

Heat Conduction: From Balance Law to FTCS

(Boundary Conditions)

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1D Heat Equation (Cartesian) and a Prescribed Flux at $x = 0$

We consider the 1D heat equation on $0 < x < L$:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \quad \alpha = \frac{k}{\rho c_p}.$$

A prescribed (time-dependent) heat flux at the left boundary:

$$q(0, t) \equiv -k \left. \frac{\partial T}{\partial x} \right|_{x=0} = g(t).$$

- ▶ $g(t) > 0$ means *heat enters* the domain at $x = 0$ (inward flux).
- ▶ $g(t) = 0$ corresponds to an insulated boundary.

Finite Difference Grid and Notation

Uniform grid:

$$x_j = j \Delta x, \quad j = 0, 1, \dots, N, \quad \Delta x = \frac{L}{N}.$$

Discrete temperatures:

$$T_j^n \approx T(x_j, t^n), \quad t^n = n \Delta t.$$

Define the diffusion number:

$$\lambda = \frac{\alpha \Delta t}{\Delta x^2}.$$

Interior FTCS update ($j = 1, \dots, N - 1$):

$$T_j^{n+1} = T_j^n + \lambda (T_{j+1}^n - 2T_j^n + T_{j-1}^n).$$

Imposing a Prescribed Flux at $x = 0$: From Flux to Gradient

The boundary condition is specified as a *flux*:

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = g(t).$$

Convert to a prescribed gradient:

$$\left. \frac{\partial T}{\partial x} \right|_{x=0} = -\frac{g(t)}{k}.$$

At time level n , define

$$\gamma^n \equiv \left. \frac{\partial T}{\partial x} \right|_{x=0, t=t^n} = -\frac{g(t^n)}{k}.$$

Taylor-Series Boundary Update at $j = 0$ (Second-Order Accurate)

Taylor expansion from node 0 to node 1:

$$T_1^n = T_0^n + \Delta x \gamma^n + \frac{\Delta x^2}{2} T_{xx}|_0^n + \mathcal{O}(\Delta x^3).$$

Solve for the boundary curvature:

$$T_{xx}|_0^n \approx \frac{2}{\Delta x^2} (T_1^n - T_0^n - \Delta x \gamma^n).$$

Use the heat equation at $x = 0$:

$$\frac{dT_0}{dt} = \alpha T_{xx}|_0.$$

Forward Euler in time gives the boundary node update:

$$\boxed{T_0^{n+1} = T_0^n + 2\lambda (T_1^n - T_0^n - \Delta x \gamma^n)} \quad \left(\gamma^n = -\frac{g(t^n)}{k} \right).$$

Algorithm Summary (FTCS with Prescribed Flux at $x = 0$)

At each time step $n \rightarrow n + 1$:

1. Evaluate the prescribed flux $g(t^n)$ and compute

$$\gamma^n = -\frac{g(t^n)}{k}.$$

2. Update the left boundary node ($j = 0$):

$$T_0^{n+1} = T_0^n + 2\lambda (T_1^n - T_0^n - \Delta x \gamma^n).$$

3. Update interior nodes ($j = 1, \dots, N - 1$):

$$T_j^{n+1} = T_j^n + \lambda (T_{j+1}^n - 2T_j^n + T_{j-1}^n).$$

4. Enforce the right boundary condition at $x = L$ (Dirichlet or Robin), as given.

Boundary Conditions for the Cooling Sphere HW3

We solve the radially symmetric heat equation in a sphere:

$$\frac{\partial T}{\partial t} = \alpha \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) \right], \quad 0 < r < R.$$

Two boundary conditions are required:

► **Center symmetry (Neumann):**

$$\left. \frac{\partial T}{\partial r} \right|_{r=0} = 0 \quad \Longleftrightarrow \quad q_r(0, t) = 0.$$

► **Surface convection (Robin):**

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=R} = h(T(R, t) - T_\infty).$$

Neumann BC at $r = 0$ via a Balance Law (Center Control Volume)

Consider a small spherical control volume around the origin, with radius $\Delta r/2$.

$$\rho c_p V \frac{dT_0}{dt} = -qA,$$

where q is the outward heat flux at $r = \Delta r/2$.

Geometry:

$$V = \frac{4}{3}\pi \left(\frac{\Delta r}{2}\right)^3, \quad A = 4\pi \left(\frac{\Delta r}{2}\right)^2.$$

Approximate the conductive flux across the boundary:

$$q \approx -k \frac{T_1 - T_0}{\Delta r}.$$

Neuman BC at $r = 0$: Conduction Balances Convection

Substitute and simplify:

$$\frac{dT_0}{dt} = 6\alpha \frac{T_1 - T_0}{\Delta r^2}.$$

Forward Euler in time gives the **center update**:

$$T_0^{n+1} = T_0^n + 6\lambda (T_1^n - T_0^n), \quad \lambda = \frac{\alpha \Delta t}{\Delta r^2}.$$

Robin BC at $r = R$: Conduction Balances Convection

At the surface, the boundary condition is

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=R} = h (T(R, t) - T_{\infty}).$$

Interpretation:

- ▶ $-k \partial T / \partial r$ is the **conductive heat flux leaving the solid**
- ▶ $h(T - T_{\infty})$ is the **convective heat flux into the air**

This is a **Robin (mixed) boundary condition** because it couples:

temperature at the boundary and temperature gradient at the boundary.

In a finite difference method, we enforce this condition at each time step to determine the surface behavior.

Robin BC Implementation: Ghost Node at the Surface

Approximate the surface derivative using a centered difference:

$$\left. \frac{\partial T}{\partial r} \right|_{r=R} \approx \frac{T_{N+1}^n - T_{N-1}^n}{2\Delta r}.$$

Insert into the Robin condition:

$$-k \frac{T_{N+1}^n - T_{N-1}^n}{2\Delta r} = h(T_N^n - T_\infty).$$

Solve for the ghost value T_{N+1}^n :

$$T_{N+1}^n = T_{N-1}^n - 2\Delta r \frac{h}{k} (T_N^n - T_\infty).$$

Then the surface node update uses the same stencil as interior nodes, with T_{N+1}^n substituted where T_{N+1} appears.

Summary: How We Enforce Boundary Conditions

- ▶ **Center ($r = 0$), Neumann / symmetry:**

$$T_0^{n+1} = T_0^n + 6\lambda (T_1^n - T_0^n).$$

- ▶ **Surface ($r = R$), Robin / convection:**

$$-k \left. \frac{\partial T}{\partial r} \right|_{r=R} = h(T_N - T_\infty), \quad T_{N+1}^n = T_{N-1}^n - 2\Delta r \frac{h}{k} (T_N^n - T_\infty).$$

- ▶ **Interior ($i = 1, \dots, N - 1$):** use the standard spherical FTCS stencil.

Takeaway: boundary conditions are enforced *at every time step*, and they determine the updates at nodes where the standard interior stencil does not apply.