Human-Oriented Robotics

Supervised Learning

Part 2/3

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Non-Probabilistic Discriminant Functions

- So far, we have considered **probabilistic classifiers** that compute a posterior probability distribution $p(\mathbf{w}|\mathbf{x})$ over the world state, for example, a discrete distribution over different class labels
- We can also learn the **discriminant function** $y = f(\mathbf{x})$ **directly** (even more "directly" than a probabilistic discriminant classifier). For instance, in a two-class problem, $f(.)$ might be binary-valued such that $f(\mathbf{x}) = 0$ represents class $\mathcal{C}_1$ and $f(\mathbf{x}) = 1$ represents class $\mathcal{C}_2$
- Inference and decision stages are **combined**
- Choosing a model for $f(.)$ and using training data to learn $y = f(\mathbf{x})$ corresponds to **learning the decision boundary directly**
- This is unlike probabilistic classifiers where the decision boundary **followed indirectly** from our choices for the involved models

Non-Probabilistic Discriminant Functions

- Let us consider **linear discriminant functions** $y = f(\mathbf{x})$. This choice implies the assumption that our data are **linearly separable**
- Let us again consider a **binary classification** problem, $y \in \{-1, +1\}$
- The representation of a linear function is

$$y = f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + b$$

where $\mathbf{w}$ is the normal to the hyperplane (sometimes called weight vector) and b is called bias

- The hyperplane itself is described by $\mathbf{w}^T\mathbf{x} + b = 0$
- The perpendicular distance from the plane to the origin is $\frac{b}{\|\mathbf{w}\|}$

- (Notice the change in notation: in this section, we adopt the standard notion $\mathbf{w}$ to denote the **normal** to the hyperplane, not the world state)

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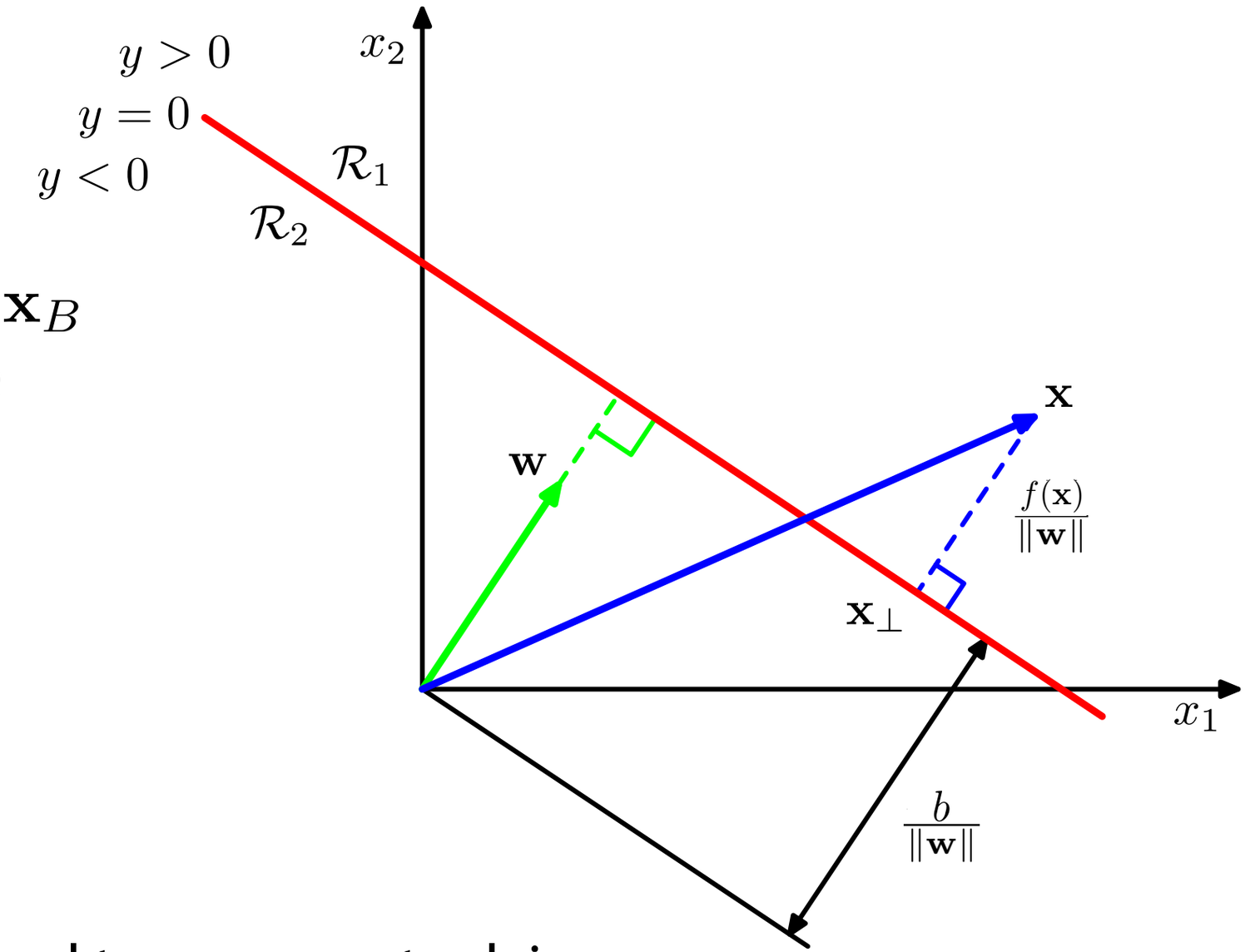
Non-Probabilistic Discriminant Functions

- Figure shows geometry of $f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + b$ in 2 dimensions



- Consider two points $\mathbf{x}_A, \mathbf{x}_B$ that both lie on the plane

$$f(\mathbf{x}_A) = \mathbf{w}^T\mathbf{x}_A + b = 0$$
$$f(\mathbf{x}_B) = \mathbf{w}^T\mathbf{x}_B + b = 0$$

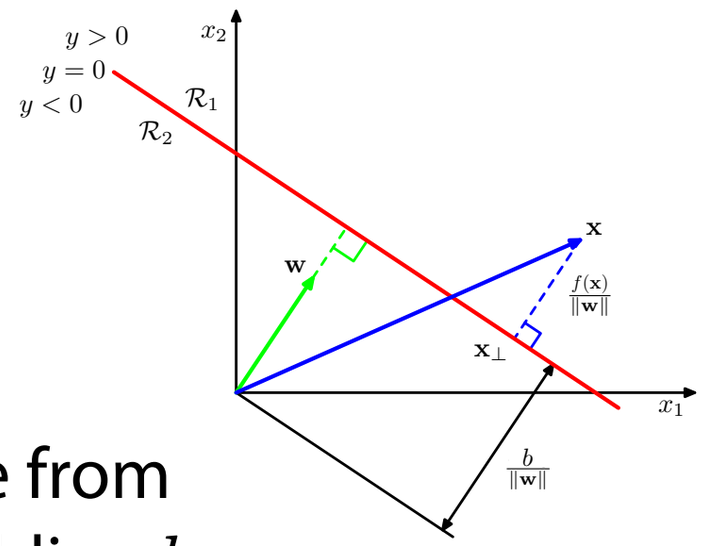
$$\mathbf{w}^T\mathbf{x}_A + b = \mathbf{w}^T\mathbf{x}_B + b$$
$$\mathbf{w}^T(\mathbf{x}_A - \mathbf{x}_B) = 0$$

- Thus, vector $\mathbf{w}$ is orthogonal to every vector lying within the hyperplane, and so $\mathbf{w}$ determines the orientation of the plane

Non-Probabilistic Discriminant Functions

- Consider a point $\mathbf{x}$ and its orthogonal projection onto the plane $\mathbf{x}_\perp$. Then

$$\mathbf{x} = \mathbf{x}_\perp + r\frac{\mathbf{w}}{\|\mathbf{w}\|}$$



- Let us solve for r, the signed perpendicular distance from $\mathbf{x}$ to the plane. Multiplying both sides by $\mathbf{w}^T$ and adding b

$$\mathbf{w}^T\mathbf{x} + b = \mathbf{w}^T\mathbf{x}_\perp + \mathbf{w}^T r\frac{\mathbf{w}}{\|\mathbf{w}\|} + b = \mathbf{w}^T\mathbf{x}_\perp + b + r\frac{\mathbf{w}^T\mathbf{w}}{\|\mathbf{w}\|} = r\frac{\mathbf{w}^T\mathbf{w}}{\|\mathbf{w}\|}$$

$$f(\mathbf{x}) = r\frac{\|\mathbf{w}\|^2}{\|\mathbf{w}\|} = r\,\|\mathbf{w}\| \quad \Leftrightarrow \quad r = \frac{f(\mathbf{x})}{\|\mathbf{w}\|}$$

- Note that distance r is **signed**
- For $\mathbf{x} = 0$, the perpendicular distance from the plane to the origin is $\frac{b}{\|\mathbf{w}\|}$

Non-Probabilistic Discriminant Functions

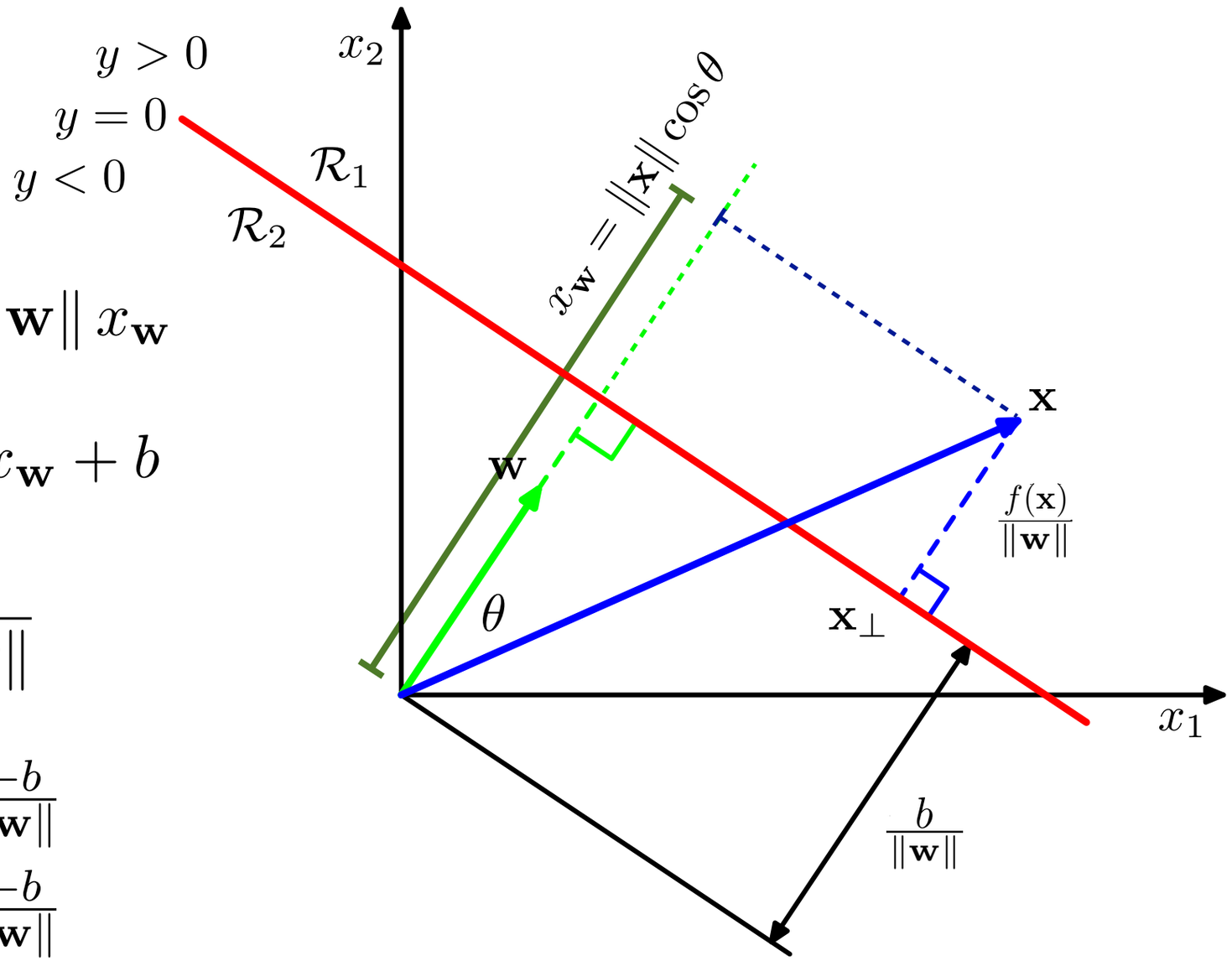
- This can also be seen from the definition of the dot product

$$\mathbf{w}^T\mathbf{x} = \|\mathbf{w}\|\,\|\mathbf{x}\|\cos\theta = \|\mathbf{w}\|\,x_{\mathbf{w}}$$

$$f(\mathbf{x}) = \mathbf{w}^T\mathbf{x} + b = \|\mathbf{w}\|\,x_{\mathbf{w}} + b$$

$$r = \frac{f(\mathbf{x})}{\|\mathbf{w}\|} = x_{\mathbf{w}} + \frac{b}{\|\mathbf{w}\|}$$

$$r \begin{cases} > 0 & \text{if } x_{\mathbf{w}} > \frac{-b}{\|\mathbf{w}\|} \\ = 0 & \text{if } x_{\mathbf{w}} = \frac{-b}{\|\mathbf{w}\|} \\ < 0 & \text{if } x_{\mathbf{w}} < \frac{-b}{\|\mathbf{w}\|} \end{cases}$$



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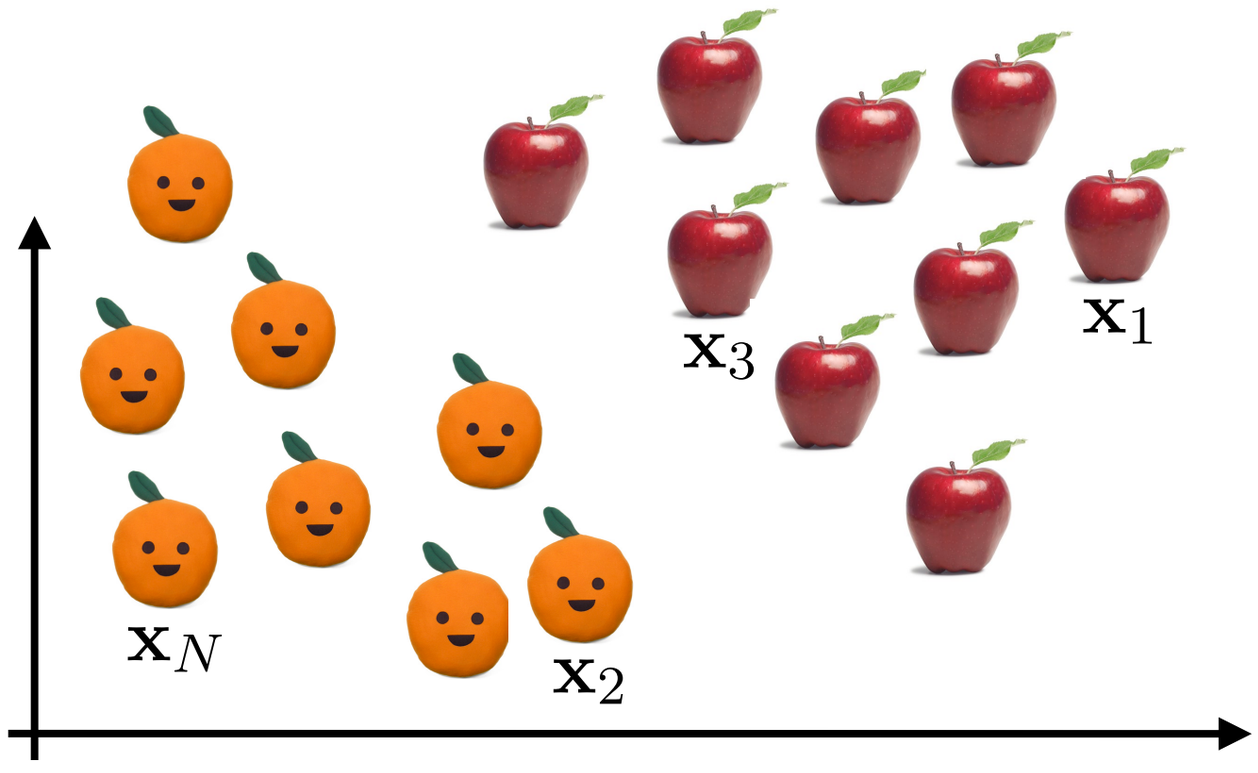
Non-Probabilistic Discriminant Functions

- Consider a linearly separable classification problem with two classes and outputs $y \in \{-1, +1\}$



$$
\begin{aligned}
y_1 &= +1 \\
y_2 &= -1 \\
y_3 &= +1 \\
&\vdots \\
y_N &= -1
\end{aligned}
$$

- How to separate the classes?

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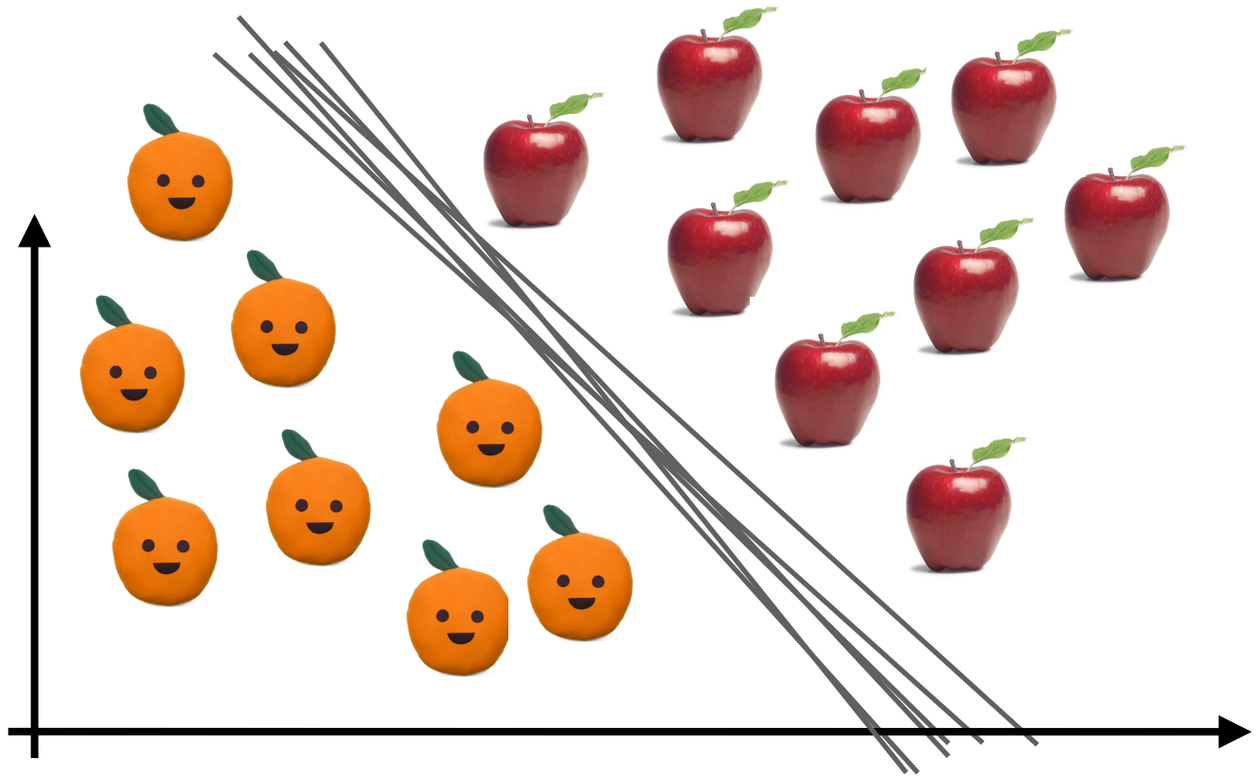
Non-Probabilistic Discriminant Functions

- Consider a linearly separable classification problem with two classes and outputs $y \in \{-1, +1\}$



- There is an **infinite number** of decision boundaries that perfectly separate the classes in the training set
- Which one to choose?

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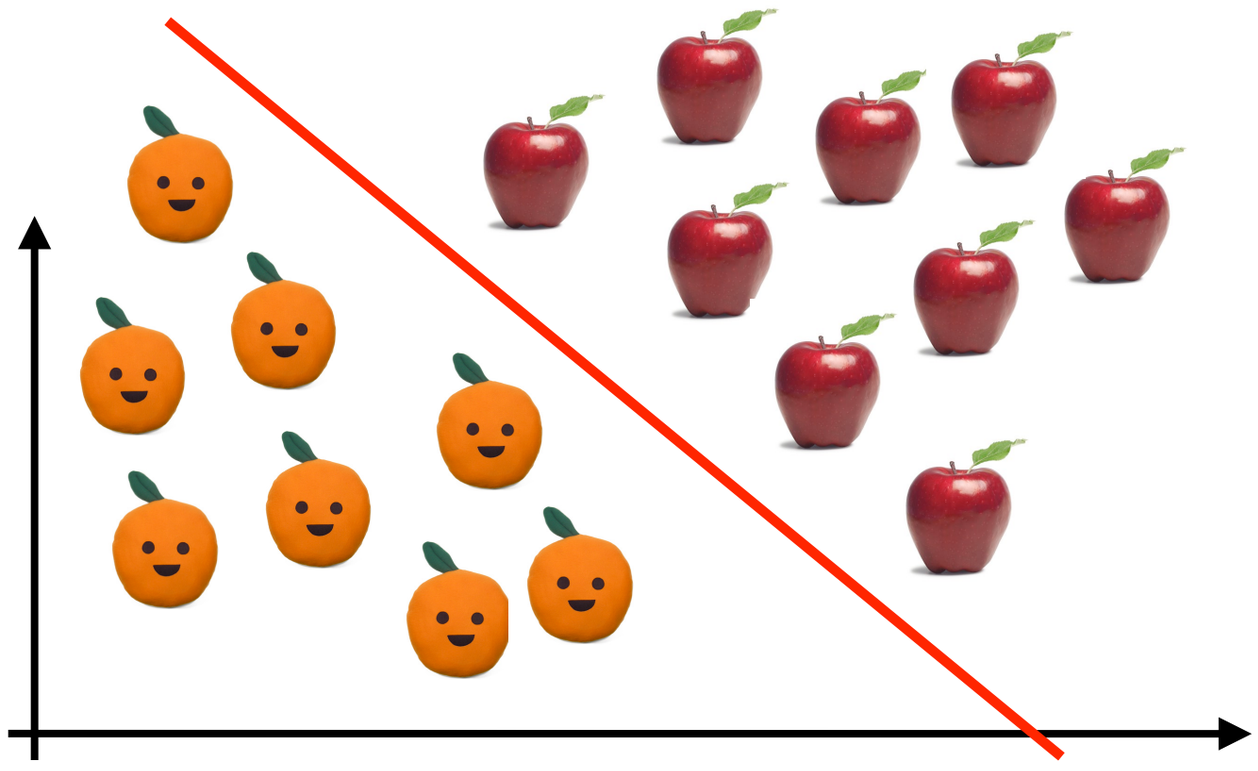
Non-Probabilistic Discriminant Functions

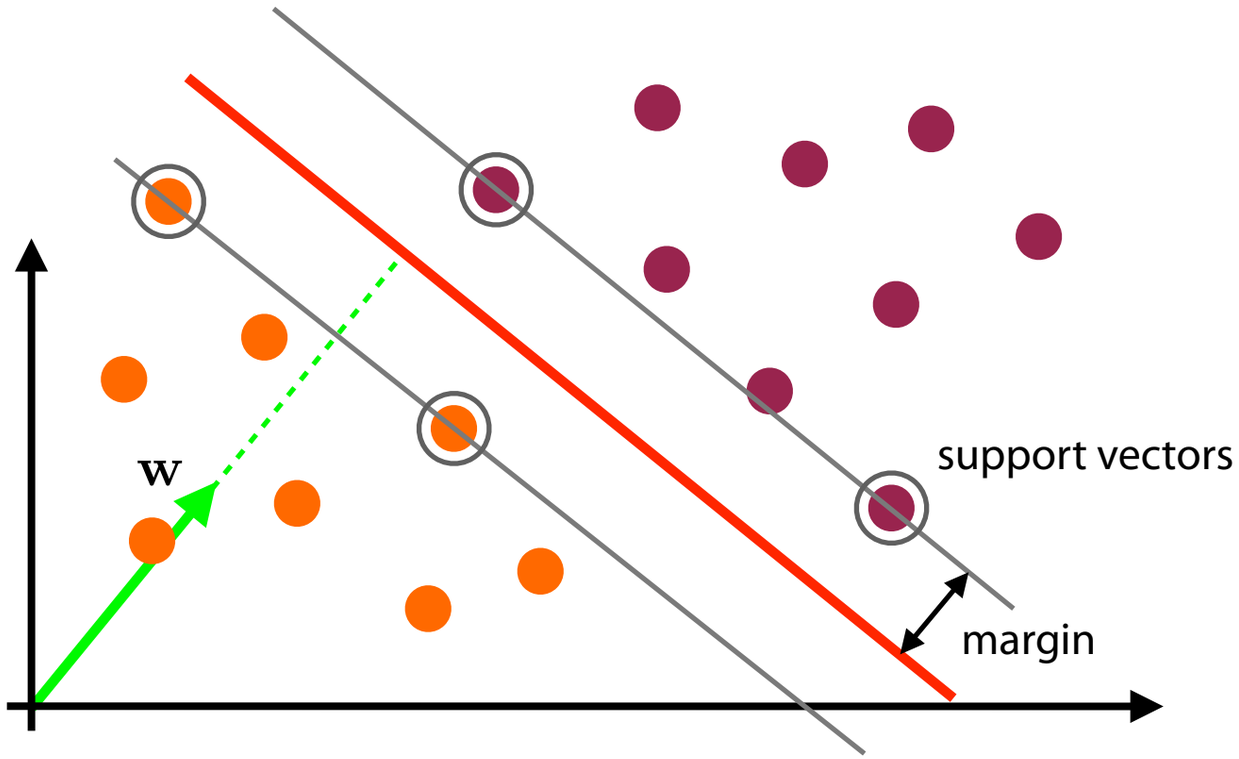
- The one with the **smallest generalization error!**



- This is what **Support Vector Machines** (SVM) do. The approach to minimize the generalization error is to **maximize the margin**

Margin and Support Vectors

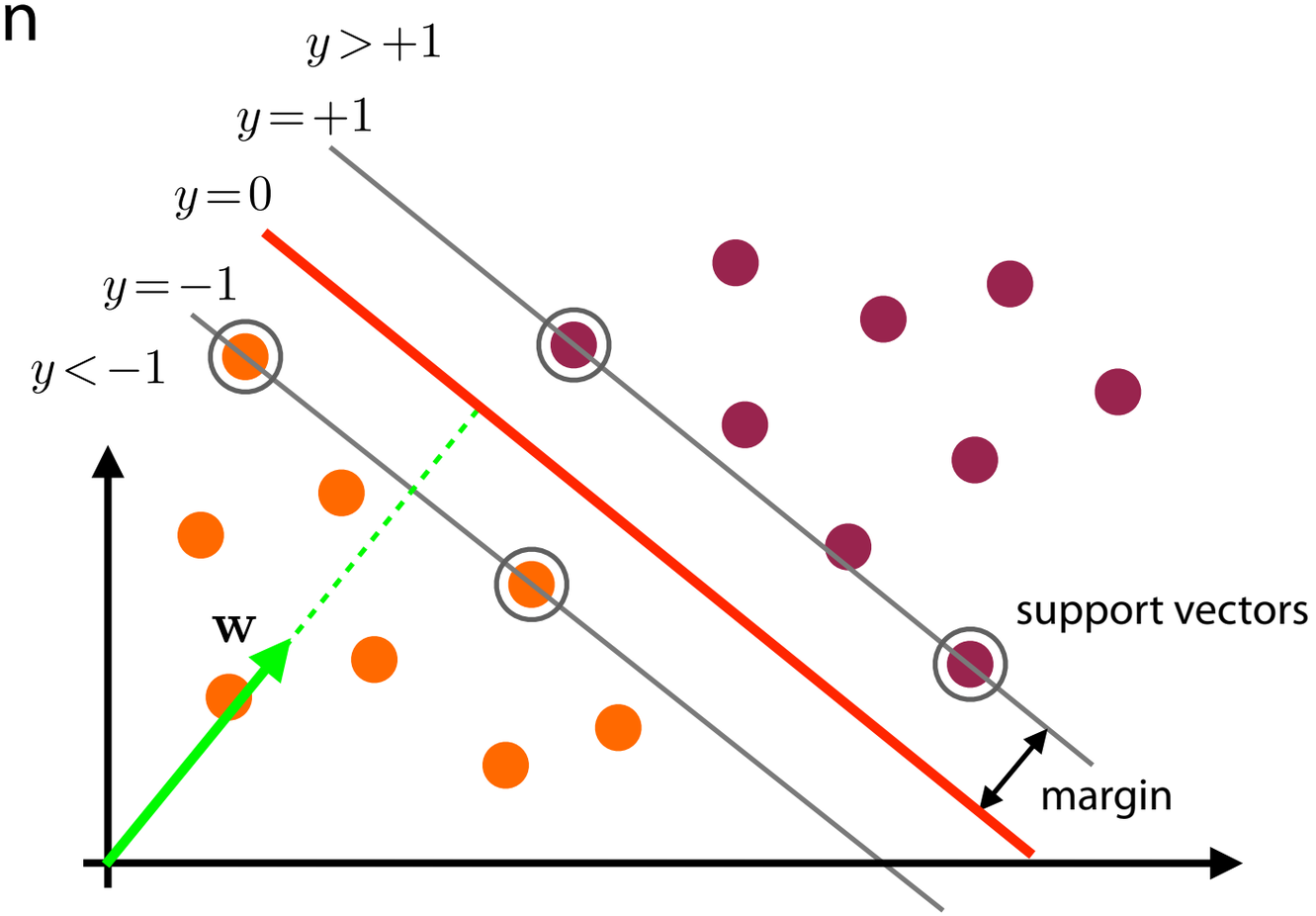
- The margin is defined as the **perpendicular distance between the decision boundary and the closest data points**



- The closest data points are called **support vectors**
- The aim of Support Vector Machines is to orientate a hyperplane in such a way as to be as far as possible from the support vectors of **both classes**

Margin and Support Vectors

- This amounts to the estimation of the normal vector $\mathbf{w}$ and the bias b
- We have seen that $\mathbf{w}$ determines the **orientation** of the hyperplane and the ratio $\frac{b}{\|\mathbf{w}\|}$ its **position from the origin**
- Thus, in addition to the direction of $\mathbf{w}$ and the value for b, there is one more degree of freedom, namely the **magnitude** of the normal vector $\|\mathbf{w}\|$
- We can thus define $\|\mathbf{w}\|$ in a way that, without loss of generality, for support vectors $|f(\mathbf{x})| = |y| = 1$ holds



Margin and Support Vectors

- We then define two planes H_1, H_2 through the support vectors. They are described by

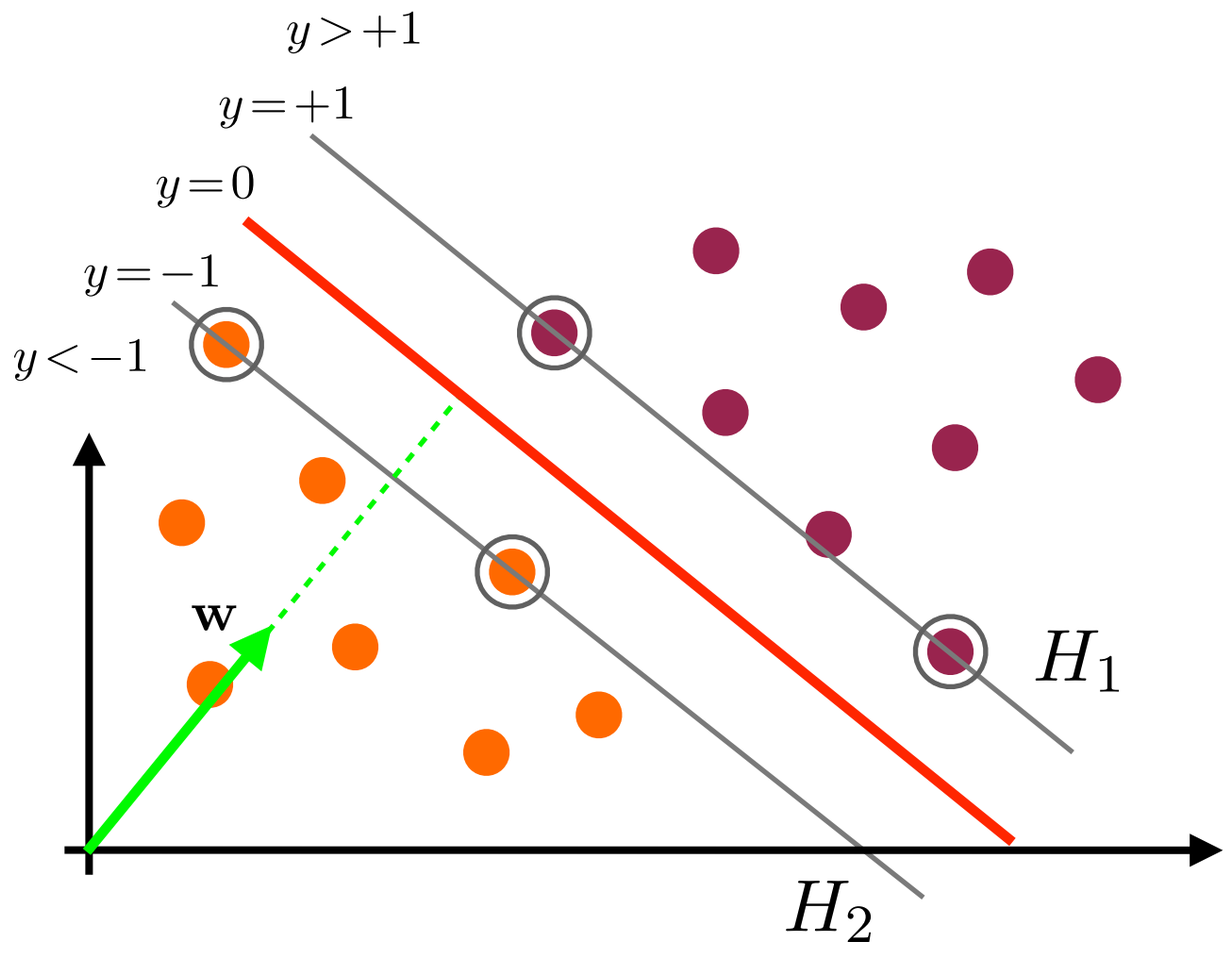
$$H_1: \quad \mathbf{w}^T\mathbf{x} + b = +1$$
$$H_2: \quad \mathbf{w}^T\mathbf{x} + b = -1$$

- Our training data $(\mathbf{x}_i, y_i)$ for all i can thus be described by

$$\mathbf{w}^T\mathbf{x}_i + b \geq +1 \quad \text{for } y_i = +1$$
$$\mathbf{w}^T\mathbf{x}_i + b \leq -1 \quad \text{for } y_i = -1$$

which can be combined to

$$y_i\left(\mathbf{w}^T\mathbf{x}_i + b\right) - 1 \geq 0$$

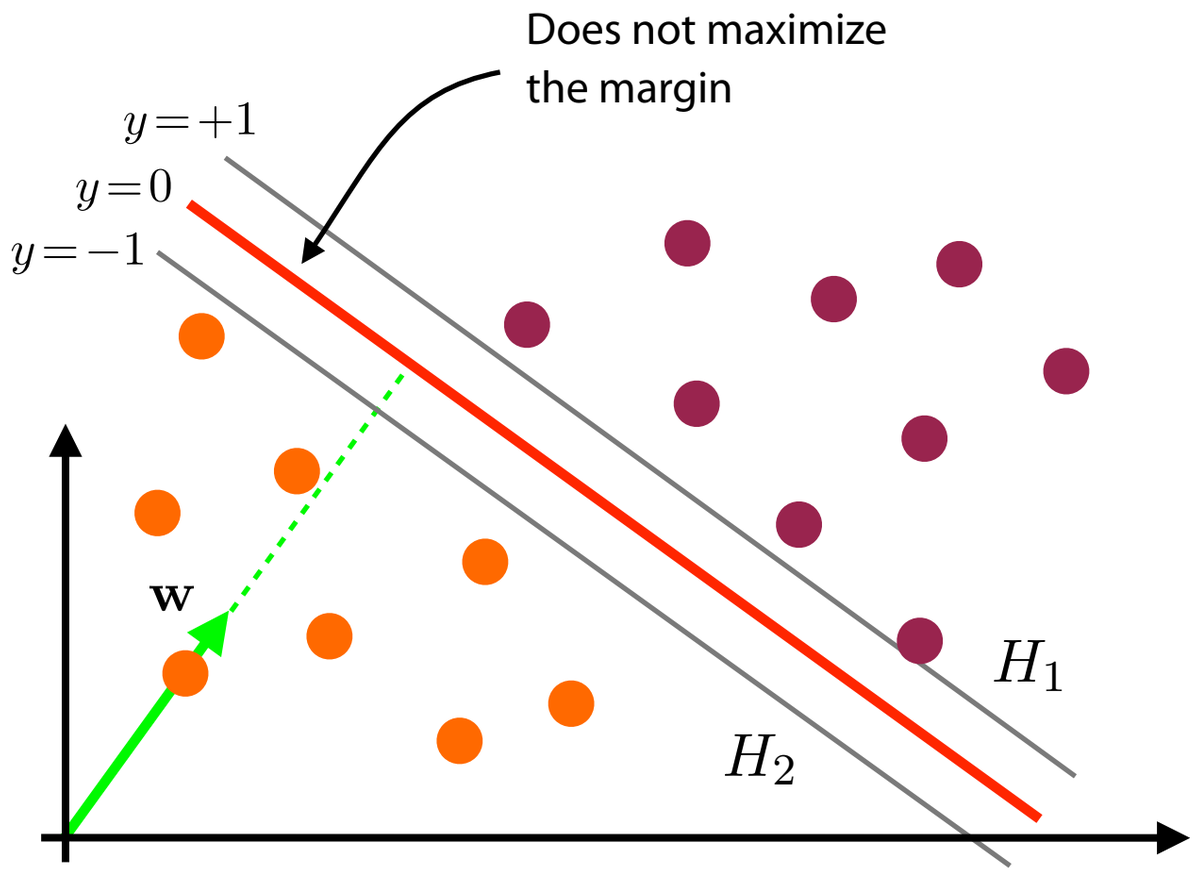


Margin and Support Vectors

- Let us look at this expression

$$y_i \left(\mathbf{w}^T \mathbf{x}_i + b \right) - 1 \geq 0$$

- It is a set of N **constraints** on $\mathbf{w}$ and b to be satisfied during the learning phase
- However, the constraints alone do not maximize the margin



- From our choice of $\|\mathbf{w}\|$ it follows that the margin is

$$r = \frac{f(\mathbf{x})}{\|\mathbf{w}\|} = \frac{1}{\|\mathbf{w}\|}$$

- Thus, **maximizing the margin** is equivalent to **minimizing** $\|\mathbf{w}\|$

Learning

- SVM **learning** consists in minimizing $\|\mathbf{w}\|$ subject to the constraints

$$y_i \left(\mathbf{w}^T \mathbf{x}_i + b\right) - 1 \geq 0$$

- Instead of minimizing $\|\mathbf{w}\|$ we can also minimize $\frac{1}{2}\|\mathbf{w}\|^2$ which leads to the formulation

$$\arg\min_{\mathbf{w},b} \frac{1}{2}\|\mathbf{w}\|^2 \qquad \text{s.t.} \qquad y_i \left(\mathbf{w}^T \mathbf{x}_i + b\right) - 1 \geq 0$$

- This is a **quadratic programming** problem in which we are trying to minimize a quadratic function subject to a set of linear inequality constraints
- In order to solve this constrained optimization problem, we will need to introduce **Lagrange multipliers**

Lagrange Multipliers

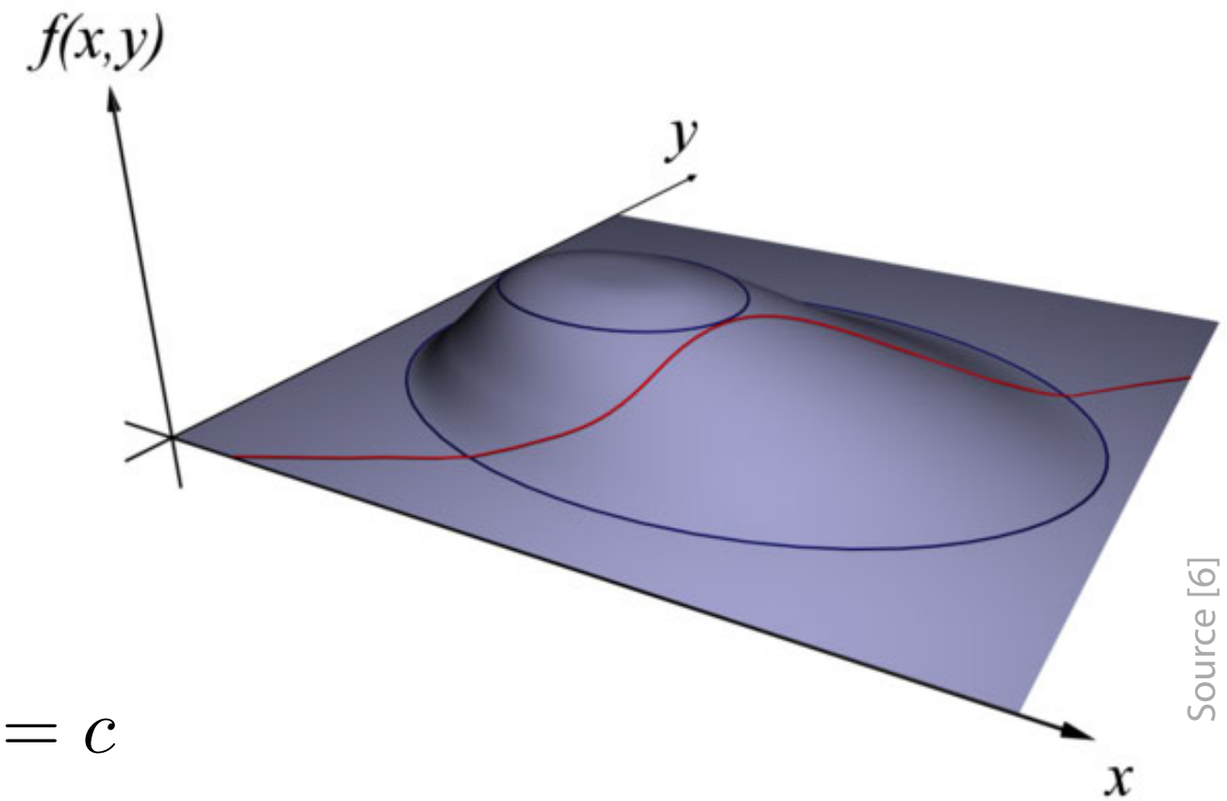
- The **method of Lagrange multipliers** is a strategy for finding the local maxima and minima of a function subject to equality constraints
- Consider, for instance, the constraint optimization problem

  maximize $f(x, y)$
  subject to $g(x, y) = c$

- Let us visualize contours of f given by

  $$f(x, y) = d$$

  for various values of f and the contour of g given by $g(x, y) = c$



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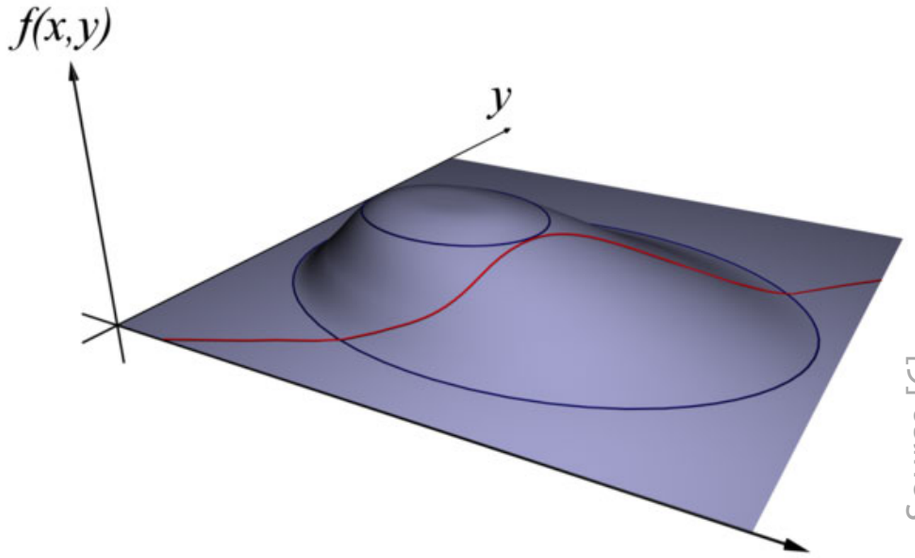
Support Vector Machines

Lagrange Multipliers

- Following the contour lines of $g = c$, we want to find the point on it with the largest value of f. Then, f will be **stationary** as we move along $g = c$
- In general, contour lines of $g = c$ will cross/intersect the contour lines of f. This is equivalent to saying that the value of f varies while moving along $g = c$
- Only when the line for $g = c$ meets contour lines of f **tangentially**, that is, the lines touch but do not cross, the value of f does not change along $g = c$



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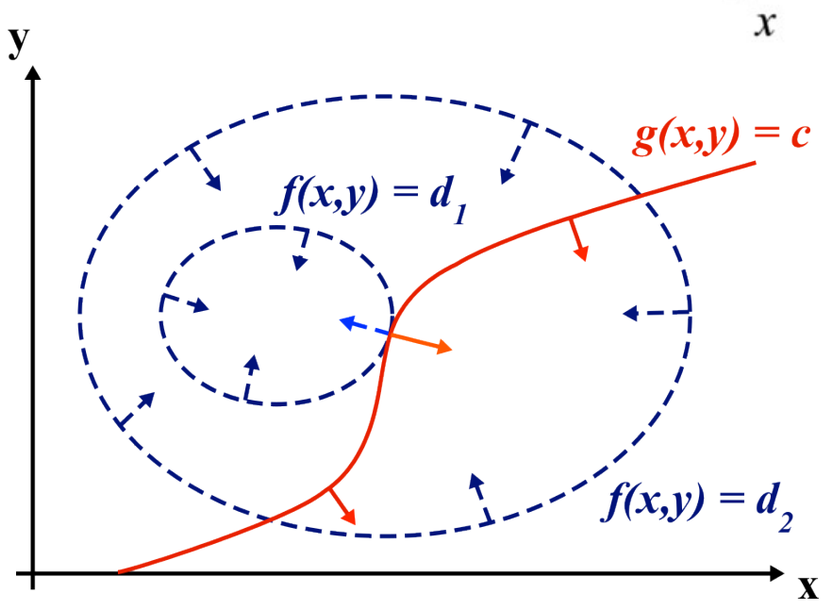

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Support Vector Machines

Lagrange Multipliers

- Contour lines touch when their tangent vectors are parallel. This is the same as saying that the **gradients** are **parallel**, because the gradient is always perpendicular to the contour
- This can be formally expressed as

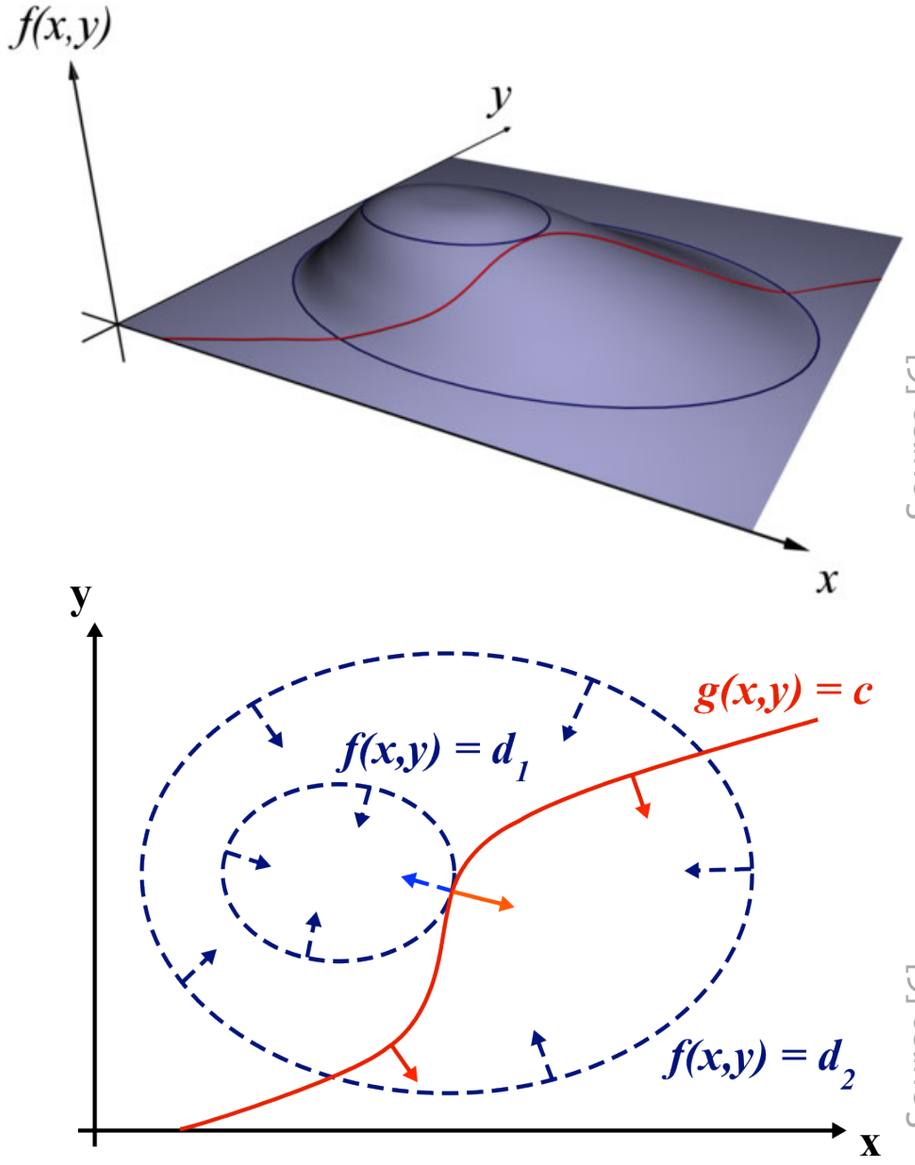
$$\nabla_{x,y} f = -\lambda \nabla_{x,y} g$$

with

$$\nabla_{x,y} f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) \qquad \nabla_{x,y} g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y} \right)$$

- In general

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = -\lambda \nabla_{\mathbf{x}} g(\mathbf{x})$$



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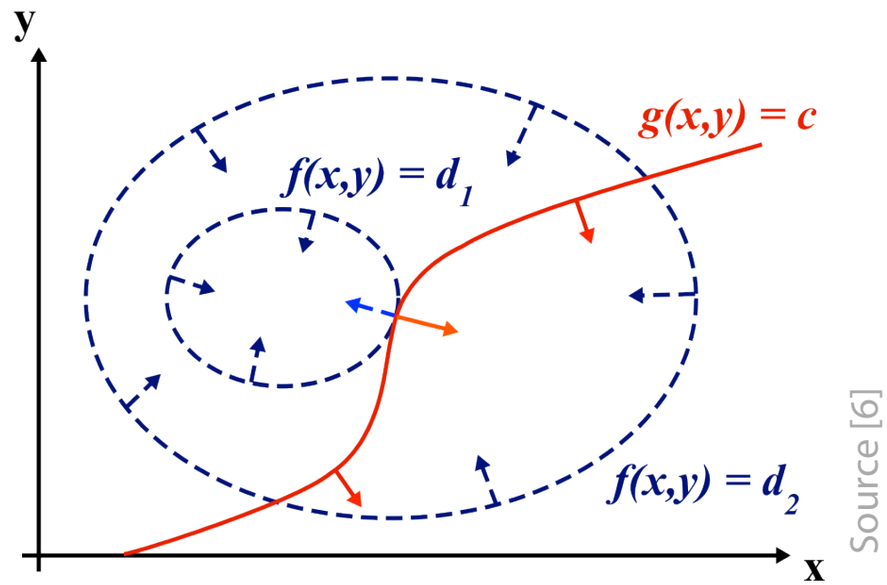
Lagrange Multipliers

- The constant λ is required because magnitudes and directions of the gradient vectors are generally not equal
- Rearranging $\nabla_{x,y} f = -\lambda \nabla_{x,y} g$ gives

$$\nabla_{x,y} f + \lambda \nabla_{x,y} g = 0$$



Source [6]

- If we were to define the function

$$L(x, y, \lambda) = f(x, y) + \lambda \cdot (g(x, y) - c)$$

we could write the above condition compactly as

$$\nabla_{x,y,\lambda} L(x, y, \lambda) = 0$$

- This is the **method of Lagrange multipliers**

Lagrange Multipliers

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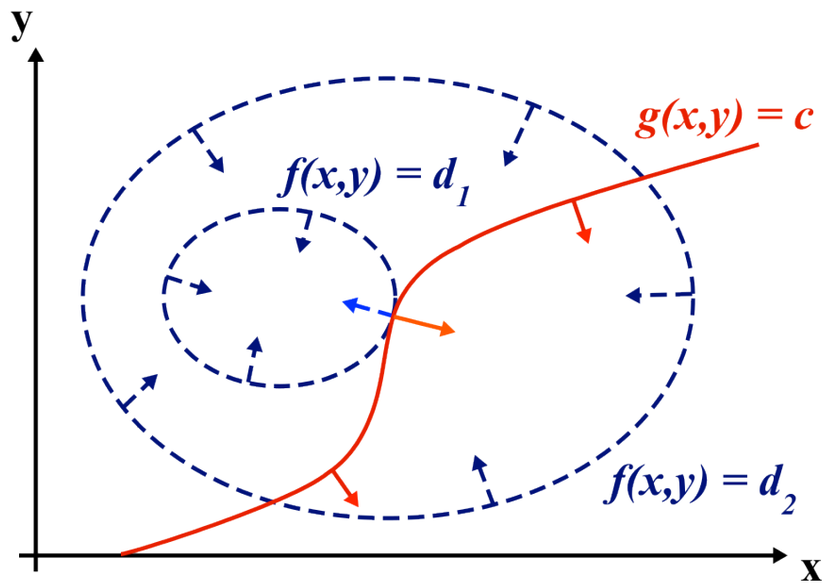

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- If we were to define the function

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Lagrange multiplier

Lagrange function or Lagrangian

we could write the above condition compactly as

$$\nabla_{x,y,\lambda} L(x, y, \lambda) = 0$$

- This is the **method of Lagrange multipliers**

Lagrange Multipliers

- The partial derivatives w.r.t. x, y recover the parallel-gradient equation, while the partial derivative w.r.t. λ recovers the constraint
- Solving the Lagrange function for its **unconstrained stationary points** generates exactly the same stationary points as solving for the stationary points of f under the constraint g
- We are looking for **stationary points** of the Lagrange function
  - Recall, **stationary points** are points of a differentiable function where the derivative is zero (i.e. where the function stops increasing or decreasing, hence the name)
- However, **not all stationary points** yield a solution of the original optimization problem
- Thus, the method of Lagrange multipliers yields only **necessary conditions** for optimality and we have to **evaluate** f at the stationary points to find our solution

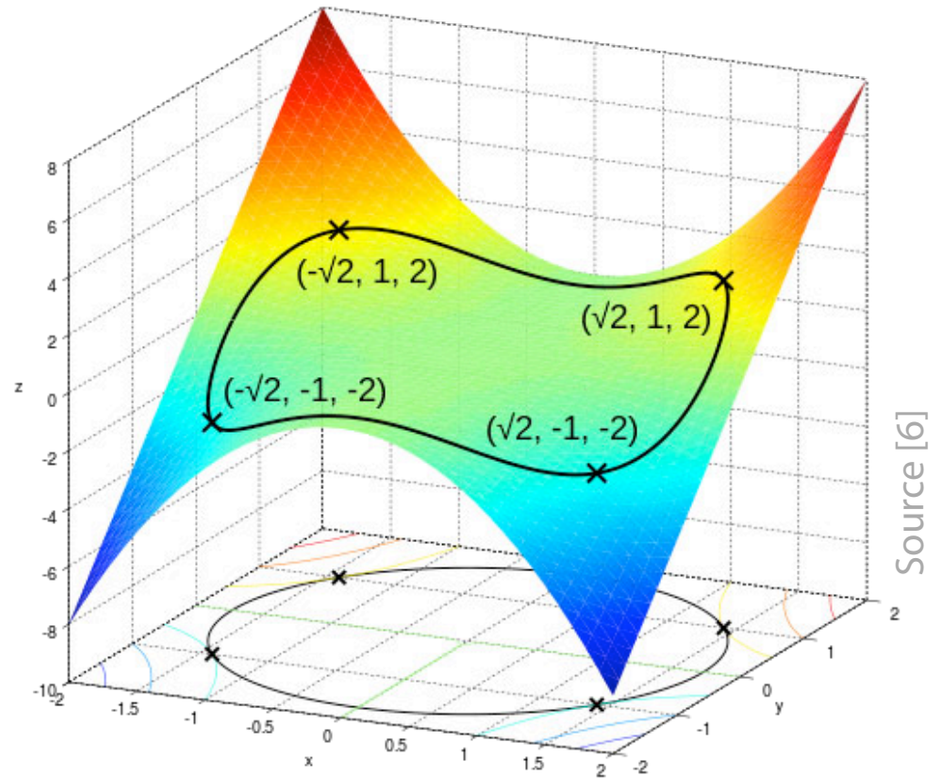
Support Vector Machines

Lagrange Multipliers

- Let us make an **example**:

  maximize $f(x, y) = x^2 y$
  subject to $g(x, y) = x^2 + y^2 = 3$

  i.e. maximize f with the constraint that the x and y coordinates lie on the circle around the origin with radius $\sqrt{3}$



Source [6]

- The Lagrangian is

$$L(x, y, \lambda) = f(x, y) + \lambda \cdot (g(x, y) - c) = x^2 y + \lambda \cdot (x^2 + y^2 - 3)$$

- Let us partially derivate L with respect to x, y and λ
- Note that, as mentioned above, $\nabla_\lambda L(x, y, \lambda) = 0$ gives the original constraint $g(x, y) = c$

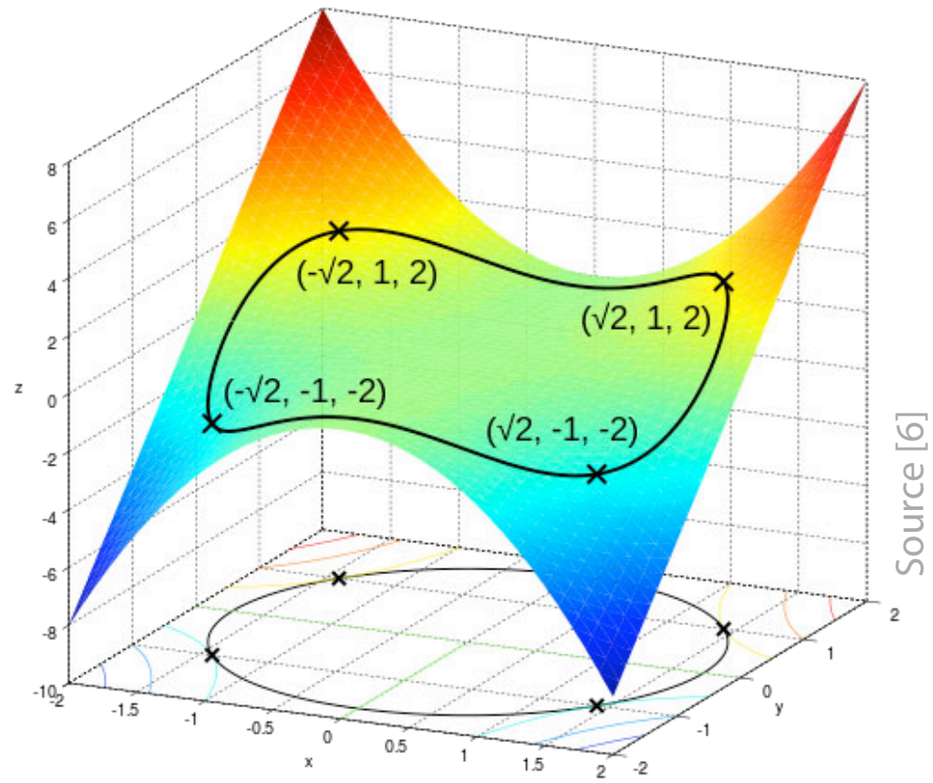
Lagrange Multipliers

- The partial derivatives are

$$\frac{\partial L}{\partial x} = 2xy + 2\lambda x = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = x^2 + 2\lambda y = 0 \quad (2)$$

$$\frac{\partial L}{\partial \lambda} = x^2 + y^2 - 3 = 0 \quad (3)$$



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- Eq. (1) implies either $x = 0$ or $\lambda = -y$. In the former case, it follows by eq. (3) that $y = \pm\sqrt{3}$ and $\lambda = 0$
- In the case $\lambda = -y$, it follows $x^2 = 2y^2$ by eq. (2). Substitution into (3) yields $y = \pm 1$ and $x = \pm\sqrt{2}$
- Thus, there are **six stationary points** of the Lagrangian

$$(\sqrt{2}, 1), \quad (-\sqrt{2}, 1), \quad (\sqrt{2}, -1), \quad (-\sqrt{2}, -1), \quad (0, \sqrt{3}), \quad (0, -\sqrt{3})$$

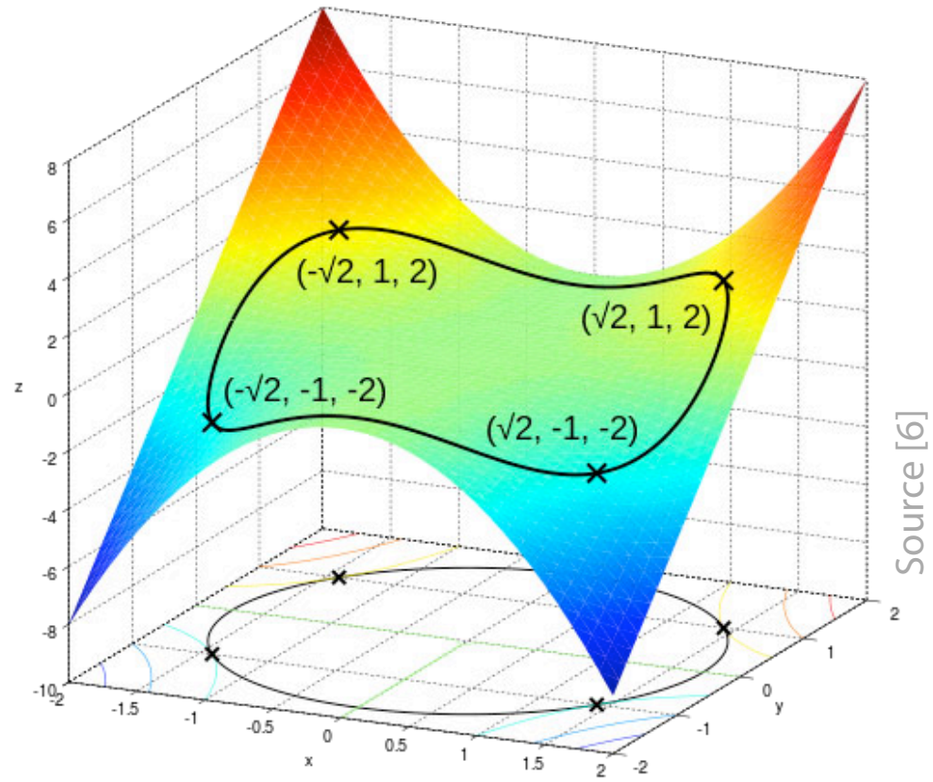
Lagrange Multipliers

- Evaluation of f at the **stationary points**

$$(\sqrt{2}, 1), \quad (-\sqrt{2}, 1), \quad (\sqrt{2}, -1),$$
$$(-\sqrt{2}, -1), \quad (0, \sqrt{3}), \quad (0, -\sqrt{3})$$

yields

$$\begin{aligned} f(\pm\sqrt{2}, 1) &= 2, \\ f(\pm\sqrt{2}, -1) &= -2, \\ f(0, \pm\sqrt{3}) &= 0 \end{aligned}$$



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- Therefore, the objective function attains the **global maximum**, **subject to the constraint**, at $(\pm\sqrt{2}, 1)$

Lagrange Multipliers

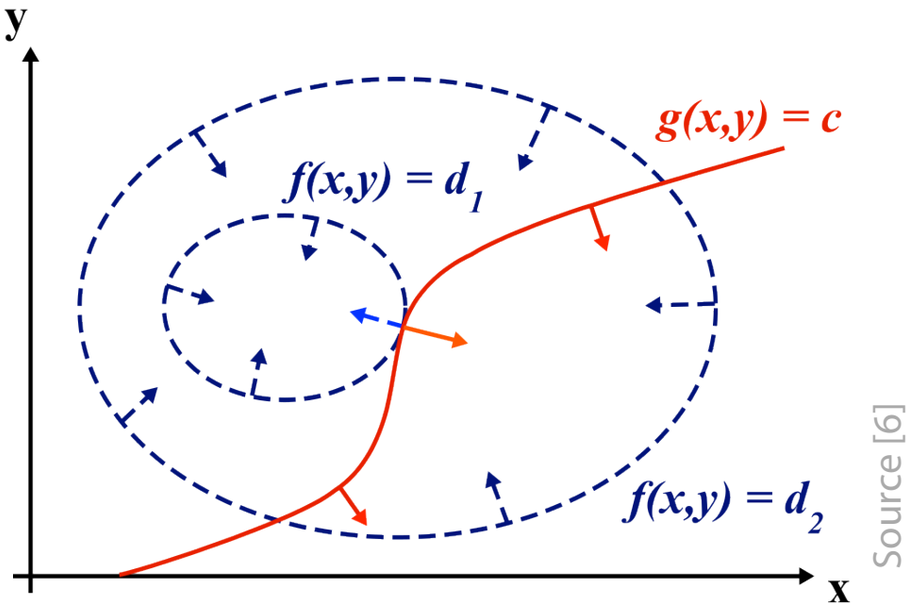
- In the case of **multiple constraints**, the same reasoning applies
- Let us recap: in the presence of a constraint, $\nabla_{\mathbf{x}} f(\mathbf{x})$ does not have to be zero at $\mathbf{x}$, but it has to be **entirely contained** in the (1-dimensional) subspace spanned by $\nabla_{\mathbf{x}} g(\mathbf{x})$

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = -\lambda \nabla_{\mathbf{x}} g(\mathbf{x})$$

- This generalizes to multiple constraints: for N constraints $g_i(\mathbf{x}) = 0$ we have

$$\nabla_{\mathbf{x}} f(\mathbf{x}) = -\sum_{i=1}^{N} \lambda_i \nabla_{\mathbf{x}} g_i(\mathbf{x})$$

- The subspace is now a **linear combination** of the gradients $\nabla_{\mathbf{x}} g_i(\mathbf{x})$ with weights λ_i



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Lagrange Multipliers

- Thus, the Lagrangian for **multiple constraints** is

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{N} \lambda_i \, g_i(\mathbf{x})$$

where $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$

- Again, we partially derivate the Lagrangian

$$\nabla_{\mathbf{x},\boldsymbol{\lambda}} \, L(\mathbf{x}, \boldsymbol{\lambda}) = 0$$

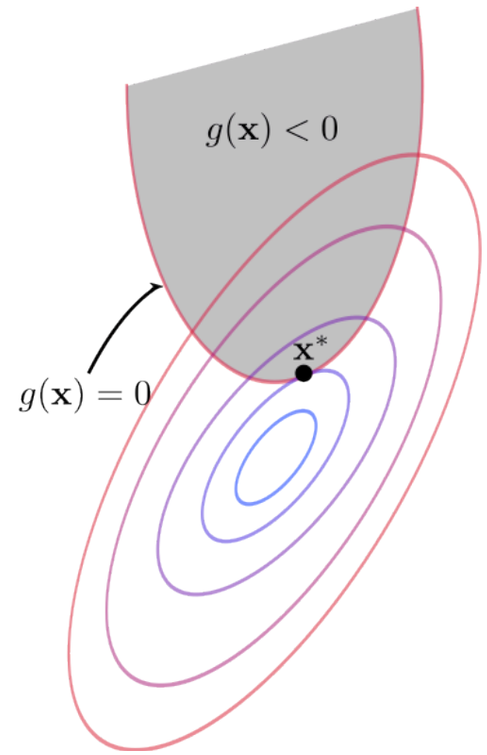
and solve for its **stationary points**, evaluate f at those points

- Again, the partial derivatives w.r.t. $\mathbf{x}$ recover the parallel-gradient equation, while the partial derivative w.r.t. λ recovers the constraint

Karush–Kuhn–Tucker Conditions

- Now, assume we also have **inequality constraints**
- The constraint optimization problem is then

maximize $f(\mathbf{x})$
subject to $g_i(\mathbf{x}) = 0$ for $i \in \{1,...,N\}$
and to $h_i(\mathbf{x}) \leq 0$ for $i \in \{1,...,M\}$



Source [6]

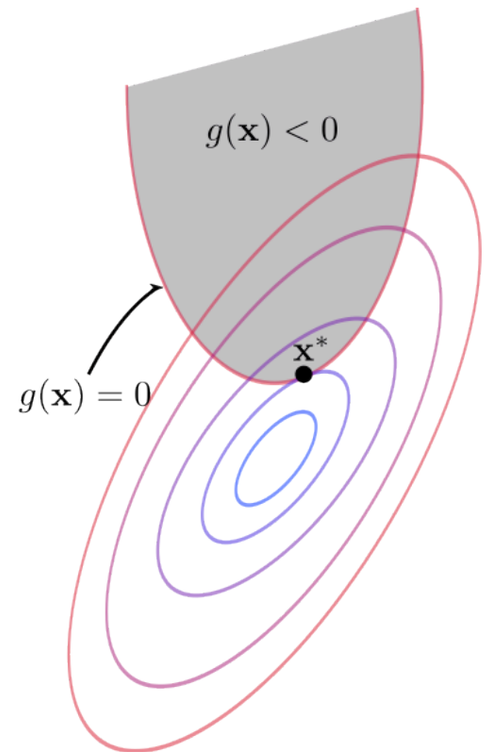
- The problem can be solved via the **general Lagrangian**

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{N} \lambda_i \, g_i(\mathbf{x}) + \sum_{i=1}^{M} \mu_i \, h_i(\mathbf{x})$$

- The **stationary points** of the general Lagrangian are again the same than the **constraint stationary points** of f

Karush–Kuhn–Tucker Conditions

- However, inequality constraints are different than equality constraints and our previously made considerations are not sufficient anymore
- We require a set of **additional conditions** (or constraints) to guarantee optimality of solutions
- The combined set of constraints is called **Karush–Kuhn–Tucker (KKT) conditions**
- Allowing inequality constraints, the KKT approach **generalizes** the method of Lagrange multipliers, which allows only equality constraints



Source [6]

- We will not go deeper at this point, but will return to SVM learning

Learning

- In the case of SVM learning, we have

$$\arg\min_{\mathbf{w},b} \frac{1}{2}\|\mathbf{w}\|^2 \qquad \text{s.t.} \qquad y_i\left(\mathbf{w}^T\mathbf{x}_i + b\right) - 1 \geq 0$$

  with a set of N inequality constraints

- Thus, the **Karush–Kuhn–Tucker (KKT) conditions** apply
- We allocate Lagrange multipliers $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \ldots, \lambda_N\}$

$$L(\mathbf{w}, b, \boldsymbol{\lambda}) \quad = \quad \frac{1}{2}\|\mathbf{w}\|^2 - \sum_{i=1}^{N} \lambda_i \left(y_i\left(\mathbf{w}^T\mathbf{x}_i + b\right) - 1\right)$$

  where now $\lambda_i \geq 0 \ \ \forall i$ (which is one of the KKT conditions)

- The minus sign comes from the KKT problem statement $h_i(\mathbf{x}) \leq 0$
- L is minimized if we minimize it w.r.t. $\mathbf{w}$, b and maximize it w.r.t. $\boldsymbol{\lambda}$

Learning

- Note that the Lagrangian is a function of $\mathbf{w}, b$ (and this is the general "$\mathbf{x}$" from the Lagrange subsection)
- Derivation of L with respect to $\mathbf{w}, b$ gives

$$\frac{\partial L}{\partial \mathbf{w}} = 0 \quad \Leftrightarrow \quad \boxed{\mathbf{w} = \sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i}$$

$$\frac{\partial L}{\partial b} = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{N} \lambda_i \, y_i = 0$$

- Instead of solving for the stationary points of L directly, let us substitute these expressions back into the Lagrange function (eliminating $\mathbf{w}, b$)

$$\begin{aligned} L(\mathbf{w}, b, \boldsymbol{\lambda}) &= \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^{N} \lambda_i \left(y_i \left(\mathbf{w}^T \mathbf{x}_i + b \right) - 1 \right) \\ &= \frac{1}{2} \|\mathbf{w}\|^2 - \sum_{i=1}^{N} \lambda_i \, y_i \left(\mathbf{w}^T \mathbf{x}_i + b \right) + \sum_{i=1}^{N} \lambda_i \end{aligned}$$

Learning

- Working the new expression for the normal...

$$\mathbf{w}^T = \sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i^T$$

$$\|\mathbf{w}\|^2 = \mathbf{w}^T\mathbf{w} = \left(\sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i^T\right) \left(\sum_{j=1}^{N} \lambda_j \, y_j \, \mathbf{x}_j\right) = \sum_{i,j}^{N} \lambda_i\lambda_j \, y_i \, y_j \, \mathbf{x}_i^T\mathbf{x}_j$$

- Substitution into

$$L(\mathbf{w}, b, \boldsymbol{\lambda}) = \frac{1}{2}\|\mathbf{w}\|^2 - \sum_{i=1}^{N} \lambda_i \, y_i \, (\mathbf{w}^T\mathbf{x}_i + b) + \sum_{i=1}^{N} \lambda_i$$

yields

$$L(\mathbf{w}, b, \boldsymbol{\lambda}) = -\frac{1}{2}\sum_{i,j=1}^{N} \lambda_i\lambda_j \, y_i \, y_j \, \mathbf{x}_i^T\mathbf{x}_j + \sum_{i=1}^{N} \lambda_i - b\sum_{i=1}^{N} \lambda_i \, y_i$$

Learning

- This gives the **dual form** $L(\boldsymbol{\lambda})$ of the **primary** $L(\mathbf{w}, b, \boldsymbol{\lambda})$

$$L(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \lambda_i \lambda_j \, y_i \, y_j \, \mathbf{x}_i^T \mathbf{x}_j$$

- We came here by minimizing the original Lagrangian w.r.t. $\mathbf{w}$, b. What remains to do is to maximize it w.r.t. $\boldsymbol{\lambda}$. This leads to the following **dual optimization problem**

  maximize $L(\boldsymbol{\lambda})$
  subject to $\lambda_i \geq 0 \;\; \forall i$ and $\sum_i \lambda_i \, y_i = 0$

- We can solve the **dual** optimization problem in lieu of the **primal** problem
- Note that the dual form requires only the **inner product of each input vector** to be calculated. This will be important for the **kernel trick**

Learning

- The dual optimization problem takes the form of a **quadratic programming problem** in which we optimize a quadratic function of the λ_i's subject to a set of inequality constraints
- There are many **QP solvers** for this purpose (such as Matlab's `quadprog`)
- We then obtain the Lagrange multipliers $\boldsymbol{\lambda}$ and can compute

$$\mathbf{w} = \sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i$$

- Substitution into the discriminative function model yields the **dual version** of the classifier

$$f(\mathbf{x}) = \mathbf{w}^T \mathbf{x} + b$$

Primal version

$$f(\mathbf{x}) = \sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i^T \mathbf{x} + b$$

Dual version

Learning

- For the computation of the normal or the dual version of the classifier, we do not need to sum over all N training pairs. It follows from the KKT conditions that **only support vectors have non-zero** λ_i's
- This is how we can find the support vectors among the training samples
- This is noteworthy and the reason why SVM are also called **sparse** kernel machines. The learned classifier only depends **sparsely** on the training set
- What remains to do is to **calculate the bias** b
- Remember our N inequality constraints

$$y_i \left(\mathbf{w}^T \mathbf{x}_i + b\right) - 1 \geq 0$$

- We have defined the normal in a way that for support vectors

$$y_s \left(\mathbf{w}^T \mathbf{x}_s + b\right) = 1 \quad s \in \mathcal{S}$$

Set of indices of the support vectors

Learning

- Substituting the dual version of the classifier leads to

$$y_{s_i} \left(\sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \mathbf{x}_{s_j}^T \mathbf{x}_{s_i} + b \right) = 1$$

- Multiplication with the label on both sides gives

$$y_{s_i}^2 \left(\sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \mathbf{x}_{s_j}^T \mathbf{x}_{s_i} + b \right) = y_{s_i}$$

- Using $y_{s_i}^2 = 1$ and solving for b

$$b = y_{s_i} - \sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \mathbf{x}_{s_j}^T \mathbf{x}_{s_i} \quad \rightarrow \quad b = \frac{1}{N_{\mathcal{S}}} \sum_{s_i \in \mathcal{S}} \left(y_{s_i} - \sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \mathbf{x}_{s_j}^T \mathbf{x}_{s_i} \right)$$

- Although we can solve this equation for b using an arbitrary support vector s_i, it is numerically more stable to take an **average** over all support vectors

Inference and Decision

- We now have the variables $\mathbf{w}$ and b that define our separating hyperplane's optimal orientation and hence our Support Vector Machine
- For **classification**, each new input $\mathbf{x}'$ is predicted by

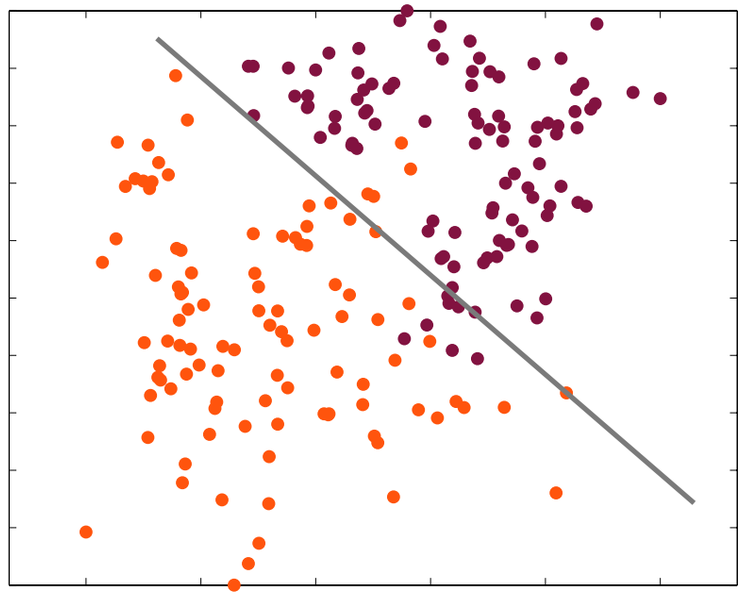
$$y' = sign(\mathbf{w}^T \mathbf{x}' + b)$$

- Note the resemblance of the **dual version** to the k-NN classifier

$$y' = sign(\sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i^T \mathbf{x}' + b)$$

Soft-Margin SVM

- So far, we have assumed that the training data points are linearly separable in feature space. But often, the class-conditional distributions **overlap** in which case **exact separation** is **not** possible



- We now modify the approach so that data points are allowed to be **"on the wrong side"** of the decision boundary
- We introduce a **penalty** that increases with the distance from that boundary. The penalty is a linear function of this distance
- To this end, we introduce a slack variable $\xi_i \geq 0 \;\; \forall i \in \{1, \ldots, N\}$ for each training sample (ξ or "xi" is pronounced zī like "high")
- They are defined to be **zero** for data points **on or inside** the "right side" of the boundary, and $\xi_i = |y_i - f(\mathbf{x}_i)|$ for other points

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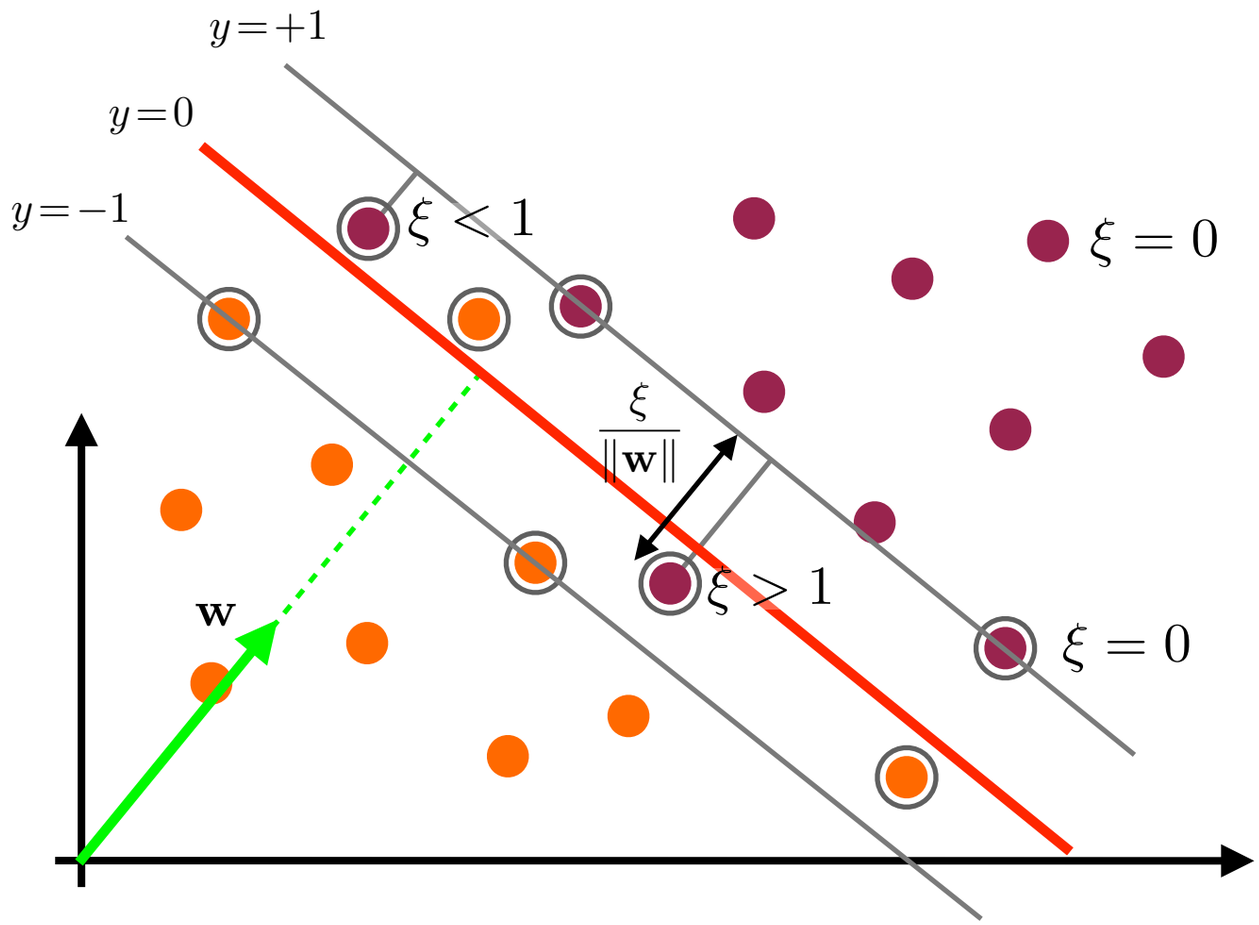
Soft-Margin SVM

- Let us visualize ξ_i
- The relationship $\xi_i = |y_i - f(\mathbf{x}_i)|$ implies that points on the boundary have $\xi_i = 1$
- Misclassified points receive $\xi_i > 1$
- The set of N constraints that describe our training data $(\mathbf{x}_i, y_i)$ is now



$$\mathbf{w}^T\mathbf{x}_i + b \geq +1 - \xi_i \quad \text{for } y_i = +1$$
$$\mathbf{w}^T\mathbf{x}_i + b \leq -1 + \xi_i \quad \text{for } y_i = -1$$

- Points with $\xi_i > 0$ that violate the margin are called **non-margin support vectors**. They are also considered support vectors

Soft-Margin SVM

- They can be combined into

$$y_i \left(\mathbf{w}^T \mathbf{x}_i + b\right) - 1 + \xi_i \geq 0 \qquad \xi_i \geq 0 \quad \forall i$$

- Notice the set of new constraints on the slack variables
- While before, in the non-overlapping case, the optimization objective was

$$\arg\min_{\mathbf{w},b} \frac{1}{2} \|\mathbf{w}\|^2 \qquad \text{s.t.} \qquad y_i \left(\mathbf{w}^T \mathbf{x}_i + b\right) - 1 \geq 0$$

our goal is now to also **reduce the number of misclassified data points**

- This is done – in addition to the maximization of the margin – by **softly penalizing** data points on the **wrong side** of the decision boundary

$$\arg\min_{\mathbf{w},b} \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^{N} \xi_i \qquad \text{s.t.} \qquad \begin{aligned} y_i \left(\mathbf{w}^T \mathbf{x}_i + b\right) - 1 + \xi_i \geq 0 \quad \forall i \\ \xi_i \geq 0 \quad \forall i \end{aligned}$$

Soft-Margin SVM

- Parameter $C > 0$ is called **stiffness parameter** and controls the trade-off between slack variable penalty and the size of the margin
- The method tries to splits the training data as cleanly as possible, while still **maximizing the distance to the nearest cleanly split samples**
- The corresponding Lagrangian is

$$L(\mathbf{w}, b, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \frac{1}{2} \|\mathbf{w}\|^2 + C \sum_{i=1}^{N} \xi_i - \sum_{i=1}^{N} \lambda_i \left(y_i \left(\mathbf{w}^T \mathbf{x}_i + b \right) - 1 + \xi_i \right) - \sum_{i=1}^{N} \mu_i \, \xi_i$$

  where $\lambda_i \geq 0, \ \mu_i \geq 0 \ \forall i$ (KKT conditions) are the Lagrange multipliers

- The corresponding **extended set** of KKT conditions collects all constraints
- We need to minimize L w.r.t. $\mathbf{w}$, b and ξ_i and maximize it w.r.t. $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$
- We proceed as before...

Soft-Margin SVM

- Differentiating w.r.t. $\mathbf{w}$, b and ξ_i and setting the derivatives to zero

$$\frac{\partial L}{\partial \mathbf{w}} = 0 \quad \Leftrightarrow \quad \mathbf{w} = \sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i$$

$$\frac{\partial L}{\partial b} = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{N} \lambda_i \, y_i = 0$$

$$\frac{\partial L}{\partial \xi_i} = 0 \quad \Leftrightarrow \quad C = \lambda_i + \mu_i$$

- Substitution into the Lagrangian eliminates $\mathbf{w}$, b and ξ_i from L and we obtain the **dual form** – which is **identical** to the **non-overlapping** case

$$L(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \lambda_i \lambda_j \, y_i \, y_j \, \mathbf{x}_i^T \mathbf{x}_j$$

Soft-Margin SVM

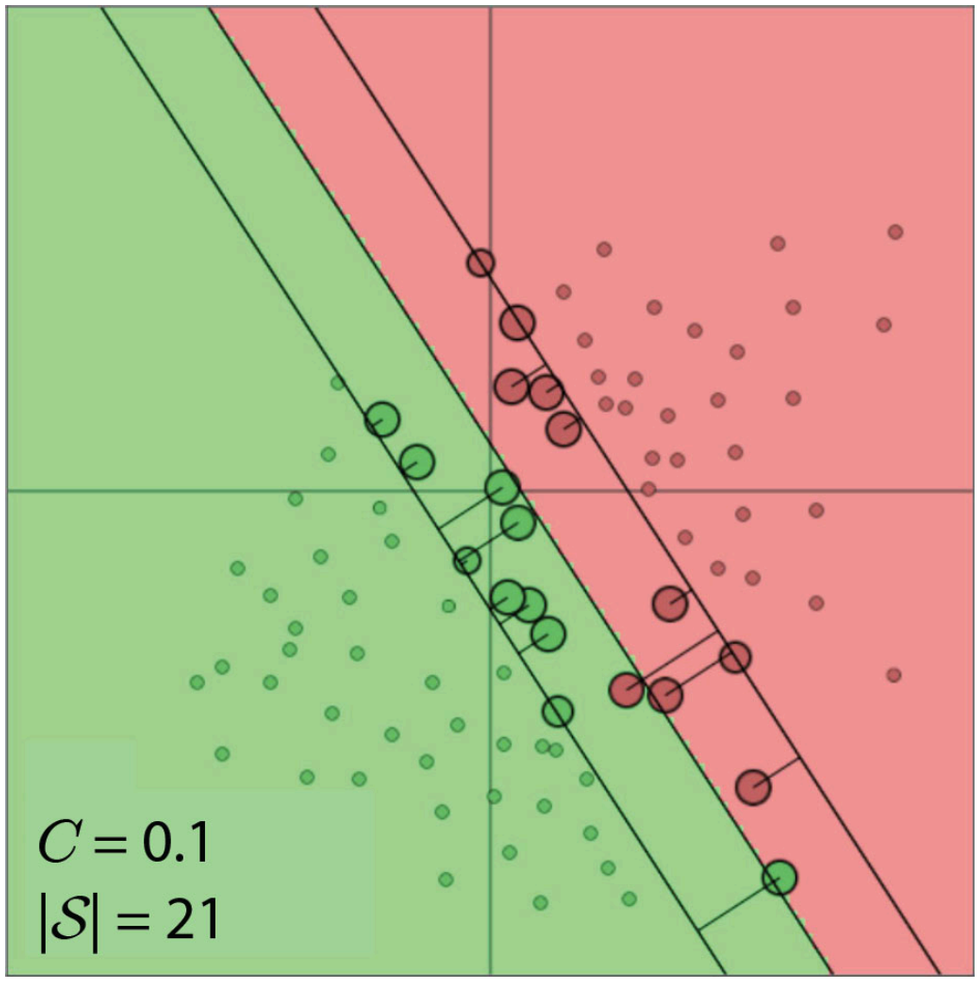
- However, the constraints are **different**. From $C = \lambda_i + \mu_i$ and $\mu_i \geq 0 \ \forall i$ follows $\lambda_i \leq C$
- The **dual optimization problem** is then

  maximize $L(\boldsymbol{\lambda})$
  subject to $0 \leq \lambda_i \leq C \ \forall i$ and $\sum_i \lambda_i \, y_i = 0$

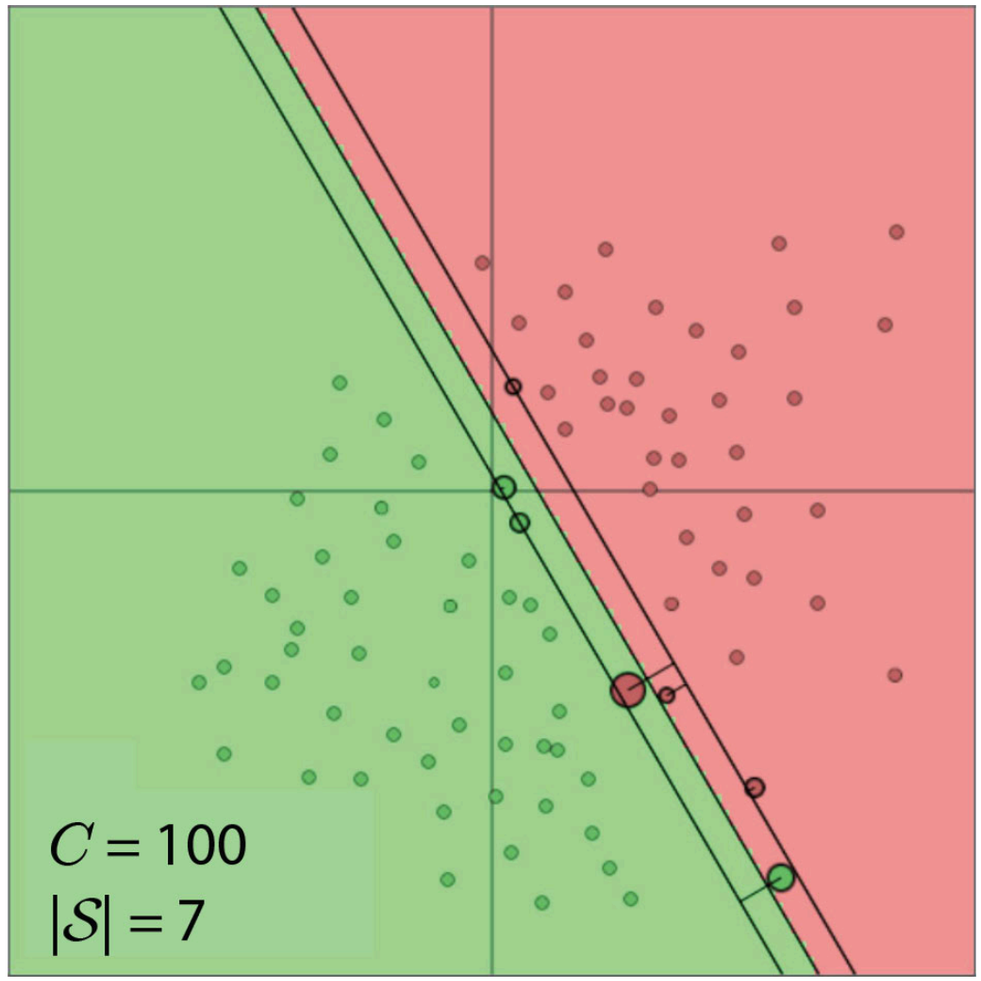
- Again, we can use standard **QP solvers** for this optimization task
- **Support vectors** are now found via the **condition** $0 < \lambda_i \leq C$
- What remains to do is to **calculate the bias** b. This is done in the same way as before using an average over all support vectors
- Class prediction (**inference** and **decision**) is then made by

$$y' = sign(\mathbf{w}^T \mathbf{x}' + b)$$

Soft-Margin SVM



Source [5]



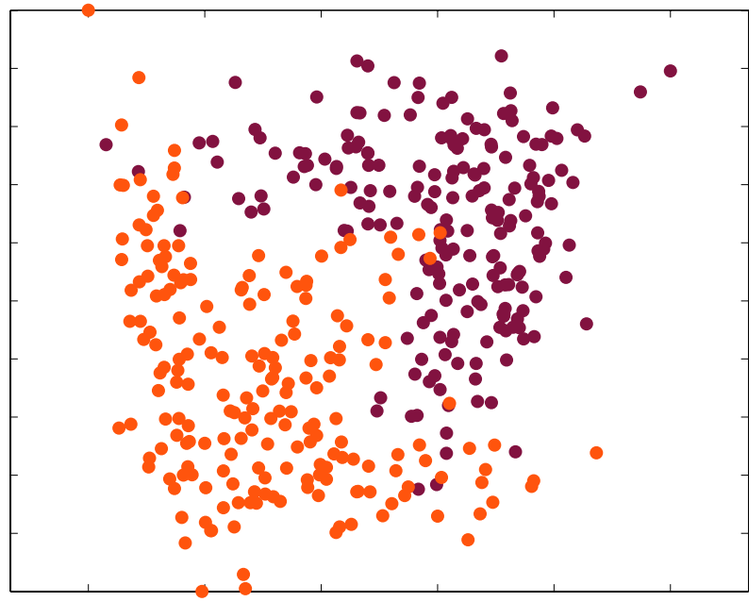
Source [5]

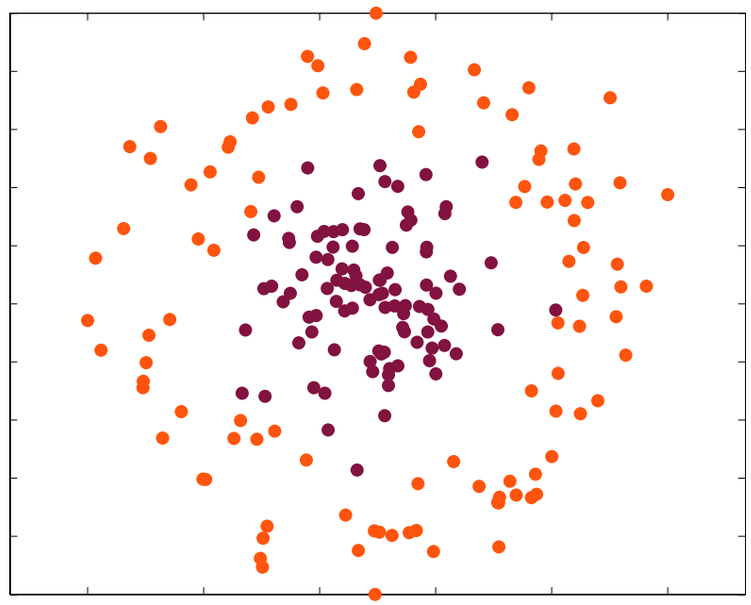
- **Increasing** C places more weight on the slack variables ξ_i leading to a stricter separation of the classes and a smaller margin. **Reducing** C leads to a larger margin and more misclassified points

Non-Linear SVM

- So far, we looked at classification problems with linearly separable class distributions (up to some extent of overlapping)
- When data are not linearly separable, we have a **non-linear classification problem**
- How can we solve such problems using Support Vector Machines
- **Idea:** make the data linearly separable by mapping them into a higher dimensional space

$$\mathbf{x} \rightarrow \phi(\mathbf{x}) \qquad \mathbb{R}^m \rightarrow \mathbb{R}^d$$





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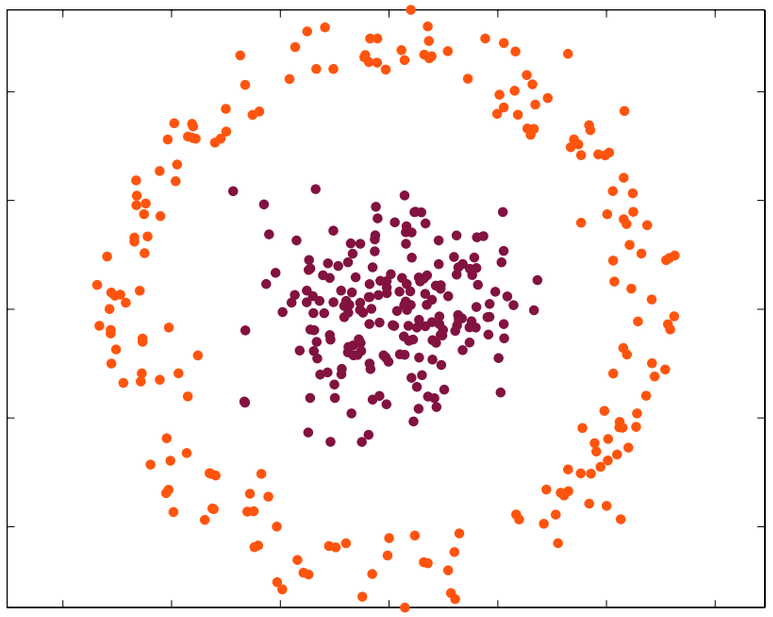
Non-Linear SVM

- Consider the following mapping

$$\phi: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}\, x_1 x_2 \end{pmatrix} \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



Non-Linear SVM

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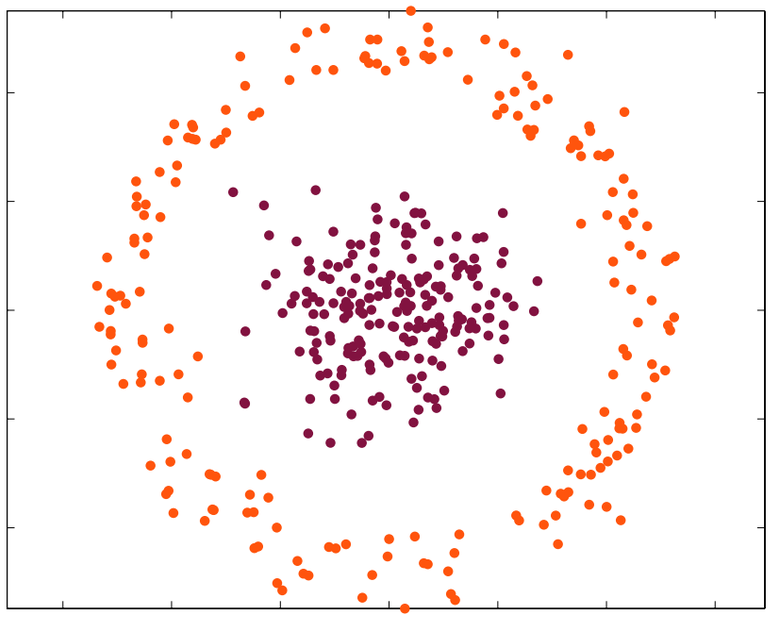
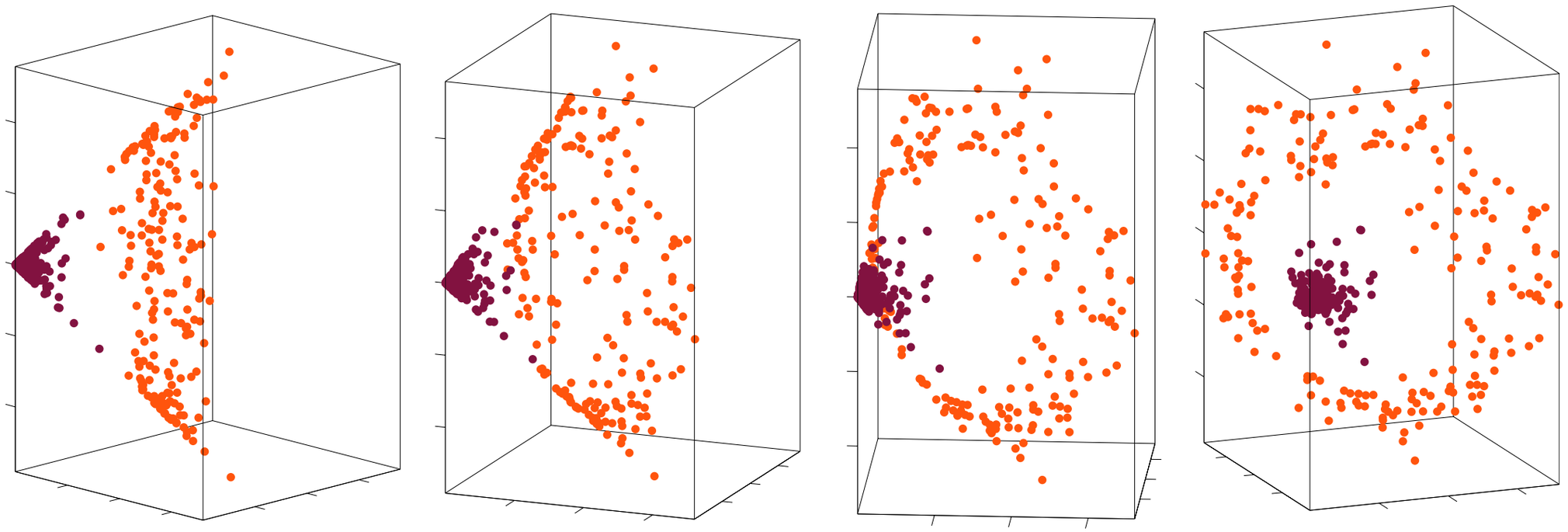
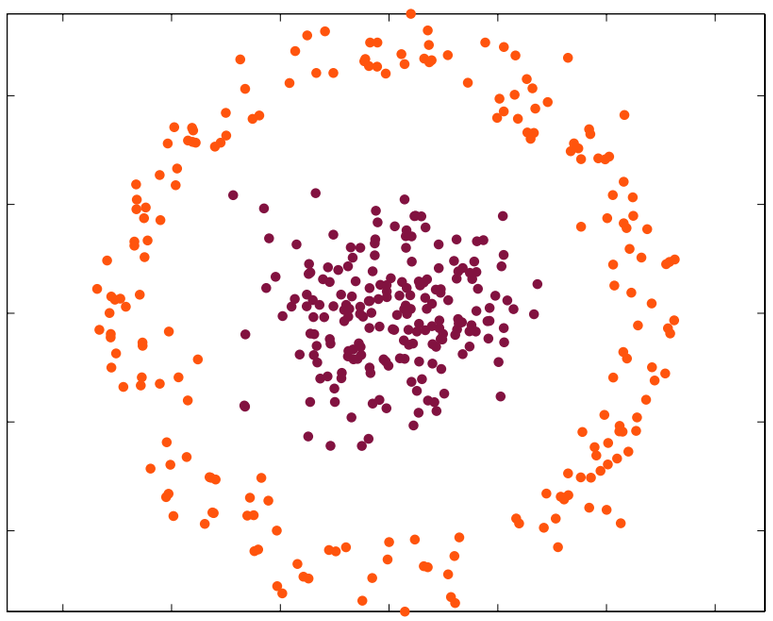




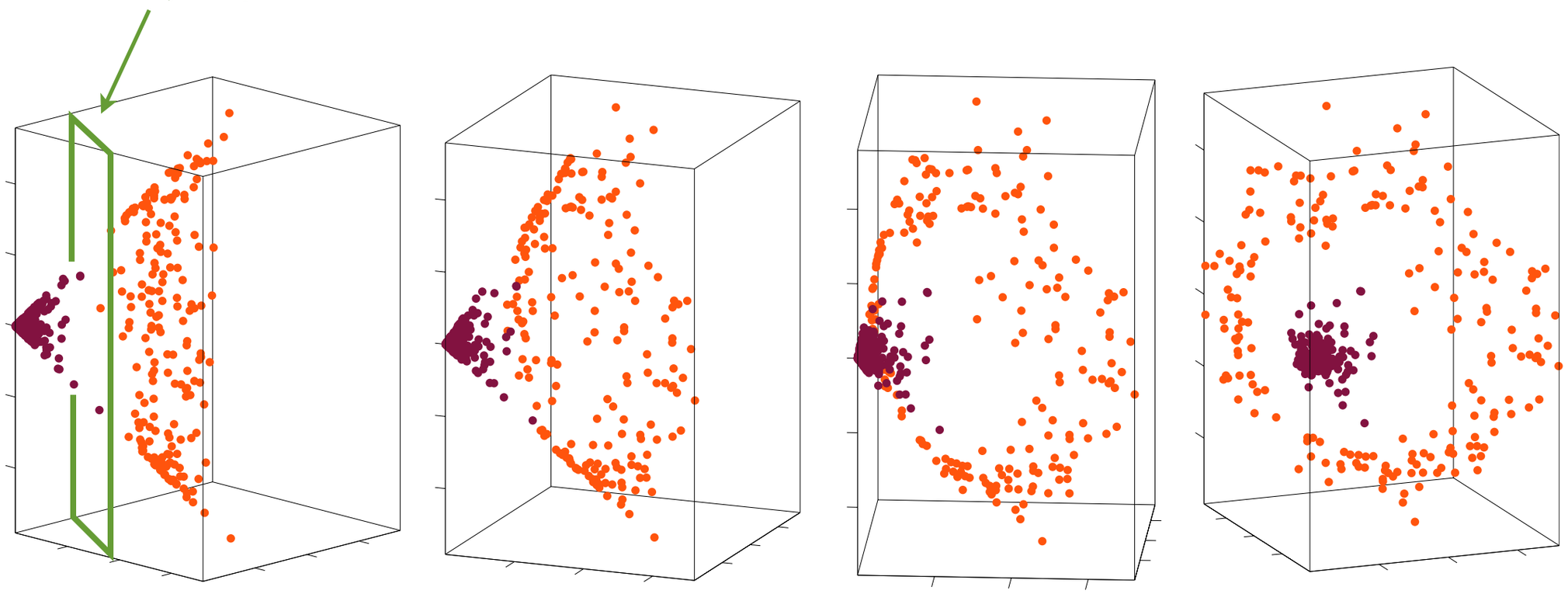
Non-Linear SVM

- Consider the following mapping

$$\phi: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}\, x_1 x_2 \end{pmatrix} \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^3$$



Linearly separable!



Non-Linear SVM

- Data may be **linearly separable** in the high dimensional space although in the original feature space they are not linearly separable
- This phenomenon is actually fairly general: if data are mapped into a space of **sufficiently high dimension**, then they will **almost always be linearly separable**
- For example, **four** dimensions suffice for linearly separating a **circle** anywhere in the plane (not just at the origin), and **five** dimensions suffice to linearly separate any **ellipse**
- In general (up to some exceptions), when we have N data points then they will always be **separable in spaces of** N–1 **dimensions** or more
- In order to frame the non-linear problem as a linear classification problem in the ϕ-space, we go over our learning and inference algorithms and **replace** $\mathbf{x}$ **everywhere by** $\phi(\mathbf{x})$:

Non-Linear SVM

- In our **Lagrange function** in dual form

$$L(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \lambda_i \lambda_j \, y_i \, y_j \, \mathbf{x}_i^T \mathbf{x}_j$$

in the expression for the **bias** b

$$b = \frac{1}{N_{\mathcal{S}}} \sum_{s_i \in \mathcal{S}} (y_{s_i} - \sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \mathbf{x}_{s_j}^T \mathbf{x}_{s_i})$$

and in the **dual version** of the **classifier**

$$y' = sign(\sum_{i=1}^{N} \lambda_i \, y_i \, \mathbf{x}_i^T \mathbf{x}' + b)$$

Non-Linear SVM

- In our **Lagrange function** in dual form

$$L(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \lambda_i \lambda_j \, y_i \, y_j \, \phi(\mathbf{x}_i)^T \phi(\mathbf{x}_j)$$

in the expression for the **bias** b

$$b = \frac{1}{N_{\mathcal{S}}} \sum_{s_i \in \mathcal{S}} (y_{s_i} - \sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \phi(\mathbf{x}_{s_j})^T \phi(\mathbf{x}_{s_i}))$$

and in the **dual version** of the **classifier**

$$y' = sign(\sum_{i=1}^{N} \lambda_i \, y_i \, \phi(\mathbf{x}_i)^T \phi(\mathbf{x}') + b)$$

Non-Linear SVM

- In our **Lagrange function** in dual form

$$L(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \lambda_i \lambda_j \, y_i \, y_j \, \phi(\mathbf{x}_i)^T \phi(\mathbf{x}_j)$$

in the expression for the **bias** b

$$b = \frac{1}{N_{\mathcal{S}}} \sum_{s_i \in \mathcal{S}} (y_{s_i} - \sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, \phi(\mathbf{x}_{s_j})^T \phi(\mathbf{x}_{s_i}))$$

and in the **dual version** of the **classifier**

$$y' = sign(\sum_{i=1}^{N} \lambda_i \, y_i \, \phi(\mathbf{x}_i)^T \phi(\mathbf{x}') + b)$$

- Vectors $\mathbf{x}$ or $\phi(\mathbf{x})$ enter only in the form of **inner products!**

Non-Linear SVM

- The fact that we can express our algorithm in terms of these inner products is key for the **kernel trick**
- A **kernel** is defined as

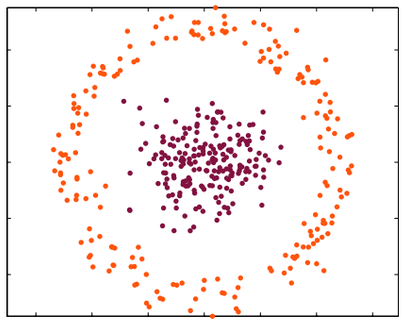
$$k(\mathbf{x}_i, \mathbf{x}_j) = \phi(\mathbf{x}_i)^T \phi(\mathbf{x}_j)$$

- Given $\phi(\mathbf{x})$, we could easily compute $k(\mathbf{x}_i, \mathbf{x}_j)$ by finding $\phi(\mathbf{x}_i)^T$ and $\phi(\mathbf{x}_j)$ and taking their inner product
- But dimension d may be extremely **large**. When the transformed space is high-dimensional, it may be **very costly** to compute the vectors $\phi(\mathbf{x})$ explicitly and then compute the inner product
- Interestingly, computing $k(\mathbf{x}_i, \mathbf{x}_j)$ may be **very inexpensive to calculate**, even though $\phi(\mathbf{x})$ itself may be very expensive to calculate

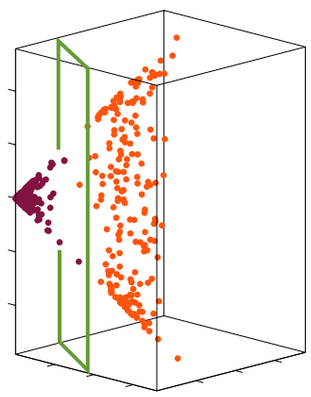
Non-Linear SVM

- Thus, with an efficient way to calculate $k(\mathbf{x}_i, \mathbf{x}_j)$, we can get SVMs to learn in the high dimensional feature space given by ϕ, but **without ever having to explicitly find or represent** vectors $\phi(\mathbf{x})$
- Let us exemplify this

$$\phi: \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1^2 \\ x_2^2 \\ \sqrt{2}\, x_1 x_2 \end{pmatrix}$$



$$\begin{aligned} \phi(\mathbf{x})^T \phi(\mathbf{z}) &= \begin{pmatrix} x_1^2 & x_2^2 & \sqrt{2}\, x_1 x_2 \end{pmatrix} \begin{pmatrix} z_1^2 \\ z_2^2 \\ \sqrt{2}\, z_1 z_2 \end{pmatrix} \\ &= x_1^2 z_1^2 + x_2^2 z_2^2 + 2\, x_1 x_2 z_1 z_2 \\ &= (x_1 z_1 + x_2 z_2)^2 \\ &= (\mathbf{x}^T \mathbf{z})^2 \end{aligned}$$



Support Vector Machines

Non-Linear SVM

- Thus, we could have used the kernel $k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^T\mathbf{z})^2$ without explicitly computing $\phi(\mathbf{x})$
- Let us look at this more systematically with a feature mapping involving all monomials of the form $x_i x_j$ (the previous one had visualization purposes). Assume again $m = 2$

$$\phi: \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \rightarrow \begin{pmatrix} x_1x_1 \\ x_1x_2 \\ x_2x_1 \\ x_2x_2 \end{pmatrix} \qquad \mathbb{R}^2 \rightarrow \mathbb{R}^4$$

- The **cost of computing** the high-dimensional $\phi(\mathbf{x})$ is $O(m^2)$

Non-Linear SVM

- The inner product $\phi(\mathbf{x})^T \phi(\mathbf{z})$ leads to the same kernel

$$\begin{aligned}
\phi(\mathbf{x})^T \phi(\mathbf{z}) &= (x_1 x_1 \;\; x_1 x_2 \;\; x_2 x_1 \;\; x_2 x_2) \begin{pmatrix} z_1 z_1 \\ z_1 z_2 \\ z_2 z_1 \\ z_2 z_2 \end{pmatrix} \\
&= x_1^2 z_1^2 + x_1 x_2 z_1 z_2 + x_2 x_1 z_2 z_1 + x_2^2 z_2^2 \\
&= (x_1 z_1 + x_2 z_2)^2 \\
&= (\mathbf{x}^T \mathbf{z})^2
\end{aligned}$$

- The **cost of computing** the kernel is only $O(m)$

Non-Linear SVM

- Let us convince ourselves that this kernel can be written as the inner product $\phi(\mathbf{x})^T \phi(\mathbf{z})$ for **general** input vector dimensions m

$$
\begin{aligned}
k(\mathbf{x}, \mathbf{z}) &= (\mathbf{x}^T \mathbf{z})^2 \\
&= \left(\sum_{i=1}^{m} x_i z_i \right) \left(\sum_{j=1}^{m} x_j z_j \right) \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j z_i z_j \\
&= \sum_{i,j=1}^{m} (x_i x_j)(z_i z_j)
\end{aligned}
$$

- This is indeed $\phi(\mathbf{x})^T \phi(\mathbf{z})$ with $\phi(\mathbf{x})$ as defined above

Non-Linear SVM

- Consider now the kernel $k(\mathbf{x}, \mathbf{z}) = (\mathbf{x}^T\mathbf{z} + c)^2$
- It happens to contain an inner product of $\mathbf{x}$ and $\mathbf{z}$ but this is not a requirement. Kernels are **general functions** of $\mathbf{x}$ and $\mathbf{z}$ (or $\mathbf{x}_i$ and $\mathbf{x}_j$)
- It can be shown that

$$
\begin{aligned}
k(\mathbf{x}, \mathbf{z}) &= (\mathbf{x}^T\mathbf{z} + c)^2 \\
&= \sum_{i,j=1}^{m} (x_i x_j)(z_i z_j) + \sum_{i=1}^{m} (\sqrt{2}c x_i)(\sqrt{2}c z_i) + c^2
\end{aligned}
$$

$$
\phi(\mathbf{x}) = \begin{bmatrix}
x_1x_1 \\ x_1x_2 \\ x_1x_3 \\ x_2x_1 \\ x_2x_2 \\ x_2x_3 \\ x_3x_1 \\ x_3x_2 \\ x_3x_3 \\ \sqrt{2}cx_1 \\ \sqrt{2}cx_2 \\ \sqrt{2}cx_3 \\ c
\end{bmatrix}
$$

and that this result corresponds to $\phi(\mathbf{x})^T\phi(\mathbf{z})$ with the feature mapping $\phi(\mathbf{x})$ shown on the right for $m = 3$

- Note the cost difference: $O(m)$ for kernel vs. $O(m^2)$ for $\phi(\mathbf{x})$

Non-Linear SVM

- Kernels do not transform the input data into the ϕ-space and then take an inner product. Kernels are **regular functions** of $\mathbf{x}$
- However, as shown in the examples, kernels **correspond** to a transformation to some ϕ-space and taking an inner product there **without ever explicitly computing** feature vectors in this high-dimensional space
- This is called the **kernel trick**
- Not every function has this property. Given some candidate kernel $k(\mathbf{x}, \mathbf{z})$, how do we know if it corresponds to a scalar product in some space?
- A kernel is a **valid kernel** if the following holds (Mercer kernels)
  - **Symmetry**: $k(\mathbf{x}_i, \mathbf{x}_j) = k(\mathbf{x}_j, \mathbf{x}_i)$
  - **Positive semi-definiteness**: let K be the N x N Kernel matrix $K_{ij} = k(\mathbf{x}_i, \mathbf{x}_j)$, then K has to be positive semi-definite, i.e. $v^T K v \geq 0 \quad \forall\, v \in \mathbb{R}^N$
- For example, $k(\mathbf{x}, \mathbf{z}) = \mathbf{x}^T \mathbf{z}$ is a valid kernel, $k(\mathbf{x}, \mathbf{z}) = \mathbf{x} - \mathbf{x}^T \mathbf{z}$ is not

Non-Linear SVM

- Popular examples of valid kernels include the **linear** kernel

$$k(\mathbf{x}_i, \mathbf{x}_j) = \mathbf{x}_i^T \mathbf{x}_j$$

- The degree p **polynomial kernel**, $p > 0$

$$k(\mathbf{x}_i, \mathbf{x}_j) = (\mathbf{x}_i^T \mathbf{x}_j + 1)^p$$

- The **Radial Basis function** (RBF) or **Gaussian kernel**, $\sigma > 0$

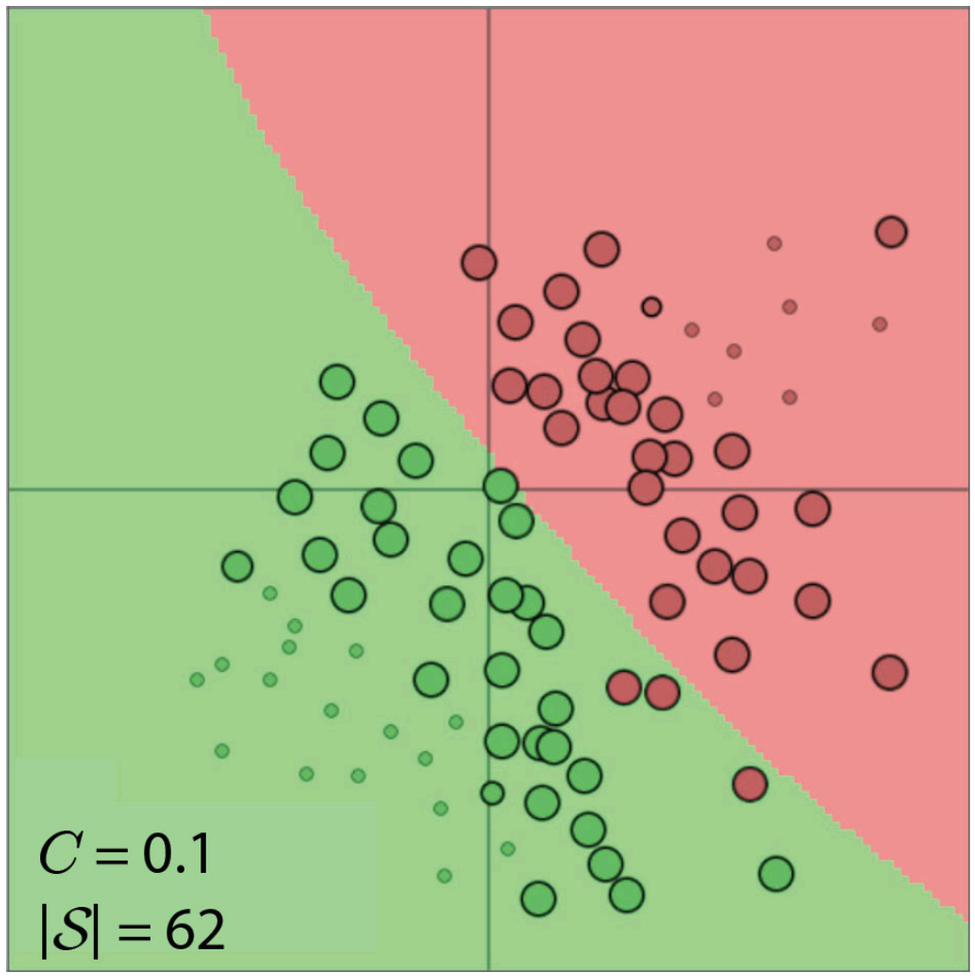
$$k(\mathbf{x}_i, \mathbf{x}_j) = \exp\left(-\frac{(\mathbf{x}_i - \mathbf{x}_j)^T(\mathbf{x}_i - \mathbf{x}_j)}{2\sigma^2}\right)$$

- The Gaussian kernel induces an **infinite dimensional** feature space (decomposition into x_i's and x_j's is done in a Taylor expansion of $\exp$)

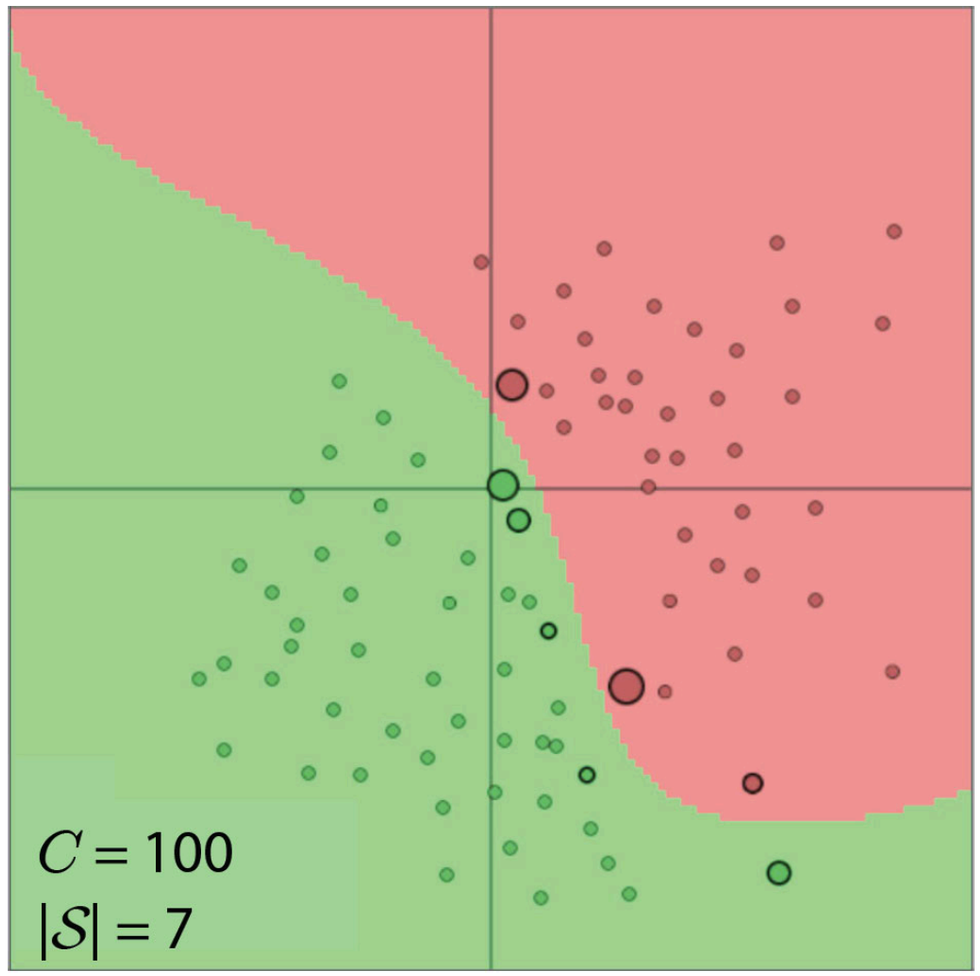
Non-Linear SVM

- The idea of kernels has **significantly broader applicability** than SVMs and is used in many learning algorithm that can be written in terms of only inner products
  - Examples include: perceptrons, kernel-PCA, kernel logistic regression, etc.
- There are many kernel functions, including ones that act upon **symbolic inputs** (as opposed to real-valued) and are defined over graphs, sets, strings or text documents
- Unless domain knowledge suggest the use of a specific kernel, the **Gaussian kernel** is a **good generic choice** for many practical classification tasks
- The concepts of SVMs using **kernels** (kernelized SVMs) and **soft-margin SVMs** can be readily **combined**

Example Classifications



Source [5]



Source [5]

- **Gaussian** (RBF) kernel, σ = 3.2

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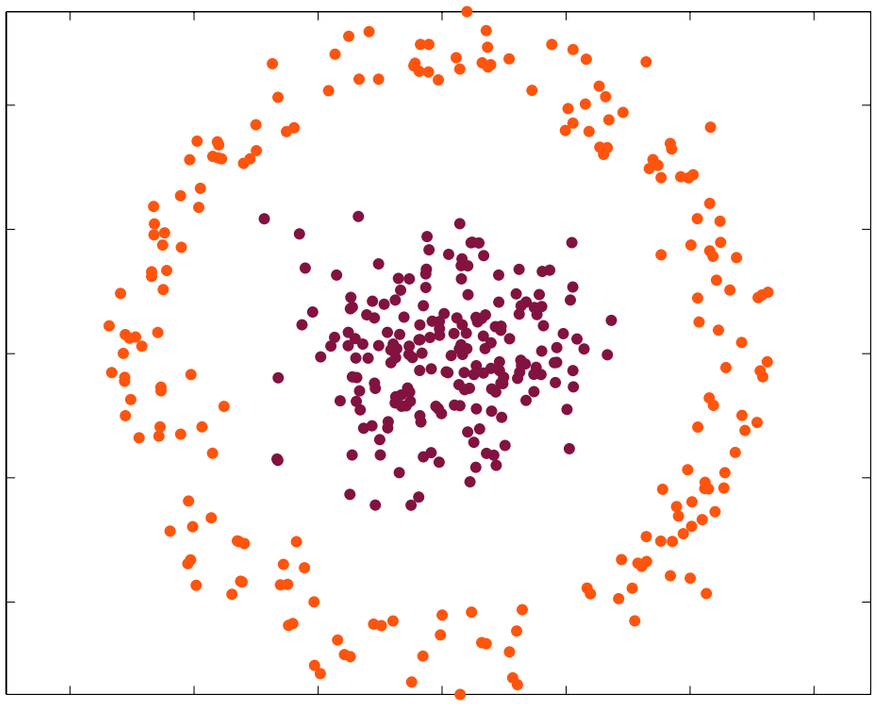
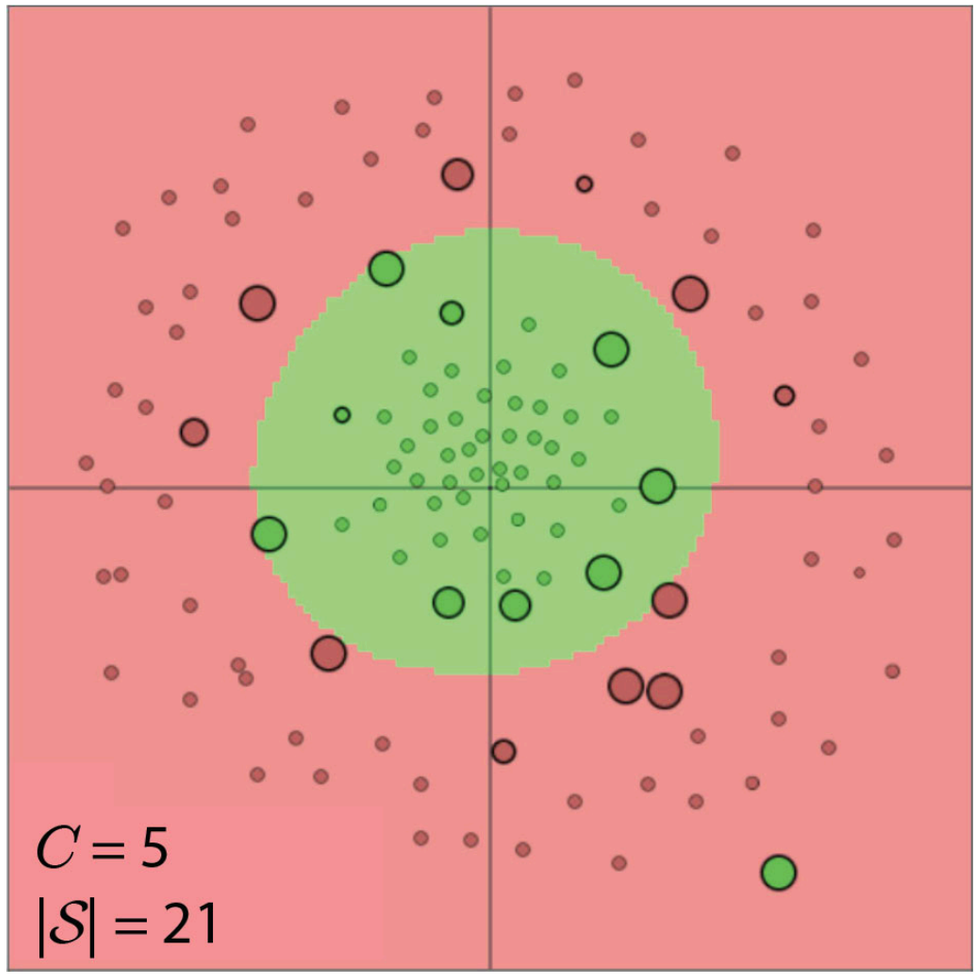
Example Classifications

- Circular class distributions





Source [5]

- **Gaussian** (RBF) kernel, σ = 3.2
- Kernel type, kernel parameters and stiffness parameter are usually determined by **cross validation** (later in this course)

Algorithm Summary

- **Learning**

  1. Find the **Lagrange multipliers** so that

$$L(\boldsymbol{\lambda}) = \sum_{i=1}^{N} \lambda_i - \frac{1}{2} \sum_{i,j=1}^{N} \lambda_i \lambda_j \, y_i \, y_j \, k(\mathbf{x}_i, \mathbf{x}_j)$$

     is maximized subject to $0 \leq \lambda_i \leq C \;\; \forall i$ and $\sum_i \lambda_i \, y_i = 0$ using a QP solver

  2. Determine the set of support vectors $\mathcal{S}$ by finding the indices such that $0 < \lambda_i \leq C \;\; \forall i$
  3. Calculate the **bias** $b = \frac{1}{N_{\mathcal{S}}} \sum_{s_i \in \mathcal{S}} (y_{s_i} - \sum_{s_j \in \mathcal{S}} \lambda_{s_j} \, y_{s_j} \, k(\mathbf{x}_{s_j}, \mathbf{x}_{s_i})$

- **Inference and decision**

  4. **Predict class** for new points $\mathbf{x}'$ by evaluating

$$y' = sign(\sum_{i=1}^{N} \lambda_i \, y_i \, k(\mathbf{x}_i, \mathbf{x}') + b)$$

Summary SVM

- A Support Vector Machine is a **non-probabilistic** discriminative classifier
- Its approach to minimize the generalization error is to **maximize the margin** (it's an instance of a maximum margin classifier)
- Learning is framed as a **constraint quadratic optimization** problem
- The learned classifier only depends **sparsely** on the training set
- **Non-linear SVM** transform input data which are not linearly separable into a higher dimensional feature space and apply linear separation there
- The **kernel trick** is an efficient transformation of input data to some space and taking an inner product in that space **without ever going there**. Works even for **infinite** dimensional feature spaces
- For non-linearly separable data there are two cases: for **outliers** use **soft-margin SVM**, for data with **inherently non-linear** class distributions, use non-linear, **kernelized SVM**

Summary SVM

- **Advantages**
  - Kernel-based framework is very **powerful**
  - Quadratic optimization problem is **convex** and has a **unique** solution (as opposed to other classifiers such as NN, RVM)
  - Efficient inference due to **sparsity**
  - SVM classifiers work usually **very well** in practice
- **Drawbacks**
  - **Not probabilistic**
  - **Binary classifier**, extension to multi-class not straightforward
  - Learning may be very **slow** for large training sets
  - Constraint QP may run into **numerical instabilities**

Sources and Further Reading

These slide contain material by Russell and Norvig [2] (chapter 18), Bischop [1] (chapter 7 and 9), Ng's lecture notes on SVM [3] and Fletcher [4]. Several images were produced using Karpathy's nice and very instructive SVM applet [5].

[1] S. Russell, P. Norvig, "Artificial Intelligence: A Modern Approach", 3rd edition, Prentice Hall, 2009. See http://aima.cs.berkeley.edu

[2] C.M. Bischop, "Pattern Recognition and Machine Learning", Springer, 2nd ed., 2007. See http://research.microsoft.com/en-us/um/people/cmbishop/prml

[3] A. Ng, "Part V: Support Vector Machines", Lecture Notes CS229 Machine Learning, Stanford University, 2012

[4] T. Fletcher, "Support Vector Machines Explained," Tutorial Paper, UCL, 2009, http://www.tristanfletcher.co.uk

[5] A. Karpathy, "svmjs: SVMs in Javascript", online: http://cs.stanford.edu/people/karpathy/svmjs/demo (Dec 2013)

[6] Wikipedia, articles on Lagrange multipliers and Karush–Kuhn–Tucker conditions: http://en.wikipedia.org/wiki/Lagrange_multiplier / Karush–Kuhn–Tucker_conditions

Supervised Learning

To be continued in Supervised Learning, part 3/3