Numerical Methods

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1. Introduction

***Definition* 1.1** (Numerical Methods)**.** Numerical Methods are algorithmic approaches to numerically solve mathematical problems. We use it often it is hard/difficult/impossible to solve analytically.

1.1 Taylor series

Given a function $f : \mathbb{R} \to \mathbb{R}$ (that is hard to evaluate for some $x \in \mathbb{R}$), but f and $f^{(n)}$ are known for a value c, which is close to x. Can we use this information to approximate $f(x)$?

We know values for $\cos^{(n)}(0)$.

$$\begin{cases} f(0) = \cos(0) = 1 \\ f'(0) = -\sin(0) = 0 \\ f''(0) = -\cos(0) = -1 \end{cases} \qquad \text{for } c = 0$$

Can we get $\cos(0.1)$ from this?

***Definition* 1.2** (Taylor series)**.** Let $f : \mathbb{R} \to \mathbb{R}$, differentiable infinitely many times at $c \in \mathbb{R}$. So we have $f^{(k)}(c),\ k = 1, 2, \dots$. Then the Taylor series of f at c is:

$$f(x) \approx f(c) + \frac{f(c)}{1!}(x-c)^1 + \frac{f''(c)}{2!}(x-c)^2 + \cdots = \sum_{k=0}^{\infty} \frac{f^{(k)}}{k!}(x-c)^k$$

Remark. Taylor series is a power series.

Remark. For $c = 0$ also known as Maclaurin series

Remark. A power series has an interval/radius of convergence. You can only evaluate the series if $x \in$ interval of convergence.

Example 1. What is the Taylor series for $f(x) = e^x$ at $c = 0$? We have $f^{(k)}(x) = e^x$, so $f^{(k)}(0) = 1$. Thus:

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k$$

and the radius of convergence is ∞.

I.e. for any $x \in \mathbb{R}$:

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

For an algorithm we need a finite amount of terms. For example,

$$e^x \approx \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 = 1 + x + \frac{x^2}{2}$$

This is a polynomial!

Example 2. Let's calculate Taylor series of a polynomial.

$$f(x) = 4x^2 + 5x + 7,\ c = 2$$
$$f(2) = 33,\ f'(2) = 8x + 5\Big|_{x=2} = 21,\ f''(2) = 8$$

Taylor series:

$$33 + 21(x-2) + \frac{8}{2}(x-2)^2 = 4x^2 + 5x + 7 = f(x)$$

Taylor series of a polynomial is itself.

Theorem 1.1 (Taylor theorem)**.** Let $f \in \mathrm{C}^{n+1}([a,b])$ (i.e. f is $(n+1)$-times continuously differentiable). Then for any $x \in [a,b]$ we have that

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi_x)}{(n+1)!}(x-c)^{n+1}$$

where ξ_x is a point that depends on x and which is between

The first sum is called *truncated Taylor series*, the error is called *the remainder*.