

Heat Conduction: From Balance Law to FTCS

(building directly on the divergence theorem)

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MMAE 450

Where we are headed (10 minutes)

- ▶ Start with a **global energy balance** on a body Ω with boundary Γ .
- ▶ Use the **divergence theorem** to convert boundary fluxes to a volume statement.
- ▶ Close the model with **Fourier's law** to obtain the heat equation.
- ▶ Specialize to 1D and build the **FTCS** explicit update.

Global energy balance (integral form)

Let $T(\mathbf{x}, t)$ be temperature, ρ density, c specific heat, \mathbf{q} conductive heat flux, and S volumetric heat generation (power per unit volume).

Total thermal energy:

$$E(t) = \int_{\Omega} \rho c T(\mathbf{x}, t) dV.$$

Balance law (rate of change = net inflow through boundary + internal generation):

$$\frac{d}{dt} \int_{\Omega} \rho c T dV = - \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} dS + \int_{\Omega} S dV.$$

Divergence theorem \Rightarrow local balance

Assuming sufficient smoothness, move the time derivative inside:

$$\int_{\Omega} \rho c \frac{\partial T}{\partial t} dV = - \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} dS + \int_{\Omega} S dV.$$

Use the divergence theorem on the boundary flux term:

$$\int_{\Gamma} \mathbf{q} \cdot \mathbf{n} dS = \int_{\Omega} \nabla \cdot \mathbf{q} dV.$$

Therefore,

$$\int_{\Omega} \left(\rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q} - S \right) dV = 0 \quad \Rightarrow \quad \rho c \frac{\partial T}{\partial t} + \nabla \cdot \mathbf{q} - S = 0 \quad \text{in } \Omega.$$

Constitutive law: Fourier's law

For isotropic conduction,

$$\mathbf{q} = -k \nabla T,$$

where k is thermal conductivity.

Substitute into the local balance:

$$\rho c \frac{\partial T}{\partial t} = \nabla \cdot (k \nabla T) + S.$$

If k is constant:

$$\rho c \frac{\partial T}{\partial t} = k \nabla^2 T + S, \quad \frac{\partial T}{\partial t} = \alpha \nabla^2 T + \frac{S}{\rho c}, \quad \alpha = \frac{k}{\rho c}.$$

1D specialization and boundary conditions

On $0 \leq x \leq L$:

$$\frac{\partial T}{\partial t}(x, t) = \alpha \frac{\partial^2 T}{\partial x^2}(x, t) + \frac{S(x, t)}{\rho c}.$$

Typical boundary conditions:

- ▶ Dirichlet: $T(0, t) = T_L(t)$, $T(L, t) = T_R(t)$.
- ▶ Neumann (prescribed flux): $-k \frac{\partial T}{\partial x}(0, t) = q_n^*(t)$.
- ▶ Insulated end is the special case $q_n^*(t) = 0$.

Key idea: the PDE says “rate of temperature change” = “net diffusion in/out” + “sources.”

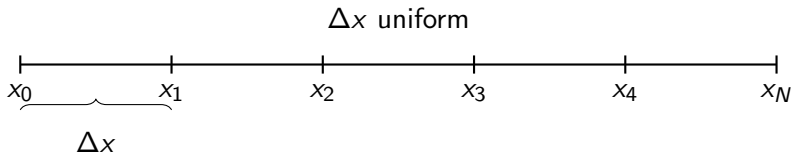
Grid and notation (space–time lattice)

Uniform grid:

$$x_i = i \Delta x, \quad i = 0, 1, \dots, N, \quad t^n = n \Delta t, \quad n = 0, 1, 2, \dots$$

Store nodal values:

$$T_i^n \approx T(x_i, t^n).$$



FTCS: Forward Time, Centered Space

Approximate derivatives at (x_i, t^n) :

$$\left. \frac{\partial T}{\partial t} \right|_{i,n} \approx \frac{T_i^{n+1} - T_i^n}{\Delta t}, \quad \left. \frac{\partial^2 T}{\partial x^2} \right|_{i,n} \approx \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2}.$$

Insert into the 1D heat equation:

$$\frac{T_i^{n+1} - T_i^n}{\Delta t} = \alpha \frac{T_{i+1}^n - 2T_i^n + T_{i-1}^n}{(\Delta x)^2} + \frac{S_i^n}{\rho c}.$$

Define the diffusion number

$$\lambda = \frac{\alpha \Delta t}{(\Delta x)^2}.$$

Explicit update (interior nodes):

$$T_i^{n+1} = T_i^n + \lambda (T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t \frac{S_i^n}{\rho c}.$$

Interpretation: stencil = local energy balance

Rewrite the update to highlight “in minus out”:

$$T_i^{n+1} = (1 - 2\lambda)T_i^n + \lambda T_{i-1}^n + \lambda T_{i+1}^n + \Delta t \frac{S_i^n}{\rho c}.$$

- ▶ $\lambda T_{i\pm 1}^n$ are contributions from neighbors (*heat diffuses in*).
- ▶ $-2\lambda T_i^n$ is the net *outflow* contribution.
- ▶ $\Delta t S/(\rho c)$ raises temperature when $S > 0$.

If $S = 0$, the update is a weighted average (plus the old value) *only if* λ is not too large.

Stability: why FTCS needs a time-step restriction

For the 1D diffusion equation with $S = 0$, von Neumann analysis gives

$$0 \leq \lambda \leq \frac{1}{2} \quad \Longleftrightarrow \quad \Delta t \leq \frac{(\Delta x)^2}{2\alpha}.$$

- ▶ Physically: you cannot let diffusion move information “too far” in one explicit step.
- ▶ Numerically: if λ is too large, high-frequency modes grow \Rightarrow oscillation/blow-up.

Rule of thumb for class demos: pick $\lambda \approx 0.25$.

Algorithm slide (what students will code)

Given α , Δx , Δt , N and initial data T_i^0 :

1. Compute $\lambda = \alpha \Delta t / (\Delta x)^2$ and ensure $\lambda \leq 1/2$.
2. For $n = 0, 1, 2, \dots$:
 - 2.1 Enforce boundary conditions to obtain T_0^n and T_N^n .
 - 2.2 For $i = 1, \dots, N - 1$ update

$$T_i^{n+1} = T_i^n + \lambda (T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t \frac{S_i^n}{\rho c}.$$

Next lecture: implicit methods \Rightarrow linear systems (tridiagonal) and the Thomas algorithm.

One-slide recap

Balance law: $\frac{d}{dt} \int_{\Omega} \rho c T \, dV = - \int_{\Gamma} \mathbf{q} \cdot \mathbf{n} \, dS + \int_{\Omega} S \, dV$

Divergence theorem: $\int_{\Gamma} \mathbf{q} \cdot \mathbf{n} \, dS = \int_{\Omega} \nabla \cdot \mathbf{q} \, dV \Rightarrow \rho c T_{,t} + \nabla \cdot \mathbf{q} - S = 0$

Fourier law: $\mathbf{q} = -k \nabla T \Rightarrow T_{,t} = \alpha \nabla^2 T + \frac{S}{\rho c}$

FTCS (1D): $T_i^{n+1} = T_i^n + \lambda (T_{i+1}^n - 2T_i^n + T_{i-1}^n) + \Delta t \frac{S_i^n}{\rho c}, \quad \lambda \leq \frac{1}{2}$