Analysis 3

Lev Leontev

February 9, 2023

Contents

1. Measure

1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of $\mathbb{R}$. Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

Theorem 1. There exist no such function $\mu : 2^{[0,1]} \to [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

Proof. First, several definitions:

Step 1. Let's define the following equivalence relation: if x, y are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an N exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets N_R are congruent to N and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists!\, y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as N contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu\big([1/3, 2/3]\big) \leqslant \mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum_{r \in \mathbb{Q} \cap [0,2/3]} \mu(N_r) \leqslant 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 1]$?! ◻

Consequence. We cannot generalize the notion of length to all subsets of real numbers.

1.2 Lebesgue Outer Measure

***Definition* 1.** If $I \subset \mathbb{R}$ is an interval, then $l(I) =$ the length of I. If I is unbounded, then $l(I) = \infty$.

***Definition* 2** (Outer Measure)**.**

$$m^* : 2^{\mathbb{R}} \to [0, +\infty]$$
$$m^*(A) = \inf\Big\{\sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j\Big\}$$

In words, it's the infimum of all *countable* covers of A. (A countable sum either converges or diverges to infinity).

***Remark*.** This is certainly not a measure — otherwise, it would contradict Theorem 1.

Example. If A is countable, then $m^*(A) = 0$.

Proof. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leqslant 2\varepsilon$. Let's choose a cover of the points with segments of lengths ε, $\varepsilon/2$, $\varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf\{\dots\} \leqslant \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

Proposition 1. If A is an interval, then $m^*(A) = l(A)$.

Proof. a) A is a closed interval, $A = [a, b]$.

1. $m^*(A) \leqslant b - a$. To prove this, we can cover A with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

   Now take $\varepsilon \to 0$.

2. $m^*(A) \geqslant b - a$. Suppose we an infinite cover of A by open intervals. Since A is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

b) If A is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

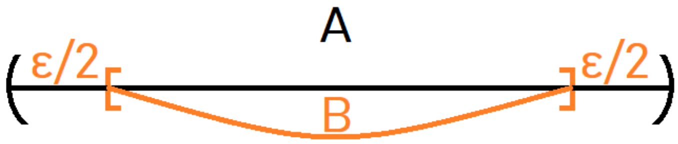
c) If A is an open or semiclosed interval, we can approximate it from both sides by closed intervals. Let's denote the closure of A by $\bar{A}$. Since we're adding points, the Outer Measure will not decrease:

$$A \subset \overline{A} \implies m^*(A) \leqslant m^*(\overline{A}) = l(a)$$

Now suppose we have a closed interval B strictly inside A. Then we get

$$m^*(A) \geqslant m^*(B) = l(B) = l(A) - \varepsilon$$

Now take $\varepsilon \to 0 \implies m^*(A) \geqslant l(A)$.



□

Lemma. m^* is translation-invariant.

Proof. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

Proposition 2 (Countable subadditivity)**.** For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leqslant \sum_{k=1}^{\infty} m^*(E_k)$$

Remark. We don't ask for the sets E_k to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that m^* is a measure, which we proved does not exist in Theorem 1.

Proof. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} E_{k,i} \text{ } (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leqslant k, i < \infty\}$ covers each of the E_k, and thus it's a cover of $\cup_{k=1}^{\infty} E_k$. Then

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \overset{\text{it's a cover}}{\leqslant} \sum_{1 \leqslant k,i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon\Big(\frac{1}{2} + \frac{1}{4} + \dots\Big) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

Now take $\varepsilon \to 0$. □

Remark. Here we assume that all of the E_k have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leqslant \infty$ which is "true".

1.3 The σ-algebra of Lebesgue-measurable sets.

***Definition* 1.** A set E is (Lebesgue) measurable if for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \qquad E^C = \mathbb{R} \setminus E$$

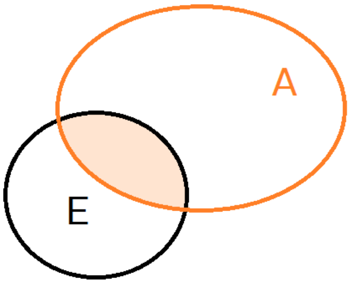


Figure 1: The set E "splits" A into two parts

Remark. We already have the $\leqslant$ sign from countable subadditivity.

Remark. Motivation: If $A \cap B = \varnothing$ and A (or B) is measurable, then

$$m^*(A \cup B) = m^*\big((A \cup B) \cap A\big) + m^*\big((A \cup B) \cap A^C\big) = m^*(A) + m^*(B)$$

Proposition 1. If $m^*(E) = 0$, then E is measurable.

Proof. For all A we have:

$$\begin{gathered} m^*(A \cap E) \leqslant m^*(E) = 0 \implies m^*(A \cap E) = 0 \\ m^*(A) \geqslant m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C) \end{gathered}$$

As we noted earlier, the inequality in the other side follows from countable subadditivity. $\square$

Proposition 2. If $E_1, \dots, E_n$ are measurable, then $\cup_1^n E_k$ is measurable.

Proof. Case $n = 2$: for all A we have

$$\begin{aligned} & m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C) = \\ & = m^*(A \cap E_1) + m^*\big((A \cap E_1^C) \cap E_2\big) + m^*\big((A \cap E_1^C) \cap E_2^C\big) = (*) \\ & X \coloneqq A \cap E_1, \ Y \coloneqq (A \cap E_1^C) \cap E_2, \ Z \coloneqq (A \cap E_1^C) \cap E_2^C \end{aligned}$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply countable subadditivity to X and Y. Then we get:

$$(*) \geqslant m^*\big(A \cap (E_1 \cup E_2)\big) + m^*\big(A \cap (E_1 \cup E_2)^C\big)$$

Yet again, the inequality in the other side follows from countable subadditivity.

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k$, E_n. $\square$

Definition 2 (Algebra)**.** Let X be a non-empty set. $\Omega \subset 2^X$ is an algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *finite* unions.

Remark. It follows that Ω is also closed under intersections:

$$\left(X_1^C \cup \dots \cup X_n^C\right)^C = X_1 \cap \dots \cap X_n$$

Definition 3 (σ-algebra)**.** Let X be a non-empty set. $\Omega \subset 2^X$ is a σ-algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *countable* unions.

Remark. Every σ-algebra is an algebra, but not vice versa.

Corollary 1. The collection $\mathcal{M}$ of all measurable subsets of $\mathbb{R}$ is an algebra.

Proof. For the proof, we'll need to show that:

1. $\mathbb{R}$ is measurable.
$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\varnothing)$$
2. It is closed under complements. It follows from the symmetry of the definition of a measurable set.
3. It is closed under unions. We have already proved this one.

□

Proposition 3. $\{E_k\}_1^n$ — disjoint measurable sets. Then for every set A

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for $A = \mathbb{R}$ we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

Proof. Induction on n.

Base $n = 1$ is obvious.

Step $n - 1 \to n$. Take $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$. Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with E_n^C is equivalent to subtracting E_n from $\hat{A}$, and since $\{E_k\}$ are disjoint, no other parts of $\hat{A}$ except E_n will be removed. Then:

$$m^*(\hat{A}) \overset{E_n \text{ is measurable}}{=\!=\!=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) =$$
$$= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \overset{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k)$$

□

Proposition 4. The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

Proof. If $A = \cup_1^\infty A_k$, define $\hat{A}_1 := A_1$ and $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$. As $\mathcal{M}$ is an algebra, all $\hat{A}_k$ are measurable, and $A = \sqcup_1^\infty \hat{A}_k$, which is what we wanted. □

Theorem 1. $\mathcal{M}$ is a σ-algebra.

Proof. We need to show that if all $\{E_k\}_1^\infty$ are measurable sets, then $E = \cup_1^\infty E_k$ is measurable. By Proposition 4, without the loss of generality, assume that E_k are all pairwise disjoint. Let $F_n := \cup_1^n E_k$, then $F_n \in \mathcal{M}$ (as a finite union). As $F_n \subset E$, we have $E^C \subset F_n^C$.

Let A be any set. Then:

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^C) \geqslant m^*(A \cap F_n) + m^*(A \cap E^C) \overset{\text{Proposition 3}}{=}$$
$$= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E_C)$$

Now take $n \to \infty$:

$$m(A) \geqslant \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E_C) \overset{\text{countable subadditivity}}{\geqslant} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity). □

Proposition 5 (Countable additivity)**.** If $\{E_k\}_1^\infty \subset \mathcal{M}$ — collection of disjoint sets, then $\cup_1^\infty E_k \in \mathcal{M}$ and

$$m^*\Big(\bigcup_1^\infty E_k\Big) = \sum_1^\infty m^*(E_k)$$

Proof. We know that:

1. $$m^*\Big(\bigcup_1^\infty E_k\Big) \leqslant \sum_1^\infty m^*(E_k) \text{ (countable \textit{sub}additivity)}$$

2. $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant m^*\Big(\bigcup_1^n E_k\Big) \overset{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

   Take $n \to \infty$, then

   $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant \sum_1^\infty m^*(E_k)$$

   Which is what we wanted.

□

***Definition* 4.** The restriction of m^* on $\mathcal{M}$ is called the Lebesgue measure and denoted by m.

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

***Definition* 5.** If X is a non-empty set and $\mathcal{A}$ is a σ-algebra on X, then any function $\mu : \mathcal{A} \to [0, +\infty]$ is called the measure on $(X, \mathcal{A})$, if:

1. $\mu(\varnothing) = 0$.

2. μ is countable additive.