

# Two problems concerning Hausdorff measures and the Lebesgue measure

Tatsuya Goto

Kobe University

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# Introduction of me

- Tatsuya Goto
- a doctoral student 1st year at Kobe University in Japan
- Twitter: @goto\_math
- web page: <https://t-goto.jp/>

# Table of contents

- ① Cardinal invariants on Hausdorff measures
- ② Goldstern's theorem

1 Cardinal invariants on Hausdorff measures

2 Goldstern's theorem

# Motivation

Cardinal invariants defined by the null ideal and the meager ideal have been well studied for a long time and are summarized in Cichoń's diagram:

$$\begin{array}{ccccccc}
 \text{cov}(\mathcal{N}) & \rightarrow & \text{non}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{N}) \longrightarrow 2^{\aleph_0} \\
 \uparrow & & \uparrow & \longrightarrow & \uparrow & & \uparrow \\
 & & \mathfrak{b} & & \mathfrak{d} & & \\
 \uparrow & & \uparrow & & \uparrow & & \\
 \aleph_1 & \longrightarrow & \text{add}(\mathcal{N}) & \rightarrow & \text{add}(\mathcal{M}) & \rightarrow & \text{cov}(\mathcal{M}) \rightarrow \text{non}(\mathcal{N})
 \end{array}$$

We would like to consider cardinal invariants defined by Hausdorff measures, which do not appear in Cichoń's diagram, and investigate their relationships.

# Ideals defined by Hausdorff measures

We consider the Cantor space  $(2^\omega, d)$ , where

$$d(x, y) = 2^{-\min\{n: x(n) \neq y(n)\}} \quad (\text{for } x \neq y).$$

For a gauge function  $f$ , we define the  $f$ -Hausdorff measure 0 ideal by

$$\mathcal{N}^f = \{A \subseteq 2^\omega : \mathcal{H}^f(A) = 0\}.$$

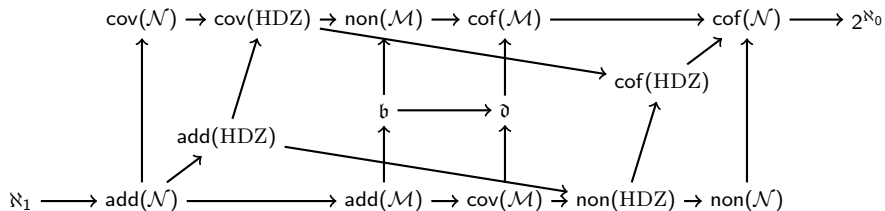
And we put

$$\text{HDZ} := \{A \subseteq 2^\omega : \dim_{\text{H}}(A) = 0\} = \bigcap_{s>0} \mathcal{N}^{\text{pow}_s},$$

where the gauge function  $\text{pow}_s$  ( $s > 0$ ) is given by

$$\text{pow}_s(x) = x^s.$$

# Expansion of Cichoń's diagram



# Relations with Yorioka ideals

- There are ideals called Yorioka ideals, which are parametrized by reals and have a combinatorial definition. We showed a relation between Hausdorff measure 0 ideals and Yorioka ideals:
  - $\forall f \exists g (\mathcal{I}_g \subseteq \mathcal{N}_f)$
  - $\forall g \exists f (\mathcal{N}_f \subseteq \mathcal{I}_g)$
- Using prior studies on Yorioka ideals and this fact, we got the following:

## Theorem (G.)

$\aleph_1$  many cardinals of the form  $\text{cov}(\mathcal{N}_f)$  can be separated and  $\aleph_1$  many cardinals of the form  $\text{non}(\mathcal{N}_f)$  can be separated.

Also we showed  $\mathcal{I}_{\text{id}} \subseteq \text{HDZ}$ .



## Further results

Shelah and Steprāns showed for every  $s > 0$ ,  $\text{non}(\mathcal{N}_s)$  and  $\text{non}(\mathcal{N})$  can be separated. Here  $\mathcal{N}^s = \mathcal{N}^{\text{pow}_s}$  is  $s$ -dimensional Hausdorff measure 0 ideal.

We showed the following:

### Theorem (G.)

This separation can be done by iterated Mathias forcing.

And we also showed that  $\text{cov}(\text{HDZ})$  and  $\text{non}(\text{HDZ})$  are stable under changing underlying space from  $2^\omega$  into  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  for each  $n$ .

# Open questions

- Can we generalize the underlying spaces of HDZ further?
- Does ZFC prove that  $\text{cov}(\text{HDZ}) = \text{cov}(\mathcal{I}_{\text{id}})$  and  $\text{non}(\text{HDZ}) = \text{non}(\mathcal{I}_{\text{id}})$ ?
- Does ZFC prove that  $\text{add}(\text{HDZ}) \leq \mathfrak{b}$  and  $\mathfrak{d} \leq \text{cof}(\text{HDZ})$ ?
- Does ZFC prove that  $\text{add}(\text{HDZ}) = \text{add}(\mathcal{N})$  and  $\text{cof}(\text{HDZ}) = \text{cof}(\mathcal{N})$ ?
- Does ZFC prove that for every  $0 < s < t < 1$ ,  $\text{non}(\mathcal{N}_s) = \text{non}(\mathcal{N}_t)$ ? (Shelah–Steprāns)

① Cardinal invariants on Hausdorff measures

② Goldstern's theorem

# Goldstern's theorem

**(full domination order)** For  $x, x' \in \omega^\omega$ , define a relation  $x \leq x'$  by  $(\forall n \in \omega)(x(n) \leq x'(n))$ .

In 1993, Martin Goldstern proved the following theorem.

## Goldstern's theorem (ZF + CC)

Let  $(Y, \mu)$  be a Polish probability space. Let  $A \subseteq \omega^\omega \times Y$  be a  $\Sigma_1^1$  set. Assume that for each  $x \in \omega^\omega$ ,

$$A_x := \{y \in Y : (x, y) \in A\}$$

has measure 0. Also, assume

$(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$ . Then  $\bigcup_{x \in \omega^\omega} A_x$  has also measure 0.

# The principle $\text{GP}(\Gamma)$

## Definition

Let  $\Gamma$  be a pointclass. Then  $\text{GP}(\Gamma)$  means the following statement: Let  $(Y, \mu)$  be a Polish probability space and  $A \subseteq \omega^\omega \times Y$  be in  $\Gamma$ . Assume that for each  $x \in \omega^\omega$ ,  $A_x$  has  $\mu$ -measure 0. Also suppose that  $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$ . Then  $\bigcup_{x \in \omega^\omega} A_x$  has also  $\mu$ -measure 0.

Goldstern's theorem says that  $\text{GP}(\Sigma_1^1)$  holds.

# Main Result

The symbol “all” denotes the class of all subsets of Polish spaces.

## Theorem (G.)

$\text{GP}(\text{all})$  is independent from ZFC.

# Consistency of $\neg \text{GP}(\text{all})$

In fact, the consistency of the negation follows from:

**Theorem (G.)**

Assume CH. Then  $\neg \text{GP}(\text{all})$  holds.

# Consistency of $\neg \text{GP}(\text{all})$

Refining the last theorem, we get the following theorem.

## Theorem (G.)

Assume that at least one of the following three conditions holds:

$$\text{add}(\mathcal{N}) = \mathfrak{b}, \text{non}(\mathcal{N}) = \mathfrak{b} \text{ or } \text{non}(\mathcal{N}) = \mathfrak{d}.$$

Then  $\neg \text{GP}(\text{all})$  holds.

$$\text{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$$

$$\text{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$$

$$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F) f <^* g\}$$

$$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F) g <^* f\}$$



# Consistency of $\neg \text{GP}(\text{all})$

Assume that at least one of the following three conditions holds:  $\text{add}(\mathcal{N}) = \mathfrak{b}$ ,  $\text{non}(\mathcal{N}) = \mathfrak{b}$  or  $\text{non}(\mathcal{N}) = \mathfrak{d}$ . Then  $\neg \text{GP}(\text{all})$  holds.

$\text{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$

$\text{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$

$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F) f <^* g\}$

$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F) g <^* f\}$

$$\begin{array}{ccccccc}
 & & \text{cov}(\mathcal{N}) & \rightarrow & \text{non}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{M}) & \rightarrow & \text{cof}(\mathcal{N}) & \longrightarrow & 2^{\aleph_0} \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 & & & & \mathfrak{b} & \longrightarrow & \mathfrak{d} & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
 \aleph_1 & \longrightarrow & \text{add}(\mathcal{N}) & \rightarrow & \text{add}(\mathcal{M}) & \rightarrow & \text{cov}(\mathcal{M}) & \rightarrow & \text{non}(\mathcal{N}) & & 
 \end{array}$$

$$V = L \text{ implies } \neg \text{GP}(\Delta_2^1)$$

Refining the last proof in another way again, we get the following theorem.

**Theorem (G.)**

$V = L \text{ implies } \neg \text{GP}(\Delta_2^1).$

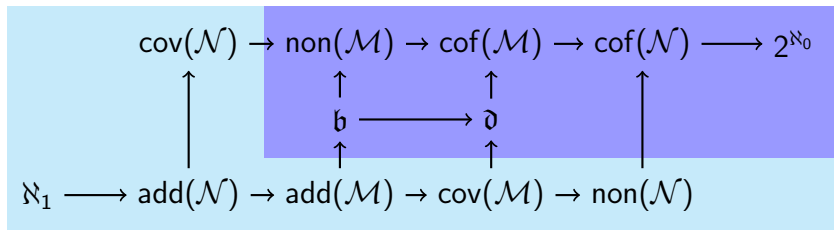
# Consistency of GP(all)

## Theorem (G.)

If ZFC is consistent then so is ZFC + GP(all).

In fact, “The Laver model” satisfies GP(all).

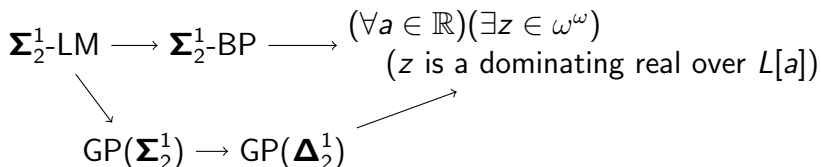
equal to  $\aleph_2$  in the Laver model



# Connections with $\Sigma_2^1$ regularity

## Theorem (G.)

$\Sigma_2^1$ -LM implies  $\text{GP}(\Sigma_2^1)$ . And  $\text{GP}(\Delta_2^1)$  implies  $(\forall a \in \mathbb{R})(\exists z \in \omega^\omega)(z \text{ is a dominating real over } L[a])$ .



# AD and the Solovay model

## Theorem (G.)

Assume  $\text{ZF} + \text{AD}$ . Then  $\text{GP}(\text{all})$  holds.

## Theorem (G.)

In the Solovay model,  $\text{GP}(\text{all})$  holds.

# Open questions

- ① Does  $V = L$  imply  $\neg \text{GP}(\Pi_1^1)$ ?
- ② Is  $\text{ZFC} + (\mathfrak{c} > \aleph_2) + \text{GP}(\text{all})$  consistent?
- ③ Is  $\text{ZFC} + (\mathfrak{b} < \mathfrak{d}) + \text{GP}(\text{all})$  consistent?
- ④ Is there a model of ZF satisfying that every set of reals are measurable and  $\neg \text{GP}(\text{all})$ ?
- ⑤ Is it possible to separate  $\text{GP}(\Sigma_{n+1}^1)$  and  $\text{GP}(\Sigma_n^1)$  for some (or every)  $n \geq 2$  (without large cardinals)?

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