

Analysis 3

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1. Measure

1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of \mathbb{R} . Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

Theorem 1. There exist no such function $\mu : 2^{[0,1]} \rightarrow [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

Proof. First, several definitions:

Step 1. Let's define the following equivalence relation: if x, y are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an N exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets N_R are congruent to N and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists! y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as N contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu([1/3, 2/3]) \leq \mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum_{r \in \mathbb{Q} \cap [0, 2/3]} \mu(N_r) \leq 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 1]$?! □

Consequence. We cannot generalize the notion of length to all subsets of real numbers.

1.2 Lebesgue Outer Measure

Definition 1. If $I \subset \mathbb{R}$ is an interval, then $l(I)$ = the length of I . If I is unbounded, then $l(I) = \infty$.

Definition 2 (Outer Measure).

$$m^* : 2^{\mathbb{R}} \rightarrow [0, +\infty]$$

$$m^*(A) = \inf \left\{ \sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

In words, it's the infimum of all *countable* covers of A . (A countable sum either converges or diverges to infinity).

Remark. This is certainly not a measure — otherwise, it would contradict Theorem 1.

Example. If A is countable, then $m^*(A) = 0$.

Proof. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leq 2\varepsilon$. Let's choose a cover of the points with segments of lengths $\varepsilon, \varepsilon/2, \varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf \{ \dots \} \leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

Proposition 1. If A is an interval, then $m^*(A) = l(A)$.

Proof. a) A is a closed interval, $A = [a, b]$.

1. $m^*(A) \leq b - a$. To prove this, we can cover A with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

Now take $\varepsilon \rightarrow 0$.

2. $m^*(A) \geq b - a$. Suppose we an infinite cover of A by open intervals. Since A is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

- b) If A is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

- c) If A is an open or semiclosed interval, we can approximate it from both sides by closed intervals.

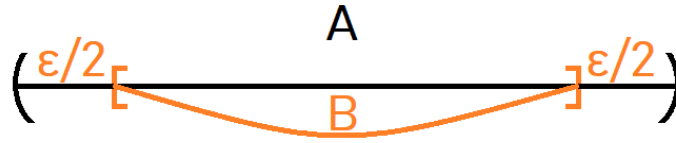
Let's denote the closure of A by \bar{A} . Since we're adding points, the Outer Measure will not decrease:

$$A \subset \bar{A} \implies m^*(A) \leq m^*(\bar{A}) = l(a)$$

Now suppose we have a closed interval B strictly inside A . Then we get

$$m^*(A) \geq m^*(B) = l(B) = l(A) - \varepsilon$$

Now take $\varepsilon \rightarrow 0 \implies m^*(A) \geq l(A)$.



□

Lemma. m^* is translation-invariant.

Proof. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

Proposition 2 (Countable subadditivity). For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

Remark. We don't ask for the sets E_k to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that m^* is a measure, which we proved does not exist in Theorem 1.

Proof. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} I_{k,i} \quad (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leq k, i < \infty\}$ covers each of the E_k , and thus it's a cover of $\bigcup_{k=1}^{\infty} E_k$. Then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \overset{\text{it's a cover}}{\leq} \sum_{1 \leq k, i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

Now take $\varepsilon \rightarrow 0$. □

Remark. Here we assume that all of the E_k have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leq \infty$ which is “true”.

1.3 The σ -algebra of Lebesgue-measurable sets.

Definition 1. A set E is (Lebesgue) measurable if for any set A ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \quad E^C = \mathbb{R} \setminus E$$

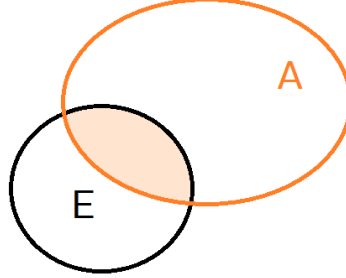


Figure 1: The set E “splits” A into two parts

Remark. We already have the \leq sign from [countable subadditivity](#).

Remark. Motivation: If $A \cap B = \emptyset$ and A (or B) is measurable, then

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^C) = m^*(A) + m^*(B)$$

Proposition 1. If $m^*(E) = 0$, then E is measurable.

Proof. For all A we have:

$$\begin{aligned} m^*(A \cap E) &\leq m^*(E) = 0 \implies m^*(A \cap E) = 0 \\ m^*(A) &\geq m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C) \end{aligned}$$

As we noted earlier, the inequality in the other side follows from [countable subadditivity](#). □

Proposition 2. If E_1, \dots, E_n are measurable, then $\cup_1^n E_k$ is measurable.

Proof. Case $n = 2$: for all A we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^C) = \\ &= m^*(A \cap E_1) + m^*((A \cap E_1^C) \cap E_2) + m^*((A \cap E_1^C) \cap E_2^C) = (*) \\ X &:= A \cap E_1, \quad Y := (A \cap E_1^C) \cap E_2, \quad Z := (A \cap E_1^C) \cap E_2^C \end{aligned}$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply [countable subadditivity](#) to X and Y . Then we get:

$$(*) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^C)$$

Yet again, the inequality in the other side follows from [countable subadditivity](#).

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k, E_n$. □

Definition 2 (Algebra). Let X be a non-empty set. $\Omega \subset 2^X$ is an algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *finite* unions.

Remark. It follows that Ω is also closed under intersections:

$$(X_1^C \cup \dots \cup X_n^C)^C = X_1 \cap \dots \cap X_n$$

Definition 3 (σ -algebra). Let X be a non-empty set. $\Omega \subset 2^X$ is a σ -algebra, if:

1. $X \in \Omega$;
2. Ω is closed under the formation of complements in X and *countable* unions.

Remark. Every σ -algebra is an algebra, but not vice versa.

Corollary 1. The collection \mathcal{M} of all measurable subsets of \mathbb{R} is an algebra.

Proof. For the proof, we'll need to show that:

1. \mathbb{R} is measurable.

$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\emptyset)$$

2. It is closed under complements. It follows from the symmetry of the [definition of a measurable set](#).
3. It is closed under unions. We [have already proved](#) this one.

□

Proposition 3. $\{E_k\}_1^n$ — disjoint measurable sets. Then for every set A

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for $A = \mathbb{R}$ we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

Proof. Induction on n .

Base $n = 1$ is obvious.

Step $n - 1 \rightarrow n$. Take $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$. Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with E_n^C is equivalent to subtracting E_n from \hat{A} , and since $\{E_k\}$ are disjoint, no other parts of \hat{A} except E_n will be removed. Then:

$$\begin{aligned} m^*(\hat{A}) &\stackrel{E_n \text{ is measurable}}{=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) = \\ &= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \stackrel{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k) \end{aligned}$$

□

Proposition 4. The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

Proof. If $A = \cup_1^\infty A_k$, define $\hat{A}_1 := A_1$ and $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$. As \mathcal{M} is an algebra, all \hat{A}_k are measurable, and $A = \sqcup_1^\infty \hat{A}_k$, which is what we wanted. \square

Theorem 1. \mathcal{M} is a σ -algebra.

Proof. We need to show that if all $\{E_k\}_1^\infty$ are measurable sets, then $E = \cup_1^\infty E_k$ is measurable. By Proposition 4, without the loss of generality, assume that E_k are all pairwise disjoint. Let $F_n := \cup_1^n E_k$, then $F_n \in \mathcal{M}$ (as a finite union). As $F_n \subset E$, we have $E^C \subset F_n^C$.

Let A be any set. Then:

$$\begin{aligned} m^*(A) &= m^*(A \cap F_n) + m^*(A \cap F_n^C) \geq m^*(A \cap F_n) + m^*(A \cap E^C) \stackrel{\text{Proposition 3}}{=} \\ &= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E^C) \end{aligned}$$

Now take $n \rightarrow \infty$:

$$m(A) \geq \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E^C) \stackrel{\text{countable subadditivity}}{\geq} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity). \square

Proposition 5 (Countable additivity). If $\{E_k\}_1^\infty \subset \mathcal{M}$ — collection of disjoint sets, then $\cup_1^\infty E_k \in \mathcal{M}$ and

$$m^*\left(\bigcup_1^\infty E_k\right) = \sum_1^\infty m^*(E_k)$$

Proof. We know that:

1.

$$m^*\left(\bigcup_1^\infty E_k\right) \leq \sum_1^\infty m^*(E_k) \text{ (countable subadditivity)}$$

2.

$$m^*\left(\bigcup_1^\infty E_k\right) \geq m^*\left(\bigcup_1^n E_k\right) \stackrel{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

Take $n \rightarrow \infty$, then

$$m^*\left(\bigcup_1^\infty E_k\right) \geq \sum_1^\infty m^*(E_k)$$

Which is what we wanted. \square

Definition 4. The restriction of m^* on \mathcal{M} is called the Lebesgue measure and denoted by m .

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

Definition 5. If X is a non-empty set and \mathcal{A} is a σ -algebra on X , then any function $\mu : \mathcal{A} \rightarrow [0, +\infty]$ is called the measure on (X, \mathcal{A}) , if:

1. $\mu(\emptyset) = 0$.
2. μ is countable additive.

Definition 6 (Measurable space). A *measurable space* is a tuple (X, \mathcal{A}) , where:

1. X is a set.
2. \mathcal{A} is a σ -algebra on X .

Definition 7 (Measure space). A *measure space* is a triple (X, \mathcal{A}, μ) , where:

1. X is a set.
2. \mathcal{A} is a σ -algebra on X .
3. μ is a measure on (X, \mathcal{A}) .

Example 1. $\{\emptyset, X\}$ is a σ -algebra. Any μ , such that $\mu(\emptyset) = 0$ and $\mu(X) \geq 0$ will be a measure.

Example 2. 2^X is a σ -algebra. We can have the following measures:

- a) $\mu(E) = |E|$ is called a *counting measure*. Here $|E|$ denotes the cardinality of E (number of elements in E).
- b) δ -measure (also called Dirac measure):

$$\mu(E) = \begin{cases} 1, & 0 \in E \\ 0, & \text{otherwise} \end{cases}$$

Definition 8. A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *ascending* if $E_k \subset E_{k+1}$.

Definition 9. A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *descending* if $E_k \supset E_{k+1}$.

Theorem 2 (Continuity of measure).

1. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and the sequence is ascending, then

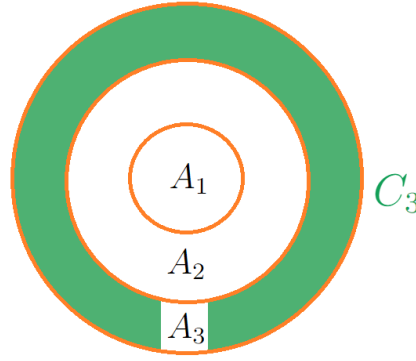
$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k \rightarrow \infty} \mu(A_k)$$

2. If $\{B_k\}_{k=1}^{\infty} \subset \mathcal{A}$, the sequence is descending and $\mu(B_1) < \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k)$$

Proof. 1. Let $C_k := A_k \setminus A_{k-1}$. Then we have:

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \mu\left(\bigsqcup_{k=1}^{\infty} C_k\right) = \sum_{k=1}^{\infty} \mu(C_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(C_k) = \lim_{n \rightarrow \infty} \mu(A_n)$$



2. Let $D_k := B_1 \setminus B_k$. Since B_k is descending, it follows that D_k is an ascending sequence. Then from the part 1 of the theorem it follows that:

$$\begin{aligned} \mu\left(\bigcup_{k=1}^{\infty} D_k\right) &= \lim_{k \rightarrow \infty} \mu(D_k) & \bigcup_{k=1}^{\infty} D_k &= B_1 \setminus \bigcap_{k=1}^{\infty} B_k \\ \mu\left(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\right) &= \lim_{k \rightarrow \infty} (\mu(B_1) - \mu(B_k)) = \mu(B_1) - \lim_{k \rightarrow \infty} \mu(B_k) \\ \mu\left(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\right) &= \mu(B_1) - \mu\left(\bigcap_{k=1}^{\infty} B_k\right) \implies \mu\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k \rightarrow \infty} \mu(B_k) \end{aligned}$$

□

Definition 10. We say that a statement (property) holds for *almost all* $x \in X$ with respect to a measure μ , if $\exists N \in \mathcal{A}$, such that $\mu(N) = 0$ and the statement (property) holds for all $x \in X \setminus N$.

Lemma (Borel–Cantelli). Let (X, \mathcal{A}, μ) be a measure space. Let $\{E_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then *almost all* $x \in X$ belong to at most finitely many E_k .

Proof. Let $B_n = \bigcup_{k=n}^{\infty} E_k$. It's easy to see that B_n is a descending sequence. At the same time,

$$\mu(B_1) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k) < \infty$$

By definition of B_n , $\bigcap_{n=1}^{\infty} B_n$ contains all the points that are contained in infinitely many E_k 's. But, by [continuity of measure](#) for $\{B_n\}_{n=1}^{\infty}$ we have:

$$\mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \rightarrow \infty} \mu(B_n) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$$

□

How large is the Lebesgue σ -algebra \mathcal{M} ?

Proposition 6. Every interval is Lebesgue-measurable.

Proof. Proof idea:

$$E \in \mathcal{M} \iff \forall A : m(A) = m(A \cap E) + m(A \cap E^C)$$

Assume $E = (-\infty, a)$. If we prove that such intervals lie in \mathcal{M} , then we'll prove everything (since \mathcal{M} is a σ -algebra). We already have $m(A) \leq m(A \cap E) + m(A \cap E^C)$ from [countable subadditivity](#).

Let's assume $a \notin A$ (since removing one point does not change the measure). Every cover of A can be split into two covers with the same sum of interval lengths: of $A \cap (-\infty, a)$ and $A \cap (a, +\infty)$. Every interval in those covers, that contains a , can be split into two. Therefore, from the [definition of Lebesgue measure](#), $m(A) \geq m(A \cap E) + m(A \cap E^C)$, so we've proved the inequality in both sides. \square

Definition 11. For any $\mathcal{X} \in 2^{\mathbb{R}}$ let $\mathcal{A}(\mathcal{X})$ be the smallest σ -algebra containing \mathcal{X} .

Lemma. $\mathcal{A}(\mathcal{X})$ always exists and is the intersection of all σ -algebras containing \mathcal{X} .

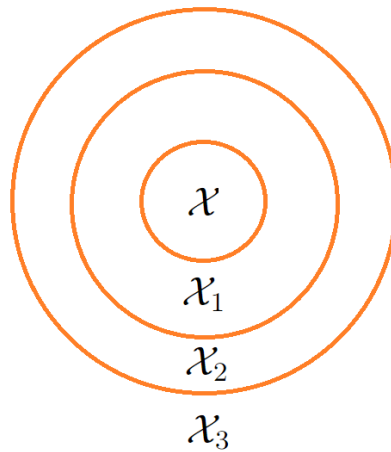
Proof. We have to prove that if we intersect a bunch σ -algebras, we still get a σ -algebra.

1. Such an intersection is closed under complements: if a set belongs to the intersection of σ -algebras, then it belongs to each of the σ -algebras, then its complement belongs to each of the σ -algebras, and thus its complement belongs to the intersection of σ -algebras.
2. In a similar way, such an intersection is closed under countable unions: if a number of sets all belong to the intersection of σ -algebras, then they all belong to each of the σ -algebras, then their countable union belongs to each of the σ -algebras, and their countable union belongs to the intersection of σ -algebras.

\square

Remark. We can try to construct $\mathcal{A}(\mathcal{X})$ in a different way. Say, \mathcal{X} is not a σ -algebra. Let's enlarge it: first by including all the complements. Then let's enlarge it by all countable unions. Let's call such a set \mathcal{X}_1 . But after such operation, \mathcal{X}_1 may be non-closed under complements. So we repeat such a procedure.

And, in general: \mathcal{X}_{n+1} is obtained from \mathcal{X}_n is obtained by including into \mathcal{X}_n all complements of the sets from \mathcal{X}_n and then including all countable unions of the obtained sets.



It is tempting to think that $\cup_1^\infty \mathcal{X}_i$ is $\mathcal{A}(\mathcal{X})$. Is it true? No, not necessarily. If the sequence $\{\mathcal{X}_i\}$ eventually stabilizes, then such a construction works. Let's now assume that every next \mathcal{X}_i is larger than the previous one. Then we can take A from \mathcal{X} , A_1 from $\mathcal{X}_1 \setminus \mathcal{X}$, A_2 from $\mathcal{X}_2 \setminus \mathcal{X}_1$, and so on.

Now let's look at $\cup_1^\infty A_i$. As a countable union, it must be contained in $\mathcal{A}(\mathcal{X}) = \cup_1^\infty \mathcal{X}_i$, thus, there exist an n , such that $\cup_1^\infty A_i \in \mathcal{X}_n$. But $A_{n+1} \in \mathcal{X}_{n+1} \setminus \mathcal{X}_n$!

Definition 12 (Topological space). A *topological space* is a set X and a collection of subsets O of X (called *open sets*), such that $\emptyset, X \in O$, and:

1. A union of (possibly infinitely many) sets from O is in O .
2. The intersection of finitely many sets from O is in O .

The complements of open sets are called *closed sets*.

Definition 13. A function $f : X \rightarrow Y$ between two topological spaces is *continuous* if the preimage of every open set is open.

Remark. It is possible to check that for \mathbb{R} this definition is equivalent to the usual one.

Definition 14 (Borel σ -algebra). For a topological space X its *Borel σ -algebra* \mathcal{B}_X is the smallest σ -algebra on X that contains all open sets.

Remark. If it's obvious from the context which set we are talking about, we will just write \mathcal{B} (without a subscript).

Theorem 3. $\mathcal{B}_{\mathbb{R}} \subset \mathcal{M}$ (all of the sets in $\mathcal{B}_{\mathbb{R}}$ are measurable).

Proposition 7. \mathcal{B} is the smallest σ -algebra that contains all open intervals.

If we prove the proposition, the theorem will follow easily. We know that [all the intervals are Lebesgue-measurable](#). We know that the Lebesgue-measurable sets (\mathcal{M}) [are a \$\sigma\$ -algebra](#). Thus, if we take the smallest σ -algebra that contains all open intervals, it will be a subset of \mathcal{M} .

Proof of Proposition 7. We will prove that every open set $O \subset \mathbb{R}$ is a finite or countable union of open intervals.

For every point $x \in O$ let I_x be the largest open interval, such that $x \in I_x$ and $I_x \subset O$. It exists as a union of all such intervals. Since O is open, x lies in O with an open neighborhood, thus, I_x is non-empty.

$$\forall x \in O : x \in I_x \implies O = \bigcup_{x \in O} I_x$$

Let's prove that $I_x \cap I_y \neq \emptyset \implies I_x = I_y$. If the intervals around x and y intersect, then $I_x \cup I_y$ is an interval as well, and $I_x \cup I_y \in O$ as $I_x \in O$ and $I_y \in O$. Since I_x and I_y are the largest such intervals, it follows that $I_x = I_x \cup I_y = I_y$.

Let's say that two points x and y are equivalent if $I_x = I_y$. Since there's a lot of same intervals in $O = \bigcup_{x \in O} I_x$, we can take just a single point from every equivalence class and still get O as a union. Particularly, every open interval contains at least one rational point (as rational numbers are dense). Therefore, there's a rational point in every equivalence class. Thus,

$$O = \bigcup_{x \in O \cap \mathbb{Q}} I_x$$

Since the set of rational numbers is countable, we have represented O as a countable union of open intervals, which is what we wanted. \square

Remark. A topological space is called *separable*, if it contains a countable dense subset.

Remark. We have proved that the Lebesgue measure exists on $\mathcal{B}_{\mathbb{R}}$, so we have a lot of measurable sets.

Remark. The Lebesgue measure can be generalized to \mathbb{R}^n .