

Von Neumann Stability for FTCS (1D Heat Equation)

Why Fourier modes, amplification factors, and the $\lambda \leq \frac{1}{2}$ restriction

What “stability” means for time stepping

A time-stepping method is *stable* if small perturbations (round-off, truncation, imperfect data) remain bounded as the computation advances.

For diffusion problems, stable schemes typically *damp* perturbations; unstable schemes may amplify them until they dominate the solution (nonphysical oscillations / blow-up).

Model problem and FTCS update

1D heat equation (no source):

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}. \quad (1)$$

Uniform grid with spacing Δx and time step Δt . Define

$$\lambda = \frac{\alpha \Delta t}{(\Delta x)^2}.$$

FTCS update (interior points):

$$T_j^{n+1} = (1 - 2\lambda) T_j^n + \lambda (T_{j+1}^n + T_{j-1}^n). \quad (2)$$

Big picture: why Fourier modes

Key idea (from the notebook):

- ▶ Any (reasonable) grid temperature profile can be written as a **sum of Fourier modes**.
- ▶ FTCS is **linear** with **constant coefficients** \Rightarrow it advances **each mode independently**.
- ▶ For a single mode, one step multiplies its amplitude by an **amplification factor**

$$G(\theta) = \frac{A^{n+1}}{A^n}. \quad (3)$$

- ▶ Stability requires **no mode grows**: $|G(\theta)| \leq 1$ for all $\theta \in [0, \pi]$.

A single Fourier mode on a grid

Assume a Fourier mode on the grid:

$$T_j^n = A^n e^{i\theta j}, \quad (4)$$

where θ is the nondimensional wavenumber (phase advance per grid point).

Interpretation:

- ▶ j labels grid points.
- ▶ The mode oscillates in space; A^n carries the time dependence.

Real temperature fields and conjugate pairs

A real-valued temperature field is built from conjugate pairs:

$$A_m e^{ik_m x} + A_m^* e^{-ik_m x} \in \mathbb{R}.$$

Equivalently on a grid:

$$A e^{i\theta j} + A^* e^{-i\theta j} = 2 \operatorname{Re}(A) \cos(\theta j) - 2 \operatorname{Im}(A) \sin(\theta j).$$

Takeaway: complex exponentials are a convenient algebra tool; real solutions come from taking the real part (or using cosine/sine series).

Superposition: building an arbitrary profile

A general profile on N grid points can be written as a sum of modes:

$$T_j = \sum_{m=0}^{N-1} A_m e^{i\theta_m j}, \quad \theta_m = \frac{2\pi m}{N}.$$

In practice, a truncated sum often captures the main shape:

$$T_j \approx A_0 + \sum_{m=1}^M \left(\tilde{a}_m \cos(\theta_m j) + \tilde{b}_m \sin(\theta_m j) \right).$$

Notebook demo: a smooth “bump” can be reconstructed by adding a handful of low-frequency modes.

Plug the mode into FTCS

Insert $T_j^n = A^n e^{i\theta j}$ into

$$T_j^{n+1} = (1 - 2\lambda) T_j^n + \lambda(T_{j+1}^n + T_{j-1}^n).$$

You get:

$$A^{n+1} e^{i\theta j} = (1 - 2\lambda) A^n e^{i\theta j} + \lambda A^n \left(e^{i\theta(j+1)} + e^{i\theta(j-1)} \right).$$

Divide by $A^n e^{i\theta j}$:

$$\frac{A^{n+1}}{A^n} = 1 + \lambda \left(e^{i\theta} - 2 + e^{-i\theta} \right). \quad (5)$$

Simplify the exponentials

Use Euler's identity:

$$e^{i\theta} + e^{-i\theta} = 2 \cos \theta.$$

So

$$e^{i\theta} - 2 + e^{-i\theta} = 2 \cos \theta - 2 = 2(\cos \theta - 1).$$

Therefore,

$$G(\theta) = \frac{A^{n+1}}{A^n} = 1 + 2\lambda(\cos \theta - 1) = 1 - 4\lambda \sin^2\left(\frac{\theta}{2}\right). \quad (6)$$

Stability condition and worst-case mode

Stability requires:

$$|G(\theta)| \leq 1 \quad \text{for all } \theta \in [0, \pi].$$

Since $\sin^2(\theta/2) \in [0, 1]$, the most restrictive case is the highest-frequency grid mode:

$$\theta = \pi \quad \Rightarrow \quad \sin^2\left(\frac{\theta}{2}\right) = 1.$$

Then

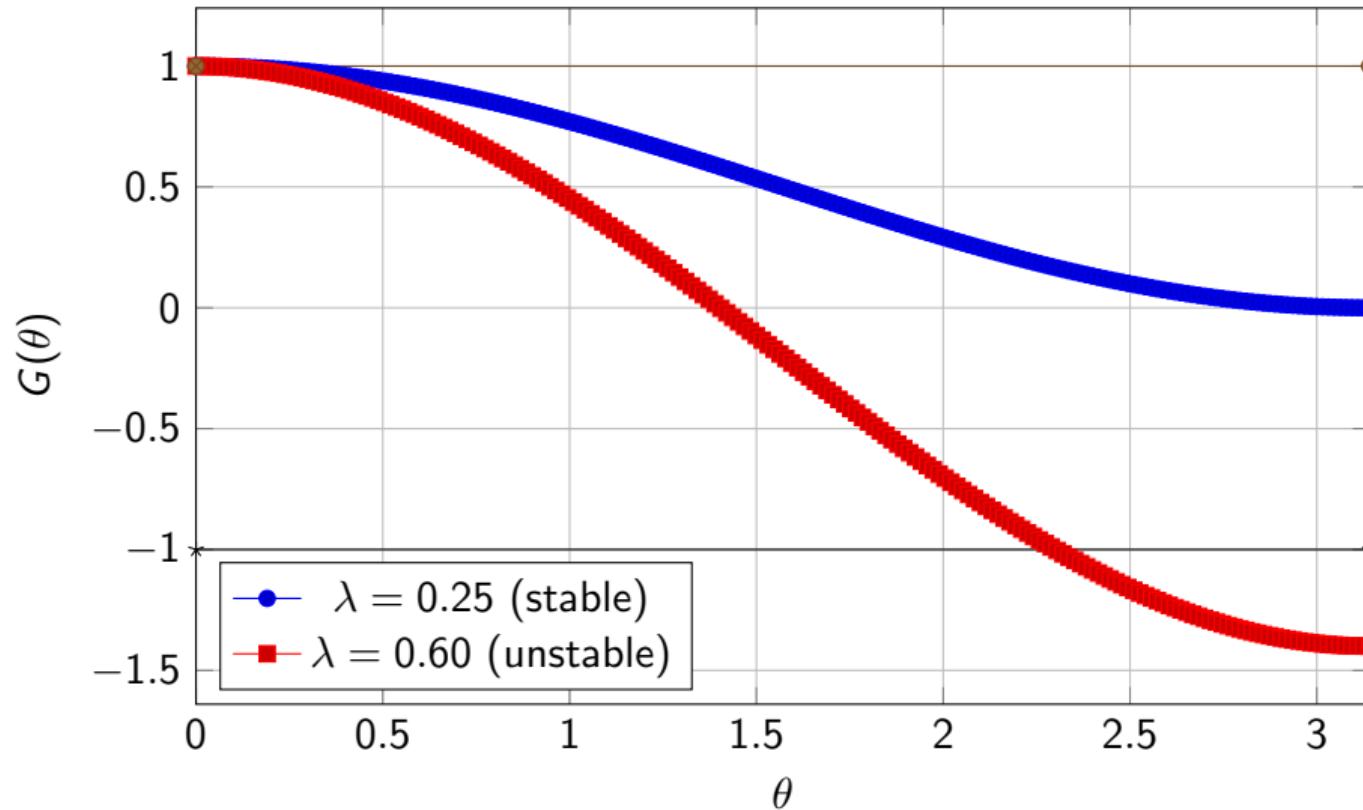
$$G(\pi) = 1 - 4\lambda.$$

Require $-1 \leq G(\pi) \leq 1$:

$$-1 \leq 1 - 4\lambda \Rightarrow \lambda \leq \frac{1}{2}.$$

$$\boxed{\Delta t \leq \frac{(\Delta x)^2}{2\alpha}}$$

Visual: $G(\theta)$ vs. θ



Stable requires the entire curve to stay between -1 and $+1$.

Notebook figure: single mode stays a single mode

What the notebook shows:

- ▶ Start with a single Fourier mode T_j^0 (blue).
- ▶ After n steps, FTCS gives

$$T_j^n = G(\theta)^n T_j^0,$$

so the **shape is unchanged** and the **amplitude scales** by $G(\theta)^n$.

- ▶ If $|G(\theta)| < 1$ the mode decays; if $|G(\theta)| > 1$ it grows.

Takeaway: von Neumann stability is about ensuring every Fourier mode decays (or at least does not grow).

Connecting back to engineering intuition

- ▶ Diffusion should smooth the solution \Rightarrow high-frequency content should decay fastest.
- ▶ FTCS is explicit: too large Δt causes the discrete update to overreact.
- ▶ The most dangerous mode is the grid-scale oscillation ($\theta = \pi$): alternating $+/-$ from node to node.

Design rule:

$$\Delta t \leq \frac{(\Delta x)^2}{2\alpha} \quad \Rightarrow \quad \text{refine grid } (\Delta x \downarrow) \Rightarrow \Delta t \downarrow \text{ quadratically.}$$

Summary

- ▶ Assume a Fourier mode $T_j^n = A^n e^{i\theta j}$.
- ▶ FTCS advances each mode independently:

$$\frac{A^{n+1}}{A^n} = G(\theta) = 1 + 2\lambda(\cos \theta - 1).$$

- ▶ Stability requires $|G(\theta)| \leq 1$ for all θ .
- ▶ Worst case $\theta = \pi$ gives:

$$0 \leq \lambda \leq \frac{1}{2} \iff \Delta t \leq \frac{(\Delta x)^2}{2\alpha}.$$