Analysis 3

Lev Leontev

February 9, 2023

Contents

1. Measure

Lemma. m^* is translation-invariant.

Proof. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

Proposition 1 (Countable subadditivity)**.** For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leqslant \sum_{k=1}^{\infty} m^*(E_k)$$

Remark. We don't ask for the sets E_k to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that m^* is a measure, which we proved does not exist in Theorem **??**.

Proof. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} E_{k,i} \ (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leqslant k, i < \infty\}$ covers each of the E_k, and thus it's a cover of $\cup_{k=1}^{\infty} E_k$. Then

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \overset{\text{it's a cover}}{\leqslant} \sum_{1 \leqslant k, i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon\Big(\frac{1}{2} + \frac{1}{4} + \dots\Big) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

Now take $\varepsilon \to 0$. □

Remark. Here we assume that all of the E_k have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leqslant \infty$ which is "true".

1.1 The σ-algebra of Lebesgue-measurable sets.

***Definition* 1.** A set E is (Lebesgue) measurable if for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \qquad E^C = \mathbb{R} \setminus E$$

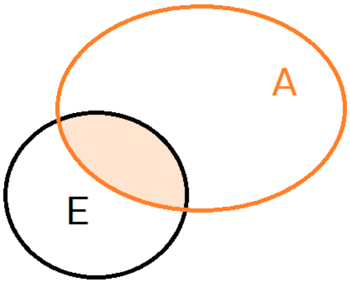

Figure 1: The set E "splits" A into two parts

Remark. We already have the $\leqslant$ sign from countable subadditivity.

Remark. Motivation: If $A \cap B = \varnothing$ and A (or B) is measurable, then

$$m^*(A \cup B) = m^*\big((A \cup B) \cap A\big) + m^*\big((A \cup B) \cap A^C\big) = m^*(A) + m^*(B)$$

Proposition 1. If $m^*(E) = 0$, then E is measurable.

Proof. For all A we have:

$$m^*(A \cap E) \leqslant m^*(E) = 0 \implies m^*(A \cap E) = 0$$
$$m^*(A) \geqslant m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C)$$

As we noted earlier, the inequality in the other side follows from countable subadditivity. □

Proposition 2. If $E_1, \dots, E_n$ are measurable, then $\cup_1^n E_k$ is measurable.

Proof. Case $n = 2$: for all A we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C) =$$
$$= m^*(A \cap E_1) + m^*\big((A \cap E_1^C) \cap E_2\big) + m^*\big((A \cap E_1^C) \cap E_2^C\big) = (*)$$
$$X := A \cap E_1,\ Y := (A \cap E_1^C) \cap E_2,\ Z := (A \cap E_1^C) \cap E_2^C$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply countable subadditivity to X and Y. Then we get:

$$(*) \geqslant m^*\big(A \cap (E_1 \cup E_2)\big) + m^*\big(A \cap (E_1 \cup E_2)^C\big)$$

Yet again, the inequality in the other side follows from countable subadditivity.

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k$, E_n. □

Definition 2 (Algebra)**.** Let X be a non-empty set. $\Omega \subset 2^X$ is an algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *finite* unions.

Remark. It follows that Ω is also closed under intersections:

$$\left(X_1^C \cup \cdots \cup X_n^C\right)^C = X_1 \cap \cdots \cap X_n$$

Definition 3 (σ-algebra)**.** Let X be a non-empty set. $\Omega \subset 2^X$ is a σ-algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *countable* unions.

Remark. Every σ-algebra is an algebra, but not vice versa.

Corollary 1. The collection $\mathcal{M}$ of all measurable subsets of $\mathbb{R}$ is an algebra.

Proof. For the proof, we'll need to show that:

1. $\mathbb{R}$ is measurable.
$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\varnothing)$$
2. It is closed under complements. It follows from the symmetry of the definition of a measurable set.
3. It is closed under unions. We have already proved this one.

□

Proposition 3. $\{E_k\}_1^n$ — disjoint measurable sets. Then for every set A

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for $A = \mathbb{R}$ we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

Proof. Induction on n.

Base $n = 1$ is obvious.

Step $n - 1 \to n$. Take $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$. Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with E_n^C is equivalent to subtracting E_n from $\hat{A}$, and since $\{E_k\}$ are disjoint, no other parts of $\hat{A}$ except E_n will be removed. Then:

$$m^*(\hat{A}) \overset{E_n \text{ is measurable}}{=\!=\!=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) =$$
$$= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \overset{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k)$$

□

Proposition 4. The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

Proof. If $A = \cup_1^\infty A_k$, define $\hat{A}_1 := A_1$ and $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$. As $\mathcal{M}$ is an algebra, all $\hat{A}_k$ are measurable, and $A = \sqcup_1^\infty \hat{A}_k$, which is what we wanted. □

Theorem 1. $\mathcal{M}$ is a σ-algebra.

Proof. We need to show that if all $\{E_k\}_1^\infty$ are measurable sets, then $E = \cup_1^\infty E_k$ is measurable. By Proposition 4, without the loss of generality, assume that E_k are all pairwise disjoint. Let $F_n := \cup_1^n E_k$, then $F_n \in \mathcal{M}$ (as a finite union). As $F_n \subset E$, we have $E^C \subset F_n^C$.

Let A be any set. Then:

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^C) \geqslant m^*(A \cap F_n) + m^*(A \cap E^C) \overset{\text{Proposition 3}}{=}$$
$$= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E_C)$$

Now take $n \to \infty$:

$$m(A) \geqslant \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E_C) \overset{\text{countable subadditivity}}{\geqslant} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity). □

Proposition 5 (Countable additivity)**.** If $\{E_k\}_1^\infty \subset \mathcal{M}$ — collection of disjoint sets, then $\cup_1^\infty E_k \in \mathcal{M}$ and

$$m^*\Big(\bigcup_1^\infty E_k\Big) = \sum_1^\infty m^*(E_k)$$

Proof. We know that:

1. $$m^*\Big(\bigcup_1^\infty E_k\Big) \leqslant \sum_1^\infty m^*(E_k) \text{ (countable } \textit{sub}\text{additivity)}$$

2. $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant m^*\Big(\bigcup_1^n E_k\Big) \overset{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

   Take $n \to \infty$, then

   $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant \sum_1^\infty m^*(E_k)$$

   Which is what we wanted.

□

***Definition* 4.** The restriction of m^* on $\mathcal{M}$ is called the Lebesgue measure and denoted by m.

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

***Definition* 5.** If X is a non-empty set and $\mathcal{A}$ is a σ-algebra on X, then any function $\mu : \mathcal{A} \to [0, +\infty]$ is called the measure on $(X, \mathcal{A})$, if:

1. $\mu(\varnothing) = 0$.
2. μ is countable additive.