

# Analysis 3

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# 1. Measure

## 1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of  $\mathbb{R}$ . Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

**Theorem 1.** There exist no such function  $\mu : 2^{[0,1]} \rightarrow [0, +\infty)$  that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

**Proof.** First, several definitions:

Step 1. Let's define the following equivalence relation: if  $x, y$  are from the unit interval, we'll say that  $x \sim y$  if  $x - y \in \mathbb{Q}$ .

Step 2. Let's choose  $N \subset [0, 1/3]$  such that it contains *precisely one* element from each equivalence class. (Such an  $N$  exists if the axiom of choice holds true).

Step 3. For all  $r \in \mathbb{Q}$  define  $N_r = N + r$ .

Claim 1. The sets  $N_R$  are congruent to  $N$  and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that  $x \in N_{r_1} \cap N_{r_2}$  for some  $r_1, r_2 \in \mathbb{Q}$ . Then  $x - r_1 \in N$ ,  $x - r_2 \in N$ , but  $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$ .

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r$$

Proof. If  $x \in [1/3, 2/3]$ , then  $\exists! y \in N$  such that  $x = y + q$  for some  $q \in \mathbb{Q}$ , as  $N$  contains exactly one representative from each of the equivalence classes. It is easy to see that such  $q \in [0, 2/3]$ .

We arrive at the following conclusion:

$$\frac{1}{3} = \mu([1/3, 2/3]) \leq \mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum_{r \in \mathbb{Q} \cap [0, 2/3]} \mu(N_r) \leq 1$$

What is  $\mu(N)$  then? If  $\mu(N) = 0$ , then

$$\mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum 0 = 0$$

If  $\mu(N) = \varepsilon > 0$ , then the sum is  $+\infty$ . But it's supposed to be in  $[1/3, 1]$ ?! □

**Consequence.** We cannot generalize the notion of length to all subsets of real numbers.

## 1.2 Lebesgue Outer Measure

**Definition 1.** If  $I \subset \mathbb{R}$  is an interval, then  $l(I)$  = the length of  $I$ . If  $I$  is unbounded, then  $l(I) = \infty$ .

**Definition 2** (Outer Measure).

$$m^* : 2^{\mathbb{R}} \rightarrow [0, +\infty]$$

$$m^*(A) = \inf \left\{ \sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

In words, it's the infimum of all *countable* covers of  $A$ . (A countable sum either converges or diverges to infinity).

**Remark.** This is certainly not a measure — otherwise, it would contradict Theorem 1.

**Example.** If  $A$  is countable, then  $m^*(A) = 0$ .

**Proof.** Let's choose an arbitrary  $\varepsilon > 0$  and prove that  $m^*(A) \leq 2\varepsilon$ . Let's choose a cover of the points with segments of lengths  $\varepsilon, \varepsilon/2, \varepsilon/2^2$ , and so on. Then

$$m^*(A) = \inf \{ \dots \} \leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

**Proposition 1.** If  $A$  is an interval, then  $m^*(A) = l(A)$ .

**Proof.** a)  $A$  is a closed interval,  $A = [a, b]$ .

1.  $m^*(A) \leq b - a$ . To prove this, we can cover  $A$  with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

Now take  $\varepsilon \rightarrow 0$ .

2.  $m^*(A) \geq b - a$ . Suppose we an infinite cover of  $A$  by open intervals. Since  $A$  is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

- b) If  $A$  is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

- c) If  $A$  is an open or semiclosed interval, we can approximate it from both sides by closed intervals.

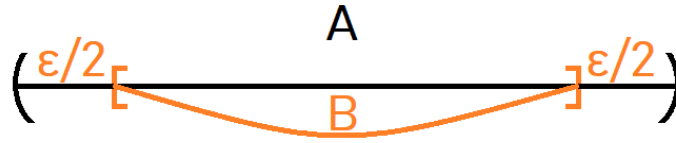
Let's denote the closure of  $A$  by  $\bar{A}$ . Since we're adding points, the Outer Measure will not decrease:

$$A \subset \bar{A} \implies m^*(A) \leq m^*(\bar{A}) = l(a)$$

Now suppose we have a closed interval  $B$  strictly inside  $A$ . Then we get

$$m^*(A) \geq m^*(B) = l(B) = l(A) - \varepsilon$$

Now take  $\varepsilon \rightarrow 0 \implies m^*(A) \geq l(A)$ .



□

**Lemma.**  $m^*$  is translation-invariant.

**Proof.** If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

**Proposition 2** (Countable subadditivity). For any countable collection of sets  $\{E_k\}_{k=1}^{\infty}$  we have

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

**Remark.** We don't ask for the sets  $E_k$  to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that  $m^*$  is a measure, which we proved does not exist in Theorem 1.

**Proof.** Choose open intervals  $I_{k,i}$ , such that

$$E_k \subset \bigcup_{i=1}^{\infty} I_{k,i} \quad (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand,  $\{I_{k,i} \mid 1 \leq k, i < \infty\}$  covers each of the  $E_k$ , and thus it's a cover of  $\bigcup_{k=1}^{\infty} E_k$ . Then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \overset{\text{it's a cover}}{\leq} \sum_{1 \leq k, i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

Now take  $\varepsilon \rightarrow 0$ . □

**Remark.** Here we assume that all of the  $E_k$  have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get  $\infty \leq \infty$  which is “true”.

### 1.3 The $\sigma$ -algebra of Lebesgue-measurable sets.

**Definition 1.** A set  $E$  is (Lebesgue) measurable if for any set  $A$ ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \quad E^C = \mathbb{R} \setminus E$$

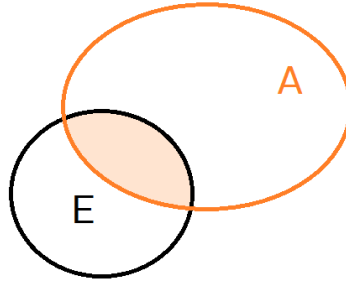


Figure 1: The set  $E$  “splits”  $A$  into two parts

**Remark.** We already have the  $\leq$  sign from [countable subadditivity](#).

**Remark.** Motivation: If  $A \cap B = \emptyset$  and  $A$  (or  $B$ ) is measurable, then

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^C) = m^*(A) + m^*(B)$$

**Proposition 1.** If  $m^*(E) = 0$ , then  $E$  is measurable.

**Proof.** For all  $A$  we have:

$$\begin{aligned} m^*(A \cap E) &\leq m^*(E) = 0 \implies m^*(A \cap E) = 0 \\ m^*(A) &\geq m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C) \end{aligned}$$

As we noted earlier, the inequality in the other side follows from [countable subadditivity](#). □

**Proposition 2.** If  $E_1, \dots, E_n$  are measurable, then  $\cup_1^n E_k$  is measurable.

**Proof.** Case  $n = 2$ : for all  $A$  we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^C) = \\ &= m^*(A \cap E_1) + m^*((A \cap E_1^C) \cap E_2) + m^*((A \cap E_1^C) \cap E_2^C) = (*) \\ X &:= A \cap E_1, \quad Y := (A \cap E_1^C) \cap E_2, \quad Z := (A \cap E_1^C) \cap E_2^C \end{aligned}$$

With Venn diagrams it's possible to prove that  $Z = A \cap (E_1 \cup E_2)^C$ ,  $X \cup Y = A \cap (E_1 \cup E_2)$ . Now let's apply [countable subadditivity](#) to  $X$  and  $Y$ . Then we get:

$$(*) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^C)$$

Yet again, the inequality in the other side follows from [countable subadditivity](#).

Induction step: Apply case  $n = 2$  to the sets  $\cup_1^{n-1} E_k, E_n$ . □

**Definition 2** (Algebra). Let  $X$  be a non-empty set.  $\Omega \subset 2^X$  is an algebra, if:

1.  $X \in \Omega$ ;
2.  $\Omega$  is closed under the formation of complements in  $X$  and *finite* unions.

**Remark.** It follows that  $\Omega$  is also closed under intersections:

$$(X_1^C \cup \dots \cup X_n^C)^C = X_1 \cap \dots \cap X_n$$

**Definition 3** ( $\sigma$ -algebra). Let  $X$  be a non-empty set.  $\Omega \subset 2^X$  is a  $\sigma$ -algebra, if:

1.  $X \in \Omega$ ;
2.  $\Omega$  is closed under the formation of complements in  $X$  and *countable* unions.

**Remark.** Every  $\sigma$ -algebra is an algebra, but not vice versa.

**Corollary 1.** The collection  $\mathcal{M}$  of all measurable subsets of  $\mathbb{R}$  is an algebra.

**Proof.** For the proof, we'll need to show that:

1.  $\mathbb{R}$  is measurable.

$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\emptyset)$$

2. It is closed under complements. It follows from the symmetry of the [definition of a measurable set](#).
3. It is closed under unions. We [have already proved](#) this one.

□

**Proposition 3.**  $\{E_k\}_1^n$  — disjoint measurable sets. Then for every set  $A$

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for  $A = \mathbb{R}$  we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

**Proof.** Induction on  $n$ .

Base  $n = 1$  is obvious.

Step  $n - 1 \rightarrow n$ . Take  $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$ . Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with  $E_n^C$  is equivalent to subtracting  $E_n$  from  $\hat{A}$ , and since  $\{E_k\}$  are disjoint, no other parts of  $\hat{A}$  except  $E_n$  will be removed. Then:

$$\begin{aligned} m^*(\hat{A}) &\stackrel{E_n \text{ is measurable}}{=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) = \\ &= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \stackrel{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k) \end{aligned}$$

□

**Proposition 4.** The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

**Proof.** If  $A = \cup_1^\infty A_k$ , define  $\hat{A}_1 := A_1$  and  $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$ . As  $\mathcal{M}$  is an algebra, all  $\hat{A}_k$  are measurable, and  $A = \sqcup_1^\infty \hat{A}_k$ , which is what we wanted.  $\square$

**Theorem 1.**  $\mathcal{M}$  is a  $\sigma$ -algebra.

**Proof.** We need to show that if all  $\{E_k\}_1^\infty$  are measurable sets, then  $E = \cup_1^\infty E_k$  is measurable. By Proposition 4, without the loss of generality, assume that  $E_k$  are all pairwise disjoint. Let  $F_n := \cup_1^n E_k$ , then  $F_n \in \mathcal{M}$  (as a finite union). As  $F_n \subset E$ , we have  $E^C \subset F_n^C$ .

Let  $A$  be any set. Then:

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap F_n^C) \geq m^*(A \cap F_n) + m^*(A \cap E^C) \stackrel{\text{Proposition 3}}{=} \\ &= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E^C) \end{aligned}$$

Now take  $n \rightarrow \infty$ :

$$m(A) \geq \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E^C) \stackrel{\text{countable subadditivity}}{\geq} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity).  $\square$

**Proposition 5** (Countable additivity). If  $\{E_k\}_1^\infty \subset \mathcal{M}$  — collection of disjoint sets, then  $\cup_1^\infty E_k \in \mathcal{M}$  and

$$m^*\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty m^*(E_k)$$

**Proof.** We know that:

1.

$$m^*\left(\bigcup_1^\infty E_k\right) \leq \sum_1^\infty m^*(E_k) \text{ (countable subadditivity)}$$

2.

$$m^*\left(\bigcup_1^\infty E_k\right) \geq m^*\left(\bigcup_1^n E_k\right) \stackrel{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

Take  $n \rightarrow \infty$ , then

$$m^*\left(\bigcup_1^\infty E_k\right) \geq \sum_1^\infty m^*(E_k)$$

Which is what we wanted.  $\square$

**Definition 4.** The restriction of  $m^*$  on  $\mathcal{M}$  is called the Lebesgue measure and denoted by  $m$ .

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

**Definition 5.** If  $X$  is a non-empty set and  $\mathcal{A}$  is a  $\sigma$ -algebra on  $X$ , then any function  $\mu : \mathcal{A} \rightarrow [0, +\infty]$  is called the measure on  $(X, \mathcal{A})$ , if:

1.  $\mu(\emptyset) = 0$ .
2.  $\mu$  is countable additive.