Two problems concerning Hausdorff measures and the Lebesgue measure

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Introduction of me

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Motivation

Cardinal invariants defined by the null ideal and the meager ideal have been well studied for a long time and are summarized in Cichoń's diagram:

$$
\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

We would like to consider cardinal invariants defined by Hausdorff measures, which do not appear in Cichoń's diagram, and investigate their relationships.

Ideals defined by Hausdorff measures

We consider the Cantor space $(2^\omega, d)$, where

$$d(x,y) = 2^{-\min\{n : x(n) \neq y(n)\}} \quad (\text{for } x \neq y).$$

For a gauge function f, we define the f-Hausdorff measure 0 ideal by

$$\mathcal{N}^f = \{A \subseteq 2^\omega : \mathcal{H}^f(A) = 0\}.$$

And we put

$$\mathrm{HDZ} := \{A \subseteq 2^\omega : \dim_{\mathrm{H}}(A) = 0\} = \bigcap_{s>0} \mathcal{N}^{\mathsf{pow}_s},$$

where the gauge function pow_s $(s > 0)$ is given by

$$\mathsf{pow}_s(x) = x^s.$$

Expansion of Cichoń's diagram

$$
\begin{tikzcd}
& \mathsf{cov}(\mathcal{N}) \arrow[r] & \mathsf{cov}(\mathrm{HDZ}) \arrow[r] \arrow[drrr] & \mathsf{non}(\mathcal{M}) \arrow[r] & \mathsf{cof}(\mathcal{M}) \arrow[rr] & & \mathsf{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
& & & & & \mathsf{cof}(\mathrm{HDZ}) \arrow[ur] & & \\
& & & \mathfrak{b} \arrow[r] \arrow[uu] & \mathfrak{d} \arrow[uu] & & & \\
& & \mathsf{add}(\mathrm{HDZ}) \arrow[uuu] \arrow[drrr] & & & & & \\
\aleph_1 \arrow[r] & \mathsf{add}(\mathcal{N}) \arrow[rr] \arrow[uuuu] \arrow[ur] & & \mathsf{add}(\mathcal{M}) \arrow[r] \arrow[uu] & \mathsf{cov}(\mathcal{M}) \arrow[r] \arrow[uu] & \mathsf{non}(\mathrm{HDZ}) \arrow[r] \arrow[uuu] & \mathsf{non}(\mathcal{N}) \arrow[uuuu] &
\end{tikzcd}
$$

Relations with Yorioka ideals

- There are ideals called Yorioka ideals, which are parametrized by reals and have a combinatorial definition. We showed a relation between Hausdorff measure 0 ideals and Yorioka ideals:
  - $\forall f\ \exists g\ (\mathcal{I}_g \subseteq \mathcal{N}_f)$
  - $\forall g\ \exists f\ (\mathcal{N}_f \subseteq \mathcal{I}_g)$
- Using prior studies on Yorioka ideals and this fact, we got the following:

Theorem (G.)

$\aleph_1$ many cardinals of the form $\mathrm{cov}(\mathcal{N}_f)$ can be separeted and $\aleph_1$ many cardinals of the form $\mathrm{non}(\mathcal{N}_f)$ can be separated.

Also we showed $\mathcal{I}_{\mathrm{id}} \subseteq \mathrm{HDZ}$.

Further results

Shelah and Steprāns showed for every $s > 0$, $\mathrm{non}(\mathcal{N}_s)$ and $\mathrm{non}(\mathcal{N})$ can be separated. Here $\mathcal{N}^s = \mathcal{N}^{\mathsf{pow}_s}$ is s-dimensional Hausdorff measure 0 ideal.
We showed the following:

Theorem (G.)

This separation can be done by iterated Mathias forcing.

And we also showed that $\mathrm{cov}(\mathrm{HDZ})$ and $\mathrm{non}(\mathrm{HDZ})$ are stable under changing underlying space from 2^ω into n-dimensional Euclidean space $\mathbb{R}^n$ for each n.

Open questions

- Can we generalize the underlying spaces of HDZ further?
- Does ZFC prove that $\mathsf{cov}(\mathrm{HDZ}) = \mathsf{cov}(\mathcal{I}_{\mathrm{id}})$ and $\mathsf{non}(\mathrm{HDZ}) = \mathsf{non}(\mathcal{I}_{\mathrm{id}})$?
- Does ZFC prove that $\mathsf{add}(\mathrm{HDZ}) \leq \mathfrak{b}$ and $\mathfrak{d} \leq \mathsf{cof}(\mathrm{HDZ})$?
- Does ZFC prove that $\mathsf{add}(\mathrm{HDZ}) = \mathsf{add}(\mathcal{N})$ and $\mathsf{cof}(\mathrm{HDZ}) = \mathsf{cof}(\mathcal{N})$?
- Does ZFC prove that for every $0 < s < t < 1$, $\mathsf{non}(\mathcal{N}_s) = \mathsf{non}(\mathcal{N}_t)$? (Shelah–Steprāns)

Goldstern's theorem

(full domination order) For $x, x' \in \omega^\omega$, define a relation $x \leq x'$ by $(\forall n \in \omega)(x(n) \leq x'(n))$.

In 1993, Martin Goldstern proved the following theorem.

Goldstern's theorem $(\mathrm{ZF} + \mathrm{CC})$

Let (Y, μ) be a Polish probability space. Let $A \subseteq \omega^\omega \times Y$ be a $\mathbf{\Sigma}^1_1$ set. Assume that for each $x \in \omega^\omega$,

$$A_x := \{y \in Y : (x, y) \in A\}$$

has measure 0. Also, assume $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also measure 0.

The principle GP(Γ)

Definition

Let Γ be a pointclass. Then GP(Γ) means the following statement: Let (Y, μ) be a Polish probability space and $A \subseteq \omega^\omega \times Y$ be in Γ. Assume that for each $x \in \omega^\omega$, A_x has μ-measure 0. Also suppose that $(\forall x, x' \in \omega^\omega)(x \leq x' \Rightarrow A_x \subseteq A_{x'})$. Then $\bigcup_{x \in \omega^\omega} A_x$ has also μ-measure 0.

Goldstern's theorem says that $\mathrm{GP}(\boldsymbol{\Sigma}^1_1)$ holds.

Main Result

The symbol "all" denotes the class of all subsets of Polish spaces.

Theorem (G.)

GP(all) is independent from ZFC.

Consistency of $\neg$ GP(all)

In fact, the consistency of the negation follows from:

Theorem (G.)

Assume CH. Then $\neg$ GP(all) holds.

Consistency of $\neg$ GP(all)

Refining the last theorem, we get the following theorem.

Theorem (G.)

Assume that at least one of the following three conditions holds:

$$\mathrm{add}(\mathcal{N}) = \mathfrak{b},\ \mathrm{non}(\mathcal{N}) = \mathfrak{b} \text{ or } \mathrm{non}(\mathcal{N}) = \mathfrak{d}.$$

Then $\neg$ GP(all) holds.

$\mathrm{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$

$\mathrm{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$

$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F)\ f <^* g\}$

$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F)\ g <^* f\}$

Consistency of $\neg\,\mathrm{GP}(\mathrm{all})$

Assume that at least one of the following three conditions holds: $\mathrm{add}(\mathcal{N}) = \mathfrak{b}$, $\mathrm{non}(\mathcal{N}) = \mathfrak{b}$ or $\mathrm{non}(\mathcal{N}) = \mathfrak{d}$. Then $\neg\,\mathrm{GP}(\mathrm{all})$ holds.

$\mathrm{add}(\mathcal{N}) := \min\{\kappa : \text{the null ideal is not } \kappa\text{-additive}\}$

$\mathrm{non}(\mathcal{N}) := \min\{|A| : A \subseteq 2^\omega, A \text{ does not have measure } 0\}$

$\mathfrak{b} := \min\{|F| : F \subseteq \omega^\omega, \neg(\exists g \in \omega^\omega)(\forall f \in F)\ f <^* g\}$

$\mathfrak{d} := \min\{|F| : F \subseteq \omega^\omega, (\forall g \in \omega^\omega)(\exists f \in F)\ g <^* f\}$

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

$V = L$ implies $\neg\,\mathrm{GP}(\Delta^1_2)$

Refining the last proof in another way again, we get the following theorem.

Theorem (G.)

$V = L$ implies $\neg\,\mathrm{GP}(\Delta^1_2)$.

Consistency of GP(all)

Theorem (G.)

If ZFC is consistent then so is $\mathrm{ZFC} + \mathsf{GP(all)}$.

In fact, "The Laver model" satisfies GP(all).

equal to $\aleph_2$ in the Laver model

$$
\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathrm{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathrm{add}(\mathcal{N}) \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}
$$

equal to $\aleph_1$ in the Laver model

Connections with $\boldsymbol{\Sigma}^1_2$ regularity

Theorem (G.)

$\boldsymbol{\Sigma}^1_2$-LM implies $\mathrm{GP}(\boldsymbol{\Sigma}^1_2)$. And $\mathrm{GP}(\boldsymbol{\Delta}^1_2)$ implies $(\forall a \in \mathbb{R})(\exists z \in \omega^\omega)(z \text{ is a dominating real over } L[a])$.

$$\begin{tikzcd}
\boldsymbol{\Sigma}^1_2\text{-LM} \arrow[r] \arrow[d] & \boldsymbol{\Sigma}^1_2\text{-BP} \arrow[r] & (\forall a \in \mathbb{R})(\exists z \in \omega^\omega)(z \text{ is a dominating real over } L[a]) \\
\mathrm{GP}(\boldsymbol{\Sigma}^1_2) \arrow[r] & \mathrm{GP}(\boldsymbol{\Delta}^1_2) \arrow[ur] &
\end{tikzcd}$$

AD and the Solovay model

Theorem (G.)

Assume $\mathrm{ZF} + \mathrm{AD}$. Then GP(all) holds.

Theorem (G.)

In the Solovay model, GP(all) holds.

Open questions

1. Does $V = L$ imply $\neg\,\mathrm{GP}(\Pi^1_1)$?
2. Is $\mathrm{ZFC} + (\mathfrak{c} > \aleph_2) + \mathrm{GP}(\text{all})$ consistent?
3. Is $\mathrm{ZFC} + (\mathfrak{b} < \mathfrak{d}) + \mathrm{GP}(\text{all})$ consistent?
4. Is there a model of ZF satisfying that every set of reals are measurable and $\neg\,\mathrm{GP}(\text{all})$?
5. Is it possible to separate $\mathrm{GP}(\boldsymbol{\Sigma}^1_{n+1})$ and $\mathrm{GP}(\boldsymbol{\Sigma}^1_n)$ for some (or every) $n \geq 2$ (without large cardinals)?

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