

Analysis 3

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1 Measure

1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of \mathbb{R} . Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

Theorem 1. There exist no such function $\mu : 2^{[0,1]} \rightarrow [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

Proof. First, several definitions:

Step 1. Let's define the following equivalence relation: if x, y are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an N exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets N_r are congruent to N and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists! y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as N contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu([1/3, 2/3]) \leq \mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum_{r \in \mathbb{Q} \cap [0, 2/3]} \mu(N_r) \leq 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 1]$?! □

Consequence. We cannot generalize the notion of length to all subsets of real numbers.

1.2 Lebesgue Outer Measure

Definition 1. If $I \subset \mathbb{R}$ is an interval, then $l(I)$ = the length of I . If I is unbounded, then $l(I) = \infty$.

Definition 2 (Outer Measure).

$$m^* : 2^{\mathbb{R}} \rightarrow [0, +\infty]$$

$$m^*(A) = \inf \left\{ \sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

In words, it's the infimum of all *countable* covers of A . (A countable sum either converges or diverges to infinity).

Remark. This is certainly not a measure — otherwise, it would contradict Theorem 1.

Example. If A is countable, then $m^*(A) = 0$.

Proof. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leq 2\varepsilon$. Let's choose a cover of the points with segments of lengths $\varepsilon, \varepsilon/2, \varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf \{ \dots \} \leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

Proposition 1. If A is an interval, then $m^*(A) = l(A)$.

Proof. a) A is a closed interval, $A = [a, b]$.

1. $m^*(A) \leq b - a$. To prove this, we can cover A with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

Now take $\varepsilon \rightarrow 0$.

2. $m^*(A) \geq b - a$. Suppose we an infinite cover of A by open intervals. Since A is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

- b) If A is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

- c) If A is an open or semiclosed interval, we can approximate it from both sides by closed intervals.

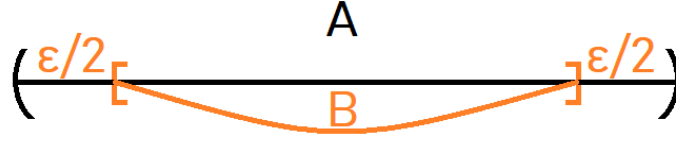
Let's denote the closure of A by \bar{A} . Since we're adding points, the Outer Measure will not decrease:

$$A \subset \bar{A} \implies m^*(A) \leq m^*(\bar{A}) = l(a)$$

Now suppose we have a closed interval B strictly inside A . Then we get

$$m^*(A) \geq m^*(B) = l(B) = l(A) - \varepsilon$$

Now take $\varepsilon \rightarrow 0 \implies m^*(A) \geq l(A)$.



□

Lemma. m^* is translation-invariant.

Proof. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

Proposition 2 (Countable subadditivity). For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

Remark. We don't ask for the sets E_k to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that m^* is a measure, which we proved does not exist in Theorem 1.

Proof. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} I_{k,i} \quad (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leq k, i < \infty\}$ covers each of the E_k , and thus it's a cover of $\bigcup_{k=1}^{\infty} E_k$. Then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \text{ it's a cover } \leq \sum_{1 \leq k, i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

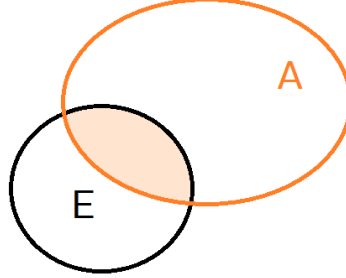
Now take $\varepsilon \rightarrow 0$. □

Remark. Here we assume that all of the E_k have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leq \infty$ which is “true”.

1.3 The σ -algebra of Lebesgue-measurable sets.

Definition 1. A set E is (Lebesgue) measurable if for any set A ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \quad E^C = \mathbb{R} \setminus E$$



The set E “splits” A into two parts

Remark. We already have the \leq sign from [countable subadditivity](#).

Remark. Motivation: If $A \cap B = \emptyset$ and A (or B) is measurable, then

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^C) = m^*(A) + m^*(B)$$

Proposition 1. If $m^*(E) = 0$, then E is measurable.

Proof. For all A we have:

$$\begin{aligned} m^*(A \cap E) &\leq m^*(E) = 0 \implies m^*(A \cap E) = 0 \\ m^*(A) &\geq m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C) \end{aligned}$$

As we noted earlier, the inequality in the other side follows from [countable subadditivity](#). □

Proposition 2. If E_1, \dots, E_n are measurable, then $\cup_1^n E_k$ is measurable.

Proof. Case $n = 2$: for all A we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^C) = \\ &= m^*(A \cap E_1) + m^*((A \cap E_1^C) \cap E_2) + m^*((A \cap E_1^C) \cap E_2^C) = (*) \\ X &:= A \cap E_1, \quad Y := (A \cap E_1^C) \cap E_2, \quad Z := (A \cap E_1^C) \cap E_2^C \end{aligned}$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply [countable subadditivity](#) to X and Y . Then we get:

$$(*) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^C)$$

Yet again, the inequality in the other side follows from [countable subadditivity](#).

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k, E_n$. □

Definition 2 (Algebra). Let X be a non-empty set. $\Omega \subset 2^X$ is an algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *finite* unions.

Remark. It follows that Ω is also closed under intersections:

$$(X_1^C \cup \dots \cup X_n^C)^C = X_1 \cap \dots \cap X_n$$

Definition 3 (σ -algebra). Let X be a non-empty set. $\Omega \subset 2^X$ is a σ -algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *countable* unions.

Remark. Every σ -algebra is an algebra, but not vice versa.

Corollary 1. The collection \mathcal{M} of all measurable subsets of \mathbb{R} is an algebra.

Proof. For the proof, we'll need to show that:

1. \mathbb{R} is measurable.

$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\emptyset)$$

2. It is closed under complements. It follows from the symmetry of the [definition of a measurable set](#).
3. It is closed under unions. We [have already proved](#) this one.

□

Proposition 3. $\{E_k\}_1^n$ — disjoint measurable sets. Then for every set A

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for $A = \mathbb{R}$ we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

Proof. Induction on n .

Base $n = 1$ is obvious.

Step $n - 1 \rightarrow n$. Take $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$. Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with E_n^C is equivalent to subtracting E_n from \hat{A} , and since $\{E_k\}$ are disjoint, no other parts of \hat{A} except E_n will be removed. Then:

$$\begin{aligned} m^*(\hat{A}) &\stackrel{E_n \text{ is measurable}}{=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) = \\ &= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \stackrel{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k) \end{aligned}$$

□

Proposition 4. The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

Proof. If $A = \cup_1^\infty A_k$, define $\hat{A}_1 := A_1$ and $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$. As \mathcal{M} is an algebra, all \hat{A}_k are measurable, and $A = \sqcup_1^\infty \hat{A}_k$, which is what we wanted. \square

Theorem 1. \mathcal{M} is a σ -algebra.

Proof. We need to show that if all $\{E_k\}_1^\infty$ are measurable sets, then $E = \cup_1^\infty E_k$ is measurable. By Proposition 4, without the loss of generality, assume that E_k are all pairwise disjoint. Let $F_n := \cup_1^n E_k$, then $F_n \in \mathcal{M}$ (as a finite union). As $F_n \subset E$, we have $E^C \subset F_n^C$.

Let A be any set. Then:

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap F_n^C) \geq m^*(A \cap F_n) + m^*(A \cap E^C) \stackrel{\text{Proposition 3}}{=} \\ &= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E^C) \end{aligned}$$

Now take $n \rightarrow \infty$:

$$m(A) \geq \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E^C) \stackrel{\text{countable subadditivity}}{\geq} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity). \square

Proposition 5 (Countable additivity). If $\{E_k\}_1^\infty \subset \mathcal{M}$ — collection of disjoint sets, then $\cup_1^\infty E_k \in \mathcal{M}$ and

$$m^*\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty m^*(E_k)$$

Proof. We know that:

1.

$$m^*\left(\bigcup_1^\infty E_k\right) \leq \sum_1^\infty m^*(E_k) \text{ (countable subadditivity)}$$

2.

$$m^*\left(\bigcup_1^\infty E_k\right) \geq m^*\left(\bigcup_1^n E_k\right) \stackrel{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

Take $n \rightarrow \infty$, then

$$m^*\left(\bigcup_1^\infty E_k\right) \geq \sum_1^\infty m^*(E_k)$$

Which is what we wanted. \square

Definition 4. The restriction of m^* on \mathcal{M} is called the Lebesgue measure and denoted by m .

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

Definition 5. If X is a non-empty set and \mathcal{A} is a σ -algebra on X , then any function $\mu : \mathcal{A} \rightarrow [0, +\infty]$ is called the measure on (X, \mathcal{A}) , if:

1. $\mu(\emptyset) = 0$.
2. μ is countable additive.

Definition 6 (Measurable space). A *measurable space* is a tuple (X, \mathcal{A}) , where:

1. X is a set.
2. \mathcal{A} is a σ -algebra on X .

Definition 7 (Measure space). A *measure space* is a triple (X, \mathcal{A}, μ) , where:

1. X is a set.
2. \mathcal{A} is a σ -algebra on X .
3. μ is a measure on (X, \mathcal{A}) .

Example 1. $\{\emptyset, X\}$ is a σ -algebra. Any μ , such that $\mu(\emptyset) = 0$ and $\mu(X) \geq 0$ will be a measure.

Example 2. 2^X is a σ -algebra. We can have the following measures:

- a) $\mu(E) = |E|$ is called a *counting measure*. Here $|E|$ denotes the cardinality of E (number of elements in E).
- b) δ -measure (also called Dirac measure):

$$\mu(E) = \begin{cases} 1, & 0 \in E \\ 0, & \text{otherwise} \end{cases}$$

1.4 Continuity of measure

Definition 1. A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *ascending* if $E_k \subset E_{k+1}$.

Definition 2. A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *descending* if $E_k \supset E_{k+1}$.

Theorem 1 (Continuity of measure).

1. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and the sequence is ascending, then

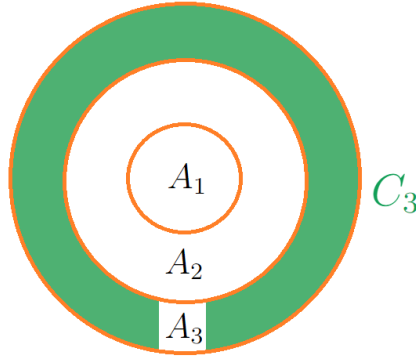
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

2. If $\{B_k\}_{k=1}^{\infty} \subset \mathcal{A}$, the sequence is descending and $\mu(B_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k)$$

Proof. 1. Let $C_k := A_k \setminus A_{k-1}$. Then we have:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigsqcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} \mu(C_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(C_k) = \lim_{n \rightarrow \infty} \mu(A_n)$$



2. Let $D_k := B_1 \setminus B_k$. Since B_k is descending, it follows that D_k is an ascending sequence. Then from the part 1 of the theorem it follows that:

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{\infty} D_k\right) &= \lim_{k \rightarrow \infty} \mu(D_k) & \bigcup_{k=1}^{\infty} D_k &= B_1 \setminus \bigcap_{k=1}^{\infty} B_k \\ \mu\left(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\right) &= \lim_{k \rightarrow \infty} (\mu(B_1) - \mu(B_k)) = \mu(B_1) - \lim_{k \rightarrow \infty} \mu(B_k) \\ \mu\left(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\right) &= \mu(B_1) - \mu\left(\bigcap_{k=1}^{\infty} B_k\right) \implies \mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k) \end{aligned}$$

□

Definition 3. We say that a statement (property) holds for *almost all* $x \in X$ with respect to a measure μ , if $\exists N \in \mathcal{A}$, such that $\mu(N) = 0$ and the statement (property) holds for all $x \in X \setminus N$.

Lemma (Borel–Cantelli). Let (X, \mathcal{A}, μ) be a measure space. Let $\{E_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then *almost all* $x \in X$ belong to at most finitely many E_k .

Proof. Let $B_n = \bigcup_{k=n}^{\infty} E_k$. It's easy to see that B_n is a descending sequence. At the same time,

$$\mu(B_1) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k) < \infty$$

By definition of B_n , $\bigcap_{n=1}^{\infty} B_n$ contains all the points that are contained in infinitely many E_k 's. But, by [continuity of measure](#) for $\{B_n\}_{n=1}^{\infty}$ we have:

$$\mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$$

□

1.5 How large is the Lebesgue σ -algebra \mathcal{M} ?

Proposition 1. Every interval is Lebesgue-measurable.

Proof. Proof idea:

$$E \in \mathcal{M} \iff \forall A : m(A) = m(A \cap E) + m(A \cap E^C)$$

Assume $E = (-\infty, a)$. If we prove that such intervals lie in \mathcal{M} , then we'll prove everything (since \mathcal{M} is a σ -algebra). We already have $m(A) \leq m(A \cap E) + m(A \cap E^C)$ from [countable subadditivity](#).

Let's assume $a \notin A$ (since removing one point does not change the measure). Every cover of A can be split into two covers with the same sum of interval lengths: of $A \cap (-\infty, a)$ and $A \cap (a, +\infty)$. Every interval in those covers, that contains a , can be split into two. Therefore, from the [definition of Lebesgue measure](#), $m(A) \geq m(A \cap E) + m(A \cap E^C)$, so we've proved the inequality in both sides. \square

Definition 1. For any $\mathcal{X} \in 2^{\mathbb{R}}$ let $\mathcal{A}(\mathcal{X})$ be the smallest σ -algebra containing \mathcal{X} .

Lemma. $\mathcal{A}(\mathcal{X})$ always exists and is the intersection of all σ -algebras containing \mathcal{X} .

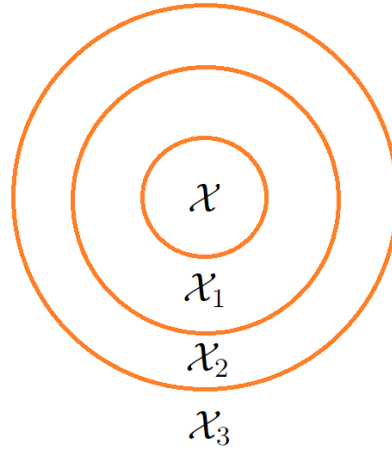
Proof. We have to prove that if we intersect a bunch σ -algebras, we still get a σ -algebra.

1. Such an intersection is closed under complements: if a set belongs to the intersection of σ -algebras, then it belongs to each of the σ -algebras, then its complement belongs to each of the σ -algebras, and thus its complement belongs to the intersection of σ -algebras.
2. In a similar way, such an intersection is closed under countable unions: if a number of sets all belong to the intersection of σ -algebras, then they all belong to each of the σ -algebras, then their countable union belongs to each of the σ -algebras, and their countable union belongs to the intersection of σ -algebras.

\square

Remark. We can try to construct $\mathcal{A}(\mathcal{X})$ in a different way. Say, \mathcal{X} is not a σ -algebra. Let's enlarge it: first by including all the complements. Then let's enlarge it by all countable unions. Let's call such a set \mathcal{X}_1 . But after such operation, \mathcal{X}_1 may be non-closed under complements. So we repeat such a procedure.

And, in general: \mathcal{X}_{n+1} is obtained from \mathcal{X}_n is obtained by including into \mathcal{X}_n all complements of the sets from \mathcal{X}_n and then including all countable unions of the obtained sets.



It is tempting to think that $\cup_1^\infty \mathcal{X}_i$ is $\mathcal{A}(\mathcal{X})$. Is it true? No, not necessarily. If the sequence $\{\mathcal{X}_i\}$ eventually stabilizes, then such a construction works. Let's now assume that every next \mathcal{X}_i is larger than the previous one. Then we can take A from \mathcal{X} , A_1 from $\mathcal{X}_1 \setminus \mathcal{X}$, A_2 from $\mathcal{X}_2 \setminus \mathcal{X}_1$, and so on.

Now let's look at $\cup_1^\infty A_i$. As a countable union, it must be contained in $\mathcal{A}(\mathcal{X}) = \cup_1^\infty \mathcal{X}_i$, thus, there exist an n , such that $\cup_1^\infty A_i \in \mathcal{X}_n$. But $A_{n+1} \in \mathcal{X}_{n+1} \setminus \mathcal{X}_n$!

Definition 2 (Topological space). A *topological space* is a set X and a collection of subsets O of X (called *open sets*), such that $\emptyset, X \in O$, and:

1. A union of (possibly infinitely many) sets from O is in O .
2. The intersection of finitely many sets from O is in O .

The complements of open sets are called *closed sets*.

Definition 3. A function $f : X \rightarrow Y$ between two topological spaces is *continuous* if the preimage of every open set is open.

Remark. It is possible to check that for \mathbb{R} this definition is equivalent to the usual one.

Definition 4 (Borel σ -algebra). For a topological space X its *Borel σ -algebra* \mathcal{B}_X is the smallest σ -algebra on X that contains all open sets.

Remark. If it's obvious from the context which set we are talking about, we will just write \mathcal{B} (without a subscript).

Theorem 1. $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$ (all of the sets in $\mathcal{B}_{\mathbb{R}}$ are measurable).

Proposition 2. \mathcal{B} is the smallest σ -algebra that contains all open intervals.

If we prove the proposition, the theorem will follow easily. We know that [all the intervals are Lebesgue-measurable](#). We know that the Lebesgue-measurable sets (\mathcal{M}) [are a \$\sigma\$ -algebra](#). Thus, if we take the smallest σ -algebra that contains all open intervals, it will be a subset of \mathcal{M} .

Proof of Proposition 2. We will prove that every open set $O \subset \mathbb{R}$ is a finite or countable union of open intervals.

For every point $x \in O$ let I_x be the largest open interval, such that $x \in I_x$ and $I_x \subset O$. It exists as a union of all such intervals. Since O is open, x lies in O with an open neighborhood, thus, I_x is non-empty.

$$\forall x \in O : x \in I_x \implies O = \bigcup_{x \in O} I_x$$

Let's prove that $I_x \cap I_y \neq \emptyset \implies I_x = I_y$. If the intervals around x and y intersect, then $I_x \cup I_y$ is an interval as well, and $I_x \cup I_y \in O$ as $I_x \in O$ and $I_y \in O$. Since I_x and I_y are the largest such intervals, it follows that $I_x = I_x \cup I_y = I_y$.

Let's say that two points x and y are equivalent if $I_x = I_y$. Since there's a lot of same intervals in $O = \bigcup_{x \in O} I_x$, we can take just a single point from every equivalence class and still get O as a union. Particularly, every open interval contains at least one rational point (as rational numbers are dense). Therefore, there's a rational point in every equivalence class. Thus,

$$O = \bigcup_{x \in O \cap \mathbb{Q}} I_x$$

Since the set of rational numbers is countable, we have represented O as a countable union of open intervals, which is what we wanted. \square

Remark. A topological space is called *separable*, if it contains a countable dense subset.

Remark. We have proved that the Lebesgue measure exists on $\mathcal{B}_{\mathbb{R}}$, so we have a lot of measurable sets.

Remark. The Lebesgue measure can be generalized to \mathbb{R}^n .

1.6 Other criteria of measurability for Lebesgue measure

As we remember, [the definition of a measurable set](#) is difficult to check. Thus, we would like to have better criteria.

Theorem 1. $E \subset \mathbb{R}$ is Lebesgue measurable if and only if one of the following holds:

1. For every $\varepsilon > 0$ there exists an open set O , such that $E \subset O$ and $m^*(O \setminus E) < \varepsilon$.
2. There exists a G_δ -set G , such that $E \subset G$ and $m^*(G \setminus E) = 0$.
(A G_δ -set is a countable intersection of open sets.)
3. For every $\varepsilon > 0$ there exists a closed set F , such that $F \subset E$ and $m^*(E \setminus F) < \varepsilon$.
4. There exists a F_σ -set F , such that $F \subset E$ and $m^*(E \setminus F) = 0$.
(A F_σ -set is a countable union of closed sets.)

Proof.

- E is measurable \implies 1.

If $m^*(E) < \infty$, then from the definition of m^* we can find O — a finite union of open intervals, such that $m^*(O) < m^*(E) + \varepsilon$. Since O is an open set, it's measurable (as we proved [earlier](#)). Therefore, both E and O are measurable, thus

$$\varepsilon > m^*(O) - m^*(E) \stackrel{E, O \in \mathcal{M}}{=} m^*(O \setminus E)$$

If $m^*(E) = \infty$, let's split the set E into a countable number of sets with finite measure. For example, by splitting the real line into segments of length 1. So, $E = \bigcup_1^\infty E_k$, where $m^*(E_k) < \infty$. Then let's use geometrically decreasing ε 's for the covers of each E_k : $\varepsilon/2$ for E_1 , $\varepsilon/4$ for E_2 , and so on. When we sum up the inequalities, the fractions will sum up to ε . So, we obtained our O , now continue like in the previous case.

Definition 1. A measure μ on X is called σ -finite, if $X = \bigcup_1^\infty X_k$ and $\mu(X_k) < \infty$ for all k .

In words: if there exists a subdivision of X into a countable number of set of finite measure.

- 1 \implies 2.

From 1, $\forall k \in \mathbb{N}$, $\exists O_k$ — open, such that $E \subset O_k$ and $m^*(O_k \setminus E) < \frac{1}{k}$. Now let's take

$$G := \bigcap_1^\infty O_k \implies \forall k : m^*(G \setminus E) \leq m^*(O_k \setminus E) < \frac{1}{k} \implies m^*(G \setminus E) = 0$$

- 2 \implies E is measurable.

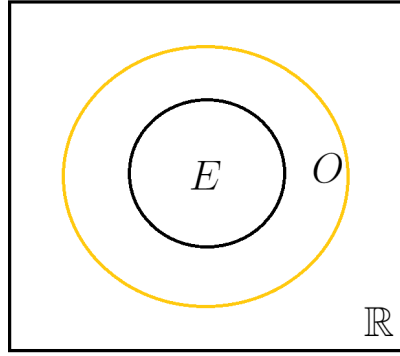
G is a G_δ -set. As a countable intersection of open sets, it's in Borel σ -algebra, and thus is Lebesgue-measurable. $m^*(G \setminus E) = 0$, then $G \setminus E$ is Lebesgue-measurable, then $E = G \setminus (G \setminus E)$ is measurable as a difference of two measurable sets.

- 3 \iff 1, 4 \iff 2.

If we assume that 3 holds for E , then, if we take $O = \overline{F}$, 1 will hold for \overline{E} . Therefore, \overline{E} is measurable, then E is measurable (as \mathcal{M} is a σ -algebra).

If 1 holds for E , then E is measurable, then \overline{E} is measurable, then 1 holds for \overline{E} . Now take $F = \overline{O}$, therefore, 3 holds for E .

In the same way, 2 and 4 are equivalent as well.



□

Theorem 2. For every $E \in \mathcal{M}$ with $m(E) < \infty$ and for every $\varepsilon < \infty$ there exists an infinite disjoint collection of open intervals $\{I_k\}_1^n$, such that $O = \cup_{k=1}^n I_k$ and $m(E \Delta O) < \varepsilon$.

(Here Δ is the symmetric difference of two sets).

Proof. From part 1 of the previous theorem, we can take such an open set U , that $E \subset U$ and $m(U \setminus E) < \varepsilon/2$.

As we proved [earlier](#), we can represent U as a countable union of disjoint open intervals I_k . Then:

$$\forall n : \bigcup_1^n I_k \subset U \implies \forall n : \sum_1^n m(I_k) \leq m(U) < \infty$$

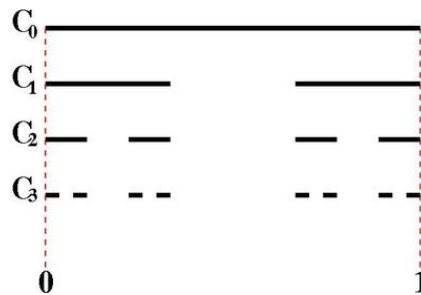
Now take n , such that $\sum_{k=1}^n m(I_k) < \varepsilon/2$, and put $O := \cup_{k=1}^n I_k$. Then $m(O \setminus E) < \varepsilon/2$ and $m(E \setminus O) < \varepsilon/2$, therefore, the measure of the symmetric difference is less than ε . □

1.7 Cantor set

Questions:

1. If $m(A) = 0$, is A countable?
2. We know that $\mathcal{B} \subset \mathcal{M}$. Is this inclusion proper?

Definition 1 (Cantor set). Let's take $[0, 1]$, split it into three parts and remove the middle part. Then continue such process. The *Cantor set* is the set $C := \cap_0^\infty C_k$.



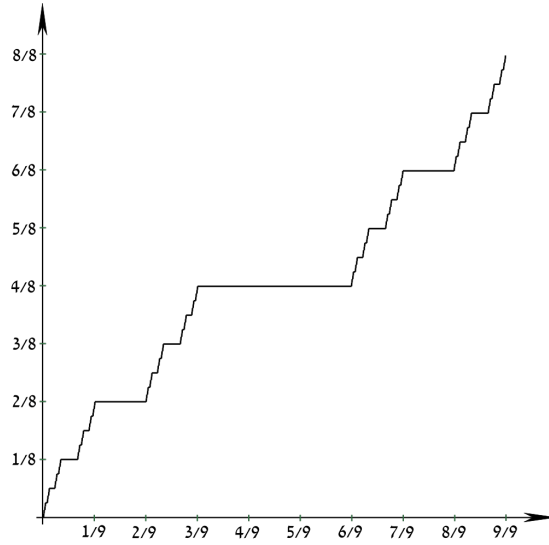
Cantor set illustration from [here](#).

Remark. The measure of C_n is $(2/3)^n = 0$. From the [continuity of measure](#), the measure of the intersection is the limit of measures of individual sets, and thus $m(C) = 0$.

Remark. The Cantor set is countable, because if we have a sequence of zeros and ones, we can traverse down-left on 0 and down-right on 1. The intersection of the corresponding intervals will be a single point of C . So, there's a bijection between C and $\{0, 1\}^{\mathbb{N}}$, therefore, C is indeed uncountable.

Remark. Usually, when we remove the middle intervals on each step, we keep the end points. If we choose to remove the end points, we essentially remove the points that correspond to sequences that end with an infinite sequence of 0's or an infinite sequence of 1's. It's clear that not all sequences of 0's and 1's are like that, so there is going to be plenty of points left in the Cantor set, anyway.

Definition 2 (The Cantor function). The Cantor function $\varphi : [0, 1] \rightarrow [0, 1]$ is defined as follows. Let's first take the unit interval $[0, 1]$, split it in three, and define φ to be $\frac{1}{2}$ on $[1/3, 2/3]$. Then let's continue the same with $[0, 1/3]$ and $[2/3, 1]$, and so on.



Now we have defined φ for all points on the Cantor set. Here's how we'll define it on all others:

$$O := [0, 1] \setminus C$$

$$\forall x \in [0, 1] \setminus C : \varphi(x) := \sup\{\varphi(t) \mid t \in C \cap [0, x]\}$$

Proposition 1. φ is increasing, continuous, surjective ($[0, 1]$ acts on $[0, 1]$), φ' exists for open set O of measure 1, $\varphi' \Big|_O \equiv 0$.

Proof. φ is increasing (that's clear), therefore, if φ is discontinuous, then it has a jump discontinuity and there will be an interval, say, $I \subset [0, 1]$, such that $\varphi([0, 1]) \cap I = \emptyset$. But

$$\varphi([0, 1]) \supset \left\{ \frac{m}{2^k} \mid k \in \mathbb{N}, m \in [0, 2^k] \right\} = K$$

And K is dense in $[0, 1]$, therefore, $K \cap I \neq \emptyset$, which is a contradiction. \square

Remark. The Cantor function is a source of a lot of counterexamples. We are going to use it to answer the second question from earlier.

Definition 3.

$$\psi : [0, 1] \rightarrow [0, 2], \quad \psi(x) = \varphi(x) + x$$

ψ is strictly increasing, continuous, surjective.

Proposition 2. 1. ψ maps C onto a measurable set of measure 1.

2. ψ maps some subset of C onto a non-measurable set.

Proof. 1. $m(C) = 0$, thus, $m(O) = m([0, 1] \setminus C) = 1$. We know that φ is constant on every part of O . Therefore, ψ looks like x on every part of O , and there's a countable number of such intervals. Therefore, O is mapped to a set of measure 1. Thus, $m(\psi(C)) = m(\psi([0, 1])) - m(\psi(O)) = 2 - 1 = 1$.

2. Since $\psi(C)$ has measure $1 > 0$, by the second homework, there exists a non-measurable set $N \subset \psi(C)$. Now take $\psi^{-1}(N)$. □

Corollary 1. There is a (measurable) subset of C that is not Borel.

Remark. Since $m(C) = 0$, the outer measure of each subset of C is also 0, thus, every subset of C is measurable. So, the "measurable" part of the corollary is obvious.

Proposition 3. If $f : E \rightarrow \mathbb{R}$ is continuous, $E \in \mathcal{M}$, then the preimage of every Borel set is measurable.

$$\forall B \in \mathcal{B} : f^{-1}(B) = \{x \in E \mid f(x) \in B\} \in \mathcal{M}$$

Let's show how we can prove the corollary if we prove the proposition:

Proof of Corollary 1. $\psi^{-1} : [0, 2] \rightarrow [0, 1]$ is continuous and bijective. Let's define $\tilde{C} = \psi^{-1}(N)$. Since $N \subset C$, we have $\tilde{C} \subset C$. Assume that \tilde{C} is a Borel set, and thus it's measurable. Then, by Proposition 3, its pre-image, N , must be measurable as well. But we know it isn't! □

Let's talk about something more general now, and later return to Proposition 3.

Definition 4. $f : X \rightarrow Y$ is continuous, if for every open set $O \subset Y$, $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is open in X .

Definition 5 (Induced topology). If X is a topological space and $Y \subset X$, the induced topology on Y is defined as follows: $O \subset Y$ is open in Y if there exists an open subset $\tilde{O} \subset X$ in X , such that $O = \tilde{O} \cap Y$.

Theorem 1 (Tietze). If $E \in \mathbb{R}$, E is closed, then any continuous function on E extends to a continuous function on \mathbb{R} .

Proof idea. The idea is that if E is closed, then its complement, i.e. the set where f is undefined, is open. As we know, an open set on \mathbb{R} can be represented as a disjoint union of a countable number of intervals. For every such interval we can simply connect the values of f on its endpoints with a straight line.

The technicality that will arise is the case if we have a countable number of intervals that accumulate at a point — that can be left as an exercise. □

Definition 6. Let (X, \mathcal{A}) , (Y, \mathcal{D}) be measurable spaces. $f : X \rightarrow Y$ is called $(\mathcal{A}, \mathcal{D})$ -measurable, if for every $E \in \mathcal{D}$ we have $f^{-1}(E) \in \mathcal{A}$.

Proposition 4. Let $f : X \rightarrow Y$ be any function. \mathcal{A} is a σ -algebra on X . Then the family of sets

$$f_*(\mathcal{A}) = \{E \subset Y \mid f^{-1}(E) \in \mathcal{A}\}$$

is a σ -algebra on Y .

Proof. To prove that $f_*(\mathcal{A})$ is a σ -algebra, we have to prove 3 properties from the definition of a σ -algebra.

1. $Y \in f_*(\mathcal{A})$, since $f^{-1}(Y) = X$. And $X \in \mathcal{A}$, since \mathcal{A} is a σ -algebra.

2. $f_*(\mathcal{A})$ is closed under complements. If $E \in f_*(\mathcal{A})$, then $f^{-1}(E^C) = (f^{-1}(E))^C$. But $f^{-1}(E) \in \mathcal{A}$ by definition of $f_*(\mathcal{A})$. Therefore, $(f^{-1}(E))^C \in \mathcal{A}$, since \mathcal{A} is a σ -algebra. Therefore, $f^{-1}(E^C) \in \mathcal{A}$, and thus $E^C \in f_*(\mathcal{A})$ by definition of $f_*(\mathcal{A})$.
3. $f_*(\mathcal{A})$ is closed under countable unions. If $\{E_n\} \subset f_*(\mathcal{A})$, then

$$f^{-1}\left(\bigcup_{k=1}^{\infty} E_k\right) = \bigcup_{k=1}^{\infty} f^{-1}(E_k) \in \mathcal{A} \implies \bigcup_{k=1}^{\infty} E_k \in f_*(\mathcal{A})$$

□

Observation. If $f_*(\mathcal{A})$ contains a generating set of \mathcal{D} , then f is $(\mathcal{A}, \mathcal{D})$ -measurable.

Proof. If $f_*(\mathcal{A})$ contains a generating set of \mathcal{D} , then, since it is a σ -algebra, it must contain the whole \mathcal{D} . That means that the preimages of all sets from \mathcal{D} lie in \mathcal{A} , which is the definition of $(\mathcal{A}, \mathcal{D})$ -measurability. □

Corollary 2. Let (X, \mathcal{A}) , (Y, \mathcal{D}) be measurable spaces. $\mathcal{E} \subset \mathcal{D}$ generates \mathcal{D} . Then $f : X \rightarrow Y$ is $(\mathcal{A}, \mathcal{D})$ -measurable if and only if $\forall E \in \mathcal{E} : f^{-1}(E) \in \mathcal{A}$.

Proof. This is just a restatement of the observation above. □

Corollary 3. Let X, Y be topological spaces. If $f : X \rightarrow Y$ is continuous, then f is $(\mathcal{B}_X, \mathcal{B}_Y)$ -measurable.

Proof. Take \mathcal{E} to be the set of all open sets in Y . \mathcal{E} is a generating set of \mathcal{B}_Y . Then, according to the Corollary 2, we have to prove that the inverse of every set in \mathcal{E} is in \mathcal{B}_X . But f^{-1} of every open set is open by the definition of a continuous function, and every open set in X is contained in \mathcal{B}_X (since the Borel σ -algebra is by definition the smallest σ -algebra containing all open sets). □

Proof of Proposition 3. Let $X = E$, $Y = \mathbb{R}$ in the previous corollary. If $f : E \rightarrow \mathbb{R}$ is continuous, then f is $(\mathcal{B}_E, \mathcal{B}_{\mathbb{R}})$ -measurable. But $\mathcal{B}_E \subset \mathcal{M}$. So, the preimage of any set in $\mathcal{B}_{\mathbb{R}}$ will be in \mathcal{M} . □

1.8 Measurable functions in $\overline{\mathbb{R}}$

Definition 1. $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty, -\infty\}$. The topology on $\overline{\mathbb{R}}$ is defined as follow: open sets are all open sets in \mathbb{R} , all intervals of the form $(a, +\infty]$, $[-\infty, a)$, as well as all possible unions (not necessarily countable) of the aforementioned sets.

Definition 2. We can say that a sequence converges to $+\infty$ or $-\infty$, if for any open set containing this infinity there exists such an index of the sequence, starting from which all elements of the sequence are contained in that set.

Definition 3. Let $E \in \mathcal{M}$ be measurable. Then $f : E \rightarrow \overline{\mathbb{R}}$ is (Lebesgue) measurable if it is $(\mathcal{M}, \mathcal{B}_{\overline{\mathbb{R}}})$ -measurable.

Example 1. All continuous functions are Lebesgue measurable.

Example 2. The Dirichlet function (1 for \mathbb{Q} , 0 for $\mathbb{R} \setminus \mathbb{Q}$) is Lebesgue measurable. That's because every preimage for the Dirichlet function is either \mathbb{R} , \mathbb{Q} , or $\mathbb{R} \setminus \mathbb{Q}$, and every one of those is measurable.

Corollary 1. Let $E \in \mathcal{M}$, $f : E \rightarrow \overline{\mathbb{R}}$ is measurable if and only if on the of the following holds:

1. For every open set $O \subset \overline{\mathbb{R}} : f^{-1}(O) \in \mathcal{M}$.

$$2. \forall c \in \overline{\mathbb{R}} \implies f^{-1}((c, +\infty]) \in \mathcal{M}.$$

$$3. \forall c \in \overline{\mathbb{R}} \implies f^{-1}([c, +\infty]) \in \mathcal{M}.$$

$$4. \forall c \in \overline{\mathbb{R}} \implies f^{-1}([-\infty, c)) \in \mathcal{M}.$$

$$5. \forall c \in \overline{\mathbb{R}} \implies f^{-1}([-\infty, c]) \in \mathcal{M}.$$