Calculus 1B: Integration Unit 1: The integral Problem Set 1 Part B

3. Generalizing MVT

You know how to find a linear approximation at $x = 0$, and that there is some error that is of order x^2, in big-"O" notation, $O(x^2)$. The quadratic approximation can be more accurate with error on the order of x^3, or $O(x^3)$. As we might expect, cubic and higher order polynomials give us better and better approximations to a function. But what is the actual size of the error? It turns out that the **MVT** is exactly the tool for giving us a bound on the size of the error. Let's see how through the following sequence of problems.

1B (5)

Recall the following about linear and quadratic approximations.

Let $P_1(x)$ be the linear, or degree 1 polynomial, approximation to $f(x)$ at $x = 0$.

$$P_1(x) = f(0) + f'(0)x$$

The quadratic, or degree 2 polynomial, $P_2(x)$ approximation, is obtained by adding a quadratic term to $P_1(x)$. In other words,

$$P_2(x) := P_1(x) + C_2x^2 = (f(0) + f'(0)x) + C_2x^2$$

The only undetermined part of $P_2(x)$ is the coefficient C_2 of the new term. To determine C_2 , we require the second derivative of $P_2(x)$ to match the second derivative of $f(x)$ at $x = 0$. That is,

$$\frac{d^2}{dx^2}P_2(x) = \frac{d^2}{dx^2}f(x)|_{x=0}$$

$$P_2(x) := P_1(x) + \frac{f''(0)}{2}x^2$$

1B (6)

Similarly, We can define the cubic, or degree 3 polynomial, approximation to $f(x)$ at $x = 0$ as the best fit cubic polynomial to $f(x)$ at $x = 0$.

We obtain the cubic approximation by adding a cubic term to $P_2(x)$.

$$P_3(x) := P_2(x) + C_3 x^3$$

Since $P_2(x)$ is already determined, the only unknown in $P_3(x)$ is the coefficient of the new term, and we find it by matching third derivatives of $P_3(x)$ and $f(x)$ at $x = 0$.

$$\frac{d^3}{dx^3} P_3(x) = \frac{d^3}{dx^3} f(x)|_{x=0}$$

$$P_3(x) := P_2(x) + \frac{f^{(3)}(0)}{3!} x^3$$

1B (8)

We can keep going and define $P_4(x)$ using $P_3(x)$.

$$P_4(x) := P_3(x) + C_4 x^4$$

Again, the only unknown in $P_4(x)$ is the coefficient of the new term, and we find it by matching fourth derivatives.

$$\frac{d^4}{dx^4} P_3(x) = \frac{d^4}{dx^4} f(x)|_{x=0}$$

$$P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2} x^2 + \frac{f^{(3)}(0)}{3!} + \frac{f^{(4)}(0)}{4!}$$

Degree n approximations

Using the above method, we can define a degree n polynomial approximation to $f(x)$ near $x = 0$ for any n. In particular, once we know $P_{n-1}(x)$, we can add a n degree term to $P_{n-1}(x)$ and then find the unknown coefficient by matching n^{th} derivatives. That is,

$$P_n(x) := P_{n-1}(x) + C_n x^n, s.t. \frac{d^n}{dx^n} P_n(x) = \frac{d^n}{dx^n} f(x)|_{x=0}$$

Generalizing MVT as the error of approximations

Let $P_n(x)$ be the degree n polynomial approximation to $f(x)$ at $x = 0$. For any degree , away from $x = 0$, $P_n(x)$ may differ from the actual value of $f(x)$. Let this difference be called the error of approximation and denoted by ϵ_n.

$$\epsilon_n(x) = f(x) - P_n(x)$$

Let us start by considering ϵ_0.

$$\epsilon_0(x) = f(x) - P_0(x) = f(x) - f(0)$$

The Mean value theorem(MVT) tells us that there is some c on the interval $0 < c < x$ such that

$$\epsilon_0(x) = xf'(x)$$

It turns out that the error of approximation by $P_n(x)$ has a similar expression. (This result is called **Taylor's Theorem**, and we'll see it again in 18.01.3x.)

$$\epsilon_1(x) = f(x) - P_1(x) = \frac{f''(c)}{2!}(x)^2$$

for some $c, 0 < c < x$.

$$\epsilon_2(x) = f(x) - P_2(x) = \frac{f'''(c)}{3!}(x)^3$$

for some $c, 0 < c < x$.

$$\vdots$$

$$\epsilon_n(x) = f(x) - P_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x)^{n+1}$$

for some $c, 0 < c < x$.