Analysis 3

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Contents

1. Measure

1.1 Other critirea of measurability for Lebesgue measure

As we remember, the definition of a measurable set is difficult to check. Thus, we would like to have better critirea.

Theorem 1. $E \subset \mathbb{R}$ is Lebesgue measurable if and only if one of the following holds:

1. For every $\varepsilon > 0$ there exists an open set O, such that $E \subset O$ and $m^*(O \setminus E) < \varepsilon$.
2. There exists a G_δ-set G, such that $E \subset G$ and $m^*(G \setminus E) = 0$.
   (A G_δ-set is a countable intersection of open sets.)
3. For every $\varepsilon > 0$ there exists a closed set F, such that $F \subset E$ and $m^*(E \setminus F) < \varepsilon$.
4. There exists a F_σ-set F, such that $F \subset E$ and $m^*(E \setminus F) = 0$.
   (A F_δ-set is a countable union of closed sets.)

Proof.

- E is measurable $\implies$ 1.

  If $m^*(E) < \infty$, then from the definition of m^* we can find O — a finite union of open intervals, such that $m^*(O) < m^*(E) + \varepsilon$. Since O is an open set, it's measurable (as we proved earlier). Therefore, both E and O are measurable, thus

  $$\varepsilon > m^*(O) - m^*(E) \overset{E,O\in\mathcal{M}}{=} m^*(O \setminus E)$$

  If $m^*(E) = \infty$, let's split the set E into a countable number of sets with finite measure. For example, by splitting the real line into segments of length 1. So, $E = \cup_1^\infty E_k$, where $m^*(E_k) < \infty$. Then let's use geometrically decreasing ε's for the covers of each E_k: $\varepsilon/2$ for E_1, $\varepsilon/4$ for E_2, and so on. When we sum up the inequalities, the fractions will sum up to ε. So, we obtained our O, now continue like in the previous case.

  ***Definition* 1.** A measure μ on X is called *σ-finite*, if $X = \cup_1^\infty X_k$ and $\mu(X_k) < \infty$ for all k.

  In words: if there exists a subdivision of X into a countable number of set of finite measure.

- $1 \implies 2$.

  From 1, $\forall k \in \mathbb{N},\ \exists O_k$ — open, such that $E \subset O_k$ and $m^*(O_k \setminus E) < \frac{1}{k}$. Now let's take

  $$G := \bigcap_1^\infty O_k \implies \forall k:\ m^*(G \setminus E) \leqslant m^*(O_k \setminus E) < \frac{1}{k} \implies m^*(G \setminus E) = 0$$

- $2 \implies E$ is measurable.

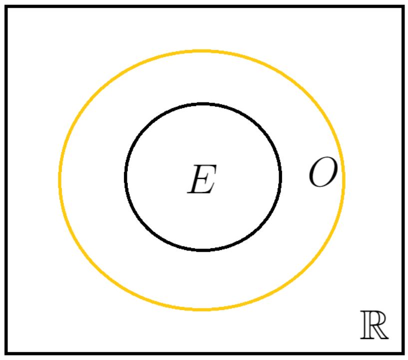
  G is a G_δ-set. As a countable intersection of open sets, it's in Borel σ-algebra, and thus is Lebesgue-measurable. $m^*(G \setminus E) = 0$, then $G \setminus E$ is Lebesgue-measurable, then $E = G \setminus (G \setminus E)$ is measurable as a difference of two measurable sets.

- $3 \iff 1,\ 4 \iff 2$.

  If we assume that 3 holds for E, then, if we take $O = \overline{F}$, 1 will hold for $\overline{E}$. Therefore, $\overline{E}$ is measurable, then E is measurable (as $\mathcal{M}$ is a σ-algebra).

  If 1 holds for E, then E is measurable, then $\overline{E}$ is measurable, then 1 holds for $\overline{E}$. Now take $F = \overline{O}$, therefore, 3 holds for E.

  In the same way, 2 and 4 are equivalent as well.



$\square$

Theorem 2. For every $E \in \mathcal{M}$ with $m(E) < \infty$ and for every $\varepsilon < \infty$ there exists an infinite disjoint collection of open intervals $\{I_k\}_1^n$, such that $O = \cup_{k=1}^n I_k$ and $m(E \triangle O) < \varepsilon$.

(Here $\triangle$ is the symmetric difference of two sets).

Proof. From part 1 of the previous theorem, we can take such an open set U, that $E \subset U$ and $m(U \setminus E) < \varepsilon/2$.

As we proved earlier, we can represent U as a countable union of disjoint open intervals I_k. Then:

$$\forall n : \bigcup_1^n I_k \subset U \implies \forall n : \sum_1^n m(I_k) \leqslant m(U) < \infty$$

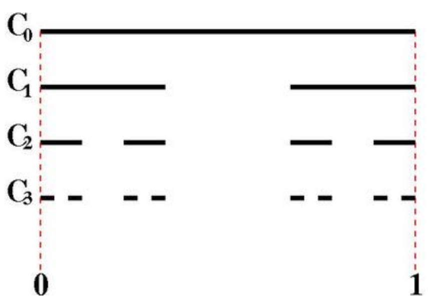
Now take n, such that $\sum_{n+1}^{\infty} m(I_k) < \varepsilon/2$, and put $O := \cup_{k=1}^n I_k$. Then $m(O \setminus E) < \varepsilon/2$ and $m(E \setminus O) < \varepsilon/2$, therefore, the measure of the symmetric difference is less that ε. $\square$

1.2 TBA

Questions:

1. If $m(A) = 0$, is A countable?
2. We know that $\mathcal{B} \subset \mathcal{M}$. Is this inclusion proper?

***Definition* 1** (Cantor set)**.** Let's take $[0, 1]$, split it into three parts and remove the middle part. Then continue such process. The *Cantor set* is the set $C := \cap_0^{\infty} C_k$.



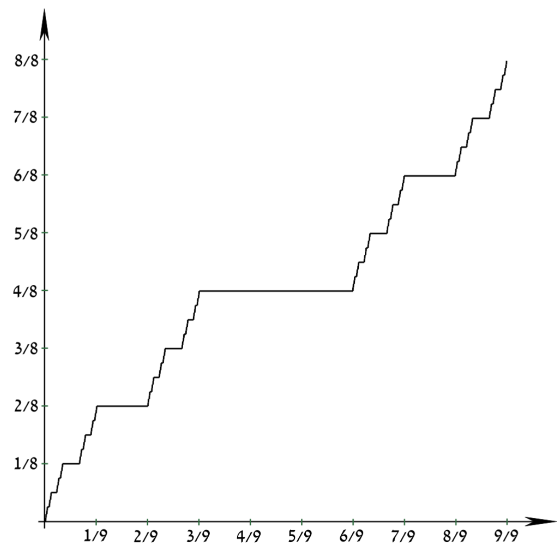
Cantor set illustration from here.

Remark. The measure of C_n is $(2/3)^n = 0$. From the continuity of measure, the measure of the intersection is the limit of measures of individual sets, and thus $m(C) = 0$.

Remark. The Cantor set is countable, because if we have a sequence of zeros and ones, we can traverse down-left on 0 and down-right on 1. The intersection of the corresponding intervals will be a single point of C. So, there's a bijection between C and $\{0,1\}^{\mathbb{N}}$, therefore, C is indeed uncountable.

Remark. Usually, when we remove the middle intervals on each step, we keep the end points. If we choose to remove the end points, we essentially remove the points that correspond to sequences that end with an infinite sequence of 0's or an infinite sequence of 1's. It's clear that not all sequences of 0's and 1's are like that, so there is going to be plenty of points left in the Cantor set, anyway.

Definition 2 (The Cantor function)**.** The Cantor function $\varphi : [0,1] \to [0,1]$ is defined as follows. Let's first take the unit interval $[0,1]$, split it in three, and define φ to be $\frac{1}{2}$ on $[1/3, 2/3]$. Then let's continue the same with $[0,1/3]$ and $[2/3,1]$, and so on.



Now we have defined φ for all points on the Cantor set. Here's how we'll define it on all others:

$$O \coloneqq C \setminus \{0,1\}$$
$$\forall x \in C \setminus \{0,1\} : \ \varphi(x) \coloneqq \sup\{\varphi(t) \mid t \in O \cap [0,x)\}$$

Proposition 1. φ is increasing, continuous, surjective ($[0,1]$ acts on $[0,1]$), φ' exists for open set O of measure 1, $\varphi'\Big|_O \equiv 0$.

Proof. φ is increasing (that's clear), therefore, if φ is discontinuous, then it has a jump discontinuity and there will be and interval, say, $I \subset [0,1]$, such that $\varphi([0,1]) \cap I = \varnothing$. But

$$\varphi([0,1]) \supset \left\{ \frac{m}{2^k} \;\middle|\; k \in \mathbb{N}, m \in [0, 2^k] \right\} = K$$

And K is dense in $[0,1]$, therefore, $K \cap I \neq \varnothing$, which is a contradiction. □

Remark. The Cantor function is a source of a lot of counterexamples. We are going to use it to answer the second question from earlier.

Definition 3.

$$\psi : [0,1] \to [0,2], \ \psi(x) = \psi(x) + x$$

ψ is strictly increasing, continuous, surjective.

Proposition 2. 1. ψ maps C onto a measurable set of measure 1.

2. ψ maps some subset of C onto a non-measurable set.

Proof. 1. $m(C) = 0$, thus, $m(O) = 1$. We know that φ is constant on every part of O. Therefore, ψ looks like x on every part of O, and there's a countable number of such intervals. Therefore, O is mapped to a set of measure 1. Thus, $m(\psi(C)) = m(\psi([0,1])) - m(\psi(O)) = 2 - 1 = 1$.

2. Since $\psi(C)$ has measure $1 > 0$, by the second homework, there exists a non-measurable set $N \supset \psi(C)$.

$\square$

Corollary 1. There is a (measurable) subset of C that is not Borel.

Remark. Since $m(C) = 0$, the outer measure of each subset of C is also 0, thus, every subset of C is measurable. So, the "measurable" part of the corollary is obvious.

Proposition 3. If $f : E \to \mathbb{R}$ is continuous, $E \in \mathcal{M}$, then the preimage of every Borel set is measurable.

$$\forall B \in \mathcal{B} : f^{-1}(B) = \{x \in E \mid f(x) \in B\} \in \mathcal{M}$$

Let's show how we can prove the corollary if we prove the proposition:

Proof of Corollary 1. $\psi^{-1} : [0,2] \to [0,1]$ is continuous and bijective. Let's define $\tilde{C} = \psi^{-1}(N)$. Since $N \subset C$, we have $\tilde{C} \subset C$. Assume that $\tilde{C}$ is a Borel set, and thus it's measurable. Then, by Proposition 3, its pre-image, N, must be measurable as well. But we know it isn't?! $\square$

Let's talk about something more general now, and later return to Proposition 3.

***Definition* 4.** $f : X \to Y$ is continuous, if for every open set $O \subset Y$, $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is open in X.

***Definition* 5** (Induced topology)**.** If X is a topological space and $Y \subset X$, the induced topology on Y is defined as follows: $O \subset Y$ is open in Y if there exists an open subset $\tilde{O} \subset X$ in X, such that $O = \tilde{O} \cap Y$.