MMAE 450 – Computational Mechanics II

Homework 5: Finite Volume Method for 1D Advection–Diffusion

Total: 20 points
Due: March 13, 11:59 pm

Objective. In this assignment you will implement the finite volume method (FVM) for the one-dimensional advection–diffusion equation and investigate the influence of the cell Peclet number on numerical behavior.

Problem Description

A contaminant is introduced into a straight river segment of length $L = 1000$ m. The river has a uniform downstream velocity v and a constant dispersion (diffusion) coefficient α.

The pollutant concentration $T(x,t)$ satisfies the one-dimensional advection–diffusion equation

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \alpha\frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < L.$$

Use

$$\alpha = 0.5 \text{ m}^2/\text{s}.$$

Initial condition (localized spill):

$$T(x,0) = \begin{cases} 1, & 0.4L \le x \le 0.6L, \\ 0, & \text{otherwise.} \end{cases}$$

Boundary conditions:

- Upstream boundary (inlet): $T(0,t) = 0$
- Downstream boundary (open outlet):

$$\frac{\partial T}{\partial x}(L,t) = 0$$

Assume the flow is steady and the pollutant does not affect the velocity field.

(a) Derivation (5 pts)

Starting from the conservative form

$$\frac{\partial T}{\partial t} + \frac{\partial}{\partial x}\left(vT - \alpha\frac{\partial T}{\partial x}\right) = 0,$$

derive the fully discrete finite volume update using:

- Central interpolation for the convective flux
- Central difference for the diffusive flux
- Forward Euler in time

Present the final update equation clearly.

(b) Implementation (10 pts)

Write a Python program that:

1. Discretizes the domain into $N = 100$ uniform cells.
2. Implements the fully discrete update derived in part (a).
3. Applies the boundary conditions correctly at each time step.
4. Advances the solution to $t = 200$ s.

Plot the solution at

$$t = 0, \quad 50, \quad 100, \quad 200 \text{ s}.$$

Plots must include labeled axes and a legend.

(c) Peclet Number Study (5 pts)

Repeat the simulation for

$$v = 1, \quad 5, \quad 10 \text{ m/s}.$$

For each case:

1. Compute the cell Peclet number
$$\mathrm{Pe}_{\Delta x} = \frac{v\Delta x}{\alpha}.$$
2. Plot the solution at $t = 200$ s.
3. Comment briefly on oscillations or instability.

Relate your observations to the magnitude of the Peclet number.

Submission Requirements

Submit:

- A single PDF containing derivations, plots, and discussion.
- Your Python code.

Code must be readable and commented.

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