Numerical Methods

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February 3, 2023

Contents

1. Introduction

***Definition* 1.1** (Numerical Methods)**.** Numerical Methods are algorithmic approaches to numerically solve mathematical problems. We use it often when it is hard/difficult/impossible to solve analytically.

1.1 Taylor series

Given a function $f : \mathbb{R} \to \mathbb{R}$ (that is hard to evaluate for some $x \in \mathbb{R}$), but f and $f^{(n)}$ are known for a value c, which is close to x. Can we use this information to approximate $f(x)$?

We know values for $\cos^{(n)}(0)$.

$$\begin{cases} f(0) = \cos(0) = 1 \\ f'(0) = -\sin(0) = 0 \\ f''(0) = -\cos(0) = -1 \end{cases} \qquad \text{for } c = 0$$

Can we get $\cos(0.1)$ from this?

***Definition* 1.2** (Taylor series)**.** Let $f : \mathbb{R} \to \mathbb{R}$, differentiable infinitely many times at $c \in \mathbb{R}$. So we have $f^{(k)}(c),\ k = 1, 2, \ldots$. Then the Taylor series of f at c is:

$$f(x) \approx f(c) + \frac{f(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(x-c)^k$$

Remark. Taylor series is a power series.

Remark. For $c = 0$ also known as Maclaurin series

Remark. A power series has an interval/radius of convergence. You can only evaluate the series if $x \in$ interval of convergence.

Example 1. What is the Taylor series for $f(x) = e^x$ at $c = 0$? We have $f^{(k)}(x) = e^x$, so $f^{(k)}(0) = 1$. Thus:

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

and the radius of convergence is ∞.

I.e. for any $x \in \mathbb{R}$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For an algorithm we need a finite amount of terms. For example,

$$e^x \approx \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 = 1 + x + \frac{x^2}{2}$$

This is a polynomial!

Example 2. Let's calculate Taylor series of a polynomial.

$$f(x) = 4x^2 + 5x + 7,\ c = 2$$
$$f(2) = 33,\ f'(2) = 8x + 5\Big|_{x=2} = 21,\ f''(2) = 8$$

Taylor series:

$$33 + 21(x-2) + \frac{8}{2}(x-2)^2 = 4x^2 + 5x + 7 = f(x)$$

Taylor series of a polynomial is itself.

Theorem 1.1 (Taylor theorem)**.** Let $f \in C^{n+1}([a,b])$ (i.e. f is $(n+1)$-times continuously differentiable). Then for any $x \in [a,b]$ we have that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-c)^{n+1}$$

where ξ_x is a point that depends on x and which is between c and x.

The first sum is called *truncated Taylor series*, the remainder is called *the error*.

Example. For $n = 0$:

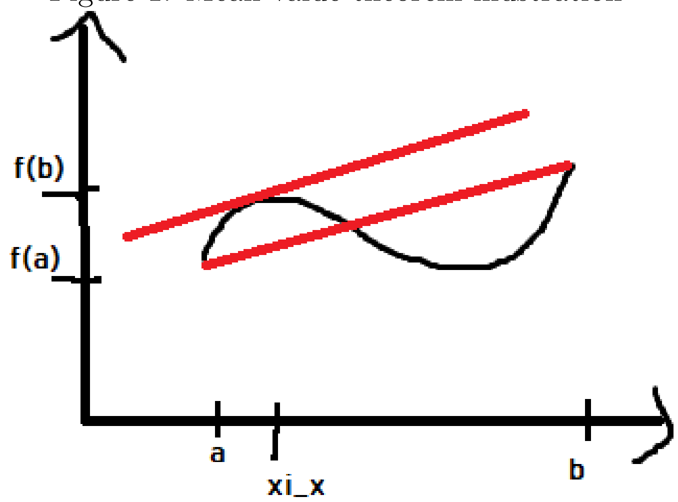
$$f(x) = f(c) + f'(\xi_x)(x-c)$$

Choose $c = a$, $x = b$:

$$f(b) = f(a) + f'(\xi_x)(b-a) \iff f'(\xi_x) = \frac{f(b)-f(a)}{b-a}$$

This is the mean value theorem!

Figure 1: Mean value theorem illustration

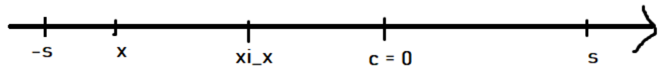


***Definition* 1.3.** We say that the Taylor series *represents* the function f at x if the Taylor series converges at that point, i.e. the remainder tends to zero as $n \to \infty$.

Example 1. Back to e^x: $f(x) = e^x$, $c = 0$, ξ_x is between c and x.

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^{\xi_x}}{(n+1)!}x^{n+1}$$

For any $x \in \mathbb{R}$ we find $s \in \mathbb{R}_0^+$ ($\mathbb{R}_0^+$ are all real, positive numbers including 0) so that $|x| \leqslant s$, and $|\xi_x| \leqslant s$ because ξ_x is between c and x.



Because e^x is monotone increasing, we have $e^{\xi_x} \leqslant e^s$, thus

$$\lim_{n\to\infty}\left|\frac{e^{\xi^x}}{(n+1)!}x^{n+1}\right| \leqslant \lim_{n\to\infty}\left|\frac{e^s}{(n+1)!}\right|s^{n+1} = e^s \lim_{n\to\infty}\frac{s^{n+1}}{(n+1)!} = 0$$

Because $(n+1)!$ will grow faster than any power of $s \implies \lim_{n\to\infty}\left|\frac{e^{\xi_x}}{(n+1)!}x^{n+1}\right| = 0$.

Thus e^x is *represented* by its Taylor series.

Example 2.

$$\begin{aligned}
&f(x) = \log(1+x),\ c = 0\\
&f'(x) = \frac{1}{1+x} = (1+x)^{-1}\\
&f''(x) = -(1+x)^{-2}\\
&f'''(x) = +2(1+x)^{-3}\\
&f^{(k)}(x) = (-1)^{k+1}(k-1)!\frac{1}{(1+x)^k}
\end{aligned}$$

So $f^{(k)}(0) = (-1)^{k-1}(k-1)!$ for $k \geqslant 1$, $f(0) = \log(1) = 0$.

Taylor series:

$$\begin{aligned}
f(x) &= \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k}x^k + \frac{(-1)^k}{n+1}\frac{1}{(1+\xi_x)^{n+1}}\cdot x^{n+1} \quad \left(\frac{n!}{(n+1)!} = \frac{1}{n+1}\right)\\
E_n(x) &= \frac{(-1)^k}{n+1}\frac{1}{(1+\xi_x)^{n+1}}\cdot x^{n+1} \text{ — the remainder}
\end{aligned}$$

Question: for which x does $\lim_{n\to\infty} E_n(x) = 0$?

$$\lim_{n\to\infty} E_n(x) = \lim_{n\to\infty}\frac{(-1)^n}{n+1}\left(\frac{x}{\xi_x+1}\right)^{n+1} \text{ for } \xi_x \in (c,x)\ (c=0)$$

Such a limit converges to 0, if the fraction is less than 1.

$$0 < \frac{x}{\xi_x+1} < 1 \iff x < \xi_x + 1 \iff x - \xi_x < 1 \text{ with } \xi_x \in (0,x) \iff x \leqslant 1$$

Consequence. $\lim_{n\to\infty} E_n(x) = 0$ if $0 < x \leqslant 1$. This means that the Taylor series represents $\log(x+1)$ for $x \in [0,1]$. We can extend this to show $x \in (-1,1]$.

Example 3. Let's compute $\cos(0.1)$. Let's approximate it with Taylor series with $c = 0$ (around zero).

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots + \text{remainder}$$

Consequence.

$$\left|\cos(x) - \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k!)}\right| = \left|(-1)^{n+1} \cos(\xi_x) \frac{x^{2(n+1)}}{\big(2(n+1)\big)!}\right| \leqslant \frac{0.1^{2(n+1)}}{2(n+1)!} \underset{n\to\infty}{\longrightarrow} 0$$

n	Taylor polynomial	$\lvert error\rvert \leqslant$
0	1	$\frac{(0.1)^2}{2} = 0.0005$
1	0.995	$\frac{0.0001}{24}$
2	0.99500416	$\frac{0.000001}{6!}$

Error depends on choice of $|x - c|$ and n.

Example 4. Compute $\log(2)$ using $f(x) = \log(x + 1)$

$$\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Keeping 8 terms (until $n = 8$) we get $\log(2) \approx 0.63452$, the actual solution is $\log(2) = 0.693147$. Not so accurate. Can we improve?

We can use Taylor series of $\log\left(\frac{1+x}{1-x}\right)$ instead, since $\log\left(\frac{1+x}{1-x}\right) = \log(1 + x) - \log(1 - x)$. We choose $x = \frac{1}{3}$ instead of $x = 1$. Since x is closer to zero, both of the logarithms converge quicker.

$$\left(\log\left(\frac{1 + 1/3}{1 - 1/3}\right) = \log(2)\right)$$

We then get

$$\log(2) = 2 \cdot \left(\frac{1}{3} + \frac{1}{3^3 \cdot 5} + \dots\right)$$

We only need 4 terms to get $\log 2 \approx 0.69313$.

Theorem 1.2 (Reformulation of Taylor's theorem)**.** $f \in \mathrm{C}^{n+1}([a, b])$. We change c to x and the old x to $x + h$ from previous version $\implies$ get for $x,\ x + h \in [a, b]$:

$$f(x + h) = \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k + \frac{f^{(n+1)}(\xi_x)}{(n + 1)!} h^{n+1} \text{ where } \xi_x \in (x, x + h),\ h > 0$$

We can write error term as

$$f(x + h) - \sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!} h^k = \mathcal{O}(h^{n+1})$$

Remark. Let's recall what the $\mathcal{O}$-notation means. $a(h) = \mathcal{O}\big(b(h)\big)$ if $\exists c > 0$ such that $\frac{a(j)}{b(j)} \leqslant c$ as $h \to 0$. So, for $n = 1$ the error decreases with h^2 (quadratic convergence).
$n = 2$: error decreases cubically, i.e. h^3, etc.