Does Random Forcing Satisfy the Knaster Property, Precaliber $\aleph_1$, σ-linked, and σ-centered?

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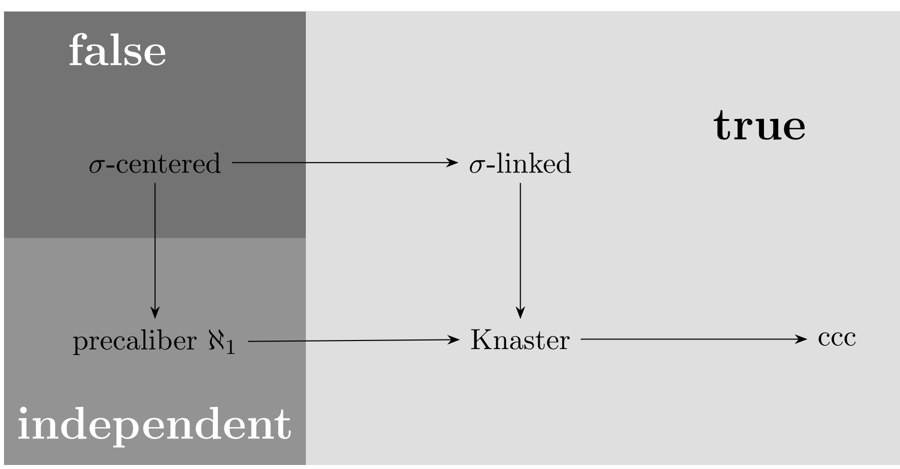
As properties of forcing notions stronger than the ccc (countable chain condition), there are σ-centered, σ-linked, precaliber $\aleph_1$, and the Knaster property. In this article, we clarify which of these are satisfied by random forcing.

$$
\begin{tikzcd}
\sigma\text{-centered} \arrow[r] \arrow[d] & \sigma\text{-linked} \arrow[d] & \\
\text{precaliber } \aleph_1 \arrow[r] & \text{Knaster} \arrow[r] & \text{ccc}
\end{tikzcd}
$$

To state the conclusion first, the situation is as shown in the following figure.



In other words, σ-centered fails; whether precaliber $\aleph_1$ holds is independent; and σ-linked (and hence the Knaster property) holds.

Let us begin with the definitions. We will review all necessary definitions, so no prior knowledge of forcing is required for this article.

Definition 1. (1) A *forcing notion* is a preordered set.

(2) A set of conditions in a forcing notion P is said to be *compatible* if it has a lower bound in P.

(3) A subset A of a forcing notion P is an *antichain* if any two distinct elements $p, q \in A$ are incompatible.

(4) A subset A of a forcing notion P is *linked* if any two elements $p, q \in A$ are compatible in P.

(5) A subset A of a forcing notion P is *centered* if every finite subset of A is compatible in P.

Definition 2. Let P be a forcing notion.

(1) P satisfies the *ccc* if every antichain in P is countable.

(2) P satisfies the *Knaster property* if for every subset $A \subseteq P$ of size $\aleph_1$, there is a subset $B \subseteq A$ of size $\aleph_1$ such that B is linked in P.

(3) P satisfies *precaliber* $\aleph_1$ if for every subset $A \subseteq P$ of size $\aleph_1$, there is a subset $B \subseteq A$ of size $\aleph_1$ such that B is centered in P.

(4) P is *σ-linked* if P can be written as a countable union of linked subsets of P.

(5) P is *σ-centered* if P can be written as a countable union of centered subsets of P.

Definition 3. Let P be a forcing notion. A subset $Q \subseteq P$ is a *dense subset* of P if for every $p \in P$ there exists $q \in Q$ such that $q \leqslant p$.

Definition 4. The random forcing $\mathbb{B}$ is the forcing notion consisting of the Borel subsets of the Cantor space 2^ω of positive Lebesgue measure, ordered by inclusion.

Later we will show that $\mathbb{B}$ is σ-linked; therefore there is no need to separately prove the ccc or the Knaster property, since they follow from σ-linked. Nevertheless, we include the ccc because it is a standard argument one should know, and we include the Knaster property because the proof is interesting.

Proposition 5. $\mathbb{B}$ satisfies the ccc.

Proof. Let $\{A_\alpha : \alpha < \omega_1\} \subseteq \mathbb{B}$. By the pigeonhole principle, there is a positive integer n such that $\aleph_1$ many of these sets have measure at least $1/n$. If among them we could not find two compatible conditions, then the total measure would be ∞, a contradiction. $\square$

In the proof of the Knaster property we use the following (somewhat wild card) tool. For a proof, see for example Jech's book (Theorem 9.7 of [Jec03]).

Theorem 6 (Erdős–Dushnik–Miller theorem)**.** $\omega_1 \to (\omega_1, \omega)^2$.

For every coloring $c\colon [\omega_1]^2 \to 2$, one of the following holds.

1. There exists $H \in [\omega_1]^{\omega_1}$ such that c is constantly 0 on $[H]^2$.
1. There exists $K \in [\omega_1]^{\omega}$ such that c is constantly 1 on $[K]^2$.

The following proof of the Knaster property was explained to me by Jorge Antonio Cruz Chapital.

Proposition 7. $\mathbb{B}$ satisfies the Knaster property.

Proof. Take an arbitrary subset $A \subseteq \mathbb{B}$ of size $\aleph_1$.

Let $A_n = \{p \in A : \mu(p) > 1/n\}$. Since $A = \bigcup_{n \in \omega \setminus 1} A_n$, there exists n such that A_n is uncountable. Hence, we may assume from the start that $A = A_n$, i.e. $\mu(p) > 1/n$ for all $p \in A$.

Define a coloring $c\colon [A]^2 \to 2$ by

$$c(p,q) = \begin{cases} 0 & \text{if } p \text{ and } q \text{ are compatible,} \\ 1 & \text{if } p \text{ and } q \text{ are incompatible.} \end{cases}$$

By the Erdős–Dushnik–Miller theorem, either clause (1) or clause (2) holds, but clause (2) is impossible.

Indeed, there can be at most n many pairwise incompatible conditions in P.

Therefore clause (1) holds: there exists $B \subseteq A$ of size $\aleph_1$ such that c is constantly 1 on $[B]^2$. By the definition of c, this B witnesses the Knaster property. □

Proposition 8. $\mathbb{B}$ is σ-linked.

Proof. For $s \in 2^{<\omega}$, let

$$L_s = \left\{ p \in \mathbb{B} : \frac{\mu(p \cap [s])}{\mu([s])} > \frac{1}{2} \right\}.$$

Each L_s is linked, since the intersection of two sets of relative measure greater than one half has positive measure.

Moreover, by the Lebesgue density theorem, $\bigcup_{s \in 2^{<\omega}} L_s$ is a dense subset of $\mathbb{B}$.

Hence, since a dense subset is σ-linked, $\mathbb{B}$ is σ-linked as well. □

Proposition 9. $\mathbb{B}$ is not σ-centered.

This follows from the fact that σ-centered forcing notions do not add random reals (see e.g. Lemma 3.7 of [Bre09]), but here we give a direct proof that does not rely on that result.

Proof. Let $\mathbb{B}' = \{K \in \mathbb{B} : K \text{ is compact}\}$. Then $\mathbb{B}'$ is a dense subset of $\mathbb{B}$, and since being σ-centered is inherited by dense subsets, it suffices to show that $\mathbb{B}'$ is not σ-centered.

Assume $\mathbb{B}'$ is σ-centered and write $\mathbb{B}' = \bigcup_{n \in \omega} C_n$, where each C_n is centered. Every finite subset $F \subseteq C_n$ has a common extension, so $\bigcap F$ has positive measure and hence is nonempty. Since in a compact space any family of closed sets with the finite intersection property has nonempty intersection, we also have $\bigcap C_n \neq \varnothing$. For each n, pick $x_n \in \bigcap C_n$.

Let $X = \{x_n : n \in \omega\}$. Since the sets C_n cover $\mathbb{B}'$, for every $K \in \mathbb{B}'$ we have $K \cap X \neq \varnothing$.

On the other hand, X is countable, hence has measure 0. Therefore $2^\omega \setminus X$ has measure 1, so we can take a compact set $K \subseteq 2^\omega \setminus X$ of positive measure. This contradicts the previous paragraph. □

Proposition 10 (525G of [Fre08])**.** $\mathbb{B}$ satisfies precaliber $\aleph_1$ if and only if $\operatorname{cov}(\mathcal{N}) > \aleph_1$. Hence, whether $\mathbb{B}$ satisfies precaliber $\aleph_1$ is independent from ZFC.

Proof. We extend the definition. A forcing notion P satisfies precaliber κ if for every subset of P of size κ there is a subset of size κ which is centered. Let $\mathsf{pc}(P)$ be the least κ such that P does not satisfy precaliber κ.

Claim 10.1. $\operatorname{cov}(\mathcal{N}) \leqslant \mathsf{pc}(\mathbb{B})$.

Proof. In the proof, we regard $\mathbb{B}$ as the forcing notion obtained from the quotient Boolean algebra $\mathsf{Borel}(2^\omega)/\mathcal{N}$ by removing the minimum element. Note that the measure of a representative is well-defined for elements of $\mathbb{B}$; we also denote it by μ, and $\mathbb{B}$ is a complete Boolean algebra. Recall also that $\mathrm{cov}(\mathcal{N})$ coincides with the Martin number for random forcing.

Note that $\mathsf{pc}(\mathbb{B}) \geqslant \aleph_1$ is immediate. So let κ be an uncountable cardinal with $\kappa < \mathrm{cov}(\mathcal{N})$, and we show that $\kappa < \mathsf{pc}(\mathbb{B})$. Let $\{a_\alpha : \alpha < \kappa\} \subseteq \mathbb{B}$. By thinning out via the pigeonhole principle, we may assume $\delta = \inf_{\alpha<\kappa} \mu(a_\alpha) > 0$. Let

$$c = \inf \left\{ \sup_{\alpha \in \kappa \smallsetminus J} a_\alpha : J \subseteq \kappa,\ |J| < \kappa \right\}.$$

Note that $\mu(c) > 0$.

Inductively choose a sequence $\langle I_\beta : \beta < \kappa \rangle$ of pairwise disjoint countable subsets of κ such that for each $\beta < \kappa$ we have $c \leqslant \sup_{\alpha \in I_\beta} a_\alpha$. Here we use that $\mathbb{B}$ is ccc, so any supremum of many elements can be obtained as the supremum of countably many of them.

For each $\beta < \kappa$ let

$$D_\beta = \{ b \in \mathbb{B}_{\leqslant c} : \exists \alpha \in I_\beta\ b \leqslant a_\alpha \}.$$

Then D_β is dense in $\mathbb{B}_{\leqslant c}$. Since $\kappa < \mathrm{cov}(\mathcal{N}) = \mathfrak{m}(\mathbb{B}) \leqslant \mathfrak{m}(\mathbb{B}_{\leqslant c})$, there exists a filter G meeting all the D_β. Let $\Gamma = \{\alpha < \kappa : \exists b \in G\ b \leqslant a_\alpha\}$. Since G is a filter, $\{a_\alpha : \alpha \in \Gamma\}$ is centered. Also, for each $\beta < \kappa$ we have $\Gamma \cap I_\beta \neq \varnothing$, hence $\{a_\alpha : \alpha \in \Gamma\}$ has size κ. //

By this claim, if $\mathrm{cov}(\mathcal{N}) > \aleph_1$ then $\mathbb{B}$ has precaliber $\aleph_1$.

Claim 10.2. $\mathsf{pc}(\mathbb{B}) \leqslant \mathrm{cov}(\mathcal{N})$.

Proof. Assume $\kappa < \mathsf{pc}(\mathbb{B})$, i.e. $\mathbb{B}$ satisfies precaliber κ. We will derive $\kappa < \mathrm{cov}(\mathcal{N})$. Assume toward a contradiction that $\mathrm{cov}(\mathcal{N}) \leqslant \kappa$. Then there exists a family $\langle N_\alpha : \alpha < \kappa \rangle$ of null sets covering the whole space. For each α choose a positive-measure compact set K_α disjoint from N_α. Since $\mathbb{B}$ has precaliber κ, there exists $\Gamma \in [\kappa]^\kappa$ such that $\{K_\alpha : \alpha \in \Gamma\}$ is centered. By compactness, we can find $x \in \bigcap_{\alpha<\kappa} K_\alpha$. Thus $x \notin \bigcup_{\alpha<\kappa} A_\alpha$, contradicting that the A_α cover 2^ω. //

By this claim, if $\mathbb{B}$ has precaliber $\aleph_1$, then $\mathrm{cov}(\mathcal{N}) > \aleph_1$. □

References

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[Fre08] D. H. Fremlin. "Measure theory, vol. 5". *Set-Theoretic Measure Theory, Parts I, II. Torres Fremlin, Colchester* (2008).

[Jec03] T. Jech. *Set theory: The third millennium edition, revised and expanded.* Springer, 2003.