Strengthening the Assumption in the Judah–Repický Preservation Theorem

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Abstract

A preservation theorem of Judah–Repický states that, for countable support iteration of proper forcing of limit length, if every initial segment does not add a dominating real, then neither does the final stage. In this note, we strengthen the assumption in this theorem by replacing proper forcing with Axiom A forcing, and we simplify the proof. As a result, we obtain a simpler proof for a weaker theorem than the original one.

First we recall the notion of Axiom A from Baumgartner's text [Bau83].

Definition 1. A forcing notion $(P, \leqslant)$ is called Axiom A forcing if there is a sequence $\langle \leqslant_n : n \in \omega \rangle$ of partial orderings on P satisfying the following properties.

(1) For all n, $\leqslant_{n+1}$ is stronger than $\leqslant_n$ and $\leqslant_n$ is stronger than $\leqslant$.

(2) For every sequence $\langle p_n : n \in \omega \rangle$ satisfying $p_{n+1} \leqslant_n p_n$ (for every n), there is q such that $q \leqslant_n p_n$ (for every n).

(3) For every antichain $A \subseteq P$, $p \in P$ and $n \in \omega$, there is $q \leqslant_n p$ such that the set $\{r \in A : r \parallel q\}$ is countable.

To handle countable support iterations of Axiom A forcing, the following notion is useful.

Definition 2. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing. For $p, q \in P_\delta$ and $F \subseteq \delta$ and $n \in \omega$, we define $q \leqslant_{F,n} p \iff q \leqslant p$ and $\forall \alpha \in F \ (q \leqslant_n p)$. A sequence $\langle (p_n, F_n) : n \in \omega \rangle$ is called a fusion sequence if

(1) $p_n \in P_\delta$, $F_n \subseteq \delta$ finite.

(2) $p_{n+1} \leqslant_{F_n,n} p_n$.

(3) $\bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} \mathrm{supt}(p_n)$.

Lemma 3. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing. Let $\langle (p_n, F_n) : n \in \omega \rangle$ be a fusion sequence. Then there is a condition q such that $q \leqslant_{F_n,n} p_n$ for all n.

Lemma 4. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing. Let $F \subseteq \delta$ be a finite set and $n \in \omega$. Let $p \in P_\delta$ and $\dot{a}$ be a P_δ-name such that $P_\delta \Vdash \dot{a} \in V$. Then there is $q \leqslant_{F,n} p$ and a countable set x such that $q \Vdash \dot{a} \in x$.

For $x, y \in \omega^\omega$, we define $x <^\infty y$ iff there exist infinitely many n such that $x(n) < y(n)$.

Now that we are prepared, we state the main theorem modified from Judah–Repický Preservation Theorem [JR95], whose assumption is strengthened but the proof is simplified.

Theorem 5. Let $\langle P_\alpha, \dot{Q}_\alpha : \alpha < \delta \rangle$ be a countable support iteration of Axiom A forcing where δ is a limit ordinal. Assume that $P_\alpha \Vdash \forall x\, \exists y \in V\ x <^\infty y$ for any $\alpha < \delta$. Then P_δ also forces it.

Proof. Let $\dot{x}$ be a P_δ-name and $p \in P_\delta$.

We create a fusion sequence $\langle p_n, F_n : n \in \omega \rangle$ inductively.

Suppose we have constructed p_n and F_n. Put $\gamma = \max F_n$. Take a sequence $\langle \dot{q}_n^m : m \in \omega \rangle$ of P_γ-names and a P_γ-name $\dot{y}_n$ such that:

$$p_n \restriction \gamma \Vdash \langle \dot{q}_n^m : m \in \omega \rangle \text{ interprets } \dot{x} \text{ as } \dot{y}_n.$$

By the assumption, we can take a P_γ-name $\dot{z}_n$ such that

$$p_n \restriction \gamma \Vdash \text{“}\dot{y}_n <^\infty \dot{z}_n \text{ and } \dot{z}_n \in V\text{”}.$$

By the property of Axiom A, we can take $q_n \leqslant_{F_n, n} p_n \restriction \gamma$ and a countable set A_n such that

$$q_n \Vdash \dot{z}_n \in A_n.$$

Let $A_n = \{w_n^l : l \in \omega\}$. Take a name $\dot{m}_n$ of a natural number such that

$$q_n \Vdash \text{“}\dot{z}_n = w_n^l \to (\dot{y}_n(\dot{m}_n) < \dot{z}_n(\dot{m}_n) \text{ and } \dot{m}_n \geqslant n + l)\text{”}.$$

Take $\dot{r}_n$ such that

$$q_n \Vdash \dot{r}_n = \dot{q}_n^{\dot{m}_n}.$$

Put $p_{n+1} = q_n \cup \dot{r}_n$. Then we have

$$p_{n+1} \Vdash \dot{x}(\dot{m}_n) < \dot{z}_n(\dot{m}_n)$$

Set F_{n+1} in a bookkeeping manner to satisfy $\bigcup_n F_n = \bigcup_n \operatorname{supt}(p_n)$ finally.

This finishes the construction of the fusion sequence. Finally, let p be the limit condition of $\langle p_n, F_n : n \in \omega \rangle$ and let $u \in \omega^\omega$ be a real dominating all elements in $\{w_n^l : l, n \in \omega\}$, in particular

$$\forall l, n\ \forall m \geqslant l + n\ (w_n^l(m) < u(m)).$$

Then we have for all n,

$$p \Vdash \dot{x}(\dot{m}_n) < u(\dot{m}_n) \text{ and } \dot{m}_n > n.$$

□

References

[Bau83] James E. Baumgartner. “Iterated forcing”. *Surveys in set theory* 87 (1983), pp. 1–59.

[JR95] Haim Judah and Miroslav Repickỳ. “No random reals in countable support iterations”. *Israel Journal of Mathematics* 92.1 (1995), pp. 349–359.