# Analysis 3

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# 1. Measure

## 1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of $\mathbb{R}$. Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

**Theorem 1.** There exist no such function $\mu : 2^{[0,1]} \to [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

**Proof**. First, several definitions:

Step 1. Let's define the following equivalence relation: if $x$, $y$ are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an $N$ exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets $N_R$ are congruent to $N$ and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists!\, y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as $N$ contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu\big([1/3, 2/3]\big) \leqslant \mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum_{r \in \mathbb{Q} \cap [0,2/3]} \mu(N_r) \leqslant 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\Big(\bigcup_{r \in \mathbb{Q} \cap [0,2/3]} N_r\Big) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 1]$?! $\square$

***Consequence.*** We cannot generalize the notion of length to all subsets of real numbers.

## 1.2 Lebesgue Outer Measure

***Definition* 1.** If $I \subset \mathbb{R}$ is an interval, then $l(I) =$ the length of $I$. If $I$ is unbounded, then $l(I) = \infty$.

***Definition* 2** (Outer Measure)**.**

$$m^* : 2^{\mathbb{R}} \to [0, +\infty]$$
$$m^*(A) = \inf\Big\{\sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j\Big\}$$

In words, it's the infimum of all *countable* covers of $A$. (A countable sum either converges or diverges to infinity).

*Remark.* This is certainly not a measure — otherwise, it would contradict Theorem 1.

**Example.** If $A$ is countable, then $m^*(A) = 0$.

**Proof**. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leqslant 2\varepsilon$. Let's choose a cover of the points with segments of lengths $\varepsilon$, $\varepsilon/2$, $\varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf\{\dots\} \leqslant \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

**Proposition 1.** If $A$ is an interval, then $m^*(A) = l(A)$.

**Proof**. a) $A$ is a closed interval, $A = [a, b]$.

1. $m^*(A) \leqslant b - a$. To prove this, we can cover $A$ with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

   Now take $\varepsilon \to 0$.

2. $m^*(A) \geqslant b - a$. Suppose we an infinite cover of $A$ by open intervals. Since $A$ is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

b) If $A$ is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

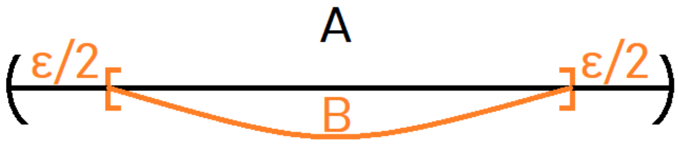
c) If $A$ is an open or semiclosed interval, we can approximate it from both sides by closed intervals. Let's denote the closure of $A$ by $\bar{A}$. Since we're adding points, the Outer Measure will not decrease:

$$A \subset \overline{A} \implies m^*(A) \leqslant m^*(\overline{A}) = l(a)$$

Now suppose we have a closed interval $B$ strictly inside $A$. Then we get

$$m^*(A) \geqslant m^*(B) = l(B) = l(A) - \varepsilon$$

Now take $\varepsilon \to 0 \implies m^*(A) \geqslant l(A)$.




□

**Lemma.** $m^*$ is translation-invariant.

**Proof**. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

**Proposition 2** (Countable subadditivity)**.** For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leqslant \sum_{k=1}^{\infty} m^*(E_k)$$

***Remark.*** We don't ask for the sets $E_k$ to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that $m^*$ is a measure, which we proved does not exist in Theorem 1.

**Proof**. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} E_{k,i} \ (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leqslant k, i < \infty\}$ covers each of the $E_k$, and thus it's a cover of $\cup_{k=1}^{\infty} E_k$. Then

$$m^*\Big(\bigcup_{k=1}^{\infty} E_k\Big) \overset{\text{it's a cover}}{\leqslant} \sum_{1 \leqslant k,i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon\Big(\frac{1}{2} + \frac{1}{4} + \dots\Big) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

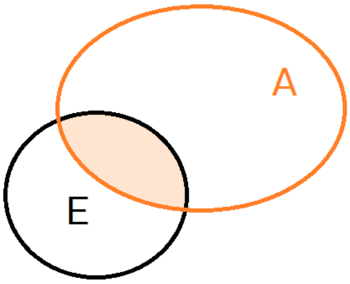
Now take $\varepsilon \to 0$. □

***Remark.*** Here we assume that all of the $E_k$ have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leqslant \infty$ which is "true".

## 1.3 The $\sigma$-algebra of Lebesgue-measurable sets.

***Definition* 1.** A set $E$ is (Lebesgue) measurable if for any set $A$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \qquad E^C = \mathbb{R} \setminus E$$




The set $E$ "splits" $A$ into two parts

*Remark.* We already have the $\leqslant$ sign from countable subadditivity.

*Remark.* Motivation: If $A \cap B = \varnothing$ and $A$ (or $B$) is measurable, then

$$m^*(A \cup B) = m^*\big((A \cup B) \cap A\big) + m^*\big((A \cup B) \cap A^C\big) = m^*(A) + m^*(B)$$

**Proposition 1.** If $m^*(E) = 0$, then $E$ is measurable.

**Proof.** For all $A$ we have:

$$m^*(A \cap E) \leqslant m^*(E) = 0 \implies m^*(A \cap E) = 0$$
$$m^*(A) \geqslant m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C)$$

As we noted earlier, the inequality in the other side follows from countable subadditivity. □

**Proposition 2.** If $E_1, \dots, E_n$ are measurable, then $\cup_1^n E_k$ is measurable.

**Proof.** Case $n = 2$: for all $A$ we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^C) =$$
$$= m^*(A \cap E_1) + m^*\big((A \cap E_1^C) \cap E_2\big) + m^*\big((A \cap E_1^C) \cap E_2^C\big) = (*)$$
$$X \coloneqq A \cap E_1,\ Y \coloneqq (A \cap E_1^C) \cap E_2,\ Z \coloneqq (A \cap E_1^C) \cap E_2^C$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply countable subadditivity to $X$ and $Y$. Then we get:

$$(*) \geqslant m^*\big(A \cap (E_1 \cup E_2)\big) + m^*\big(A \cap (E_1 \cup E_2)^C\big)$$

Yet again, the inequality in the other side follows from countable subadditivity.

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k$, $E_n$. □

***Definition 2*** (Algebra)**.** Let $X$ be a non-empty set. $\Omega \subset 2^X$ is an algebra, if:

1. $X \in \Omega$;
2. $\Omega$ is closed under the formation of complements in $X$ and *finite* unions.

***Remark*.** It follows that $\Omega$ is also closed under intersections:

$$\left(X_1^C \cup \cdots \cup X_n^C\right)^C = X_1 \cap \cdots \cap X_n$$

***Definition 3*** ($\sigma$-algebra)**.** Let $X$ be a non-empty set. $\Omega \subset 2^X$ is a $\sigma$-algebra, if:

1. $X \in \Omega$;
2. $\Omega$ is closed under the formation of complements in $X$ and *countable* unions.

***Remark*.** Every $\sigma$-algebra is an algebra, but not vice versa.

**Corollary 1.** The collection $\mathcal{M}$ of all measurable subsets of $\mathbb{R}$ is an algebra.

**Proof**. For the proof, we'll need to show that:

1. $\mathbb{R}$ is measurable.
$$m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^C) = m^*(A) + m^*(\varnothing)$$
2. It is closed under complements. It follows from the symmetry of the definition of a measurable set.
3. It is closed under unions. We have already proved this one.

□

**Proposition 3.** $\{E_k\}_1^n$ — disjoint measurable sets. Then for every set $A$

$$m^*\left(A \cap \left[\bigcup_1^n E_k\right]\right) = \sum_1^n m^*(A \cap E_k)$$

In particular, for $A = \mathbb{R}$ we have

$$m^*\left(\bigcup_1^n E_k\right) = \sum_1^n m^*(E_k)$$

**Proof**. Induction on $n$.

Base $n = 1$ is obvious.

Step $n-1 \to n$. Take $\hat{A} := A \cap \left[\bigcup_1^n E_k\right]$. Then

$$\hat{A} \cap E_n = A \cap E_n$$

We also have

$$\hat{A} \cap E_n^C = A \cap \left[\bigcup_1^{n-1} E_k\right]$$

That is true, as intersecting with $E_n^C$ is equivalent to subtracting $E_n$ from $\hat{A}$, and since $\{E_k\}$ are disjoint, no other parts of $\hat{A}$ except $E_n$ will be removed. Then:

$$m^*(\hat{A}) \overset{E_n \text{ is measurable}}{=\!=} m^*(\hat{A} \cap E_n) + m^*(\hat{A} \cap E_n^C) =$$
$$= m^*(A \cap E_n) + m^*\left(A \cap \left[\bigcup_1^{n-1} E_k\right]\right) \overset{\text{induction}}{=} m^*(A \cap E_n) + \sum_1^{n-1} m^*(A \cap E_k)$$

□

**Proposition 4.** The union of a countable collection of measurable sets is the union of a countable collection of *disjoint* measurable sets.

**Proof**. If $A = \cup_1^\infty A_k$, define $\hat{A}_1 := A_1$ and $\hat{A}_k := A_k \setminus \cup_1^{k-1} A_j$. As $\mathcal{M}$ is an algebra, all $\hat{A}_k$ are measurable, and $A = \sqcup_1^\infty \hat{A}_k$, which is what we wanted. □

**Theorem 1.** $\mathcal{M}$ is a $\sigma$-algebra.

**Proof**. We need to show that if all $\{E_k\}_1^\infty$ are measurable sets, then $E = \cup_1^\infty E_k$ is measurable. By Proposition 4, without the loss of generality, assume that $E_k$ are all pairwise disjoint. Let $F_n := \cup_1^n E_k$, then $F_n \in \mathcal{M}$ (as a finite union). As $F_n \subset E$, we have $E^C \subset F_n^C$.

Let $A$ be any set. Then:

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^C) \geqslant m^*(A \cap F_n) + m^*(A \cap E^C) \overset{\text{Proposition 3}}{=}$$
$$= \sum_1^n m^*(A \cap E_k) + m^*(A \cap E_C)$$

Now take $n \to \infty$:

$$m(A) \geqslant \sum_1^\infty m^*(A \cap E_k) + m^*(A \cap E_C) \overset{\text{countable subadditivity}}{\geqslant} m^*(A \cap E) + m^*(A \cap E^C)$$

Now we have the inequality in the difficult direction. The inequality in the other direction is obvious (again, from countable subadditivity). □

**Proposition 5** (Countable additivity). If $\{E_k\}_1^\infty \subset \mathcal{M}$ — collection of disjoint sets, then $\cup_1^\infty E_k \in \mathcal{M}$ and

$$m^*\Big(\bigcup_1^\infty E_k\Big) = \sum_1^\infty m^*(E_k)$$

**Proof**. We know that:

1. $$m^*\Big(\bigcup_1^\infty E_k\Big) \leqslant \sum_1^\infty m^*(E_k) \text{ (countable } \textit{sub}\text{additivity)}$$

2. $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant m^*\Big(\bigcup_1^n E_k\Big) \overset{\text{Proposition 3}}{=} \sum_1^n m^*(E_k)$$

   Take $n \to \infty$, then

   $$m^*\Big(\bigcup_1^\infty E_k\Big) \geqslant \sum_1^\infty m^*(E_k)$$

   Which is what we wanted.

□

***Definition* 4.** The restriction of $m^*$ on $\mathcal{M}$ is called the Lebesgue measure and denoted by $m$.

$$m(E) := m^*(E) \quad \forall E \in \mathcal{M}$$

***Definition* 5.** If $X$ is a non-empty set and $\mathcal{A}$ is a $\sigma$-algebra on $X$, then any function $\mu : \mathcal{A} \to [0, +\infty]$ is called the measure on $(X, \mathcal{A})$, if:

1. $\mu(\varnothing) = 0$.
2. $\mu$ is countable additive.

***Definition* 6** (Measurable space)**.** A *measurable space* is a tuple $(X, \mathcal{A})$, where:

1. $X$ is a set.
2. $\mathcal{A}$ is a $\sigma$-algebra on $X$.

***Definition* 7** (Measure space)**.** A *measure space* is a triple $(X, \mathcal{A}, \mu)$, where:

1. $X$ is a set.
2. $\mathcal{A}$ is a $\sigma$-algebra on $X$.
3. $\mu$ is a measure on $(X, \mathcal{A})$.

**Example 1.** $\{\varnothing, X\}$ is a $\sigma$-algebra. Any $\mu$, such that $\mu(\varnothing) = 0$ and $\mu(X) \geqslant 0$ will be a measure.

**Example 2.** $2^X$ is a $\sigma$-algebra. We can have the following measures:

a) $\mu(E) = |E|$ is called a *counting measure*. Here $|E|$ denotes the cardinality of $E$ (number of elements in $E$).

b) $\delta$-measure (also called Dirac measure):

$$\mu(E) = \begin{cases} 1, & 0 \in E \\ 0, & \text{otherwise} \end{cases}$$

## 1.4 Continuity of measure

***Definition* 1.** A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *ascending* if $E_k \subset E_{k+1}$.

***Definition* 2.** A countable collection of sets $\{E_k\}_{k=1}^{\infty}$ is called *descending* if $E_k \supset E_{k+1}$.

**Theorem 1** (Continuity of measure)**.**

1. If $\{A_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and the sequence is ascending, then

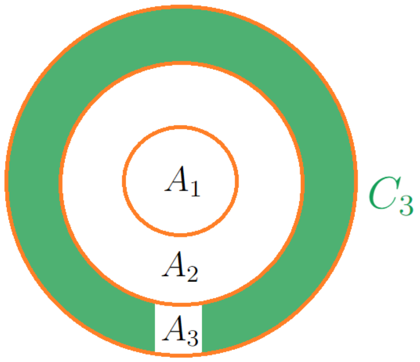
$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \lim_{k\to\infty} \mu(A_k)$$

2. If $\{B_k\}_{k=1}^{\infty} \subset \mathcal{A}$, the sequence is descending and $\mu(B_1) < \infty$, then

$$\mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) = \lim_{k\to\infty} \mu(B_k)$$

**Proof.** 1. Let $C_k \coloneqq A_k \setminus A_{k-1}$. Then we have:

$$\mu\Big(\bigcup_{k=1}^{\infty} A_k\Big) = \mu\Big(\bigsqcup_{k=1}^{\infty} C_k\Big) = \sum_{k=1}^{\infty} \mu(C_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(C_k) = \lim_{n\to\infty} \mu(A_n)$$




2. Let $D_k \coloneqq B_1 \setminus B_k$. Since $B_k$ is descending, it follows that $D_k$ is an ascending sequence. Then from the part 1 of the theorem it follows that:

$$\mu\Big(\bigcup_{k=1}^{\infty} D_k\Big) = \lim_{k\to\infty} \mu(D_k) \qquad \bigcup_{k=1}^{\infty} D_k = B_1 \setminus \bigcap_{k=1}^{\infty} B_k$$

$$\mu\Big(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\Big) = \lim_{k\to\infty} \big(\mu(B_1) - \mu(B_k)\big) = \mu(B_1) - \lim_{k\to\infty} \mu(B_k)$$

$$\mu\Big(B_1 \setminus \bigcap_{k=1}^{\infty} B_k\Big) = \mu(B_1) - \mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) \implies \mu\Big(\bigcap_{k=1}^{\infty} B_k\Big) = \lim_{k\to\infty} \mu(B_k)$$

□

***Definition 3.*** We say that a statement (property) holds for *almost all $x \in X$ with respect to a measure $\mu$*, if $\exists N \in \mathcal{A}$, such that $\mu(N) = 0$ and the statement (property) holds for all $x \in X \setminus N$.

**Lemma** (Borel–Cantelli)**.** Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $\{E_k\}_{k=1}^{\infty} \subset \mathcal{A}$ and $\sum_{k=1}^{\infty} \mu(E_k) < \infty$. Then *almost all* $x \in X$ belong to at most finitely many $E_k$.

**Proof**. Let $B_n = \cup_{k=n}^{\infty} E_k$. It's easy to see that $B_k$ is a descending measure. At the same time,

$$\mu(B_1) = \mu\Big(\bigcup_{k=1}^{\infty} E_k\Big) \leqslant \sum_{k=1}^{\infty} \mu(E_k) < \infty$$

By definition of $B_n$, $\cap_{n=1}^{\infty} B_n$ contains all the points that are contained in infinitely many $E_k$'s. But, by continuity of measure for $\{B_n\}_{n=1}^{\infty}$ we have:

$$\mu\Big(\bigcap_{n=1}^{\infty} B_n\Big) = \lim_{n\to\infty} \mu(B_n) = \lim_{n\to\infty} \Big(\bigcup_{k=n}^{\infty} E_k\Big) \leqslant \lim_{n\to\infty} \sum_{k=n}^{\infty} \mu(E_k) = 0$$

□

## 1.5 How large is the Lebesgue $\sigma$-algebra $\mathcal{M}$?

**Proposition 1.** Every interval is Lebesgue-measurable.

**Proof**. Proof idea:

$$E \in \mathcal{M} \iff \forall A : m(A) = m(A \cap E) + m(A \cap E^C)$$

Assume $E = (-\infty, a)$. If we prove that such intervals lie in $\mathcal{M}$, then we'll prove everything (since $\mathcal{M}$ is a $\sigma$-algebra). We already have $m(A) \leqslant m(A \cap E) + m(A \cap E^C)$ from countable subadditivity.

Let's assume $a \notin A$ (since removing one point does not change the measure). Every cover of $A$ can be split into two covers with the same sum of interval lengths: of $A \cap (-\infty, a)$ and $A \cap (a, +\infty)$. Every interval in those covers, that contains $a$, can be split into two. Therefore, from the definition of Lebesgue measure, $m(A) \geqslant m(A \cap E) + m(A \cap E^C)$, so we've proved the inequality in both sides. $\square$

***Definition* 1.** For any $\mathcal{X} \in 2^{\mathbb{R}}$ let $\mathcal{A}(\mathcal{X})$ be the smallest $\sigma$-algebra containing $\mathcal{X}$.

**Lemma.** $\mathcal{A}(\mathcal{X})$ always exists and is the intersection of all $\sigma$-algebras containing $\mathcal{X}$.

**Proof.** We have to prove that if we intersect a bunch $\sigma$-algebras, we still get a $\sigma$-algebra.

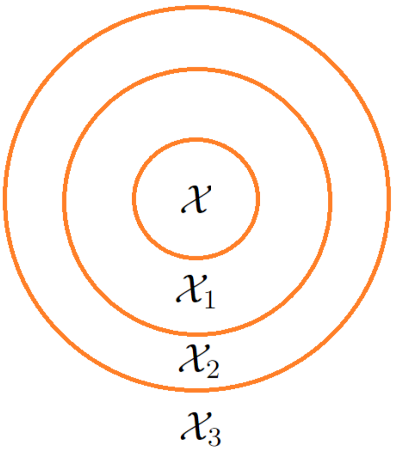
1. Such an intersection is closed under complements: if a set belongs to the intersection of $\sigma$-algebras, then it belongs to each of the $\sigma$-algebras, then its complement belongs to each of the $\sigma$-algebras, and thus its complement belongs to the intersection of $\sigma$-algebras.

2. In a similar way, such an intersection is closed under countable unions: if a number of sets all belong to the intersection of $\sigma$-algebras, then they all belong to each of the $\sigma$-algebras, then their countable union belongs to each of the $\sigma$-algebras, and their countable union belongs to the intersection of $\sigma$-algebras.

$\square$

*Remark.* We can try to construct $\mathcal{A}(\mathcal{X})$ in a different way. Say, $\mathcal{X}$ is not a $\sigma$-algebra. Let's enlarge it: first by including all the complements. Then let's enlarge it by all countable unions. Let's call such a set $\mathcal{X}_1$. But after such operation, $\mathcal{X}_1$ may be non-closed under complements. So we repeat such a procedure.

And, in general: $\mathcal{X}_{n+1}$ is obtained from $\mathcal{X}_n$ is obtained by including into $\mathcal{X}_n$ all complements of the sets from $\mathcal{X}_n$ and then including all countable unions of the obtained sets.




It is tempting to think that $\cup_1^\infty \mathcal{X}_i$ is $\mathcal{A}(\mathcal{X})$. Is it true? No, not necessarily. If the sequence $\{\mathcal{X}_i\}$ eventually stabilizes, then such a construction works. Let's now assume that every next $\mathcal{X}_i$ is larger than the previous one. Then we can take $A$ from $\mathcal{X}$, $A_1$ from $\mathcal{X}_1 \setminus \mathcal{X}$, $A_2$ from $\mathcal{X}_2 \setminus \mathcal{X}_1$, and so on.

Now let's look at $\cup_1^\infty A_i$. As a countable union, it must be contained in $\mathcal{A}(\mathcal{X}) = \cup_1^\infty \mathcal{X}_i$, thus, there exist an $n$, such that $\cup_1^\infty A_i \in \mathcal{X}_n$. But $A_{n+1} \in \mathcal{X}_{n+1} \setminus \mathcal{X}_n$?!

***Definition* 2** (Topological space)**.** A *topological space* is a set $X$ and a collection of subsets $O$ of $X$ (called *open sets*), such that $\varnothing, X \in O$, and:

1. A union of (possibly infinitely many) sets from $O$ is in $O$.

2. The intersection of finitely many sets from $O$ is in $O$.

The complements of open sets are called *closed sets*.

***Definition 3.*** A function $f : X \to Y$ between two topological spaces is *continuous* if the preimage of every open set is open.

*Remark.* It is possible to check that for $\mathbb{R}$ this definition is equivalent to the usual one.

***Definition 4*** (Borel $\sigma$-algebra)**.** For a topological space $X$ its *Borel $\sigma$-algebra* $\mathcal{B}_X$ is the smallest $\sigma$-algebra on $X$ that contains all open sets.

*Remark.* If it's obvious from the context which set we are talking about, we will just write $\mathcal{B}$ (without a subscript).

**Theorem 1.** $\mathcal{B}_\mathbb{R} \subset \mathcal{M}$ (all of the sets in $\mathcal{B}_\mathbb{R}$ are measurable).

**Proposition 2.** $\mathcal{B}$ is the smallest $\sigma$-algebra that contains all open intervals.

If we prove the proposition, the theorem will follow easily. We know that all the intervals are Lebesgue-measurable. We know that the Lebesgue-measurable sets ($\mathcal{M}$) are a $\sigma$-algebra. Thus, if we take the smallest $\sigma$-algebra that contains all open intervals, it will be a subset of $\mathcal{M}$.

*Proof of Proposition 2.* We will prove that every open set $O \subset \mathbb{R}$ is a finite or countable union of open intervals.

For every point $x \in O$ let $I_x$ be the largest open interval, such that $x \in I_x$ and $I_x \subset O$. It exists as a union of all such intervals. Since $O$ is open, $x$ lies in $O$ with an open neighborhood, thus, $I_x$ is non-empty.

$$\forall x \in O : x \in I_x \implies O = \bigcup_{x \in O} I_x$$

Let's prove that $I_x \cap I_y \neq \varnothing \implies I_x = I_y$. If the intervals around $x$ and $y$ intersect, then $I_x \cup I_y$ is an interval as well, and $I_x \cup I_y \in O$ as $I_x \in O$ and $I_y \in O$. Since $I_x$ and $I_y$ are the largest such intervals, it follows that $I_x = I_x \cup I_y = I_y$.

Let's say that two points $x$ and $y$ are equivalent if $I_x = I_y$. Since there's a lot of same intervals in $O = \cup_{x \in O} I_x$, we can take just a single point from every equivalence class and still get $O$ as a union. Particularly, every open interval contains at least one rational point (as rational numbers are dense). Therefore, there's a rational point in every equivalence class. Thus,

$$O = \bigcup_{x \in O \cap \mathbb{Q}} I_x$$

Since the set of rational numbers is countable, we have represented $O$ as a countable union of open intervals, which is what we wanted. $\square$

*Remark.* A topological space is called *separable*, if it contains a countable dense subset.

*Remark.* We have proved that the Lebesgue measure exists on $\mathcal{B}_\mathbb{R}$, so we have a lot of measurable sets.

*Remark.* The Lebesgue measure can be generalized to $\mathbb{R}^n$.

## 1.6 Other critirea of measurability for Lebesgue measure

As we remember, the definition of a measurable set is difficult to check. Thus, we would like to have better critirea.

**Theorem 1.** $E \subset \mathbb{R}$ is Lebesgue measurable if and only if one of the following holds:

1. For every $\varepsilon > 0$ there exists an open set $O$, such that $E \subset O$ and $m^*(O \setminus E) < \varepsilon$.
2. There exists a $G_\delta$-set $G$, such that $E \subset G$ and $m^*(G \setminus E) = 0$.
   (A $G_\delta$-set is a countable intersection of open sets.)
3. For every $\varepsilon > 0$ there exists a closed set $F$, such that $F \subset E$ and $m^*(E \setminus F) < \varepsilon$.
4. There exists a $F_\sigma$-set $F$, such that $F \subset E$ and $m^*(E \setminus F) = 0$.
   (A $F_\delta$-set is a countable union of closed sets.)

**Proof**.

- $E$ is measurable $\implies$ 1.

  If $m^*(E) < \infty$, then from the definition of $m^*$ we can find $O$ — a finite union of open intervals, such that $m^*(O) < m^*(E) + \varepsilon$. Since $O$ is an open set, it's measurable (as we proved earlier). Therefore, both $E$ and $O$ are measurable, thus

  $$\varepsilon > m^*(O) - m^*(E) \overset{E,O \in \mathcal{M}}{=} m^*(O \setminus E)$$

  If $m^*(E) = \infty$, let's split the set $E$ into a countable number of sets with finite measure. For example, by splitting the real line into segments of length 1. So, $E = \cup_1^\infty E_k$, where $m^*(E_k) < \infty$. Then let's use geometrically decreasing $\varepsilon$'s for the covers of each $E_k$: $\varepsilon/2$ for $E_1$, $\varepsilon/4$ for $E_2$, and so on. When we sum up the inequalities, the fractions will sum up to $\varepsilon$. So, we obtained our $O$, now continue like in the previous case.

  ***Definition* 1.** A measure $\mu$ on $X$ is called *$\sigma$-finite*, if $X = \cup_1^\infty X_k$ and $\mu(X_k) < \infty$ for all $k$.

  In words: if there exists a subdivision of $X$ into a countable number of set of finite measure.

- 1 $\implies$ 2.

  From 1, $\forall k \in \mathbb{N}$, $\exists O_k$ — open, such that $E \subset O_k$ and $m^*(O_k \setminus E) < \frac{1}{k}$. Now let's take

  $$G := \bigcap_1^\infty O_k \implies \forall k: \ m^*(G \setminus E) \leqslant m^*(O_k \setminus E) < \frac{1}{k} \implies m^*(G \setminus E) = 0$$

- 2 $\implies$ $E$ is measurable.

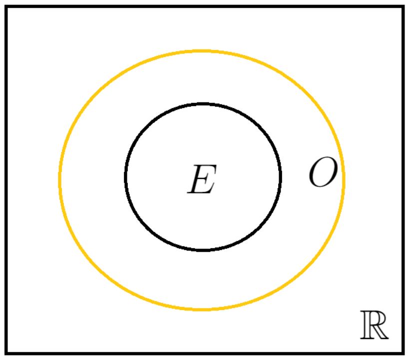
  $G$ is a $G_\delta$-set. As a countable intersection of open sets, it's in Borel $\sigma$-algebra, and thus is Lebesgue-measurable. $m^*(G \setminus E) = 0$, then $G \setminus E$ is Lebesgue-measurable, then $E = G \setminus (G \setminus E)$ is measurable as a difference of two measurable sets.

- $3 \iff 1$, $4 \iff 2$.

  If we assume that 3 holds for $E$, then, if we take $O = \overline{F}$, 1 will hold for $\overline{E}$. Therefore, $\overline{E}$ is measurable, then $E$ is measurable (as $\mathcal{M}$ is a $\sigma$-algebra).

  If 1 holds for $E$, then $E$ is measurable, then $\overline{E}$ is measurable, then 1 holds for $\overline{E}$. Now take $F = \overline{O}$, therefore, 3 holds for $E$.

  In the same way, 2 and 4 are equivalent as well.




□

**Theorem 2.** For every $E \in \mathcal{M}$ with $m(E) < \infty$ and for every $\varepsilon < \infty$ there exists an infinite disjoint collection of open intervals $\{I_k\}_1^n$, such that $O = \cup_{k=1}^n I_k$ and $m(E \triangle O) < \varepsilon$.

(Here $\triangle$ is the symmetric difference of two sets).

**Proof**. From part 1 of the previous theorem, we can take such an open set $U$, that $E \subset U$ and $m(U \setminus E) < \varepsilon/2$.

As we proved earlier, we can represent $U$ as a countable union of disjoint open intervals $I_k$. Then:

$$\forall n : \bigcup_1^n I_k \subset U \implies \forall n : \sum_1^n m(I_k) \leqslant m(U) < \infty$$

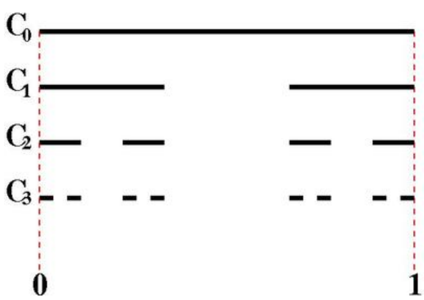
Now take $n$, such that $\sum_{n+1}^{\infty} m(I_k) < \varepsilon/2$, and put $O \coloneqq \cup_{k=1}^n I_k$. Then $m(O \setminus E) < \varepsilon/2$ and $m(E \setminus O) < \varepsilon/2$, therefore, the measure of the symmetric difference is less that $\varepsilon$. □

## 1.7 TBA

Questions:

1. If $m(A) = 0$, is $A$ countable?
2. We know that $\mathcal{B} \subset \mathcal{M}$. Is this inclusion proper?

***Definition* 1** (Cantor set)**.** Let's take $[0, 1]$, split it into three parts and remove the middle part. Then continue such process. The *Cantor set* is the set $C \coloneqq \cap_0^\infty C_k$.



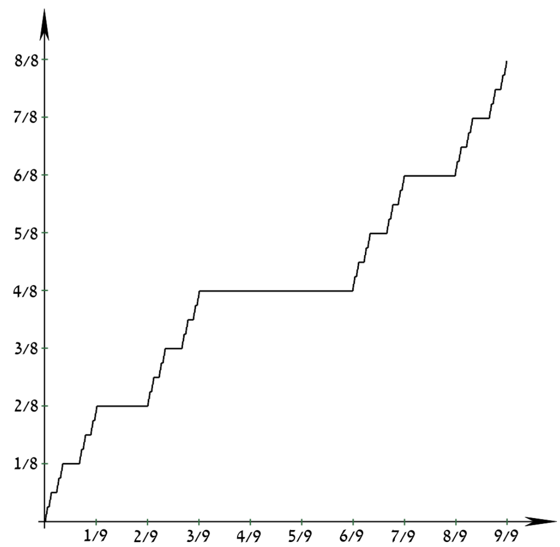

Cantor set illustration from here.

*Remark.* The measure of $C_n$ is $(2/3)^n = 0$. From the continuity of measure, the measure of the intersection is the limit of measures of individual sets, and thus $m(C) = 0$.

***Remark.*** The Cantor set is countable, because if we have a sequence of zeros and ones, we can traverse down-left on 0 and down-right on 1. The intersection of the corresponding intervals will be a single point of $C$. So, there's a bijection between $C$ and $\{0,1\}^{\mathbb{N}}$, therefore, $C$ is indeed uncountable.

***Remark.*** Usually, when we remove the middle intervals on each step, we keep the end points. If we choose to remove the end points, we essentially remove the points that correspond to sequences that end with an infinite sequence of 0's or an infinite sequence of 1's. It's clear that not all sequences of 0's and 1's are like that, so there is going to be plenty of points left in the Cantor set, anyway.

***Definition 2*** (The Cantor function)**.** The Cantor function $\varphi : [0,1] \to [0,1]$ is defined as follows. Let's first take the unit interval $[0,1]$, split it in three, and define $\varphi$ to be $\frac{1}{2}$ on $[1/3, 2/3]$. Then let's continue the same with $[0,1/3]$ and $[2/3,1]$, and so on.




Now we have defined $\varphi$ for all points on the Cantor set. Here's how we'll define it on all others:

$$O \coloneqq C \setminus \{0,1\}$$
$$\forall x \in C \setminus \{0,1\} : \ \varphi(x) \coloneqq \sup\{\varphi(t) \mid t \in O \cap [0,x)\}$$

**Proposition 1.** $\varphi$ is increasing, continuous, surjective ($[0,1]$ acts on $[0,1]$), $\varphi'$ exists for open set $O$ of measure 1, $\varphi'\Big|_O \equiv 0$.

**Proof**. $\varphi$ is increasing (that's clear), therefore, if $\varphi$ is discontinuous, then it has a jump discontinuity and there will be and interval, say, $I \subset [0,1]$, such that $\varphi([0,1]) \cap I = \varnothing$. But

$$\varphi([0,1]) \supset \left\{ \frac{m}{2^k} \ \middle| \ k \in \mathbb{N}, m \in [0, 2^k] \right\} = K$$

And $K$ is dense in $[0,1]$, therefore, $K \cap I \neq \varnothing$, which is a contradiction. □

***Remark.*** The Cantor function is a source of a lot of counterexamples. We are going to use it to answer the second question from earlier.

***Definition 3.***

$$\psi : [0,1] \to [0,2], \ \psi(x) = \psi(x) + x$$

$\psi$ is strictly increasing, continuous, surjective.

**Proposition 2.** 1. $\psi$ maps $C$ onto a measurable set of measure 1.

2. $\psi$ maps some subset of $C$ onto a non-measurable set.

**Proof.** 1. $m(C) = 0$, thus, $m(O) = 1$. We know that $\varphi$ is constant on every part of $O$. Therefore, $\psi$ looks like $x$ on every part of $O$, and there's a countable number of such intervals. Therefore, $O$ is mapped to a set of measure 1. Thus, $m(\psi(C)) = m(\psi([0,1])) - m(\psi(O)) = 2 - 1 = 1$.

2. Since $\psi(C)$ has measure $1 > 0$, by the second homework, there exists a non-measurable set $N \supset \psi(C)$.

$\square$

**Corollary 1.** There is a (measurable) subset of $C$ that is not Borel.

***Remark.*** Since $m(C) = 0$, the outer measure of each subset of $C$ is also 0, thus, every subset of $C$ is measurable. So, the "measurable" part of the corollary is obvious.

**Proposition 3.** If $f : E \to \mathbb{R}$ is continuous, $E \in \mathcal{M}$, then the preimage of every Borel set is measurable.

$$\forall B \in \mathcal{B} : f^{-1}(B) = \{x \in E \mid f(x) \in B\} \in \mathcal{M}$$

Let's show how we can prove the corollary if we prove the proposition:

*Proof of Corollary 1.* $\psi^{-1} : [0,2] \to [0,1]$ is continuous and bijective. Let's define $\tilde{C} = \psi^{-1}(N)$. Since $N \subset C$, we have $\tilde{C} \subset C$. Assume that $\tilde{C}$ is a Borel set, and thus it's measurable. Then, by Proposition 3, its pre-image, $N$, must be measurable as well. But we know it isn't?! $\square$

Let's talk about something more general now, and later return to Proposition 3.

***Definition* 4.** $f : X \to Y$ is continuous, if for every open set $O \subset Y$, $f^{-1}(O) = \{x \in X \mid f(x) \in O\}$ is open in $X$.

***Definition* 5** (Induced topology)**.** If $X$ is a topological space and $Y \subset X$, the induced topology on $Y$ is defined as follows: $O \subset Y$ is open in $Y$ if there exists an open subset $\tilde{O} \subset X$ in $X$, such that $O = \tilde{O} \cap Y$.