Analysis 3

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February 22, 2023

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1. Measure

1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of $\mathbb{R}$. Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

Theorem 1. There exist no such function $\mu : 2^{[0,1]} \to [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

Proof. First, several definitions:

Step 1. Let's define the following equivalence relation: if x, y are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an N exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets N_R are congruent to N and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists!\, y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as N contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu\big([1/3, 2/3]\big) \leqslant \mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum_{r \in \mathbb{Q} \cap [0,2/3]} \mu(N_r) \leqslant 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 1]$?! ☐

Consequence. We cannot generalize the notion of length to all subsets of real numbers.

1.2 Lebesgue Outer Measure

***Definition* 1.** If $I \subset \mathbb{R}$ is an interval, then $l(I) =$ the length of I. If I is unbounded, then $l(I) = \infty$.

***Definition* 2** (Outer Measure)**.**

$$m^* : 2^{\mathbb{R}} \to [0, +\infty]$$
$$m^*(A) = \inf\Big\{\sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j\Big\}$$

In words, it's the infimum of all *countable* covers of A. (A countable sum either converges or diverges to infinity).

Remark. This is certainly not a measure — otherwise, it would contradict Theorem 1.

Example. If A is countable, then $m^*(A) = 0$.

Proof. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leqslant 2\varepsilon$. Let's choose a cover of the points with segments of lengths ε, $\varepsilon/2$, $\varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf\{\dots\} \leqslant \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

Proposition 1. If A is an interval, then $m^*(A) = l(A)$.

Proof. a) A is a closed interval, $A = [a, b]$.

1. $m^*(A) \leqslant b - a$. To prove this, we can cover A with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

   Now take $\varepsilon \to 0$.

2. $m^*(A) \geqslant b - a$. Suppose we an infinite cover of A by open intervals. Since A is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

b) If A is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

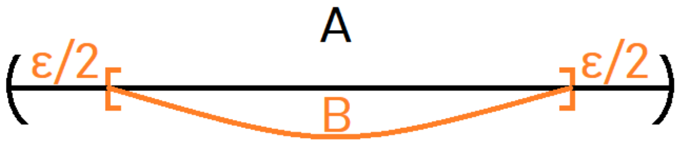
c) If A is an open or semiclosed interval, we can approximate it from both sides by closed intervals. Let's denote the closure of A by $\bar{A}$. Since we're adding points, the Outer Measure will not decrease:

$$A \subset \overline{A} \implies m^*(A) \leqslant m^*(\overline{A}) = l(a)$$

Now suppose we have a closed interval B strictly inside A. Then we get

$$m^*(A) \geqslant m^*(B) = l(B) = l(A) - \varepsilon$$

Now take $\varepsilon \to 0 \implies m^*(A) \geqslant l(A)$.



□

Lemma. m^* is translation-invariant.

Proof. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

Proposition 2 (Countable subadditivity)**.** For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leqslant \sum_{k=1}^{\infty} m^*(E_k)$$

Remark. We don't ask for the sets E_k to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that m^* is a measure, which we proved does not exist in Theorem 1.

Proof. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} E_{k,i} \ (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leqslant k, i < \infty\}$ covers each of the E_k, and thus it's a cover of $\cup_{k=1}^{\infty} E_k$. Then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \overset{\text{it's a cover}}{\leqslant} \sum_{1 \leqslant k,i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon\left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

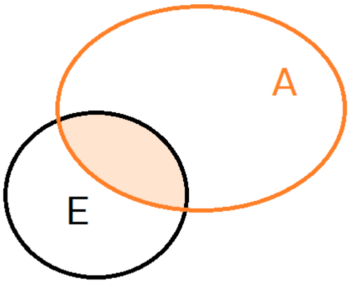
Now take $\varepsilon \to 0$. □

Remark. Here we assume that all of the E_k have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leqslant \infty$ which is "true".

1.3 The σ-algebra of Lebesgue-measurable sets.

***Definition* 1.** A set E is (Lebesgue) measurable if for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \qquad E^C = \mathbb{R} \setminus E$$



The set E "splits" A into two parts

***Remark*.** We already have the $\leqslant$ sign from countable subadditivity.

***Remark*.** Motivation: If $A \cap B = \varnothing$ and A (or B) is measurable, then

$$m^*(A \cup B) = m^*\big((A \cup B) \cap A\big) + m^*\big((A \cup B) \cap A^C\big) = m^*(A) + m^*(B)$$

Proposition 1. If $m^*(E) = 0$, then E is measurable.

Proof. For all A we have:

$$m^*(A \cap E) \leqslant m^*(E) = 0 \implies m^*(A \cap E) = 0$$
$$m^*(A) \geqslant m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C)$$

As we noted earlier, the inequality in the other side follows from countable subadditivity. □

Proposition 2. If $E_1, \ldots, E_n$ are measurable, then $\cup_1^n E_k$ is measurable.

Proof. Case $n = 2$: for all A we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C) =$$
$$= m^*(A \cap E_1) + m^*\big((A \cap E_1^C) \cap E_2\big) + m^*\big((A \cap E_1^C) \cap E_2^C\big) = (*)$$
$$X \coloneqq A \cap E_1,\ Y \coloneqq (A \cap E_1^C) \cap E_2,\ Z \coloneqq (A \cap E_1^C) \cap E_2^C$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply countable subadditivity to X and Y. Then we get:

$$(*) \geqslant m^*\big(A \cap (E_1 \cup E_2)\big) + m^*\big(A \cap (E_1 \cup E_2)^C\big)$$

Yet again, the inequality in the other side follows from countable subadditivity.

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k$, E_n. □

Definition 2 (Algebra)**.** Let X be a non-empty set. $\Omega \subset 2^X$ is an algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *finite* unions.

Remark. It follows that Ω is also closed under intersections:

$$\left(X_1^C \cup \cdots \cup X_n^C\right)^C = X_1 \cap \cdots \cap X_n$$

Definition 3 (σ-algebra)**.** Let X be a non-empty set. $\Omega \subset 2^X$ is a σ-algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *countable* unions.

Remark. Every σ-algebra is an algebra, but not vice versa.

Corollary 1. The collection $\mathcal{M}$ of all measurable subsets of $\mathbb{R}$ is an algebra.

Proof. For the proof, we'll need to show that:

1. $\mathbb{R}$ is measurable.
$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\varnothing)$$
2. It is closed under complements. It follows from the symmetry of the definition of a measurable set.
3. It is closed under unions. We have already proved this one.

□

Proposition 3. $\{E_k\}_1^n$ — disjoint measurable sets. Then for every set A

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for $A = \mathbb{R}$ we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

Proof. Induction on n.

Base $n = 1$ is obvious.

Step $n-1 \to n$. Take $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$. Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with E_n^C is equivalent to subtracting E_n from $\hat{A}$, and since $\{E_k\}$ are disjoint, no other parts of $\hat{A}$ except E_n will be removed. Then:

$$m^*(\hat{A}) \overset{E_n \text{ is measurable}}{=\!=\!=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) =$$
$$= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \overset{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k)$$

□

Proposition 4. The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

Proof. If $A = \cup_1^\infty A_k$, define $\hat{A}_1 := A_1$ and $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$. As $\mathcal{M}$ is an algebra, all $\hat{A}_k$ are measurable, and $A = \sqcup_1^\infty \hat{A}_k$, which is what we wanted. □

Theorem 1. $\mathcal{M}$ is a σ-algebra.

Proof. We need to show that if all $\{E_k\}_1^\infty$ are measurable sets, then $E = \cup_1^\infty E_k$ is measurable. By Proposition 4, without the loss of generality, assume that E_k are all pairwise disjoint. Let $F_n := \cup_1^n E_k$, then $F_n \in \mathcal{M}$ (as a finite union). As $F_n \subset E$, we have $E^C \subset F_n^C$.

Let A be any set. Then:

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^C) \geqslant m^*(A \cap F_n) + m^*(A \cap E^C) \overset{\text{Proposition 3}}{=}$$
$$= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E_C)$$

Now take $n \to \infty$:

$$m(A) \geqslant \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E_C) \overset{\text{countable subadditivity}}{\geqslant} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity). □

Proposition 5 (Countable additivity)**.** If $\{E_k\}_1^\infty \subset \mathcal{M}$ — collection of disjoint sets, then $\cup_1^\infty E_k \in \mathcal{M}$ and

$$m^*\Big(\bigcup_1^\infty E_k\Big) = \sum_1^\infty m^*(E_k)$$

Proof. We know that:

1. $$m^*\Big(\bigcup_1^\infty E_k\Big) \leqslant \sum_1^\infty m^*(E_k) \text{ (countable } \textit{sub}\text{additivity)}$$

2. $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant m^*\Big(\bigcup_1^n E_k\Big) \overset{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

   Take $n \to \infty$, then
   $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant \sum_1^\infty m^*(E_k)$$

   Which is what we wanted.

□

***Definition* 4.** The restriction of m^* on $\mathcal{M}$ is called the Lebesgue measure and denoted by m.

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

***Definition* 5.** If X is a non-empty set and $\mathcal{A}$ is a σ-algebra on X, then any function $\mu : \mathcal{A} \to [0, +\infty]$ is called the measure on $(X, \mathcal{A})$, if:

1. $\mu(\varnothing) = 0$.
2. μ is countable additive.

***Definition* 6** (Measurable space)**.** A *measurable space* is a tuple $(X, \mathcal{A})$, where:

1. X is a set.
2. $\mathcal{A}$ is a σ-algebra on X.

***Definition* 7** (Measure space)**.** A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where:

1. X is a set.
2. $\mathcal{A}$ is a σ-algebra on X.
3. μ is a measure on $(X, \mathcal{A})$.

Example 1. $\{\varnothing, X\}$ is a σ-algebra. Any μ, such that $\mu(\varnothing) = 0$ and $\mu(X) \geqslant 0$ will be a measure.

Example 2. 2^X is a σ-algebra. We can have the following measures:

a) $\mu(E) = |E|$ is called a *counting measure*. Here $|E|$ denotes the cardinality of E (number of elements in E).

b) δ-measure (also called Dirac measure):

$$\mu(E) = \begin{cases} 1, & 0 \in E \\ 0, & \text{otherwise} \end{cases}$$

1.4 Continuity of measure

***Definition* 1.** A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *ascending* if $E_k \subset E_{k+1}$.

***Definition* 2.** A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *descending* if $E_k \supset E_{k+1}$.

Theorem 1 (Continuity of measure)**.**

1. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and the sequence is ascending, then

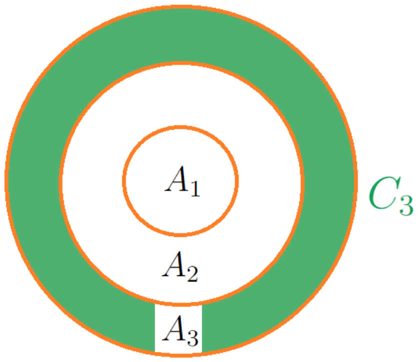
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} \mu(A_k)$$

2. If $\{B_k\}_{k=1}^{\infty} \subset \mathcal{A}$, the sequence is descending and $\mu(B_1) < \infty$, then

$$\mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) = \lim_{k\to\infty} \mu(B_k)$$

Proof. 1. Let $C_k \coloneqq A_k \setminus A_{k-1}$. Then we have:

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \mu\Big(\bigsqcup_{k=1}^{\infty} C_k\Big) = \sum_{k=1}^{\infty} \mu(C_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(C_k) = \lim_{n\to\infty} \mu(A_n)$$



2. Let $D_k \coloneqq B_1 \setminus B_k$. Since B_k is descending, it follows that D_k is an ascending sequence. Then from the part 1 of the theorem it follows that:

$$\mu\Big(\bigcup_{k=1}^{\infty} D_k\Big) = \lim_{k\to\infty} \mu(D_k) \qquad \bigcup_{k=1}^{\infty} D_k = B_1 \setminus \bigcap_{k=1}^{\infty} B_k$$

$$\mu\Big(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\Big) = \lim_{k\to\infty} \big(\mu(B_1) - \mu(B_k)\big) = \mu(B_1) - \lim_{k\to\infty} \mu(B_k)$$

$$\mu\Big(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\Big) = \mu(B_1) - \mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) \implies \mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) = \lim_{k\to\infty} \mu(B_k)$$

□

Definition 3. We say that a statement (property) holds for *almost all $x \in X$ with respect to a measure μ*, if $\exists N \in \mathcal{A}$, such that $\mu(N) = 0$ and the statement (property) holds for all $x \in X \setminus N$.

Lemma (Borel–Cantelli)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{E_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then *almost all* $x \in X$ belong to at most finitely many E_k.

Proof. Let $B_n = \cup_{k=n}^{\infty} E_k$. It's easy to see that B_k is a descending measure. At the same time,

$$\mu(B_1) = \mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leqslant \sum_{k=1}^{\infty} \mu(E_k) < \infty$$

By definition of B_n, $\cap_{n=1}^{\infty} B_n$ contains all the points that are contained in infinitely many E_k's. But, by continuity of measure for $\{B_n\}_{n=1}^{\infty}$ we have:

$$\mu\Big(\bigcap_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \Big(\bigcup_{k=n}^{\infty} E_k\Big) \leqslant \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$$

□

1.5 How large is the Lebesgue σ-algebra $\mathcal{M}$?

Proposition 1. Every interval is Lebesgue-measurable.

Proof. Proof idea:

$$E \in \mathcal{M} \iff \forall A : m(A) = m(A \cap E) + m(A \cap E^C)$$

Assume $E = (-\infty, a)$. If we prove that such intervals lie in $\mathcal{M}$, then we'll prove everything (since $\mathcal{M}$ is a σ-algebra). We already have $m(A) \leqslant m(A \cap E) + m(A \cap E^C)$ from countable subadditivity.

Let's assume $a \notin A$ (since removing one point does not change the measure). Every cover of A can be split into two covers with the same sum of interval lengths: of $A \cap (-\infty, a)$ and $A \cap (a, +\infty)$. Every interval in those covers, that contains a, can be split into two. Therefore, from the definition of Lebesgue measure, $m(A) \geqslant m(A \cap E) + m(A \cap E^C)$, so we've proved the inequality in both sides. □

***Definition* 1.** For any $\mathcal{X} \in 2^{\mathbb{R}}$ let $\mathcal{A}(\mathcal{X})$ be the smallest σ-algebra containing $\mathcal{X}$.

Lemma. $\mathcal{A}(\mathcal{X})$ always exists and is the intersection of all σ-algebras containing $\mathcal{X}$.

Proof. We have to prove that if we intersect a bunch σ-algebras, we still get a σ-algebra.

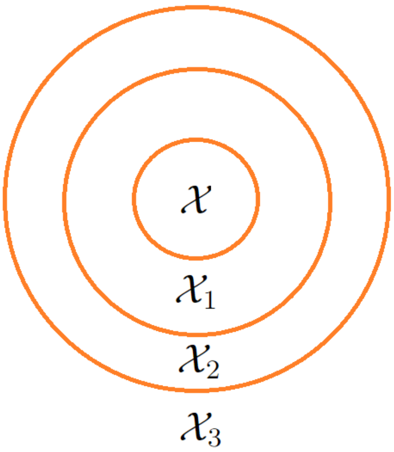
1. Such an intersection is closed under complements: if a set belongs to the intersection of σ-algebras, then it belongs to each of the σ-algebras, then its complement belongs to each of the σ-algebras, and thus its complement belongs to the intersection of σ-algebras.

2. In a similar way, such an intersection is closed under countable unions: if a number of sets all belong to the intersection of σ-algebras, then they all belong to each of the σ-algebras, then their countable union belongs to each of the σ-algebras, and their countable union belongs to the intersection of σ-algebras.

□

Remark. We can try to construct $\mathcal{A}(\mathcal{X})$ in a different way. Say, $\mathcal{X}$ is not a σ-algebra. Let's enlarge it: first by including all the complements. Then let's enlarge it by all countable unions. Let's call such a set $\mathcal{X}_1$. But after such operation, $\mathcal{X}_1$ may be non-closed under complements. So we repeat such a procedure.

And, in general: $\mathcal{X}_{n+1}$ is obtained from $\mathcal{X}_n$ is obtained by including into $\mathcal{X}_n$ all complements of the sets from $\mathcal{X}_n$ and then including all countable unions of the obtained sets.



It is tempting to think that $\cup_1^\infty \mathcal{X}_i$ is $\mathcal{A}(\mathcal{X})$. Is it true? No, not necessarily. If the sequence $\{\mathcal{X}_i\}$ eventually stabilizes, then such a construction works. Let's now assume that every next $\mathcal{X}_i$ is larger than the previous one. Then we can take A from $\mathcal{X}$, A_1 from $\mathcal{X}_1 \setminus \mathcal{X}$, A_2 from $\mathcal{X}_2 \setminus \mathcal{X}_1$, and so on.

Now let's look at $\cup_1^\infty A_i$. As a countable union, it must be contained in $\mathcal{A}(\mathcal{X}) = \cup_1^\infty \mathcal{X}_i$, thus, there exist an n, such that $\cup_1^\infty A_i \in \mathcal{X}_n$. But $A_{n+1} \in \mathcal{X}_{n+1} \setminus \mathcal{X}_n$?!

***Definition* 2** (Topological space)**.** A *topological space* is a set X and a collection of subsets O of X (called *open sets*), such that $\varnothing, X \in O$, and:

1. A union of (possibly infinitely many) sets from O is in O.

2. The intersection of finitely many sets from O is in O.

The complements of open sets are called *closed sets*.

***Definition* 3.** A function $f : X \to Y$ between two topological spaces is *continuous* if the preimage of every open set is open.

Remark. It is possible to check that for $\mathbb{R}$ this definition is equivalent to the usual one.

***Definition* 4** (Borel σ-algebra)**.** For a topological space X its *Borel σ-algebra* $\mathcal{B}_X$ is the smallest σ-algebra on X that contains all open sets.

Remark. If it's obvious from the context which set we are talking about, we will just write $\mathcal{B}$ (without a subscript).

Theorem 1. $\mathcal{B}_\mathbb{R} \subset \mathcal{M}$ (all of the sets in $\mathcal{B}_\mathbb{R}$ are measurable).

Proposition 2. $\mathcal{B}$ is the smallest σ-algebra that contains all open intervals.

If we prove the proposition, the theorem will follow easily. We know that all the intervals are Lebesgue-measurable. We know that the Lebesgue-measurable sets ($\mathcal{M}$) are a σ-algebra. Thus, if we take the smallest σ-algebra that contains all open intervals, it will be a subset of $\mathcal{M}$.

Proof of Proposition 2. We will prove that every open set $O \subset \mathbb{R}$ is a finite or countable union of open intervals.

For every point $x \in O$ let I_x be the largest open interval, such that $x \in I_x$ and $I_x \subset O$. It exists as a union of all such intervals. Since O is open, x lies in O with an open neighborhood, thus, I_x is non-empty.

$$\forall x \in O : x \in I_x \implies O = \bigcup_{x \in O} I_x$$

Let's prove that $I_x \cap I_y \neq \varnothing \implies I_x = I_y$. If the intervals around x and y intersect, then $I_x \cup I_y$ is an interval as well, and $I_x \cup I_y \in O$ as $I_x \in O$ and $I_y \in O$. Since I_x and I_y are the largest such intervals, it follows that $I_x = I_x \cup I_y = I_y$.

Let's say that two points x and y are equivalent if $I_x = I_y$. Since there's a lot of same intervals in $O = \cup_{x \in O} I_x$, we can take just a single point from every equivalence class and still get O as a union. Particularly, every open interval contains at least one rational point (as rational numbers are dense). Therefore, there's a rational point in every equivalence class. Thus,

$$O = \bigcup_{x \in O \cap \mathbb{Q}} I_x$$

Since the set of rational numbers is countable, we have represented O as a countable union of open intervals, which is what we wanted. □

Remark. A topological space is called *separable*, if it contains a countable dense subset.

Remark. We have proved that the Lebesgue measure exists on $\mathcal{B}_\mathbb{R}$, so we have a lot of measurable sets.

Remark. The Lebesgue measure can be generalized to $\mathbb{R}^n$.

1.6 Other critirea of measurability for Lebesgue measure

As we remember, the definition of a measurable set is difficult to check. Thus, we would like to have better critirea.

Theorem 1. $E \subset \mathbb{R}$ is Lebesgue measurable if and only if one of the following holds:

1. For every $\varepsilon > 0$ there exists an open set O, such that $E \subset O$ and $m^*(O \setminus E) < \varepsilon$.
2. There exists a G_δ-set G, such that $E \subset G$ and $m^*(G \setminus E) = 0$.
   (A G_δ-set is a countable intersection of open sets.)
3. For every $\varepsilon > 0$ there exists a closed set F, such that $F \subset E$ and $m^*(E \setminus F) < \varepsilon$.
4. There exists a F_σ-set F, such that $F \subset E$ and $m^*(E \setminus F) = 0$.
   (A F_δ-set is a countable union of closed sets.)

Proof.

- E is measurable $\implies$ 1.

  If $m^*(E) < \infty$, then from the definition of m^* we can find O — a finite union of open intervals, such that $m^*(O) < m^*(E) + \varepsilon$. Since O is an open set, it's measurable (as we proved earlier). Therefore, both E and O are measurable, thus

  $$\varepsilon > m^*(O) - m^*(E) \overset{E,O\in\mathcal{M}}{=} m^*(O \setminus E)$$

  If $m^*(E) = \infty$, let's split the set E into a countable number of sets with finite measure. For example, by splitting the real line into segments of length 1. So, $E = \cup_1^\infty E_k$, where $m^*(E_k) < \infty$. Then let's use geometrically decreasing ε's for the covers of each E_k: $\varepsilon/2$ for E_1, $\varepsilon/4$ for E_2, and so on. When we sum up the inequalities, the fractions will sum up to ε. So, we obtained our O, now continue like in the previous case.

  ***Definition* 1.** A measure μ on X is called *σ-finite*, if $X = \cup_1^\infty X_k$ and $\mu(X_k) < \infty$ for all k.

  In words: if there exists a subdivision of X into a countable number of set of finite measure.

- $1 \implies 2$.

  From 1, $\forall k \in \mathbb{N},\ \exists O_k$ — open, such that $E \subset O_k$ and $m^*(O_k \setminus E) < \frac{1}{k}$. Now let's take

  $$G := \bigcap_1^\infty O_k \implies \forall k:\ m^*(G \setminus E) \leqslant m^*(O_k \setminus E) < \frac{1}{k} \implies m^*(G \setminus E) = 0$$

- $2 \implies E$ is measurable.

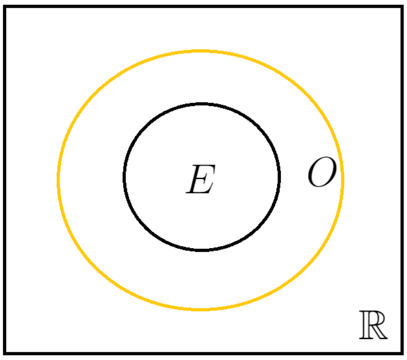
  G is a G_δ-set. As a countable intersection of open sets, it's in Borel σ-algebra, and thus is Lebesgue-measurable. $m^*(G \setminus E) = 0$, then $G \setminus E$ is Lebesgue-measurable, then $E = G \setminus (G \setminus E)$ is measurable as a difference of two measurable sets.

- $3 \iff 1,\ 4 \iff 2$.

  If we assume that 3 holds for E, then, if we take $O = \overline{F}$, 1 will hold for $\overline{E}$. Therefore, $\overline{E}$ is measurable, then E is measurable (as $\mathcal{M}$ is a σ-algebra).

  If 1 holds for E, then E is measurable, then $\overline{E}$ is measurable, then 1 holds for $\overline{E}$. Now take $F = \overline{O}$, therefore, 3 holds for E.

  In the same way, 2 and 4 are equivalent as well.



□

Theorem 2. For every $E \in \mathcal{M}$ with $m(E) < \infty$ and for every $\varepsilon < \infty$ there exists an infinite disjoint collection of open intervals $\{I_k\}_1^n$, such that $O = \cup_{k=1}^n I_k$ and $m(E \triangle O) < \varepsilon$.

(Here $\triangle$ is the symmetric difference of two sets).

Proof. From part 1 of the previous theorem, we can take such an open set U, that $E \subset U$ and $m(U \setminus E) < \varepsilon/2$.

As we proved earlier, we can represent U as a countable union of disjoint open intervals I_k. Then:

$$\forall n : \bigcup_1^n I_k \subset U \implies \forall n : \sum_1^n m(I_k) \leqslant m(U) < \infty$$

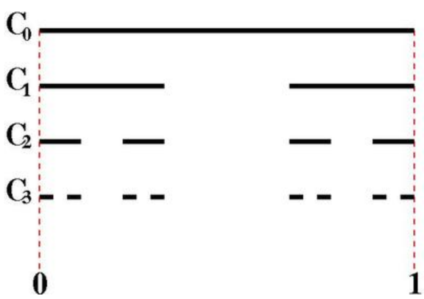
Now take n, such that $\sum_{n+1}^\infty m(I_k) < \varepsilon/2$, and put $O \coloneqq \cup_{k=1}^n I_k$. Then $m(O \setminus E) < \varepsilon/2$ and $m(E \setminus O) < \varepsilon/2$, therefore, the measure of the symmetric difference is less that ε. □

1.7 TBA

Questions:

1. If $m(A) = 0$, is A countable?
2. We know that $\mathcal{B} \subset \mathcal{M}$. Is this inclusion proper?

***Definition* 1** (Cantor set)**.** Let's take $[0, 1]$, split it into three parts and remove the middle part. Then continue such process. The *Cantor set* is the set $C \coloneqq \cap_0^\infty C_k$.



Cantor set illustration from here.

Remark. Its length is zero as $(2/3)^\infty = 0$. But it's uncountable, because if we have a sequence of zeros and ones, we can traverse down-left on 0 and down-right on 1. The intersection of the corresponding intervals will be a single point of C. So, there's a bijection between C and $\{0, 1\}^{\mathbb{N}}$, therefore, C is indeed uncountable.