# Analysis 3

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# 1. Measure

## 1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of $\mathbb{R}$. Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

**Theorem 1.1.** There exist no such function $\mu : 2^{[0,1]} \to [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The length of the unit interval is 1.

**Proof**. First, several definitions:

Step 1. Let's define the following equivalence relation: if $x$, $y$ are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an $N$ exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets $N_R$ are congruent to $N$ and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists! y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as $N$ contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu\big([1/3, 2/3]\big) \leqslant \mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum_{r \in \mathbb{Q} \cap [0,2/3]} \mu(N_r) \leqslant 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 2/3]$?! □

***Consequence.*** We cannot generalize the notion of length to all subsets of real numbers.

## 1.2 Lebesgue Outer Measure

***Definition* 1.1.** If $I \subset \mathbb{R}$ is an interval, then $l(I) =$ the length of $I$. If $I$ is unbounded, then $l(I) = \infty$.

***Definition* 1.2** (Outer Measure)**.**

$$m^* : 2^{\mathbb{R}} \to [0, +\infty]$$
$$m^*(A) = \inf\Big\{\sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j\Big\}$$

In words, it's the infimum of all *countable* covers of $A$. (A countable sum either converges or diverges to infinity).

***Remark*.** This is certainly not a measure — otherwise, it would contradict Theorem 1.1.

**Example.** If $A$ is countable, then $m^*(A) = 0$.

**Proof**. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leqslant 2\varepsilon$. Let's choose a cover of the points with segments of lengths $\varepsilon$, $\varepsilon/2$, $\varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf\{\dots\} \leqslant \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

**Proposition 1.2.** If $A$ is an interval, then $m^*(A) = l(A)$.

**Proof**. a) $A$ is a closed interval, $A = [a, b]$.

1. $m^*(A) \leqslant b - a$. To prove this, we can cover $A$ with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

. Now take $\varepsilon \to 0$.

2. $m^*(A) \geqslant b - a$. Suppose we an infinite cover of $A$ by open intervals. Since $A$ is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is obvious.

b) If $A$ is an open or semiclosed interval, we can approximate it from both sides by closed intervals.

c) If $A$ is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

□