Kinematic Wave Theory

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Outline

- **Introduction**
- Conservation law
- Initial and boundary conditions
- Homogeneous problems
  - A special case
  - General cases
- Summary

Framework

Traffic flow theory

— Dynamic macroscopic models (Continuum models)

—— Kinematic wave theory (LWR model)

- **Continuum models** assume that Δx and Δt can be replaced by differentials (i.e., dx and dt - treating finite intervals as infinitesimally small differentials), which open the door to the world of calculus; i.e. assumption that a relationship between flow and density existed at all locations and times. The traffic is viewed as a fluid with a well-defined density everywhere;
- **KWT** (Lighthill and Whitham, 1955 and Richards, 1956; thus also LWR) is the simplest one of the continuum models.
- The **goal** is to predict the evolution of traffic from initial conditions;

A description of LWR

- Lighthill and Whitham (1955) and Richards (1956) provided the first traffic flow approximation models that compared traffic flow to fluid flow. LWR is extensively used as the preferred model for representing flow dynamics from a **macroscopic** perspective.
- **CAPABLE** in providing a coarse description of main traffic features (e.g., formation and dissolution of shockwaves); **INADEQUATE** in describing complex traffic patterns such as stop-and-go waves, capacity drop phenomena.
- Based on a *hyperbolic partial differential equation* of first order, which describes the conservation of cars in time and space.
- Assumes that the relation q-k-v observed under steady states holds at all times, even when flow and density vary with time and space. This assumption suppresses all other traffic states and phase transitions not belonging on this curve.

Hyperbolic partial differential equation

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Hyperbolic partial differential equation

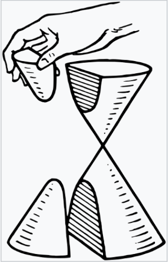
In mathematics, a **hyperbolic partial differential equation** of order n is a partial differential equation (PDE) that, roughly speaking, has a well-posed initial value problem for the first $n-1$ derivatives. More precisely, the Cauchy problem can be locally solved for arbitrary initial data along any non-characteristic hypersurface. Many of the equations of mechanics are hyperbolic, and so the study of hyperbolic equations is of substantial contemporary interest. The model hyperbolic equation is the wave equation. In one spatial dimension, this is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

The equation has the property that, if u and its first time derivative are arbitrarily specified initial data on the line $t = 0$ (with sufficient smoothness properties), then there exists a solution for all time t.

The solutions of hyperbolic equations are "wave-like". If a disturbance is made in the initial data of a hyperbolic differential equation, then not every point of space feels the disturbance at once. Relative to a fixed time coordinate, disturbances have a finite propagation speed. They travel along the characteristics of the equation. This feature qualitatively distinguishes hyperbolic equations from elliptic partial differential equations and parabolic partial differential equations. A perturbation of the initial (or boundary) data of an elliptic or parabolic equation is felt at once by essentially all points in the domain.

Although the definition of hyperbolicity is fundamentally a qualitative one, there are precise criteria that depend on the particular kind of differential equation under consideration. There is a well-developed theory for linear differential operators, due to Lars Gårding, in the context of microlocal analysis. Nonlinear differential equations are hyperbolic if their linearizations are hyperbolic in the sense of Gårding. There is a somewhat different theory for first order systems of equations coming from systems of conservation laws.



A hyperbola is an open curve with two branches, the intersection of a plane with both halves of a double cone. The plane does not have to be parallel to the axis of the cone; the hyperbola will be symmetrical in any case.

Basic ideas

The basic components of the KWT are as follows:

- Fundamental diagram;
- Conservation law;
- Initial and boundary conditions;

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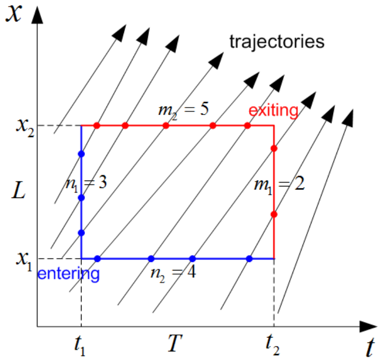
Conservation law w/o lane-changing



$$m_1 + m_2 = n_1 + n_2$$

$$\implies \quad (m_1 - n_1) + (m_2 - n_2) = 0$$

$$\implies \quad \frac{m_1 - n_1}{LT} + \frac{m_2 - n_2}{LT} = 0$$

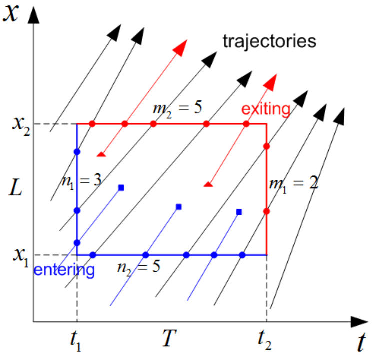
$$\implies \quad \frac{k_{t_2} - k_{t_1}}{T} + \frac{q_{x_2} - q_{x_1}}{L} = 0$$

$$\implies \quad \frac{\Delta k}{T} + \frac{\Delta q}{L} = 0 \;(\Delta : \textit{small } L \& T)$$

$$\implies \quad \frac{\partial \boldsymbol{k(t,x)}}{\partial \boldsymbol{t}} + \frac{\partial \boldsymbol{q(t,x)}}{\partial \boldsymbol{x}} = \mathbf{0} \;(\textit{smaller})$$

Hyperbolic PDE
treating finite intervals as infinitesimally small differentials

Conservation law with lane-changing

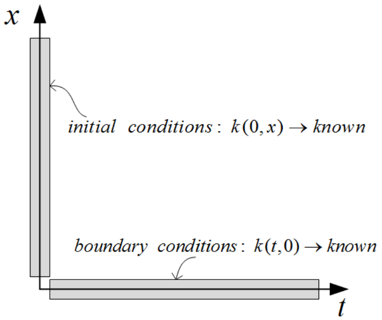


$$(n_1 + n_2) - (m_1 + m_2) = LC^{out} - LC^{in} \Longrightarrow \frac{\partial k(t,x)}{\partial t} + \frac{\partial q(t,x)}{\partial x} = \frac{LC^{in}}{LT} - \frac{LC^{out}}{LT}$$

where $\frac{LC^{in}}{LT} - \frac{LC^{out}}{LT}$ is the net lane-changing rate into the lane.

Initial and boundary conditions

- **Initial conditions**: $k(0, x)$ is known;
- **Boundary conditions**: $k(t, 0)$ is known;



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Homogeneous problem

Assume

(1) homogeneous vehicles (2) no on- or off-ramp (3) Concave FD.

Have

(i) fundamental diagram (ii) conservation law (iii) initial conditions:

$$\begin{cases} \text{(i) } q = Q(k) \\ \text{(ii) } \frac{\partial k(x,t)}{\partial t} + \frac{\partial q(x,t)}{\partial x} = 0 \\ \text{(iii) } k(x,0) = g(x) \end{cases} \longrightarrow \begin{cases} \frac{\partial k}{\partial t} + Q'(k)\frac{\partial k}{\partial x} = 0 \\ k(x,0) = g(x) \end{cases}$$

GOAL: to get $q(x,t)$, $k(x,t)$. Let's solve the system of equations

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Chain rule

In calculus, the **chain rule** is a formula that expresses the derivative of the composition of two differentiable functions f and g in terms of the derivatives of f and g. More precisely, if $h = f \circ g$ is the function such that $h(x) = f(g(x))$ for every x, then the chain rule is, in Lagrange's notation,

$$h'(x) = f'(g(x))g'(x).$$

or, equivalently,

$$h' = (f \circ g)' = (f' \circ g) \cdot g'.$$

The chain rule may also be expressed in Leibniz's notation. If a variable z depends on the variable y, which itself depends on the variable x (that is, y and z are dependent variables), then z depends on x as well, via the intermediate variable y. In this case, the chain rule is expressed as

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx},$$

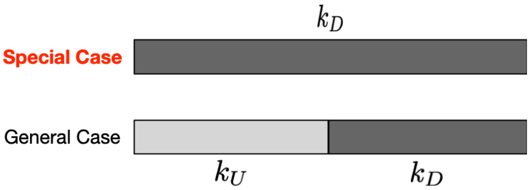
and

$$\left.\frac{dz}{dx}\right|_x = \left.\frac{dz}{dy}\right|_{y(x)} \cdot \left.\frac{dy}{dx}\right|_x,$$

for indicating at which points the derivatives have to be evaluated.

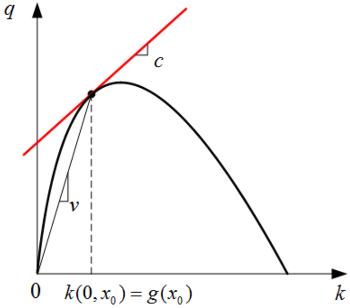
In integration, the counterpart to the chain rule is the substitution rule.

Homogeneous problem



Special case: traffic on a road is in one state

$$Q'(k) = c(k) = c, \qquad (c \text{ is a constant})$$



Special case: the advection equation

Special case:

$$\boldsymbol{Q'(k) = c(k) = c}$$

then we obtain $\boldsymbol{\frac{\partial k}{\partial t} + c\frac{\partial k}{\partial x} = 0}$, which is the advection equation.

Solution for k, given (x, t, c) - space, time, slope:

$$\boxed{\boldsymbol{k(t,x) = f(x - ct)}}$$

where $f(\cdot)$ is an arbitrary function.

Verification:

$$\begin{cases} \frac{\partial k}{\partial t} = \frac{df(y)}{dy} \cdot \frac{dy}{dt} = \frac{df(y)\cdot(-c)}{dy} \\ c\frac{\partial k}{\partial x} = c\frac{df(y)}{dy} \cdot \frac{dy}{dx} = c\frac{df(y)}{dy} \end{cases} \longrightarrow \quad \frac{\partial k}{\partial t} + c\frac{\partial k}{\partial x} = 0$$

$$y = x - ct$$

Special case: the advection equation

The solution, i.e., $k(x,t) = f(x-ct)$, means that

if $(x-ct)$ is constant, e.g.,

$$x - ct = x_0$$

density $k(x,t)$ at location x on a road at time t is constant:

$$k(x,t) = f(x-ct) = \underline{f(x_0) = \text{constant}}$$

i.e.,
density $k(x,t)$ along the following straight line is *constant*:

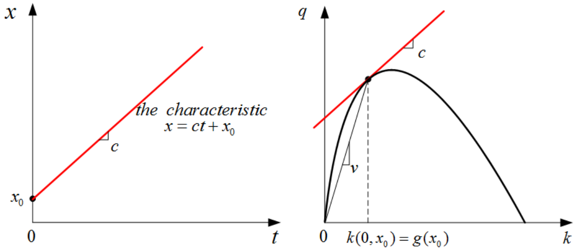
$$\underline{x - ct = x_0 \text{ in a x-t plane with slope } c}$$

where $c = Q'(k)$ is the slope of the fundamental diagram.

The line is called "characteristic"

Special case: the advection equation

In the diagrams, the characteristic is



Given initial condition $k(x_0, 0)$, we have a characteristic with a slope $c = Q'(k(x_0, 0))$. Along this characteristic, the density is the same and equal to $k(x_0, 0)$.

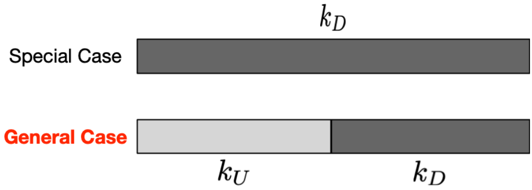
Characteristics (called in mathematics books) are also called "waves" or "signals" in the fluids literature.

The properties of characteristics are:

- Density, flow and speed are CONSTANT and equal to the values at the boundary;
- Vehicle speed is always GREATER than the characteristic speed c;
- Relations with the fundamental diagram:

traffic states	k	c	direction of characteristics
free flow	$k < k_c$	$c = Q'(k) > 0$	with traffic stream
capacity	$k = k_c$	$c = Q'(k) = 0$	stationary
congestion	$k > k_c$	$c = Q'(k) < 0$	against traffic stream

Homogeneous problem

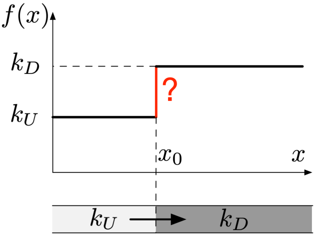


General cases: the Riemann problem

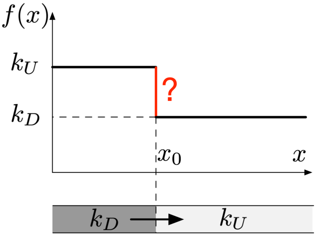
The initial condition is

$$Q'(k) = c(k) = \begin{cases} c_U, & k = k_U, \; x < x_0 \\ c_D, & k = k_D, \; x > x_0 \end{cases}$$

The goal is how the two traffic states transit:

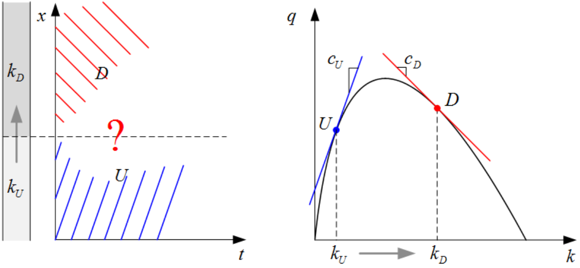


Case I

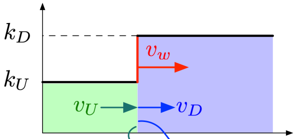


Case II

Case I: Shock waves



Case I: Shock waves



traffic ($\begin{matrix} v_U \\ v_D \end{matrix}$) : right → upstream traffic, downstream traffic

wave (v_w) : right

assume

Passing the cross-section v_w

	Relative traffic speed	Traffic flow
upstream traffic	$v_U - v_w$	$k_U\,(v_U - v_w)$
downstream traffic	$v_D - v_w$	$k_D\,(v_D - v_w)$

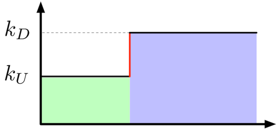
EQUAL
Passing the same cross-section
Conservation

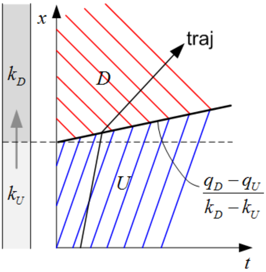
$$k_U\,(v_U - v_w) = k_D\,(v_D - v_w)$$

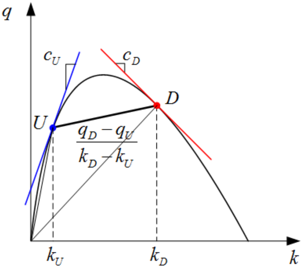
$$\boxed{v_w = \frac{k_D v_D - k_U v_U}{k_D - k_U} = \frac{q_D - q_U}{k_D - k_U}}$$

Case I: Shock Wave

$$k_U < k_D$$

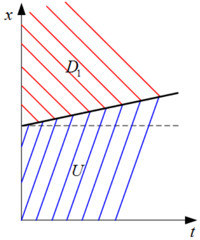




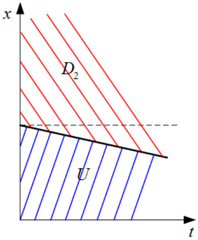


- Shock waves separate the waves from lower density to higher density
- Characteristics end in the shock wave and the traffic state shows a discontinuous changes
- Trajectories that cross a shock wave change their speed abruptly

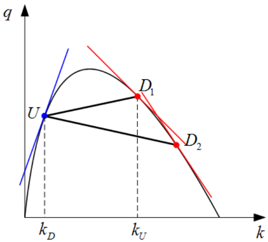
Directions of shock waves



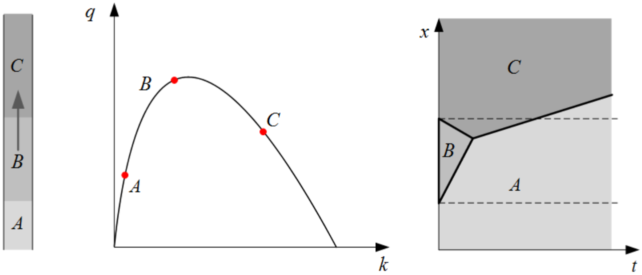
Moving with the traffic



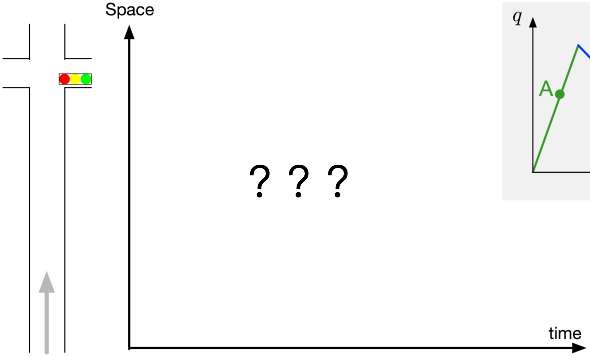
Moving against the traffic

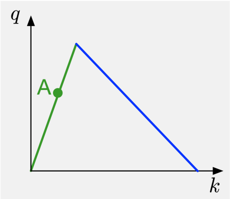


Practice-1: merging shock waves



Practice-2: approaching an intersection





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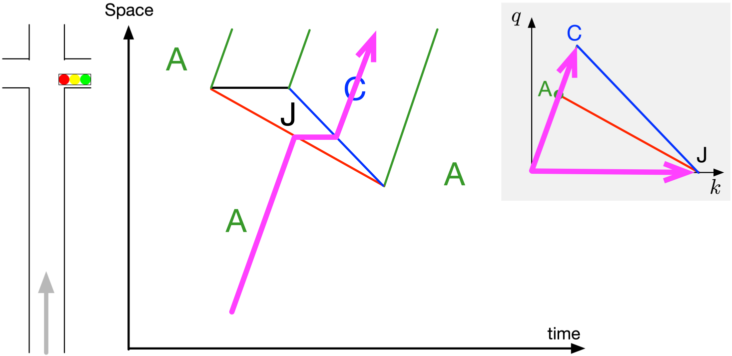
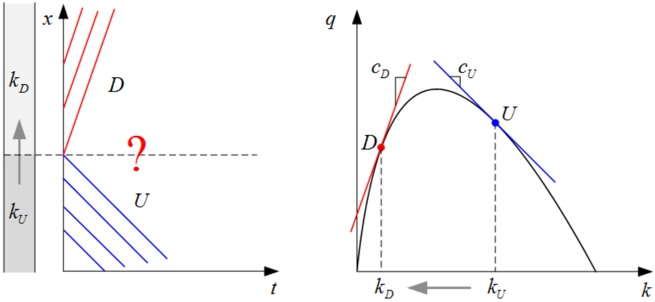
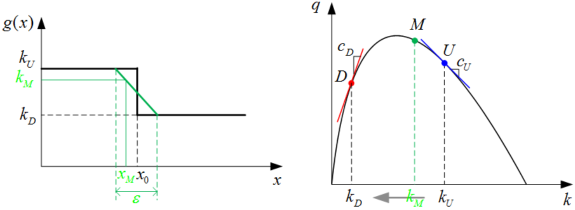


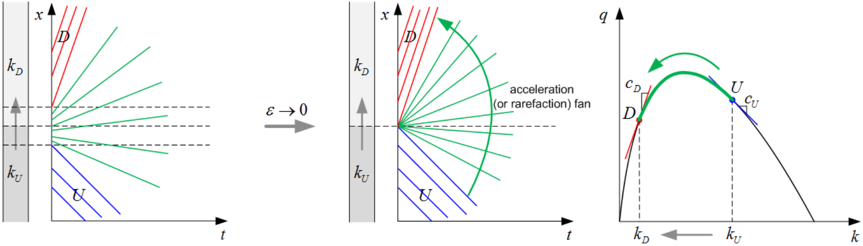
Case II: Fans



Case II: Fans

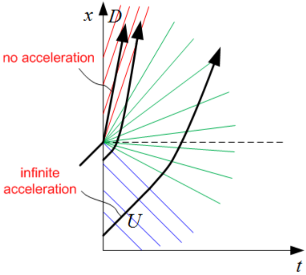
(1) expand the process - instantaneous event; (2) contract it back

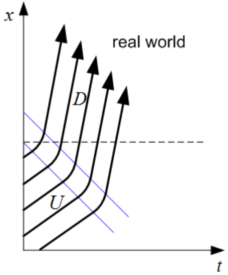




Case II: Weakness of FDs

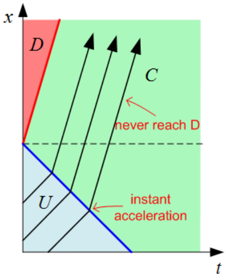
- Acceleration does not exist when $t = 0$
- Time to accelerate from v_U to v_D increases with the vehicle number

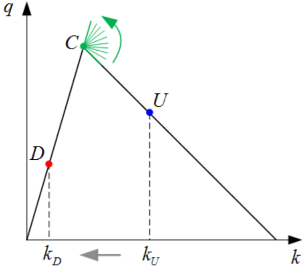




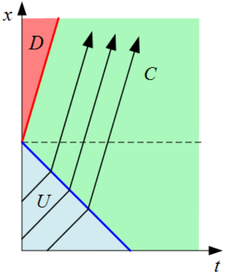
Case II: Triangular Fundamental Diagrams

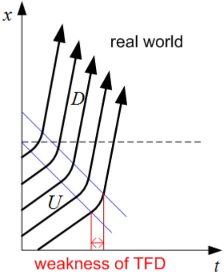
Traffic state C carries all intermediate characteristics $v_{char} \in [-w, v_f]$





Case II: Weakness of TFDs





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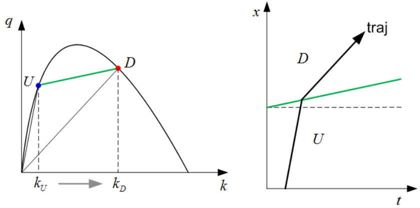
Summary

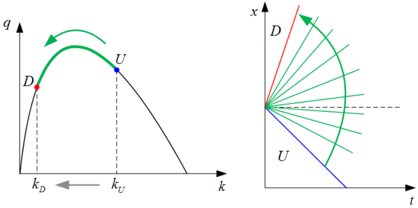
Initial Value Problem (IVP) : $\begin{cases} \partial_t k + Q'(k)\partial_x k = 0 \\ k(0,x) = g(x) \end{cases}$

- ONE traffic state: $Q'(k) = c \;\rightarrow\;$ Solution: $k(t,x) = f(x - tc)$
- TWO traffic states: decompose into a series of Riemann problems

Case I

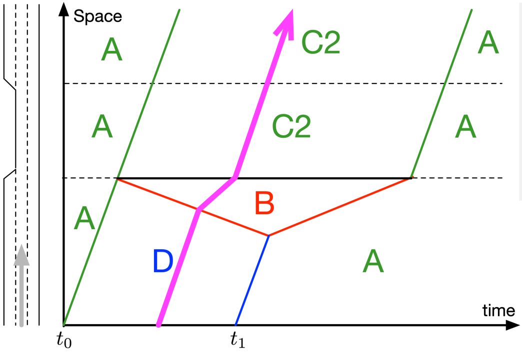


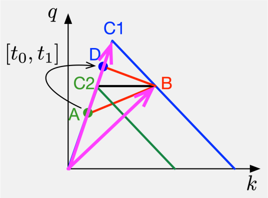
Case II



Practice-3

Facing a bottleneck (3 lanes $\rightarrow$ 2 lanes), traffic demand increases to **D** from **A** during t_0 and t_1. How will the traffic evolve?





Thank you!