = k \frac{\partial^2 T}{\partial x_j \partial x_j} + \rho S$$

- Time-dependent advection–diffusion equation

Reduction to 1D

Assume:

- $v = v_x(x)$
- $T = T(x, t)$

$$\rho c \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} \right) = k \frac{\partial^2 T}{\partial x^2} + \rho S$$

Why This Matters

- Clean bridge from continuum mechanics to numerics
- Every term maps directly to a discrete operator
- Sets the stage for FV, FD, FEM, and ML surrogates

Next Time

- Finite volume discretization of advection–diffusion
- Upwind vs. central differencing
- Stability and physical interpretation