

Numerical Methods

Lev Leontev

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1. Introduction

Definition 1 (Numerical Methods). Numerical Methods are algorithmic approaches to numerically solve mathematical problems. We use it often when it is hard/difficult/impossible to solve analytically.

1.1 Taylor series

Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ (that is hard to evaluate for some $x \in \mathbb{R}$), but f and $f^{(n)}$ are known for a value c , which is close to x . Can we use this information to approximate $f(x)$?

We know values for $\cos^{(n)}(0)$.

$$\begin{cases} f(0) = \cos(0) = 1 \\ f'(0) = -\sin(0) = 0 \\ f''(0) = -\cos(0) = -1 \end{cases} \quad \text{for } c = 0$$

Can we get $\cos(0.1)$ from this?

Definition 1 (Taylor series). Let $f : \mathbb{R} \rightarrow \mathbb{R}$, differentiable infinitely many times at $c \in \mathbb{R}$. So we have $f^{(k)}(c)$, $k = 1, 2, \dots$. Then the Taylor series of f at c is:

$$f(x) \approx f(c) + \frac{f'(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \dots = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

Remark. Taylor series is a power series.

Remark. For $c = 0$ also known as Maclaurin series

Remark. A power series has an interval/radius of convergence. You can only evaluate the series if $x \in$ interval of convergence.

Example 1. What is the Taylor series for $f(x) = e^x$ at $c = 0$? We have $f^{(k)}(x) = e^x$, so $f^{(k)}(0) = 1$. Thus:

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

and the radius of convergence is ∞ .

I.e. for any $x \in \mathbb{R}$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For an algorithm we need a finite amount of terms. For example,

$$e^x \approx \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 = 1 + x + \frac{x^2}{2}$$

This is a polynomial!

Example 2. Let's calculate Taylor series of a polynomial.

$$f(x) = 4x^2 + 5x + 7, \quad c = 2$$

$$f(2) = 33, \quad f'(2) = 8x + 5 \Big|_{x=2} = 21, \quad f''(2) = 8$$

Taylor series:

$$33 + 21(x - 2) + \frac{8}{2}(x - 2)^2 = 4x^2 + 5x + 7 = f(x)$$

Taylor series of a polynomial is itself.

Theorem 1 (Taylor theorem). Let $f \in C^{n+1}([a, b])$ (i.e. f is $(n + 1)$ -times continuously differentiable). Then for any $x \in [a, b]$ we have that

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!} (x - c)^k + \frac{f^{(n+1)}(\xi_x)}{(n + 1)!} (x - c)^{n+1}$$

where ξ_x is a point that depends on x and which is between c and x .

The first sum is called *truncated Taylor series*, the remainder is called *the error*.

Example. For $n = 0$:

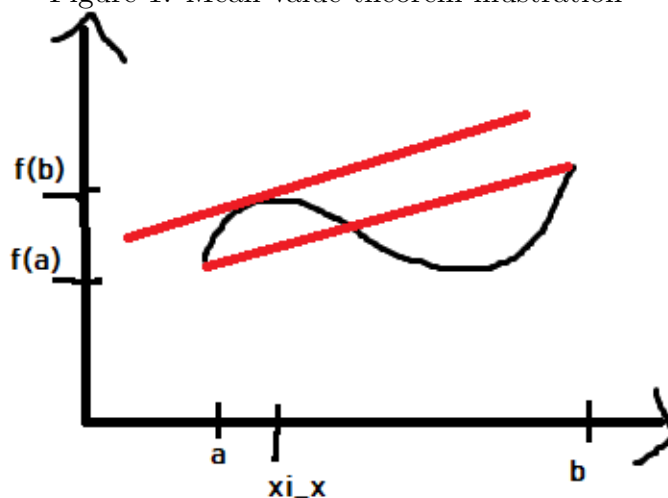
$$f(x) = f(c) + f'(\xi_x)(x - c)$$

Choose $c = a$, $x = b$:

$$f(b) = f(a) + f'(\xi_x)(b - a) \iff f'(\xi_x) = \frac{f(b) - f(a)}{b - a}$$

This is the mean value theorem!

Figure 1: Mean value theorem illustration



Definition 2. We say that the Taylor series *represents* the function f at x if the Taylor series converges at that point, i.e. the remainder tends to zero as $n \rightarrow \infty$.

Example 1. Back to e^x : $f(x) = e^x$, $c = 0$, ξ_x is between c and x .

$$e^x = \sum_{k=0}^n \frac{x^k}{k!} + \frac{e^{\xi_x}}{(n + 1)!} x^{n+1}$$

For any $x \in \mathbb{R}$ we find $s \in \mathbb{R}_0^+$ (\mathbb{R}_0^+ are all real, positive numbers including 0) so that $|x| \leq s$, and $|\xi_x| \leq s$ because ξ_x is between c and x .



Because e^x is monotone increasing, we have $e^{\xi_x} \leq e^s$, thus

$$\lim_{n \rightarrow \infty} \left| \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \right| \leq \lim_{n \rightarrow \infty} \left| \frac{e^s}{(n+1)!} s^{n+1} \right| = e^s \lim_{n \rightarrow \infty} \frac{s^{n+1}}{(n+1)!} = 0$$

Because $(n+1)!$ will grow faster than any power of $s \implies \lim_{n \rightarrow \infty} \left| \frac{e^{\xi_x}}{(n+1)!} x^{n+1} \right| = 0$.

Thus e^x is *represented* by its Taylor series.

Example 2.

$$\begin{aligned} f(x) &= \log(1+x), \quad c=0 \\ f'(x) &= \frac{1}{1+x} = (1+x)^{-1} \\ f''(x) &= -(1+x)^{-2} \\ f'''(x) &= +2(1+x)^{-3} \\ f^{(k)}(x) &= (-1)^{k+1}(k-1)! \frac{1}{(1+x)^k} \end{aligned}$$

So $f^{(k)}(0) = (-1)^{k-1}(k-1)!$ for $k \geq 1$, $f(0) = \log(1) = 0$.

Taylor series:

$$\begin{aligned} f(x) &= \sum_{k=1}^n \frac{(-1)^{k-1}}{k} x^k + \frac{(-1)^k}{n+1} \frac{1}{(1+\xi_x)^{n+1}} \cdot x^{n+1} \quad \left(\frac{n!}{(n+1)!} = \frac{1}{n+1} \right) \\ E_n(x) &= \frac{(-1)^k}{n+1} \frac{1}{(1+\xi_x)^{n+1}} \cdot x^{n+1} \text{ --- the remainder} \end{aligned}$$

Question: for which x does $\lim_{n \rightarrow \infty} E_n(x) = 0$?

$$\lim_{n \rightarrow \infty} E_n(x) = \lim_{n \rightarrow \infty} \frac{(-1)^n}{n+1} \left(\frac{x}{\xi_x + 1} \right)^{n+1} \text{ for } \xi_x \in (c, x) \quad (c=0)$$

Such a limit converges to 0, if the fraction is less than 1.

$$0 < \frac{x}{\xi_x + 1} < 1 \iff x < \xi_x + 1 \iff x - \xi_x < 1 \text{ with } \xi_x \in (0, x) \iff x \leq 1$$

Consequence. $\lim_{n \rightarrow \infty} E_n(x) = 0$ if $0 < x \leq 1$. This means that the Taylor series represents $\log(x+1)$ for $x \in [0, 1]$. We can extend this to show $x \in (-1, 1]$.

Example 3. Let's compute $\cos(0.1)$. Let's approximate it with Taylor series with $c=0$ (around zero).

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \dots + \text{remainder}$$

Consequence.

$$\left| \cos(x) - \sum_{k=0}^n (-1)^k \frac{x^{2k}}{(2k)!} \right| = \left| (-1)^{n+1} \cos(\xi_x) \frac{x^{2(n+1)}}{(2(n+1))!} \right| \leq \frac{0.1^{2(n+1)}}{2(n+1)!} \xrightarrow{n \rightarrow \infty} 0$$

n	Taylor polynomial	$ error \leq$
0	1	$\frac{(0.1)^2}{2} = 0.0005$
1	0.995	$\frac{0.0001}{24}$
2	0.99500416	$\frac{0.000001}{6!}$

Error depends on choice of $|x - c|$ and n .

Example 4. Compute $\log(2)$ using $f(x) = \log(x + 1)$

$$\log(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

Keeping 8 terms (until $n = 8$) we get $\log(2) \approx 0.63452$, the actual solution is $\log(2) = 0.693147$. Not so accurate. Can we improve?

We can use Taylor series of $\log\left(\frac{1+x}{1-x}\right)$ instead, since $\log\left(\frac{1+x}{1-x}\right) = \log(1+x) - \log(1-x)$. We choose $x = \frac{1}{3}$ instead of $x = 1$. Since x is closer to zero, both of the logarithms converge quicker.

$$\left(\log\left(\frac{1+1/3}{1-1/3}\right) = \log(2) \right)$$

We then get

$$\log(2) = 2 \cdot \left(\frac{1}{3} + \frac{1}{3^3 \cdot 5} + \dots \right)$$

We only need 4 terms to get $\log 2 \approx 0.69313$.

Theorem 2 (Reformulation of Taylor's theorem). $f \in C^{n+1}([a, b])$. We change c to x and the old x to $x + h$ from previous version \implies get for $x, x + h \in [a, b]$:

$$f(x + h) = \sum_{k=0}^n \frac{f^{(k)}(x)}{k!} h^k + \frac{f^{(n+1)}(\xi_x)}{(n+1)!} h^{n+1} \text{ where } \xi_x \in (x, x + h), h > 0$$

We can write error term as

$$f(x + h) - \sum_{k=0}^n \frac{f^{(k)}(x)}{k!} h^k = \mathcal{O}(h^{n+1})$$

Remark. Let's recall what the \mathcal{O} -notation means. $a(h) = \mathcal{O}(b(h))$ if $\exists c > 0$ such that $\frac{a(h)}{b(h)} \leq c$ as $h \rightarrow 0$. So, for $n = 1$ the error decreases with h^2 (quadratic convergence).
 $n = 2$: error decreases cubically, i.e. h^3 , etc.

Summary of Taylor series:

- Problem: Evaluate $f(x)$ with a given error bound.
- Required: $f \in C^{n+1}$, values of derivatives $f^{(k)}(a)$.
- Check interval of convergence: does Taylor series expansion work?
- Estimate the maximum error for n terms of the Taylor polynomial.
- Choose n , such that the error bound is low enough.
- Evaluate the Taylor polynomial.

1.2 Number representation

There are different error types:

1. An error in data (partly due to roundoff).
2. Roundoff errors (during computation). For example, multiplication increases the amount of needed significant digits, and we can't store them all on a computer.
3. Truncation error, that is inherent to numeric methods. For example, if we take a finite number of terms in our Taylor series.

Definition 1. Let \tilde{a} be an approximation of a . Then $|\tilde{a} - a|$ is the *absolute error*, and $\frac{|\tilde{a} - a|}{a}$ is the *relative error*. The *error bound* is the magnitude of admissible error.

Example. 0.00123 with error $0.000004 = 0.4 \cdot 10^{-5}$. The error is below $\frac{1}{2} \cdot 10^{-t}$ with $t = 5$, so there's 5 correct digits and 3 significant digits (the number of non-leading zeros).

Example. 0.00123 with error $0.000006 = 0.06 \cdot 10^{-4} > \frac{1}{2} \cdot 10^{-5}$. Only has 2 significant digits, because we have to round the error up.

Theorem 1. In addition/subtraction the bounds for absolute errors are added, in multiplication/division the relative errors are added.

Example. Solve $x^2 - 56x + 1 = 0$:

$$x = 28 - \sqrt{783} \approx 28 - 27.982 \text{ (5 significant digits)} = 0.018 \pm \frac{1}{2} \cdot 10^{-3}$$

We end up with 2 significant digits in the answer, despite that we used to have 5. That's why computers use floating point numbers, as leading zeros are bad.

Definition 2 (Error propagation). If $y(x)$ is smooth, then the derivative $|y(x)'|$ can be interpreted as the sensitivity of $y(x)$ to errors in x . We can generalize this to functions of multiple variables:

$$|\Delta y| \leq \sum \left| \frac{\partial y}{\partial x_i} \right| \cdot |\Delta x_i|, \text{ where } \Delta y = \tilde{y} - y, \Delta x_i = \tilde{x}_i - x_i$$

This is an empirical inequality that is only valid for small Δx_i . It is used a lot in physics.

Definition 3 (Base representation). Every number $x \in \mathbb{N}$ can be written in the following form as a unique expansion with the respect to the base b , where $b \in \mathbb{N} \setminus \{0\}$:

$$x = a_0 b^0 + a_1 b^1 + a_2 b^2 + \dots + a_n b^n = \sum_{i=0}^n a_i b^i$$

$$a \in \mathbb{N}_0, a_i < b, a_i \in \{0, \dots, b-1\}$$

Here b is called the *base*, a_i are called the *digits*. Humans usually use base 10. But, for example, computers can use base 2.

For a real number $x \in \mathbb{R}$ we can write:

$$x = \sum_{i=0}^n a_i b^i + \sum_{i=1}^{\infty} a_{-i} b^{-i}$$

Example. $b = 2$: $1011 = 1 \cdot 2^0 + 1 \cdot 2^1 + 0 \cdot 2^2 + 1 \cdot 2^3 = (11)_{10}$

There are different algorithms that convert number systems.

1.2.1 Euclid's algorithm

Euclid's algorithm converts $(x)_{10}$ to $(y)_b$.

1. Input $(x)_{10}$.
2. Determine the smallest n , such that $x < b^{n+1}$.
3. For $i = n$ to 0 do:

$$\begin{aligned} a_i &:= x \text{ div } b^i \text{ (integer division)} \\ x &:= x \text{ mod } b^i \text{ (the remainder)} \end{aligned}$$

4. Output result $a_n a_{n-1} a_{n-2} \dots a_0 = (y)_b$.

Example. 1. $(x)_{10} = (13)_{10} \rightarrow (y)_2$

2. $n = 3$ since $13 < 2^4$.
- 3.

$$\begin{aligned} i = 3 : a_3 &= 13 \text{ div } 2^3 = 1, x = 13 \text{ mod } 2^3 = 5 \\ i = 2 : a_2 &= 5 \text{ div } 2^2 = 1, x = 5 \text{ mod } 2^2 = 1 \\ i = 1 : a_1 &= 1 \text{ div } 2^1 = 0, x = 1 \text{ mod } 2^1 = 1 \\ i = 0 : a_0 &= 1 \text{ div } 2^0 = 1, x = 1 \text{ mod } 2^0 = 0 \end{aligned}$$

4. Output: $(1101)_2 = (13)_{10}$.

Two problems of the Euclid's algorithm:

1. Step 2 is inefficient
2. Division by large numbers can be problematic.

1.2.2 Horner's scheme

Horner's scheme is a more efficient algorithm. The idea is to represent the number as follows:

$$(a_n a_{n-1} \dots a_0)_b = a_0 + b(a_1 + b(a_2 + b(a_3 + \dots + b(a_n)))) \dots$$

The algorithm is the following:

1. Input $(x)_{10}$.
2. $i := 0$.
3. While $x > 0$ do

$$\begin{aligned} a_i &:= x \text{ mod } b \\ x &:= x \text{ div } b \end{aligned}$$

4. Output result $a_n a_{n-1} a_{n-2} \dots a_0 = (y)_b$.

Remark. The algorithm is very similar to the Euclid's algorithm — the difference is that we execute it in reverse. We no longer have divisions by large numbers, and thus it runs faster.

General remarks:

1. A number with simple representation in one base may be complicated to represent in another base. For example, $(0.1)_{10} = (0.0001100110011 \dots)_2$.
2. Base 2 is called *binary*, base 8 is *octal*, base 16 is *hexadecimal*.
3. To convert from a base b to base 10 we can just perform the following computation:

$$(42)_8 = 4 \cdot 8^1 + 2 \cdot 8^0 = (34)_{10}$$