MMAE 450
Midterm 1 Review

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What This Exam Is Really About

- Taylor series
- Linearization (Newton)
- Discretization (finite difference)
- Stability
- Boundary conditions

Everything reduces to:
Linear systems or explicit updates

Newton's Method (1 Variable)

Solve:

$$f(x) = 0$$

Update:

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$$

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Derived from Taylor series.

Newton for Systems

$$\mathbf{R}(\mathbf{x}) = 0$$

Linearize:

$$\mathbf{J}(\mathbf{x}_k)\Delta\mathbf{x} = -\mathbf{R}(\mathbf{x}_k)$$

Update:

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \Delta\mathbf{x}$$

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Residual $\rightarrow$ Jacobian $\rightarrow$ Linear solve $\rightarrow$ Update

Taylor Series (General Form)

Start with:

$$\Delta x = x - x_0$$

First-order expansion:

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Second-order expansion:

$$f(x) = f(x_0) + f'(x_0)\,\Delta x + \frac{1}{2}f''(x_0)\,(\Delta x)^2 + O((\Delta x)^3)$$

Taylor Series for Finite Differences

Choose:

$$\Delta x = h \quad \text{and} \quad \Delta x = -h$$

Then:

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)$$

$$f(x_0 - h) = f(x_0) - hf'(x_0) + \frac{h^2}{2} f''(x_0) + O(h^3)$$

Deriving the Second Derivative

Add the two expansions:

$$f(x_0 + h) + f(x_0 - h) = 2f(x_0) + h^2 f''(x_0) + O(h^4)$$

Rearrange:

$$f''(x_0) = \frac{f(x_0 + h) - 2f(x_0) + f(x_0 - h)}{h^2} + O(h^2)$$

Second-order accurate

1D Steady-State Heat

$$\frac{d^2 T}{dx^2} = 0$$

Discretization:

$$-T_{i-1} + 2T_i - T_{i+1} = 0$$

Tridiagonal linear system

1D Transient — FTCS

$$T_i^{n+1} = T_i^n + r(T_{i+1}^n - 2T_i^n + T_{i-1}^n)$$

$$r = \frac{\alpha \Delta t}{\Delta x^2}$$

1D Transient — FTCS

$$T_i^{n+1} = T_i^n + r(T_{i+1}^n - 2T_i^n + T_{i-1}^n)$$

$$r = \frac{\alpha \Delta t}{\Delta x^2}$$

Stability:

$$r \leq \frac{1}{2}$$

Explicit = conditionally stable

1D Transient — Crank–Nicolson

$$AT^{n+1} = BT^n$$

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- Implicit
- Unconditionally stable
- Must solve linear system each step

Crank–Nicolson Example (3 Interior Nodes)

1D Heat Equation:

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

Domain: $x \in [0, 1]$

$$\Delta x = 0.25, \qquad \alpha = 1, \qquad r = \frac{\alpha \Delta t}{\Delta x^2} = 1$$

Dirichlet BCs:

$$T_0 = 0, \qquad T_4 = 0$$

Initial interior values:

$$\mathbf{T}^n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Building the Matrix A

Crank–Nicolson interior equation:

$$-\frac{r}{2}T_{i-1}^{n+1} + (1+r)T_i^{n+1} - \frac{r}{2}T_{i+1}^{n+1} = \text{RHS}$$

For this example, $r = 1$:

$$-\frac{1}{2}T_{i-1}^{n+1} + 2T_i^{n+1} - \frac{1}{2}T_{i+1}^{n+1} = \text{RHS}$$

Each interior node contributes:

$$\boxed{\begin{bmatrix} -\frac{1}{2} & 2 & -\frac{1}{2} \end{bmatrix}}$$

Crank–Nicolson System

For $r = 1$, the discrete system becomes:

$$\underbrace{\begin{bmatrix} 2 & -\frac{1}{2} & 0 \\ -\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 2 \end{bmatrix}}_{A} \begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \\ T_3^{n+1} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}}_{\text{RHS}}$$

Building the Right-Hand Side

Crank–Nicolson equation:

$$-\frac{r}{2}T_{i-1}^{n+1} + (1+r)T_i^{n+1} - \frac{r}{2}T_{i+1}^{n+1} = \frac{r}{2}T_{i-1}^{n} + (1-r)T_i^{n} + \frac{r}{2}T_{i+1}^{n}$$

For $r = 1$:

$$\text{RHS} = \frac{1}{2}T_{i-1}^{n} + 0 \cdot T_i^{n} + \frac{1}{2}T_{i+1}^{n}$$

RHS depends only on known values at time level n.

Compute the RHS Vector

Given:

$$\mathbf{T}^n = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Node $i = 1$:

$$\frac{1}{2} T_2^n = \frac{1}{2}$$

Node $i = 2$:

$$\frac{1}{2}(T_1^n + T_3^n) = 0$$

Node $i = 3$:

$$\frac{1}{2} T_2^n = \frac{1}{2}$$

$$\text{RHS} = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

2D Steady-State

$$T_{i,j} = \frac{1}{4}\left(T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}\right)$$

5-point stencil

2D Transient FTCS

$$T_{i,j}^{n+1} = T_{i,j}^{n} + r_x \Delta_x^2 + r_y \Delta_y^2$$

2D Transient FTCS

$$T_{i,j}^{n+1} = T_{i,j}^{n} + r_x \Delta_x^2 + r_y \Delta_y^2$$

If $\Delta x = \Delta y$:

$$r \leq \frac{1}{4}$$

Boundary Conditions

Dirichlet	Prescribed value
Neumann	Prescribed derivative
Robin	Mixed condition

BCs modify discrete equations.

Closing Thought

Taylor series $\rightarrow$ Discretization $\rightarrow$ Linear system

Explicit scheme $\rightarrow$ Stability restriction

Newton $\rightarrow$ Linearize $\rightarrow$ Solve linear system