Divergence Theorem via 1D Heat Conduction (Constant Cross Section)

MMAE 450

Big Idea

Core principle

Local conservation laws, when integrated over a domain, become global balance laws.

- ▶ Boundary view: add what crosses the boundary
- ▶ Interior view: add what is created or destroyed inside
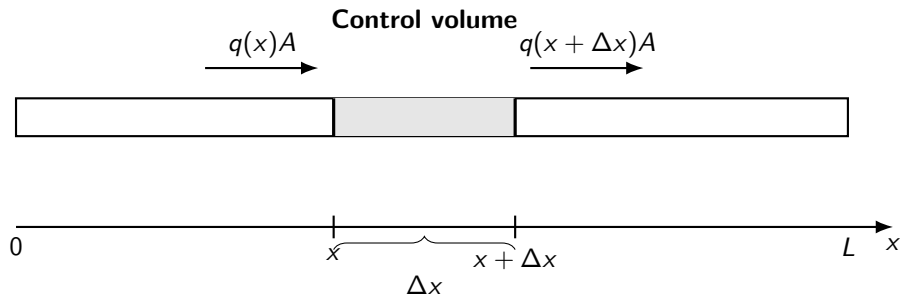
These two viewpoints are equivalent.

1D Fin Model

- Fin along $x \in [0, L]$
- Constant cross-sectional area A
- Heat flux $q(x)$ [W/m^2]
- Steady state
- No internal heat generation

Goal
Derive the local and global energy balance from first principles.

Control Volume in the Fin

Local Energy Balance (No Generation)

On the segment $[x, x + \Delta x]$, steady state with no generation:

$$q(x)A - q(x + \Delta x)A = 0$$

Divide by $A\,\Delta x$ and take the limit $\Delta x \to 0$:

$$\frac{q(x + \Delta x) - q(x)}{\Delta x} = 0 \quad \Rightarrow \quad \frac{dq}{dx} = 0$$

Interpretation
The heat flux is spatially constant when no energy is generated.

From Local to Global Balance

Integrate the local conservation law over $[0, L]$:

$$\int_0^L \frac{dq}{dx}\, dx = 0$$

Apply the Fundamental Theorem of Calculus:

$$q(L) - q(0) = 0$$

Boundary heat-rate balance

Multiplying by the area gives

$$q(0)A = q(L)A$$

Local Balance with a Source Term

Now include volumetric heat generation $S(x)$ [W/m^3].
Energy balance on $[x, x + \Delta x]$:

$$q(x)A - q(x + \Delta x)A + S(x)\, A\, \Delta x = 0$$

Divide by $A\, \Delta x$ and take the limit:

$$-\frac{dq}{dx} + S(x) = 0$$

Local conservation law
Divergence of heat flux balances internal generation.

Global Balance with Sources

Integrate the local equation over $[0, L]$:

$$\int_0^L \left(-\frac{dq}{dx} + S(x) \right) dx = 0$$

$$-\left(q(L) - q(0) \right) + \int_0^L S(x) \, dx = 0$$

Multiply by the area:

$$q(0)A - q(L)A + \int_0^L S(x) \, A \, dx = 0$$

Meaning

in $-$ out $+$ generated $= 0$

Divergence Theorem (Heat)

In three dimensions, let $\boldsymbol{q}(\boldsymbol{x})$ be the heat flux vector.

$$-\nabla \cdot \boldsymbol{q} + S = 0$$

Integrating over a volume V:

$$\int_V \nabla \cdot \boldsymbol{q} \, dV = \int_\Gamma \boldsymbol{q} \cdot \boldsymbol{n} \, dS$$

Key idea
Divergence converts surface flux into a volumetric density.

Exact Analogy: Stress Divergence

Replace heat flux with the Cauchy stress tensor $\boldsymbol{\sigma}$.

$$\int_V \nabla \cdot \boldsymbol{\sigma} \, dV = \int_\Gamma \boldsymbol{\sigma} \boldsymbol{n} \, dS$$

Linear momentum balance:

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho \boldsymbol{a}$$

- ▶ $\nabla \cdot \boldsymbol{\sigma}$: internal force per unit volume
- ▶ $\boldsymbol{b}$: body force density

One-sentence takeaway
Divergence turns boundary interactions into volume forces.

Wrap-Up

- ▶ Start from conservation on an infinitesimal segment
- ▶ Divide by volume to obtain a local law
- ▶ Integrate to recover the global balance
- ▶ Same structure appears in heat, fluids, and solids

Exit question

If $S = 0$, what must be true of the boundary heat fluxes?