MMAE 450 — Homework #3

FTCS for Transient Heat Conduction: Verification and Sphere Cooling

Due: Tuesday, February 3, 2026 (11:59 pm)

Learning goals. By completing this assignment, you will:

- implement the **Forward Time, Centered Space (FTCS)** method for diffusion,
- investigate the distinction between *stability* and *accuracy* for explicit time integration,
- enforce **Dirichlet**, **Neumann (zero flux)**, and **Robin (convection)** boundary conditions,
- apply FTCS to a physically meaningful cooling problem.

Problem 1: Code Verification on a 1D Rod (Dirichlet–Dirichlet)

Before tackling the spherical cooling problem, you will verify your FTCS implementation on the *1D Cartesian heat equation*:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \alpha \frac{\partial^2 T}{\partial x^2}, \qquad 0 < x < L,\ t > 0. \tag{1}$$

Use the following setup:

$$L = 0.10\,\mathrm{m}, \qquad T_{\mathrm{L}} = 100\,^{\circ}\mathrm{C}, \qquad T_{\mathrm{R}} = 20\,^{\circ}\mathrm{C}.$$

Initial condition:

$$T(x, 0) = 100\,^{\circ}\mathrm{C}.$$

Boundary conditions (Dirichlet at both ends):

$$T(0, t) = T_{\mathrm{L}}, \qquad T(L, t) = T_{\mathrm{R}}.$$

Material data

Use copper properties throughout this assignment:

$$k = 401\,\mathrm{W/(m \cdot K)}, \qquad \rho = 8960\,\mathrm{kg/m^3}, \qquad c_p = 385\,\mathrm{J/(kg \cdot K)}, \qquad \alpha = \frac{k}{\rho c_p}.$$

Tasks

1. Discretize the rod with $N + 1$ nodes and $\Delta x = L/N$. Implement FTCS for the interior nodes:

   $$T_i^{n+1} = T_i^n + \lambda\left(T_{i+1}^n - 2T_i^n + T_{i-1}^n\right), \qquad \lambda = \frac{\alpha\,\Delta t}{\Delta x^2}, \qquad i = 1, \dots, N-1.$$

   Enforce the Dirichlet boundary values at *every* time step.

2. Choose at least three time steps Δt (equivalently, three values of λ) including:

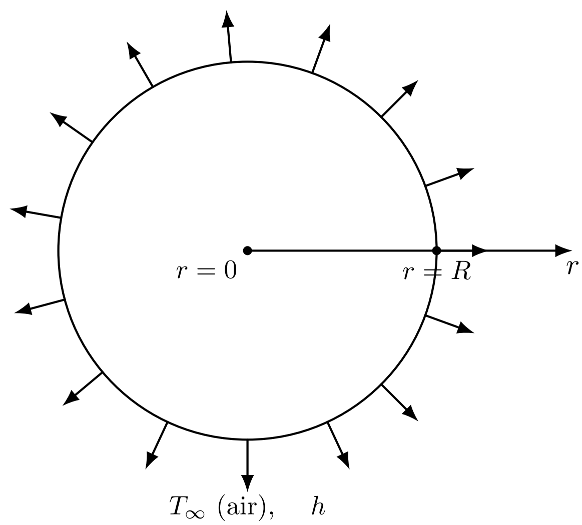
   - one that is clearly stable,

- one that is close to the stability limit,
- one that is unstable.

3. For each run, plot:
   - $T(x, t)$ profiles at several times,
   - T at the midpoint vs. time.
4. In 2–3 sentences, describe what you observe as Δt increases (stability vs. accuracy).

Problem 2: Cooling of a Copper Sphere (Neumann at Center, Convection at Surface)

A solid copper sphere of radius $R = 0.10\,\text{m}$ is initially at a uniform temperature $T_0 = 100\,°\text{C}$. At time $t = 0$ the sphere is removed from an oven and exposed to still air at $T_\infty = 20\,°\text{C}$ with a constant convective heat transfer coefficient h.



Assume:

- radial symmetry (temperature depends only on radius r and time t),
- constant material properties,
- no internal heat generation.

Material properties and convection data

The thermal diffusivity is

$$\alpha = \frac{k}{\rho c_p}.$$

Property	Symbol	Value
Thermal conductivity	k	$401\,\mathrm{W/(m\cdot K)}$
Density	ρ	$8960\,\mathrm{kg/m^3}$
Specific heat capacity	c_p	$385\,\mathrm{J/(kg\cdot K)}$
Convective heat transfer coefficient (air)	h	$10\,\mathrm{W/(m^2{\cdot}K)}$

Governing equation and conditions (given)

With radial symmetry, the transient heat equation in spherical coordinates is

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \right], \qquad 0 < r < R,\ t > 0. \tag{2}$$

Boundary conditions:

$$\left.\frac{\partial T}{\partial r}\right|_{r=0} = 0 \qquad \text{(symmetry / zero flux at the center)}, \tag{3}$$

$$-k \left.\frac{\partial T}{\partial r}\right|_{r=R} = h\,(T(R,t) - T_\infty) \qquad \text{(convection at the surface)}. \tag{4}$$

Initial condition:

$$T(r,0) = T_0. \tag{5}$$

Tasks

1. Discretize the radial domain using $N+1$ nodes:

$$r_i = i\Delta r, \qquad i = 0, 1, \ldots, N, \qquad \Delta r = \frac{R}{N}.$$

2. Implement FTCS time stepping for the sphere. Your implementation must:
   - enforce the **Neumann (zero flux)** condition at $r = 0$ (use the *center control-volume balance law* from lecture),
   - enforce the **Robin (convection)** condition at $r = R$ (e.g. via a ghost-node relation or an equivalent one-sided discretization),
   - update only the interior nodes with the spherical FTCS stencil.
3. Choose at least three time steps (or λ values) and demonstrate:
   - a stable run,
   - a run near the stability limit,
   - an unstable run (nonphysical oscillations or blow-up).
4. For your *best stable* run, produce:
   - $T(r,t)$ profiles at several times,
   - $T(0,t)$ (center temperature) vs. time,
   - $T(R,t)$ (surface temperature) vs. time.

5. Define "near steady state" as the time $t^\star$ when the center temperature satisfies

$$T(0,t^\star) - T_\infty \le 0.01\,(T_0 - T_\infty).$$

Estimate $t^\star$ and report it with appropriate units.

Submission Requirements

Submit **one PDF report** and **one code archive** (or a single Jupyter notebook) containing:

- brief descriptions of your discretization and boundary-condition enforcement,
- plots requested in Problems 1 and 2,
- a short discussion of stability vs. accuracy for FTCS,

Grading (10 points total)

- (3) Problem 1: FTCS implementation with Dirichlet–Dirichlet and stability exploration
- (5) Problem 2: FTCS for the sphere (Neumann at center, convection at surface) with correct BC handling and plots
- (2) Discussion + reported Bi and $t^\star$ (clear, physically consistent)

Notes.

- Enforce boundary conditions at *every* time step.
- When you label a run "unstable," describe the evidence (oscillations, blow-up, temperatures below T_∞, etc.).