

Two-Dimensional Heat Conduction

Finite Differences and the Five-Point Stencil

MMAE 450: Computational Mechanics II

Motivation for 2D Heat Conduction

- ▶ Many engineering problems involve temperature variation in more than one spatial direction.
- ▶ Examples:
 - ▶ Thin plates and heat sinks
 - ▶ Cooling of electronic components
 - ▶ Layered or composite materials
- ▶ Extending 1D ideas to 2D is conceptually straightforward.
- ▶ The resulting algebraic systems become *large and sparse*.

Governing Equation in Two Dimensions

For a homogeneous, isotropic material with constant properties:

$$\frac{\partial T}{\partial t} = \alpha \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)$$

Under steady-state conditions:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

- ▶ This is the 2D *Laplace equation*.
- ▶ Boundary conditions (Dirichlet/Neumann/Robin) are prescribed on $\partial\Omega$.

Rectangular Domain and Grid

Consider a rectangular plate:

- ▶ $0 \leq x \leq L_x$
- ▶ $0 \leq y \leq L_y$

Uniform grid:

$$\Delta x = \frac{L_x}{N_x - 1}, \quad \Delta y = \frac{L_y}{N_y - 1}$$

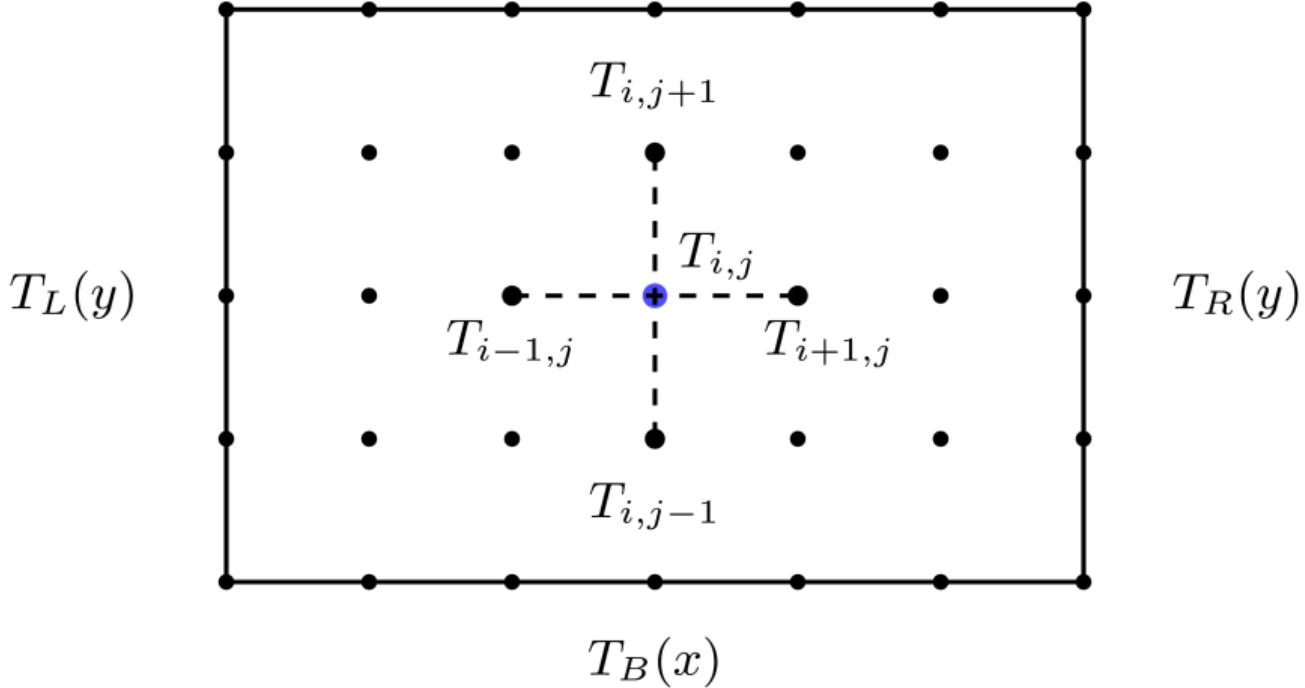
Grid points labeled by (i, j) :

$$x_i = i \Delta x, \quad i = 0, 1, \dots, N_x,$$
$$y_j = j \Delta y, \quad j = 0, 1, \dots, N_y.$$

We denote the nodal approximation by $T_{i,j}^n \approx T(x_i, y_j, t^n)$.

Five-Point Stencil

$$T_T(x)$$



- ▶ Interior node (i,j) depends on four nearest neighbors.
- ▶ This produces the classic *five-point stencil*.

Discrete Laplacian

Second derivatives using centered finite differences:

$$\frac{\partial^2 T}{\partial x^2} \Big|_{i,j} \approx \frac{T_{i+1,j} - 2T_{i,j} + T_{i-1,j}}{(\Delta x)^2},$$

$$\frac{\partial^2 T}{\partial y^2} \Big|_{i,j} \approx \frac{T_{i,j+1} - 2T_{i,j} + T_{i,j-1}}{(\Delta y)^2}.$$

If $\Delta x = \Delta y = h$, then

$$\nabla^2 T \Big|_{i,j} \approx \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j}}{h^2}.$$

Steady-State as a Linear System

At each interior node, the discrete Laplace equation gives

$$T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1} - 4T_{i,j} = 0.$$

Collecting all interior equations yields

$$\mathbf{A} \mathbf{T} = \mathbf{d},$$

where

- ▶ \mathbf{T} contains the unknown interior temperatures,
- ▶ \mathbf{A} is sparse (*at most five nonzeros per row*),
- ▶ \mathbf{d} accounts for prescribed boundary values.

Structure of the Discrete System Matrix \mathbf{A}

For the five-point stencil, the steady problem

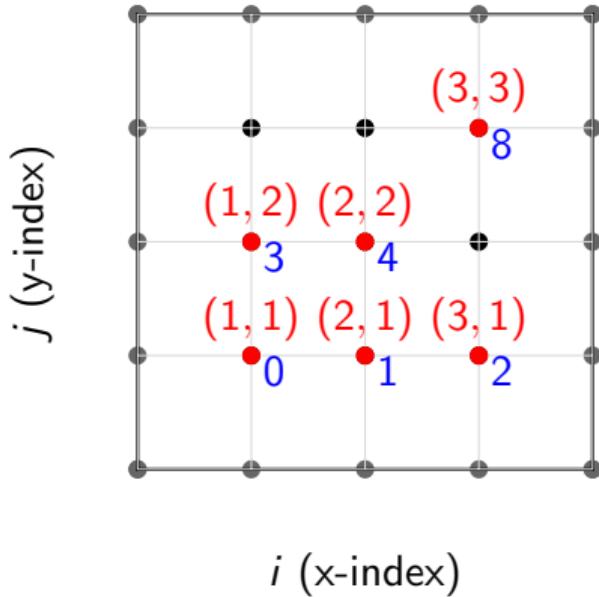
$$\mathbf{A} \mathbf{T} = \mathbf{d}$$

leads to a sparse matrix with a highly regular structure.

$$\mathbf{A} = \begin{bmatrix} \ddots & & & \\ & \ddots & & \\ & & -4 & 1 \\ & & 1 & -4 & 1 \\ & & & 1 & -4 & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

- ▶ Each row corresponds to one interior grid node
- ▶ Diagonal entry (-4) : contribution from $T_{i,j}$
- ▶ Off-diagonals $(+1)$: coupling to nearest neighbors
- ▶ Overall structure is sparse and banded

Interior unknowns: $i, j = 1, 2, 3$ ($3 \times 3 = 9$ unknowns)



$$idx(i, j) = 3(j - 1) + (i - 1)$$

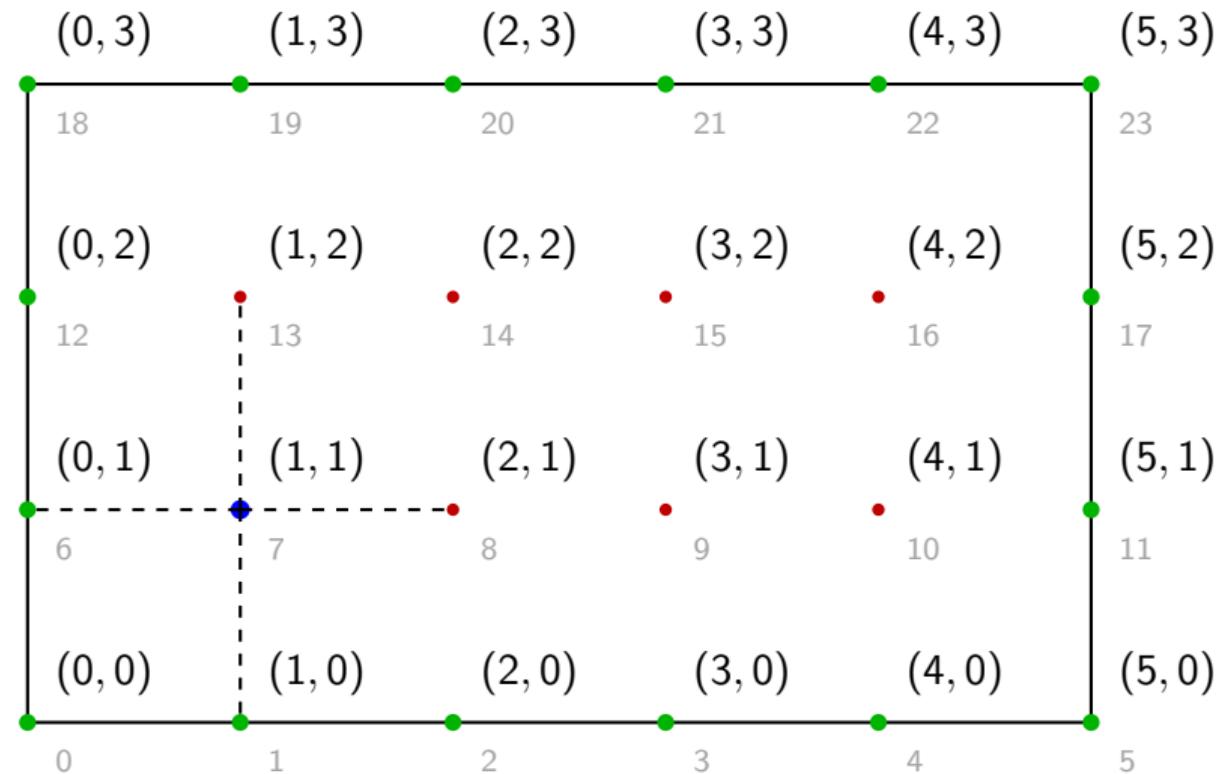
Gauss–Seidel Iteration in 2D

Update each interior node in-place using the stencil average:

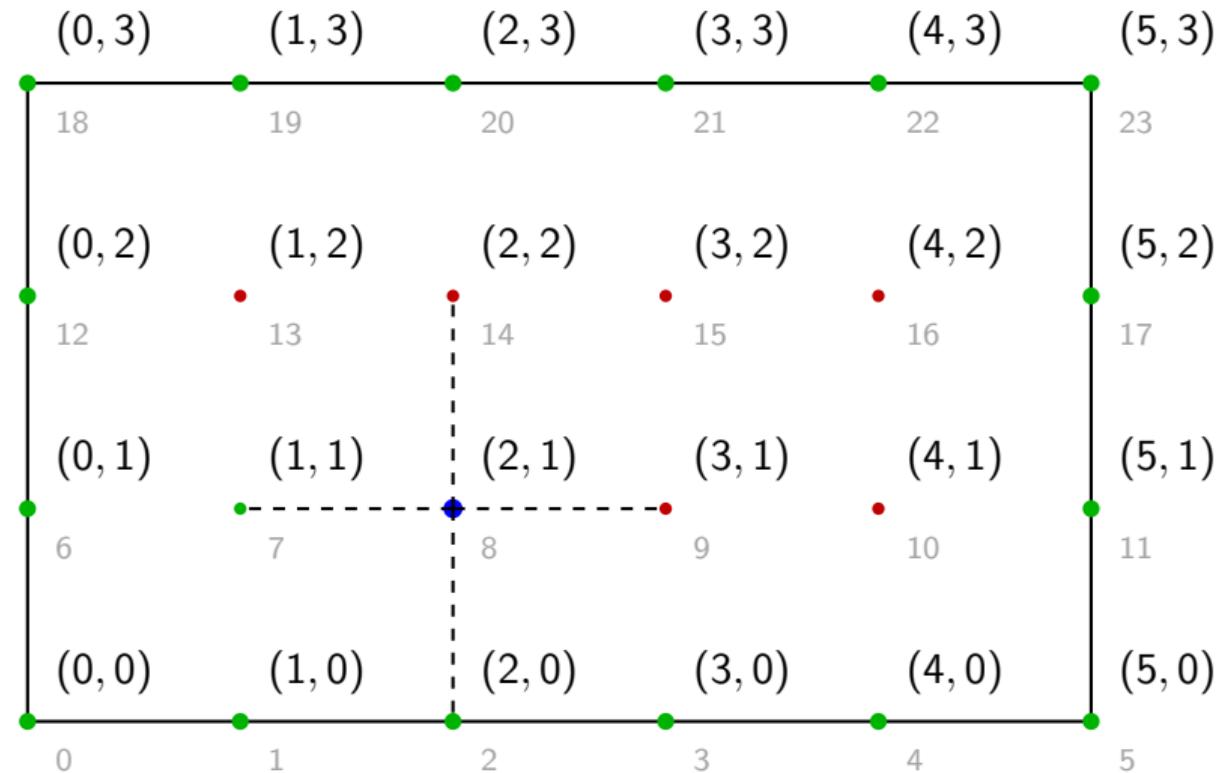
$$T_{i,j}^{(k+1)} = \frac{1}{4} \left(T_{i+1,j}^{(k)} + T_{i-1,j}^{(k+1)} + T_{i,j+1}^{(k)} + T_{i,j-1}^{(k+1)} \right).$$

- ▶ Sweep through the grid node-by-node.
- ▶ Dirichlet boundary values are imposed directly.
- ▶ Stop when the max change between sweeps is below a tolerance.

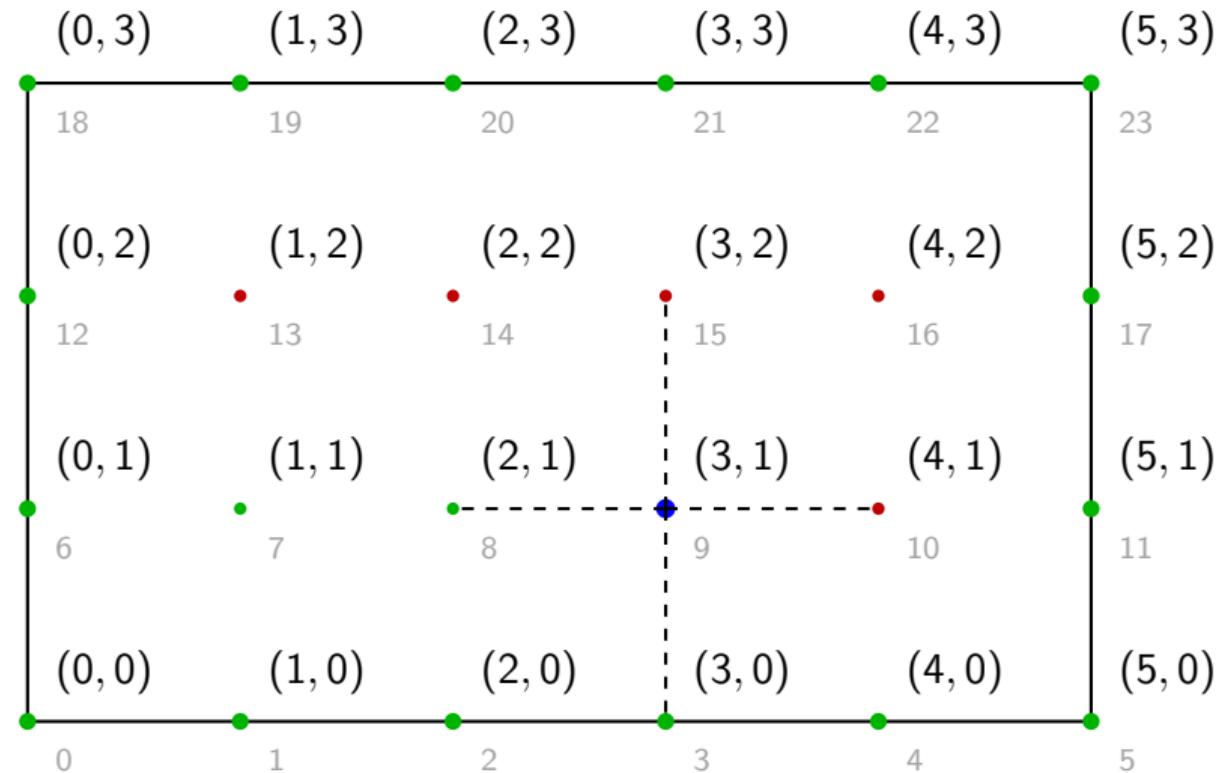
Gauss–Seidel sweep (current point $(1, 1)$)



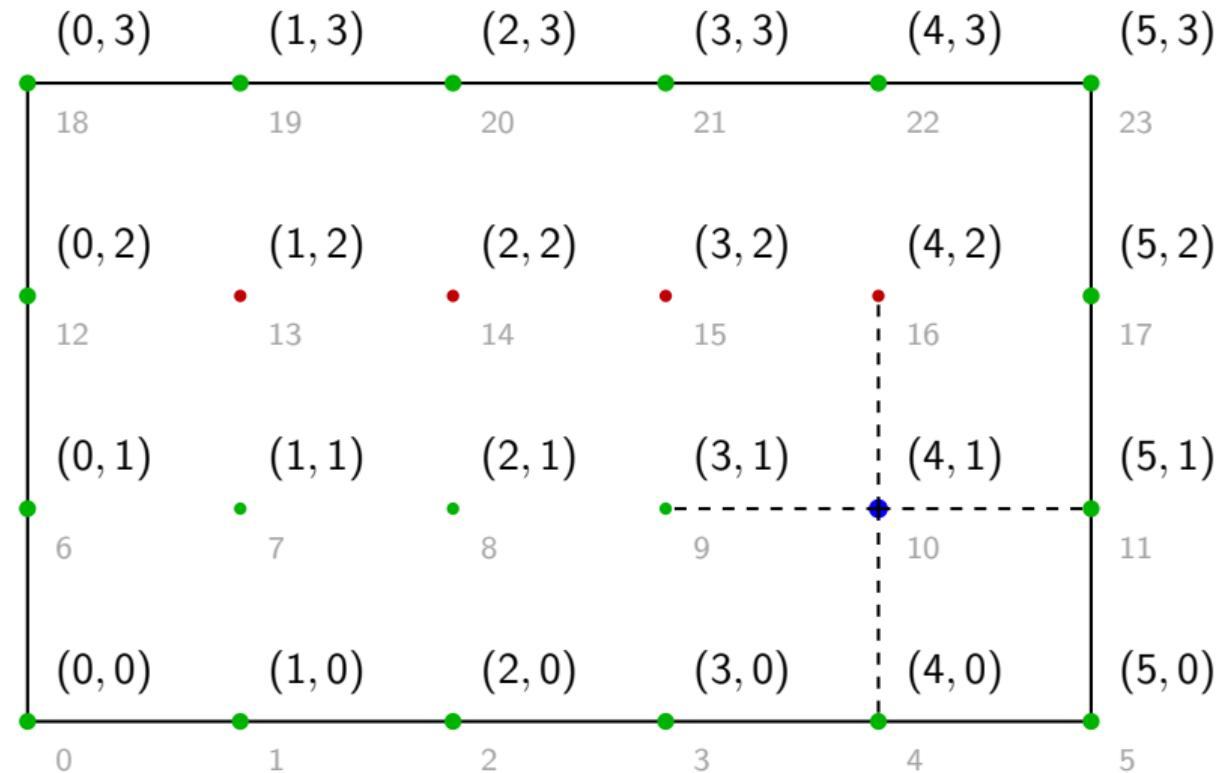
Gauss–Seidel sweep (current point $(2, 1)$)



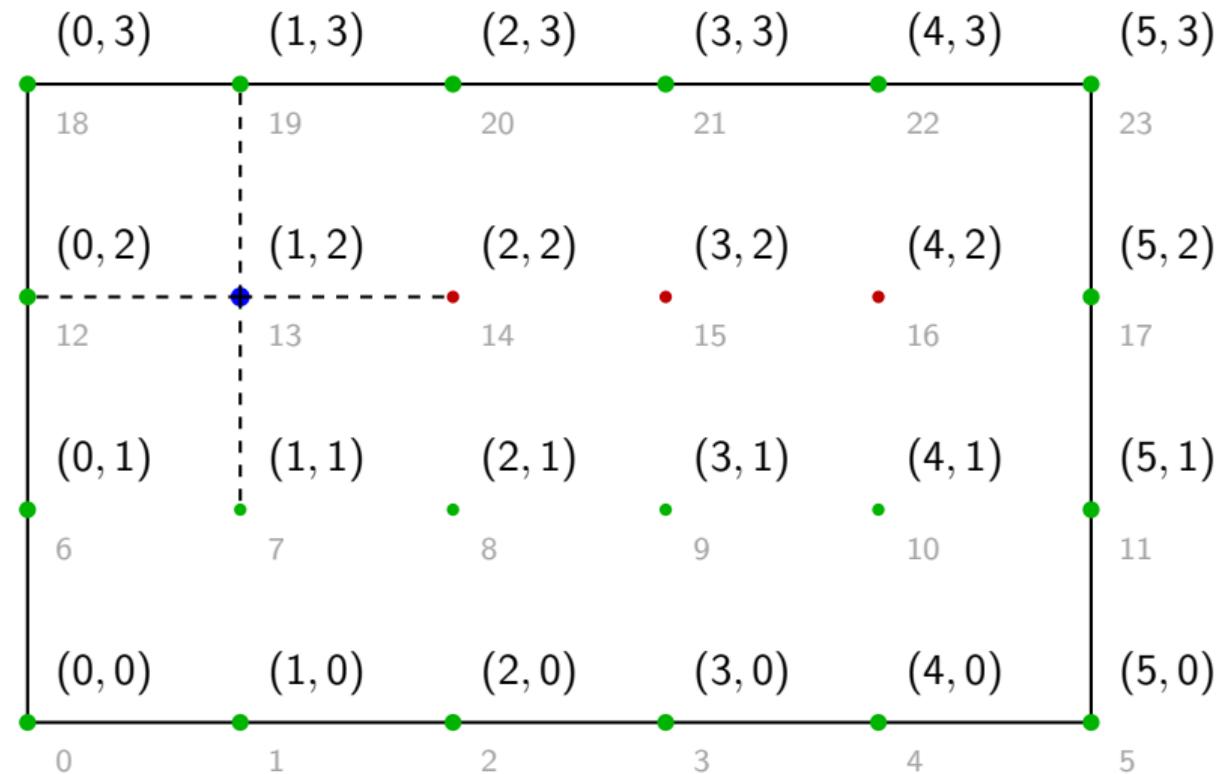
Gauss–Seidel sweep (current point $(3, 1)$)



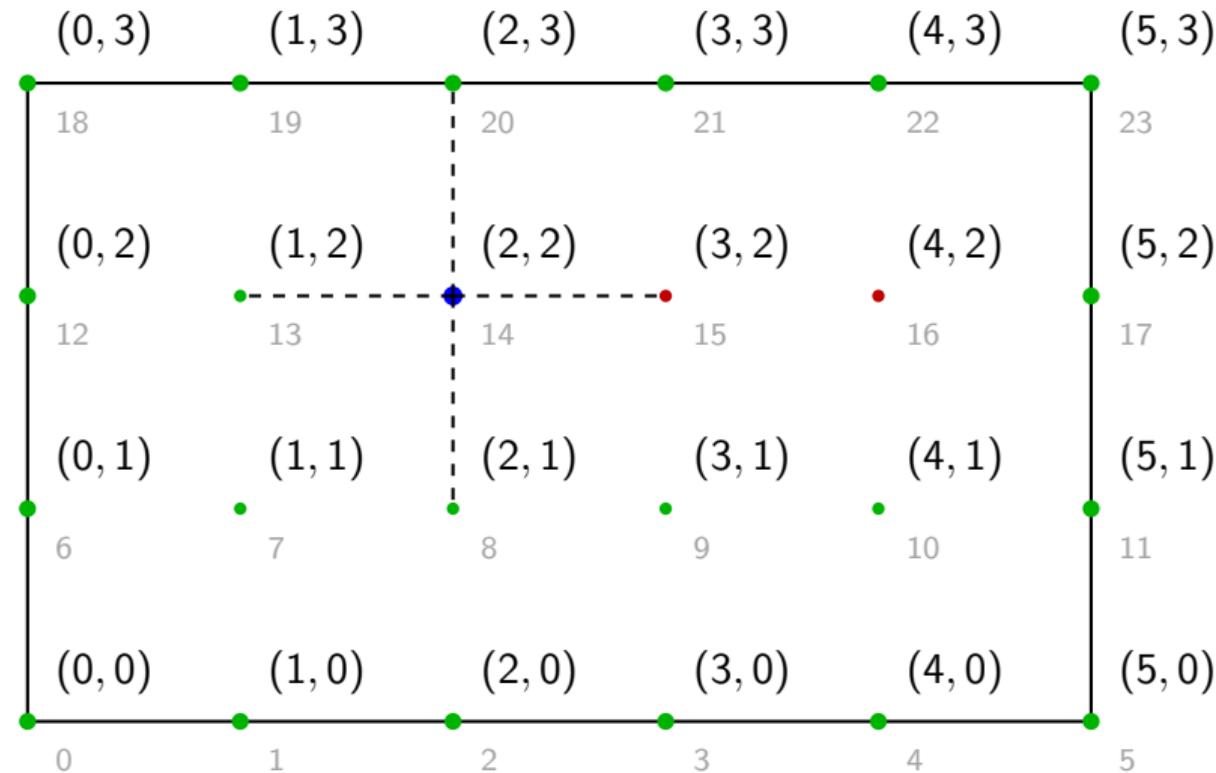
Gauss–Seidel sweep (current point $(4, 1)$)



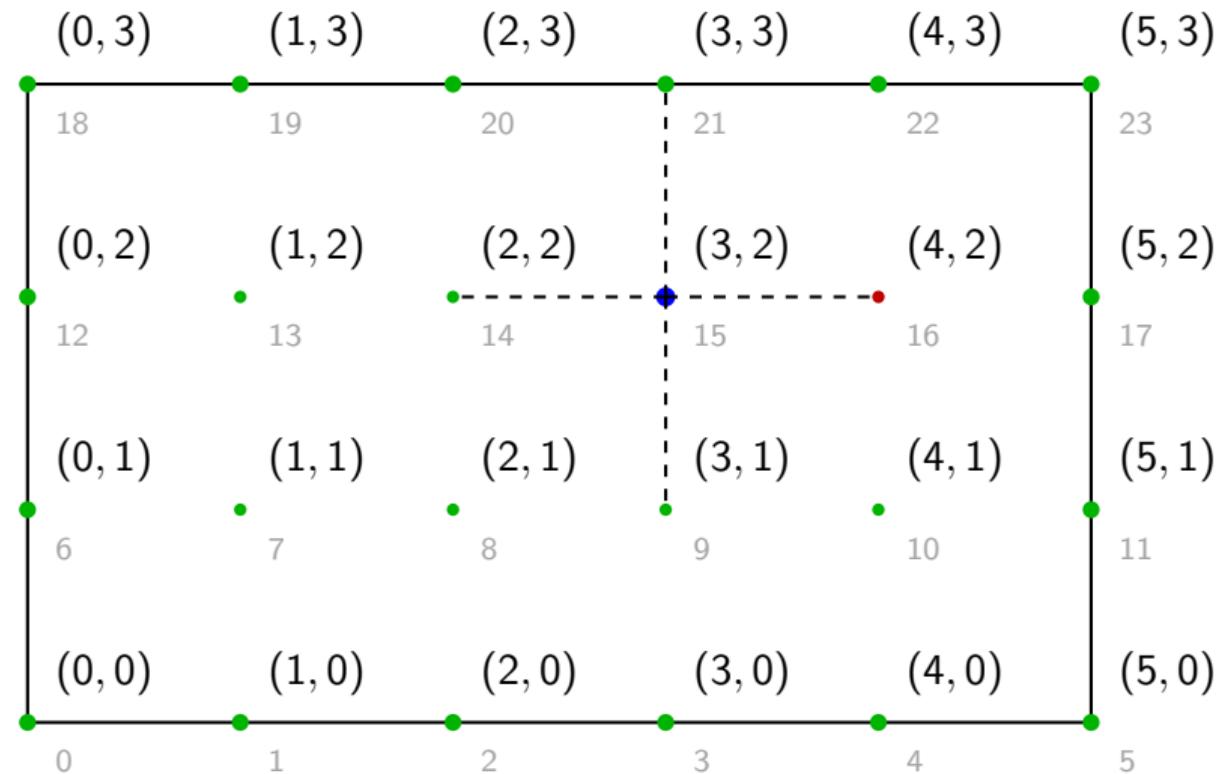
Gauss–Seidel sweep (current point $(1, 2)$)



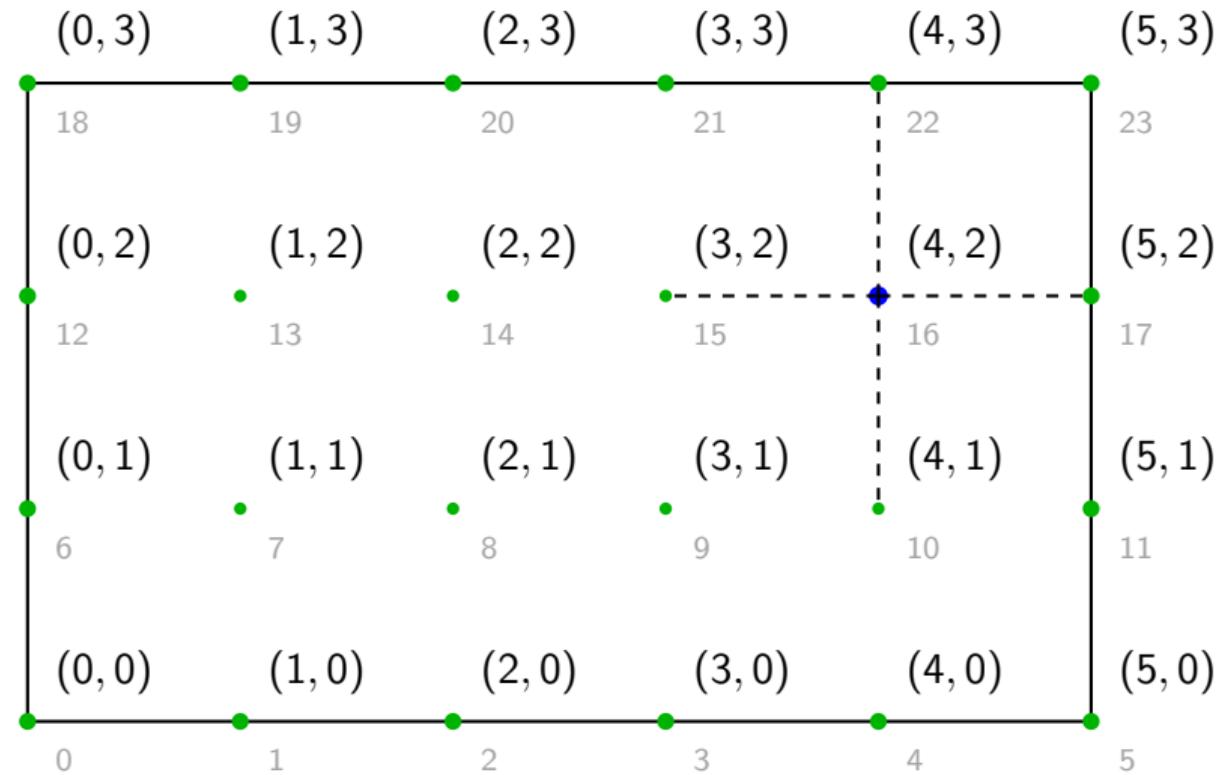
Gauss–Seidel sweep (current point $(2, 2)$)



Gauss–Seidel sweep (current point $(3, 2)$)



Gauss–Seidel sweep (current point $(4, 2)$)



Worked Example: Heated Plate

Square plate with $L_x = L_y = 1$ and Dirichlet conditions on all edges:

$$T(0, y) = 0^\circ\text{C},$$

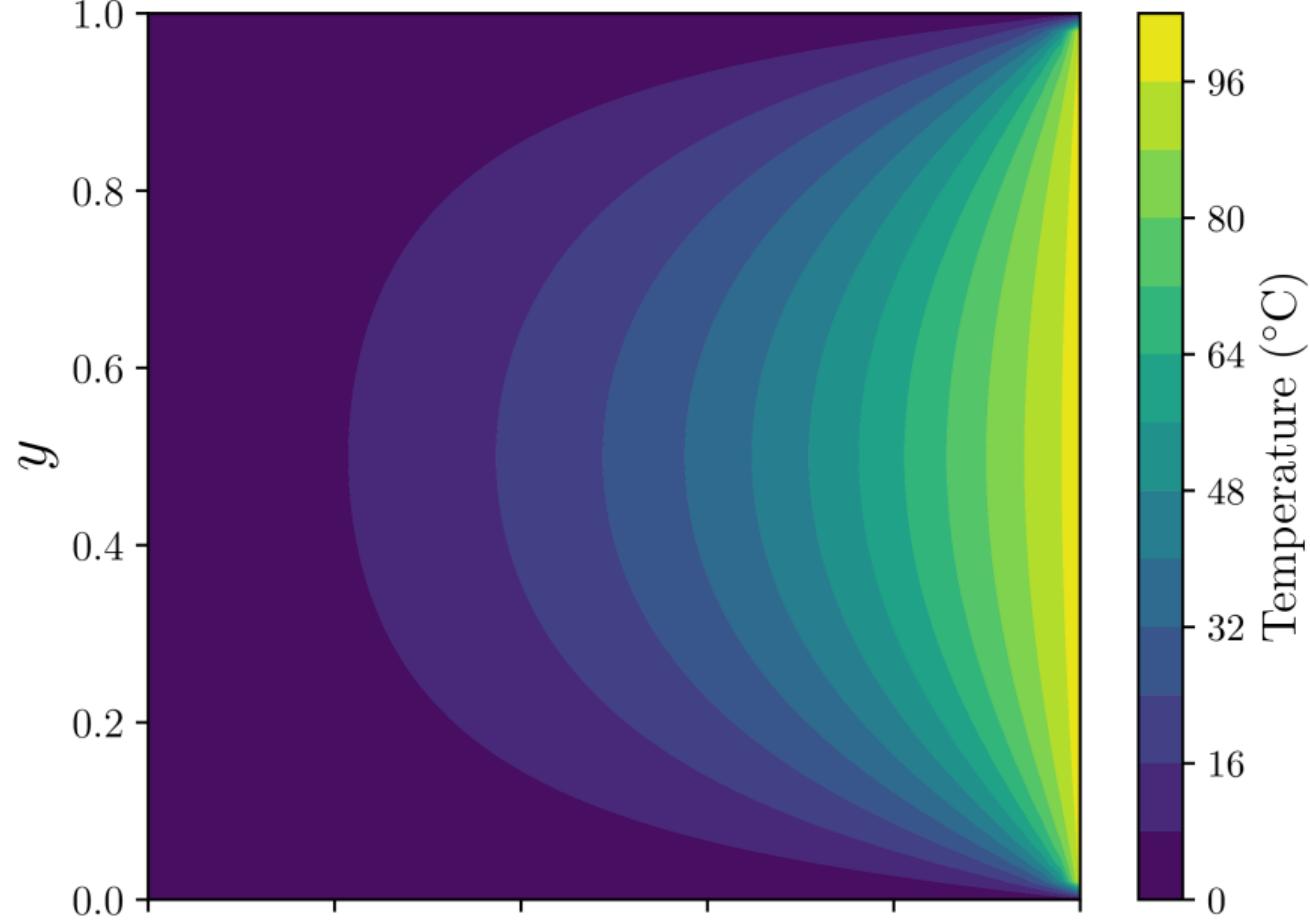
$$T(1, y) = 100^\circ\text{C},$$

$$T(x, 0) = 0^\circ\text{C},$$

$$T(x, 1) = 0^\circ\text{C}.$$

- ▶ Three cold boundaries, one hot boundary.
- ▶ Solve Laplace equation on the grid (e.g., Gauss–Seidel).
- ▶ The interior field is influenced by all boundaries.

Temperature Field Solution



Extension to Transient Problems

A 2D FTCS update (for $\Delta x = \Delta y = h$):

$$T_{i,j}^{n+1} = T_{i,j}^n + \lambda \left[T_{i+1,j}^n + T_{i-1,j}^n + T_{i,j+1}^n + T_{i,j-1}^n - 4T_{i,j}^n \right],$$

where $\lambda = \alpha \Delta t / h^2$.

- ▶ Direct analogue of 1D FTCS.
- ▶ Stability is more restrictive in 2D (smaller Δt).
- ▶ Implicit / Crank–Nicolson in 2D leads to sparse linear systems.

Key Takeaways

- ▶ 2D conduction extends 1D ideas with a local stencil in both directions.
- ▶ The five-point stencil leads to large, sparse linear systems.
- ▶ Iterative methods (Gauss–Seidel, etc.) map naturally onto grid sweeps.
- ▶ Transient schemes follow the same stencil structure (plus time stepping).

Next: Wave propagation and fundamentally different physics