

Analysis 3

Lev Leontev

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1. Measure

1.1 Introduction

We want to generalize the notion of the *length* towards all the subsets of \mathbb{R} . Such a generalized function is usually called *measure*. But, unfortunately, such a function does not exist.

Theorem 1. There exist no such function $\mu : 2^{[0,1]} \rightarrow [0, +\infty)$ that satisfies the following properties:

1. The function is non-negative;
2. It's countably additive;
3. It's monotonic: the measure of a subset is not greater than the entire set;
4. Translation does not change the measure;
5. The measure of the unit interval is 1.

Proof. First, several definitions:

Step 1. Let's define the following equivalence relation: if x, y are from the unit interval, we'll say that $x \sim y$ if $x - y \in \mathbb{Q}$.

Step 2. Let's choose $N \subset [0, 1/3]$ such that it contains *precisely one* element from each equivalence class. (Such an N exists if the axiom of choice holds true).

Step 3. For all $r \in \mathbb{Q}$ define $N_r = N + r$.

Claim 1. The sets N_R are congruent to N and are pairwise disjoint.

Proof. The sets are congruent by definition. Let's prove that they are pairwise disjoint.

Assume that $x \in N_{r_1} \cap N_{r_2}$ for some $r_1, r_2 \in \mathbb{Q}$. Then $x - r_1 \in N$, $x - r_2 \in N$, but $(x - r_1) \sim (x - r_2) \implies r_1 = r_2$.

Claim 2.

$$\left[\frac{1}{3}, \frac{2}{3}\right] \in \bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r$$

Proof. If $x \in [1/3, 2/3]$, then $\exists! y \in N$ such that $x = y + q$ for some $q \in \mathbb{Q}$, as N contains exactly one representative from each of the equivalence classes. It is easy to see that such $q \in [0, 2/3]$.

We arrive at the following conclusion:

$$\frac{1}{3} = \mu([1/3, 2/3]) \leq \mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum_{r \in \mathbb{Q} \cap [0, 2/3]} \mu(N_r) \leq 1$$

What is $\mu(N)$ then? If $\mu(N) = 0$, then

$$\mu\left(\bigcup_{r \in \mathbb{Q} \cap [0, 2/3]} N_r\right) = \sum 0 = 0$$

If $\mu(N) = \varepsilon > 0$, then the sum is $+\infty$. But it's supposed to be in $[1/3, 1]$?! □

Consequence. We cannot generalize the notion of length to all subsets of real numbers.

1.2 Lebesgue Outer Measure

Definition 1. If $I \subset \mathbb{R}$ is an interval, then $l(I)$ = the length of I . If I is unbounded, then $l(I) = \infty$.

Definition 2 (Outer Measure).

$$m^* : 2^{\mathbb{R}} \rightarrow [0, +\infty]$$

$$m^*(A) = \inf \left\{ \sum_{j=1}^{\infty} l(I_j) \mid I_j \text{ — open intervals, } A \subseteq \bigcup_{j=1}^{\infty} I_j \right\}$$

In words, it's the infimum of all *countable* covers of A . (A countable sum either converges or diverges to infinity).

Remark. This is certainly not a measure — otherwise, it would contradict Theorem 1.

Example. If A is countable, then $m^*(A) = 0$.

Proof. Let's choose an arbitrary $\varepsilon > 0$ and prove that $m^*(A) \leq 2\varepsilon$. Let's choose a cover of the points with segments of lengths $\varepsilon, \varepsilon/2, \varepsilon/2^2$, and so on. Then

$$m^*(A) = \inf \{ \dots \} \leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \dots = 2\varepsilon$$

□

Proposition 1. If A is an interval, then $m^*(A) = l(A)$.

Proof. a) A is a closed interval, $A = [a, b]$.

1. $m^*(A) \leq b - a$. To prove this, we can cover A with a single interval:

$$(a - \varepsilon, b + \varepsilon) \implies \sum l(I_j) = b - a + 2\varepsilon$$

Now take $\varepsilon \rightarrow 0$.

2. $m^*(A) \geq b - a$. Suppose we an infinite cover of A by open intervals. Since A is a compact set, we can choose a finite subcover. The case of a finite cover with open intervals is simple. We can prove it as follows: if we have two intersecting open intervals, we can replace them with a single interval of a lesser length. Then we can continue this process using induction.

- b) If A is unbounded, then all of the covers would have infinite sum, and thus the infimum will be infinite as well.

- c) If A is an open or semiclosed interval, we can approximate it from both sides by closed intervals.

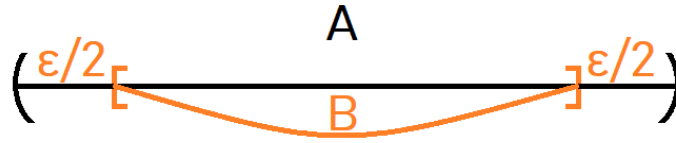
Let's denote the closure of A by \bar{A} . Since we're adding points, the Outer Measure will not decrease:

$$A \subset \bar{A} \implies m^*(A) \leq m^*(\bar{A}) = l(a)$$

Now suppose we have a closed interval B strictly inside A . Then we get

$$m^*(A) \geq m^*(B) = l(B) = l(A) - \varepsilon$$

Now take $\varepsilon \rightarrow 0 \implies m^*(A) \geq l(A)$.



□

Lemma. m^* is translation-invariant.

Proof. If we translate the set, we can translate all of its covers as well. Since translating an interval does not change its length, the lengths of the covers won't change either. □

Proposition 2 (Countable subadditivity). For any countable collection of sets $\{E_k\}_{k=1}^{\infty}$ we have

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k)$$

Remark. We don't ask for the sets E_k to be disjoint. If we proved that we have an equality sign for the disjoint case, we would have proved that m^* is a measure, which we proved does not exist in Theorem 1.

Proof. Choose open intervals $I_{k,i}$, such that

$$E_k \subset \bigcup_{i=1}^{\infty} I_{k,i} \quad (E_{k,i} \text{ are a cover of } E_k)$$

and

$$\sum_{i=1}^{\infty} l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Such intervals exist from the definition of the infimum.

On the other hand, $\{I_{k,i} \mid 1 \leq k, i < \infty\}$ covers each of the E_k , and thus it's a cover of $\bigcup_{k=1}^{\infty} E_k$. Then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \overset{\text{it's a cover}}{\leq} \sum_{1 \leq k, i < \infty} l(I_{k,i}) < \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \left(\frac{1}{2} + \frac{1}{4} + \dots\right) = \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon$$

Now take $\varepsilon \rightarrow 0$. □

Remark. Here we assume that all of the E_k have finite outer measures. Otherwise, both of the sides of the inequality would diverge to infinity, and we get $\infty \leq \infty$ which is “true”.

1.3 The σ -algebra of Lebesgue-measurable sets.

Definition 1. A set E is (Lebesgue) measurable if for any set A ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C) \quad E^C = \mathbb{R} \setminus E$$

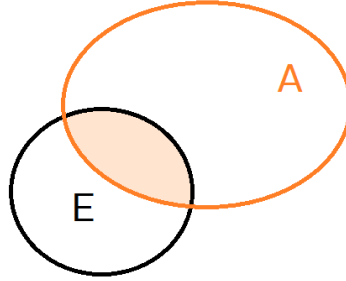


Figure 1: The set E “splits” A into two parts

Remark. We already have the \leq sign from [countable subadditivity](#).

Remark. Motivation: If $A \cap B = \emptyset$ and A (or B) is measurable, then

$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^C) = m^*(A) + m^*(B)$$

Proposition 1. If $m^*(E) = 0$, then E is measurable.

Proof. For all A we have:

$$\begin{aligned} m^*(A \cap E) &\leq m^*(E) = 0 \implies m^*(A \cap E) = 0 \\ m^*(A) &\geq m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C) \end{aligned}$$

As we noted earlier, the inequality in the other side follows from [countable subadditivity](#). □

Proposition 2. If E_1, \dots, E_n are measurable, then $\cup_1^n E_k$ is measurable.

Proof. Case $n = 2$: for all A we have

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^C) = \\ &= m^*(A \cap E_1) + m^*((A \cap E_1^C) \cap E_2) + m^*((A \cap E_1^C) \cap E_2^C) = (*) \\ X &:= A \cap E_1, \quad Y := (A \cap E_1^C) \cap E_2, \quad Z := (A \cap E_1^C) \cap E_2^C \end{aligned}$$

With Venn diagrams it's possible to prove that $Z = A \cap (E_1 \cup E_2)^C$, $X \cup Y = A \cap (E_1 \cup E_2)$. Now let's apply [countable subadditivity](#) to X and Y . Then we get:

$$(*) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^C)$$

Yet again, the inequality in the other side follows from [countable subadditivity](#).

Induction step: Apply case $n = 2$ to the sets $\cup_1^{n-1} E_k, E_n$. □