

Strengthening the Assumption in the Judah–Repický Preservation Theorem

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Abstract

A preservation theorem of Judah–Repický states that, for countable support iteration of proper forcing of limit length, if every initial segment does not add a dominating real, then neither does the final stage. In this note, we strengthen the assumption in this theorem by replacing proper forcing with Axiom A forcing, and we simplify the proof.

First we recall the notion of Axiom A from Baumgartner’s text [Bau83].

Definition 1. A forcing notion (P, \leq) is called Axiom A forcing if there is a sequence $\langle \leq_n : n \in \omega \rangle$ of partial orderings on P satisfying the following properties.

- (1) For all n , \leq_{n+1} is stronger than \leq_n and \leq_n is stronger than \leq .
- (2) For every sequence $\langle p_n : n \in \omega \rangle$ satisfying $p_{n+1} \leq_n p_n$ (for every n), there is q such that $q \leq_n p_n$ (for every n).
- (3) For every antichain $A \subseteq P$, $p \in P$ and $n \in \omega$, there is $q \leq_n p$ such that the set $\{r \in A : r \parallel q\}$ is countable.

To handle countable support iterations of Axiom A forcing, the following notion is useful.

Definition 2. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing.

For $p, q \in P_\delta$ and $F \subseteq \delta$ and $n \in \omega$, we define $q \leq_{F,n} p \iff q \leq p$ and $\forall \alpha \in F (q \leq_n p)$.

A sequence $\langle (p_n, F_n) : n \in \omega \rangle$ is called a fusion sequence if

- (1) $p_n \in P_\delta$, $F_n \subseteq \delta$ finite.
- (2) $p_{n+1} \leq_{F_n, n} p_n$.
- (3) $\bigcup_{n \in \omega} F_n = \bigcup_{n \in \omega} \text{supt}(p_n)$.

Lemma 3. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing. Let $\langle (p_n, F_n) : n \in \omega \rangle$ be a fusion sequence. Then there is a condition q such that $q \leq_{F_n, n} p_n$ for all n .

Lemma 4. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing. Let $F \subseteq \delta$ be a finite set and $n \in \omega$. Let $p \in P_\delta$ and \dot{a} be a P_δ -name such that $P_\delta \Vdash \dot{a} \in V$. Then there is $q \leq_{F, n} p$ and a countable set x such that $q \Vdash \dot{a} \in x$.

For $x, y \in \omega^\omega$, we define $x <^\infty y$ iff there exist infinitely many n such that $x(n) < y(n)$.

Now that we are prepared, we state the main theorem modified from Judah–Repický Preservation Theorem [JR95], whose assumption is strengthened but the proof is simplified.

Theorem 5. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing where δ is a limit ordinal. Assume that $P_\alpha \Vdash \forall x \exists y \in V \ x <^\infty y$ for any $\alpha < \delta$. Then P_δ also forces it.

Proof. Let \dot{x} be a P_δ -name and $p \in P_\delta$.

We create a fusion sequence $\langle p_n, F_n : n \in \omega \rangle$ inductively.

Suppose we have constructed p_n and F_n . Put $\gamma = \max F_n$. Take a sequence $\langle \dot{q}_n^m : m \in \omega \rangle$ of P_γ -names and a P_γ -name \dot{y}_n such that:

$$p_n \restriction \gamma \Vdash \langle \dot{q}_n^m : m \in \omega \rangle \text{ interprets } \dot{x} \text{ as } \dot{y}_n.$$

By the assumption, we can take a P_γ -name \dot{z}_n such that

$$p_n \restriction \gamma \Vdash “\dot{y}_n <^\infty \dot{z}_n \text{ and } \dot{z}_n \in V”.$$

By the property of Axiom A, we can take $q_n \leq_{F_n, n} p_n \restriction \gamma$ and a countable set A_n such that

$$q_n \Vdash \dot{z}_n \in A_n.$$

Let $A_n = \{w_n^l : l \in \omega\}$. Take a name \dot{m}_n of a natural number such that

$$q_n \Vdash “\dot{z}_n = w_n^l \rightarrow (\dot{y}_n(\dot{m}_n) < \dot{z}_n(\dot{m}_n) \text{ and } \dot{m}_n \geq n + l)”.$$

Take \dot{r}_n such that

$$q_n \Vdash \dot{r}_n = \dot{q}_n^{\dot{m}_n}.$$

Put $p_{n+1} = q_n \cup \dot{r}_n$. Then we have

$$p_{n+1} \Vdash \dot{x}(\dot{m}_n) < \dot{z}_n(\dot{m}_n)$$

Set F_{n+1} in a bookkeeping manner to satisfy $\bigcup_n F_n = \bigcup_n \text{supt}(p_n)$ finally.

This finishes the construction of the fusion sequence. Finally, let p be the limit condition of $\langle p_n, F_n : n \in \omega \rangle$ and let $u \in \omega^\omega$ be a real dominating all elements in $\{w_n^l : l, n \in \omega\}$, in particular

$$\forall l, n \ \forall m \geq l + n \ (w_n^l(m) < u(m)).$$

Then we have for all n ,

$$p \Vdash \dot{x}(\dot{m}_n) < u(\dot{m}_n) \text{ and } \dot{m}_n > n.$$

□

References

- [Bau83] James E. Baumgartner. “Iterated forcing”. *Surveys in set theory* 87 (1983), pp. 1–59.
- [JR95] Haim Judah and Miroslav Repický. “No random reals in countable support iterations”. *Israel Journal of Mathematics* 92.1 (1995), pp. 349–359.