Two-Dimensional Heat Conduction

Finite Differences and the Five-Point Stencil

MMAE 450: Computational Mechanics II

Motivation for 2D Heat Conduction

- Many engineering problems involve temperature variation in more than one spatial direction.
- Examples:
  - Thin plates and heat sinks
  - Cooling of electronic components
  - Layered or composite materials
- Extending 1D ideas to 2D is conceptually straightforward.
- The resulting algebraic systems become *large* and *sparse*.

Governing Equation in Two Dimensions

For a homogeneous, isotropic material with constant properties:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

Under steady-state conditions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- This is the 2D *Laplace equation*.
- Boundary conditions (Dirichlet/Neumann/Robin) are prescribed on $\partial\Omega$.

Rectangular Domain and Grid

Consider a rectangular plate:

- $0 \le x \le L_x$
- $0 \le y \le L_y$

Uniform grid:

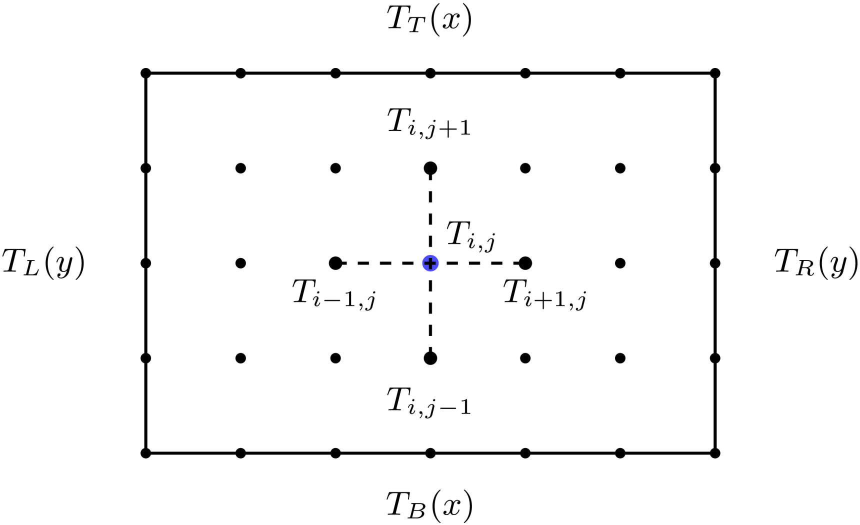
$$\Delta x = \frac{L_x}{N_x}, \qquad \Delta y = \frac{L_y}{N_y}$$

Grid points labeled by (i, j):

$$\begin{aligned} x_i &= i\,\Delta x, \quad i = 0, 1, \ldots, N_x, \\ y_j &= j\,\Delta y, \quad j = 0, 1, \ldots, N_y. \end{aligned}$$

We denote the nodal approximation by $T_{i,j}^n \approx T(x_i, y_j, t^n)$.

Five-Point Stencil



- Interior node (i, j) depends on four nearest neighbors.
- This produces the classic *five-point stencil*.

Discrete Laplacian

Second derivatives using centered finite differences:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{i,j} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2},$$
$$\left.\frac{\partial^2 T}{\partial y^2}\right|_{i,j} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}.$$

If $\Delta x = \Delta y = h$, then

$$\nabla^2 T\big|_{i,j} \approx \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j}}{h^2}.$$

Steady-State as a Linear System

At each interior node, the discrete Laplace equation gives

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0.$$

Collecting all interior equations yields

$$\mathbf{A}\,\mathbf{T} = \mathbf{d},$$

where

- $\mathbf{T}$ contains the unknown interior temperatures,
- $\mathbf{A}$ is sparse (*at most five nonzeros per row*),
- $\mathbf{d}$ accounts for prescribed boundary values.

Structure of the Discrete System Matrix $\mathbf{A}$

For the five-point stencil, the steady problem

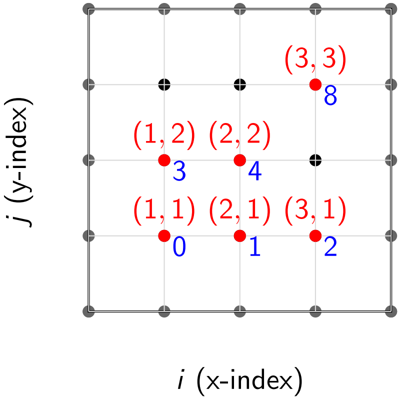
$$\mathbf{A}\,\mathbf{T} = \mathbf{d}$$

leads to a sparse matrix with a highly regular structure.

$$\mathbf{A} = \begin{bmatrix} \ddots & \ddots & & & \\ \ddots & -4 & 1 & & \\ & 1 & -4 & 1 & \\ & & 1 & -4 & \ddots \\ & & & \ddots & \ddots \end{bmatrix}$$

- Each row corresponds to one interior grid node
- Diagonal entry (-4): contribution from $T_{i,j}$
- Off-diagonals $(+1)$: coupling to nearest neighbors
- Overall structure is sparse and banded

Interior unknowns: $i, j = 1, 2, 3 \quad (3 \times 3 = 9 \text{ unknowns})$



$$idx(i, j) = 3(j - 1) + (i - 1)$$

Gauss–Seidel Iteration in 2D

Update each interior node in-place using the stencil average:

$$T_{i,j}^{(k+1)} = \frac{1}{4}\Big(T_{i+1,j}^{(k)} + T_{i-1,j}^{(k+1)} + T_{i,j+1}^{(k)} + T_{i,j-1}^{(k+1)} \Big).$$

- Sweep through the grid node-by-node.
- Dirichlet boundary values are imposed directly.
- Stop when the max change between sweeps is below a tolerance.

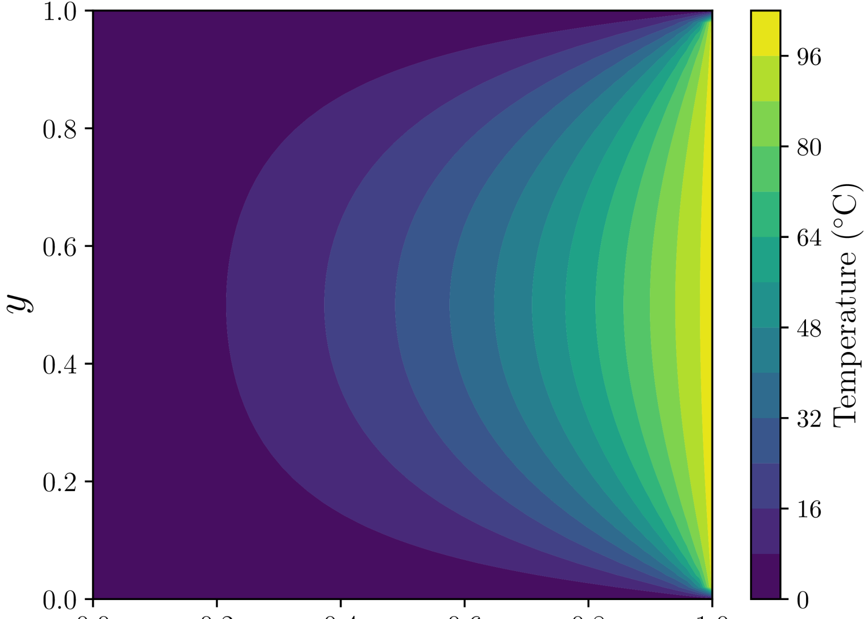
Worked Example: Heated Plate

Square plate with $L_x = L_y = 1$ and Dirichlet conditions on all edges:

$$T(0,y) = 0^\circ\mathrm{C}, \qquad T(1,y) = 100^\circ\mathrm{C},$$
$$T(x,0) = 0^\circ\mathrm{C}, \qquad T(x,1) = 0^\circ\mathrm{C}.$$

- Three cold boundaries, one hot boundary.
- Solve Laplace equation on the grid (e.g., Gauss–Seidel).
- The interior field is influenced by all boundaries.

Temperature Field Solution



Extension to Transient Problems

A 2D FTCS update (for $\Delta x = \Delta y = h$):

$$T_{i,j}^{n+1} = T_{i,j}^{n} + \lambda \Big[T_{i+1,j}^{n} + T_{i-1,j}^{n} + T_{i,j+1}^{n} + T_{i,j-1}^{n} - 4 T_{i,j}^{n} \Big],$$

where $\lambda = \alpha \, \Delta t / h^2$.

- Direct analogue of 1D FTCS.
- Stability is more restrictive in 2D (smaller Δt).
- Implicit / Crank–Nicolson in 2D leads to sparse linear systems.

Key Takeaways

- 2D conduction extends 1D ideas with a local stencil in both directions.
- The five-point stencil leads to large, sparse linear systems.
- Iterative methods (Gauss–Seidel, etc.) map naturally onto grid sweeps.
- Transient schemes follow the same stencil structure (plus time stepping).

Next: Wave propagation and fundamentally different physics