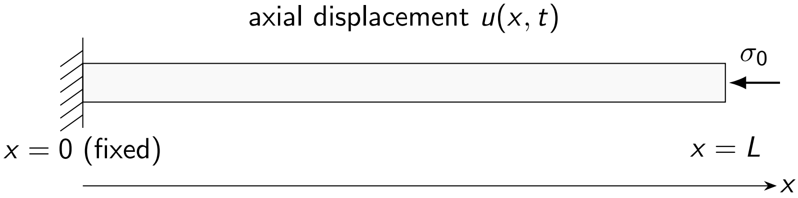
1D Wave Equation

Derivation, Finite Differences, and Stability

MMAE 450

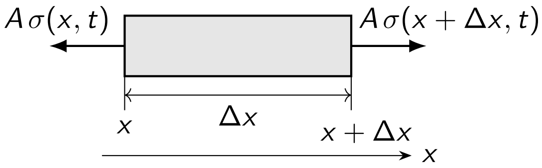
Physical model: axial waves in a slender bar



Assumptions

- No body force; constant area A and density ρ
- Linear elasticity: $\sigma = E\,\varepsilon$, with $\varepsilon = \dfrac{\mathrm{d}u}{\mathrm{d}x}$
- Small displacement gradients: $|\dfrac{\mathrm{d}u}{\mathrm{d}x}| \ll 1$ (geometric linearization)

Derivation of the 1D wave equation



- Axial stress acts on each face of the slice
- Force on a face is $A\,\sigma(x,t)$
- Mass of the slice is $\rho A\,\Delta x$

Derivation of the 1D wave equation

Linear momentum balance (Newton's 2nd law)

$$A\,\sigma(x+\Delta x, t) - A\,\sigma(x, t) = \rho A\,\Delta x\,\frac{\partial^2 u}{\partial t^2}$$

Divide through by $A\Delta x$ and take limit as $\Delta x \to 0$.

$$\frac{\partial \sigma}{\partial x} = \rho \frac{\partial^2 u}{\partial x^2}$$

Constitutive law + kinematics (small strain)

$$\varepsilon = \frac{\partial u}{\partial x}, \qquad \sigma = E\,\varepsilon = E\,\frac{\partial u}{\partial x}$$

$$\frac{\partial \sigma}{\partial x} = E\,\frac{\partial^2 u}{\partial x^2}$$

1D wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad c = \sqrt{\frac{E}{\rho}}$$

The parameter c is the *wave speed*

Finite differences: grid and discrete operators

Uniform mesh:

$$x_j = j\,\Delta x, \qquad t^n = n\,\Delta t, \qquad u_j^n \approx u(x_j, t^n)$$

Second derivatives (centered)

$$\frac{\partial^2 u}{\partial t^2}(x_j, t^n) \;\approx\; \frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2}(x_j, t^n) \;\approx\; \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Define the Courant number (wave CFL):

$$r \equiv \frac{c\,\Delta t}{\Delta x}.$$

Finite differences: explicit update (central in time and space)

Start from $u_{tt} = c^2 u_{xx}$ at (x_j, t^n) and substitute the centered formulas:

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} = c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

Explicit time-marching scheme (CTCS)

$$\boxed{u_j^{n+1} = 2u_j^n - u_j^{n-1} + r^2 \left(u_{j+1}^n - 2u_j^n + u_{j-1}^n \right)}$$

Boundary conditions: displacement and stress

Dirichlet (prescribed displacement)

At a boundary where the displacement is specified, e.g. at $x = 0$:

$$u(0, t) = u^*(t)$$

On the grid, this is enforced *strongly* at every time step:

$$u_0^n = u^*(t^n)$$

- No update equation is used at this node
- The boundary value is imposed directly
- Interior updates use this known value

Neumann boundary condition via Taylor expansion

Taylor expansion about the boundary $x = 0$

At a fixed time t, expand $u(x, t)$ about $x = 0$:

$$u(\Delta x, t) = u(0, t) + \Delta x \,\frac{\partial u}{\partial x}(0, t) + \frac{\Delta x^2}{2}\,\frac{\partial^2 u}{\partial x^2}(0, t) + \mathcal{O}(\Delta x^3)$$

Impose the Neumann condition (zero stress / zero slope):

$$\frac{\partial u}{\partial x}(0, t) = 0$$

Solve for the second derivative:

$$\frac{\partial^2 u}{\partial x^2}(0, t) \;\approx\; \frac{2\big(u(\Delta x, t) - u(0, t)\big)}{\Delta x^2}$$

Boundary update equation

On the grid:

$$\frac{\partial^2 u}{\partial x^2}(0, t^n) \approx \frac{2\left(u_1^n - u_0^n\right)}{\Delta x^2}$$

Use the centered time discretization:

$$\frac{\partial^2 u}{\partial t^2}(0, t^n) \approx \frac{u_0^{n+1} - 2u_0^n + u_0^{n-1}}{\Delta t^2}$$

Substitute into $u_{tt} = c^2 u_{xx}$ to obtain:

$$\boxed{u_0^{n+1} = 2u_0^n - u_0^{n-1} + 2r^2\left(u_1^n - u_0^n\right)}$$

$$r = \frac{c\,\Delta t}{\Delta x}$$

Von Neumann stability: Fourier mode substitution

Assume a Fourier mode on an infinite / periodic grid:

$$u_j^n = A^n e^{i\theta j}, \qquad \theta \in [0, \pi].$$

Substitute into the explicit update:

$$A^{n+1} e^{i\theta j} = 2A^n e^{i\theta j} - A^{n-1} e^{i\theta j} + r^2 A^n \left(e^{i\theta(j+1)} - 2e^{i\theta j} + e^{i\theta(j-1)} \right).$$

Divide through and simplify

Divide by $A^{n-1} e^{i\theta j}$:

$$\frac{A^{n+1}}{A^{n-1}} = 2\frac{A^n}{A^{n-1}} - 1 + r^2 \frac{A^n}{A^{n-1}} \left(e^{i\theta} - 2 + e^{-i\theta} \right).$$

Use

$$e^{i\theta} - 2 + e^{-i\theta} = -4\sin^2\frac{\theta}{2},$$

to obtain

$$\frac{A^{n+1}}{A^{n-1}} - 2\left(1 - 2r^2\sin^2\frac{\theta}{2}\right)\frac{A^n}{A^{n-1}} + 1 = 0.$$

Amplification factor

Define

$$G \equiv \frac{A^n}{A^{n-1}}.$$

Then the quadratic becomes

$$G^2 - 2\left(1 - 2r^2\sin^2\frac{\theta}{2}\right)G + 1 = 0.$$

Roots of the characteristic equation

Define

$$a(\theta) \;\equiv\; 1 - 2r^2 \sin^2 \frac{\theta}{2}.$$

Then the amplification factor satisfies

$$G^2 - 2a(\theta)\, G + 1 = 0,$$

with roots

$$G_{1,2} = a \pm \sqrt{a^2 - 1}.$$

Key fact

The constant term is $+1$, so

$$G_1 G_2 = 1.$$

Stable case: complex roots (bounded oscillations)

When do we get complex roots?

If

$$a^2 - 1 \leq 0 \qquad \iff \qquad |a| \leq 1,$$

then the roots are complex conjugates:

$$G_{1,2} = a \pm i\sqrt{1 - a^2}.$$

Magnitude of the roots

$$|G_{1,2}|^2 = a^2 + (1 - a^2) = 1 \qquad \Longrightarrow \qquad |G_1| = |G_2| = 1.$$

So each Fourier mode is bounded in time (no growth).

Unstable case: real roots (blow-up)

When are the roots real?

If

$$a^2 - 1 > 0,$$

then $G_{1,2}$ are real.

Why that implies blow-up

Because

$$G_1 G_2 = 1,$$

it is impossible for both real roots to satisfy $|G| \leq 1$ unless $|G_1| = |G_2| = 1$ (which occurs only at the boundary case $a = \pm 1$).
Therefore, when $a^2 - 1 > 0$ one root must satisfy

$$|G| > 1,$$

and that Fourier component grows exponentially with n.

CFL stability condition for the wave equation

We require stability for *all* Fourier modes $\theta \in [0, \pi]$:

$$|a(\theta)| \leq 1, \qquad a(\theta) = 1 - 2r^2 \sin^2 \frac{\theta}{2}.$$

Upper bound is automatic

Since $\sin^2(\theta/2) \geq 0$,

$$a(\theta) \leq 1 \quad \text{always.}$$

Lower bound gives the restriction

The smallest value occurs at $\sin^2(\theta/2) = 1$ (i.e., $\theta = \pi$):

$$a_{\min} = 1 - 2r^2.$$

Require $a_{\min} \geq -1$:

$$1 - 2r^2 \geq -1 \quad \Longrightarrow \quad r^2 \leq 1 \quad \Longrightarrow \quad \boxed{r = \frac{c\,\Delta t}{\Delta x} \leq 1}.$$