A Geometric Approach to Joint Source Separation and Declipping

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1 Problem

Throughout this work we assume noiseless linear mixtures. We observe F mixtures of N time points each, denoted by $\mathbf{x} \in \mathbb{R}^{F \times N}$. Each of these F observations is a linear mixture of D sources, denoted by $\mathbf{s} \in \mathbb{R}^{D \times N}$, obtained by a mixing matrix $\mathbf{A} \in \mathbb{R}^{F \times D}$, as shown in the equation below.

$$\mathbf{x} = \mathbf{A}\mathbf{s} \tag{1}$$

Clipping occurs in practice when attempting to represent a signal that contains amplitude values that saturate the maximum amplitude permitted by a system, yet most independent component analysis (ICA) algorithms neglect this. In a digital system, input samples whose absolute value exceeds a given *clipping threshold* θ have an output limited by this threshold. Formally, denote by $x_{i,n}$ the nth sample of the ith mixture before clipping and $x'_{i,n}$ the equivalent sample after clipping.

$$x'_{i,n} = \begin{cases} \operatorname{sgn}(x_{i,n})\,\theta & \text{if } |x_{i,n}| > \theta, \\ x_{i,n} & \text{if } |x_{i,n}| \le \theta. \end{cases} \tag{2}$$

Equations (1) and (2) combined describe the model of the problem:

$$x'_{i,n} = \begin{cases} \operatorname{sgn}\left(\sum_{j=1}^{D} A_{i,j} s_{j,n}\right)\theta & \text{if } \left|\sum_{j=1}^{D} A_{i,j} s_{j,n}\right| > \theta, \\ \sum_{j=1}^{D} A_{i,j} s_{j,n} & \text{if } \left|\sum_{j=1}^{D} A_{i,j} s_{j,n}\right| \le \theta. \end{cases} \tag{3}$$

Now the task is to find both $\mathbf{s}$ and $\mathbf{A}$.

2 Underdetermined source separation

Our approach can be extended to handle an increasing number of sources ($D > 2$). For this case, the optimisation step and the source estimation step are altered.

2.1 Mixing matrix estimation

Since clipping distorts the source lines, first discard all clipped points, where $\left|x'_{1,n}\right| = \theta$. All remaining time points, $\left|x'_{1,n}\right| \leq \theta$, lie on a source line. Since each source line passes through the origin, each of the remaining time points $(x'_{1,n}, x'_{2,n})^T$ has enough information to recover one of the source line slopes, $\nabla_n = (x'_{2,n} - 0)/(x'_{1,n} - 0)$. We denote by M the number of unclipped time points, and calculate the slope of all of these as shown below. The symbol $\odot$ represents the elementwise product.

$$\nabla_{\mathbf{M}} = \left(x'_{2,1}, \ldots, x'_{2,M}\right) \odot \left(\tfrac{1}{x'_{1,1}}, \ldots, \tfrac{1}{x'_{1,M}}\right) \tag{4}$$

The set $\nabla_{\mathbf{M}}$ contains M elements, some of which may be duplicates. We define the set $\nabla_{\mathbf{s}}$ to contain only the D unique entries of $\nabla_{\mathbf{M}}$. We make each value of the first row of the mixing matrix $\hat{A}_{1,1} \ldots \hat{A}_{1,D} = 1$, which means the D elements in $\nabla_{\mathbf{s}}$ can be used directly as $\hat{A}_{2,1} \ldots \hat{A}_{2,D}$, the second row of estimated mixing matrix $\hat{\mathbf{A}}$. To remove scaling indeterminacy, we normalise by dividing each element in column i of $\hat{\mathbf{A}}$ by its $l2$-norm, $\hat{\mathbf{A}}_{1:2,i} = \hat{\mathbf{A}}_{1:2,i} \frac{1}{||\hat{\mathbf{A}}_{1:2,i}||_2}$.

2.2 Quantisation of repairable time points

Information from the unclipped mixture x'_2 can be used to repair some of the clipped points in x'_1. We split the set of entries of $\nabla_{\mathbf{s}}$ into positives, denoted $\nabla^+_{\mathbf{s}}$, and negatives, denoted $\nabla^-_{\mathbf{s}}$. Denote by ∇_+ the positives in ascending order and by ∇_- the negatives in descending order as follows:

$$\nabla_+ = \left[\frac{x'_{2,i}}{x'_{1,i}} \in \nabla^+_{\mathbf{s}} : \frac{x'_{2,i}}{x'_{1,i}} \leq \frac{x'_{2,i+1}}{x'_{1,i+1}} \forall i = 1 ... \left|\nabla^+_{\mathbf{s}}\right|\right] \tag{5}$$

$$\nabla_- = \left[\frac{x'_{2,i}}{x'_{1,i}} \in \nabla^-_{\mathbf{s}} : \frac{x'_{2,i}}{x'_{1,i}} \geq \frac{x'_{2,i+1}}{x'_{1,i+1}} \forall i = 1 ... \left|\nabla^-_{\mathbf{s}}\right|\right] \tag{6}$$

We define $[\nabla_+]_i$ as the ith element of ∇_+, the list of positive slopes in ascending order, and $[\nabla_-]_i$ as the ith element of ∇_-, the list of negative slopes in descending order. Denote by p_i the value of x'_2 when the source line with positive slope $[\nabla_+]_i$ intersects the clipping threshold ($x'_1 = \theta$). Denote by q_i the same for the source line with negative slope $[\nabla_-]_i$.

$$p_i = [\nabla_+]_i\theta \tag{7}$$

$$q_i = [\nabla_-]_i\theta \tag{8}$$

A time point with index n can be repaired if it satisfies two conditions:

1. It is clipped in x'_1, the first mixture $\left(\left|x'_{1,n}\right| = \theta\right)$.
2. In x'_2, it lies between the intersection points p_1 and p_2 or q_1 and q_2 (the two least steep positive slopes and negative slopes, respectively).

If a time point satisfies both conditions, it must belong to the least steep source line. Figure 1, which shows the mixtures before clipping has occurred, shows why this is true. When we observe clipped data, we know that a point $(x'_{1,n}, x'_{2,n})^T$ on the dashed line between p_1 and p_2 is repairable. We also know that the original point $(x_{1,n}, x_{2,n})^T$ was located on a horizontal straight line, because only the first mixture coordinate $x_{1,n}$ is unknown ($x_{2,n} = x'_{2,n}$). The only source line that intersects the partially bounded region $\left|x_{1,n}\right| \geq \theta \wedge p_1 \leq \left|x_{2,n}\right| < p_2$ is the least steep, having slope $[\nabla_+]_1$. This region is shaded in Figure 1.

With knowledge of one coordinate $x'_{2,n}$, the slope $[\nabla_+]_1$ and that all source lines pass through the origin, we can calculate $x'_{1,n}$ geometrically for points in the region $\left|x_{1,n}\right| \geq \theta \wedge p_1 \leq \left|x_{2,n}\right| < p_2$. Note that this also applies to points in the region $\left|x_{1,n}\right| \geq \theta \wedge q_1 \geq \left|x_{2,n}\right| > q_2$ for reasons of symmetry:

$$x''_{1,n} = \begin{cases} x'_{2,n}/[\nabla_+]_1 & \text{if } \left|x'_{1,n}\right| \geq \theta \wedge p_1 \leq \left|x'_{2,n}\right| < p_2, \\ x'_{2,n}/[\nabla_-]_1 & \text{if } \left|x'_{1,n}\right| \geq \theta \wedge q_1 \geq \left|x'_{2,n}\right| > q_2, \\ x'_{1,n} & \text{otherwise.} \end{cases} \tag{9}$$

$$x''_{2,n} = x'_{2,n} \tag{10}$$

Figure 2 shows the result of this repair for our example.

2.3 Declipping by $\ell 1$ minimisation

We reconstruct the unrepairable clipped points by assuming they have a *sparse representation*. We assume that estimated source signals $\hat{\mathbf{s}}$ can be sufficiently

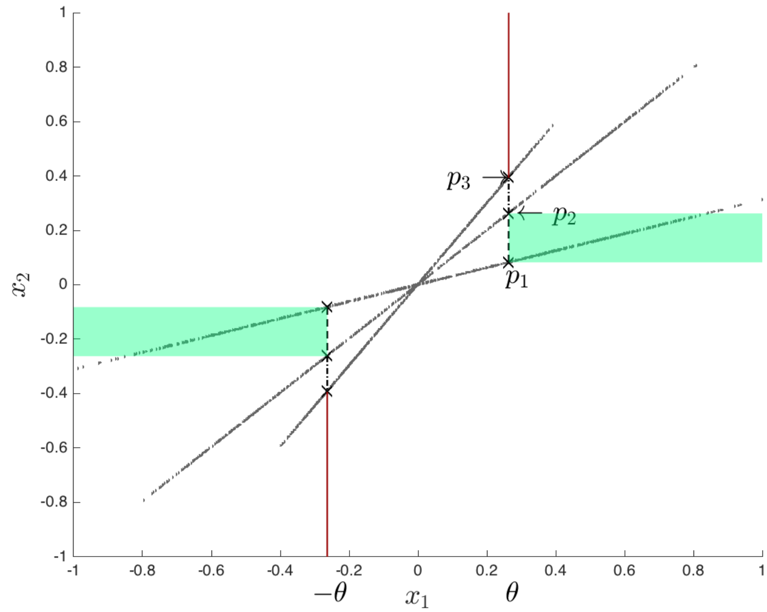

Figure 1: Possible x_2 values for the first source at clipped points.

represented by a small number of non-zero coefficients $\mathbf{r}$ in the basis matrix $\mathbf{\Psi}$, as shown below:

$$\hat{\mathbf{s}} = \mathbf{r}^T \mathbf{\Psi}^T \tag{11}$$

We choose the Discrete Cosine Transform (DCT) operator matrix, commonly used in signal processing, as $\mathbf{\Psi}$ for our example. It is likely that a sparser basis could be chosen, however the DCT will be sufficient to demonstrate our approach. We then formulate a convex optimisation problem that will give us a sparse solution with few non-zero elements. We use $\ell 1$ minimisation because the number of non-zero elements, known as the $\ell 0$ "norm", is not convex. The task is then to find a solution for the linear system below, where sparse representations $\mathbf{r} \in \mathbb{R}^{N \times D}$ and $\mathbf{x}''$ is the clipped mixtures $\mathbf{x}'$ with repairable points repaired:

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1}^{D} \|\mathbf{r}_i\|_1 \\ \text{subject to} \quad & \hat{\mathbf{A}} \mathbf{r}^T \mathbf{\Psi}^T = \mathbf{x}''. \end{aligned} \tag{12}$$

The above optimisation problem is constrained by the mixture model $\mathbf{x}'' = \hat{\mathbf{A}}\hat{\mathbf{s}} = \hat{\mathbf{A}}\mathbf{r}^T\mathbf{\Psi}^T$. We know, however, that some time points in x_1'' were unrepairable in the

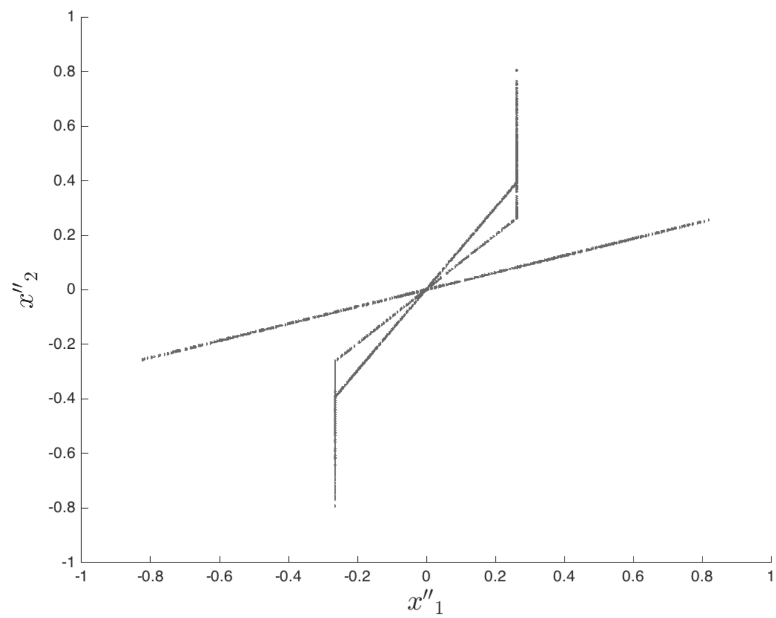

Figure 2: Clipped mixtures after repairing repairable points.

previous step and should be ignored by this constraint. We instead use a separate constraint for x_1'' and unclipped mixture x_2''. We use $\mathbf{I}_N$ to denote an $N \times N$ identity matrix. A masking matrix $\mathbf{C_u}$ is constructed, consisting of the $\mathbf{I}_N$ rows corresponding to indices of all unclipped and repairable points in x_1'', which can be used to isolate these points. Similar masking matrices, $\mathbf{C}_+$ and $\mathbf{C}_-$, contain $\mathbf{I}_N$ rows that correspond, respectively, to the positively clipped ($x'_{1,n} \geq \theta$) and negatively clipped ($x'_{1,n} \leq \theta$) points in x_1''. This pair of masking matrices is used to form constraints, similar to those developed in [1], that ensure the sparse sources $\mathbf{r}$ obey the clipping model in Equation (2). The final form of the optimisation problem, where $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ refer to the first and second rows of the matrix $\hat{\mathbf{A}}$ respectively, is as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{D} ||\mathbf{r}_i||_1 \\
\text{subject to} \quad & \hat{\mathbf{A}}_1 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_u}^T = x_1'' \mathbf{C_u}^T, \\
& \hat{\mathbf{A}}_2 \mathbf{r}^T \boldsymbol{\Psi}^T = x_2'', \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C}_-^T \leq -\theta, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C}_+^T \geq \theta.
\end{aligned} \tag{13}$$

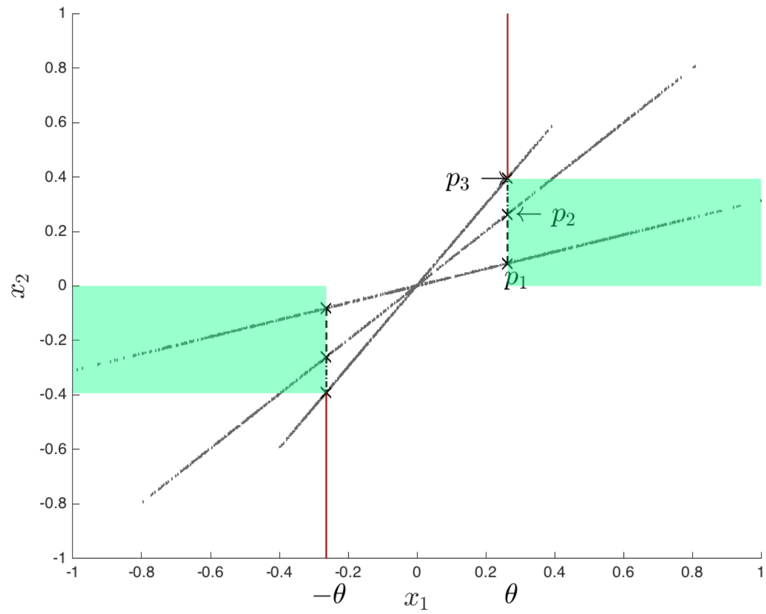

Figure 3: Example of partial information about clipped points between p_2 and p_3.

The problem can be further constrained for $D \geq 3$ sources by making additional observations about the clipping geometry. A scattergram of both unclipped mixtures, x_1 and x_2, is shown in Figure 3. In reality, we only observe x'_1 and x'_2, but the clipping geometry is easier to visualise from the unclipped mixtures. The steepest source line that intersects the partially bounded region $(x_{1,n} \geq \theta \wedge 0 < x_{2,n} < p_3) \vee (x_{1,n} \leq -\theta \wedge 0 > x_{2,n} > -p_3)$ has the second least steep positive slope, $[\nabla_+]_2$. This region is shaded in Figure 3. Generally, the steepest source line that intersects the region $(x_{1,n} \geq \theta \wedge 0 < x_{2,n} < p_i) \vee (x_{1,n} \leq -\theta \wedge 0 > x_{2,n} > -p_i)$ has the $(i-1)$th least steep positive slope, $[\nabla_+]_{i-1}$. By symmetry, this is also true for the region $(x_{1,n} \geq \theta \wedge 0 > x_{2,n} > q_i) \vee (x_{1,n} \leq -\theta \wedge 0 < x_{2,n} < -q_i)$ containing source lines with negative slopes $\geq [\nabla_-]_{i-1}$. Our example in Figure 3, however, contains no negative slopes.

After clipping, points from the region $(x_{1,n} \geq \theta \wedge 0 < x_{2,n} < p_i) \vee (x_{1,n} \leq -\theta \wedge 0 > x_{2,n} > -p_i)$ are transformed to lie on the line segment $(x'_{1,n} = \theta \wedge 0 < x'_{2,n} < p_i) \vee (x'_{1,n} = -\theta \wedge 0 > x'_{2,n} > -p_i)$. The position of this line segment when $i = 3$ is shown by the dotted and dashed lines in Figure 3. Similarly, the unrepairable clipped points with positive slopes in x''_1 lie on the line segment $(x''_{1,n} = \theta \wedge 0 < x''_{2,n} < p_i) \vee (x''_{1,n} = -\theta \wedge 0 > x''_{2,n} > -p_i)$. By symmetry, the unrepairable clipped points with negative slopes in x''_1 lie on the line segment $(x''_{1,n} = \theta \wedge 0 > x''_{2,n} >$

$q_i) \vee (x''_{1,n} = -\theta \wedge 0 < x''_{2,n} < -q_i)$. This gives the following rules that must be satisfied by the optimisation result:

$$\frac{\hat{x}_{2,n}}{\hat{x}_{1,n}} \leq [\nabla_+]_{i-1} \quad \forall i = 3...\left|\nabla_{\mathbf{s}}^+\right| \quad \text{if } (x''_{1,n} = \theta \wedge 0 < x''_{2,n} < p_i) \vee (x''_{1,n} = -\theta \wedge 0 > x''_{2,n} > -p_i) \tag{14}$$

$$\frac{\hat{x}_{2,n}}{\hat{x}_{1,n}} \geq [\nabla_-]_{i-1} \quad \forall i = 3...\left|\nabla_{\mathbf{s}}^-\right| \quad \text{if } (x''_{1,n} = \theta \wedge 0 > x''_{2,n} > q_i) \vee (x''_{1,n} = -\theta \wedge 0 < x''_{2,n} < -q_i) \tag{15}$$

Note that the rules are not needed for the two least steep source lines having a positive slope ($0 < x_{2,n} < p_2$) or the two least steep source lines having a negative slope ($0 > x_{2,n} > q_2$) because points in this region are repairable by the previous quantisation step. This is the reason the index i starts at 3 in Equations (14) and (15). Denote by $[\mathbf{C}_{++\mathbf{i}}]$ a matrix consisting of the $\mathbf{I}_N$ rows corresponding to indices of clipped points in the region $(x''_{1,n} = \theta \wedge 0 < x''_{2,n} < p_i) \vee (x''_{1,n} = -\theta \wedge 0 > x''_{2,n} > -p_i)$ and by $[\mathbf{C}_{--\mathbf{i}}]$ a matrix consisting of the $\mathbf{I}_N$ rows corresponding to indices of clipped points in the region $(x''_{1,n} = \theta \wedge 0 > x''_{2,n} > q_i) \vee (x''_{1,n} = -\theta \wedge 0 < x''_{2,n} < -q_i)$. These matrices $[\mathbf{C}_{++\mathbf{i}}]$ and $[\mathbf{C}_{--\mathbf{i}}]$ can then be used to generalise and simplify the notation of the rules given by Equations (14) and (15):

$$\hat{x}_2[\mathbf{C}_{++\mathbf{i}}](\hat{x}_1[\mathbf{C}_{++\mathbf{i}}])^{-1} \leq [\nabla_+]_{i-1} \quad \forall i = 3...\left|\nabla_{\mathbf{s}}^+\right| \tag{16}$$

$$\hat{x}_2[\mathbf{C}_{++\mathbf{i}}](\hat{x}_1[\mathbf{C}_{++\mathbf{i}}])^{-1} \geq [\nabla_-]_{i-1} \quad \forall i = 3...\left|\nabla_{\mathbf{s}}^-\right| \tag{17}$$

This pair of masking matrices is used to form constraints that ensure the sparse sources $\mathbf{r}$ obey the clipping geometry rules in Equations (16) and (17). The final form of the optimisation problem is as follows:

$$\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{D} \|\mathbf{r}_i\|_1 \\
\text{subject to} \quad & \hat{\mathbf{A}}_1 \mathbf{r}^T \mathbf{\Psi}^T \mathbf{C_u}^T = x''_1 \mathbf{C_u}^T, \\
& \hat{\mathbf{A}}_2 \mathbf{r}^T \mathbf{\Psi}^T = x''_2, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \mathbf{\Psi}^T \mathbf{C}_-^T \leq -\theta, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \mathbf{\Psi}^T \mathbf{C}_+^T \geq \theta, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \mathbf{\Psi}^T [\mathbf{C}_{++\mathbf{i}}]^T \geq x''_2 [\mathbf{C}_{++\mathbf{i}}]^T [\nabla_+]_{i-1}^{-1} \quad \forall i = 3...\left|\nabla_{\mathbf{s}}^+\right|, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \mathbf{\Psi}^T [\mathbf{C}_{--\mathbf{i}}]^T \leq x''_2 [\mathbf{C}_{--\mathbf{i}}]^T [\nabla_-]_{i-1}^{-1} \quad \forall i = 3...\left|\nabla_{\mathbf{s}}^-\right|.
\end{aligned} \tag{18}$$

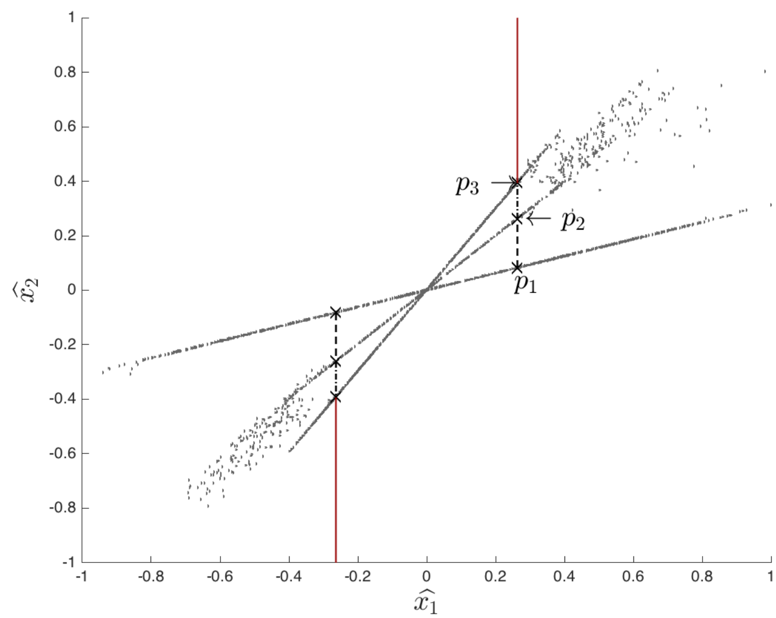

Figure 4: Clipped mixtures after clipped points have been repaired by $\ell 1$ minimisation.

After solving the optimisation problem, the following equation yields an estimate of the declipped mixture $\hat{x}_1$:

$$\hat{x}_1 = \hat{\mathbf{A}}_1 \mathbf{r}^T \mathbf{\Psi}^T \tag{19}$$

To solve the problem in Equation (18) we used CVX, a package for specifying and solving convex problems [2, 3]. The results are shown in Figure 4.

2.4 Quantisation of declipped points

The mixture points must lie on known source lines, which is not a guaranteed outcome of the optimisation problem. For this reason, we need to quantise the points so they they lie on the source lines. Since the second mixture x_2 was not clipped, we should only adjust $\hat{x}_1$. In the scattergram, this quantisation translates to horizontal adjustments of the repaired points. Each repaired point is quantised to the closest source line i using the following pair of equations:

$$l_n = \arg\min_i \left| \hat{x}_{1,n} - \frac{x_{2,n}}{\nabla_{\mathbf{s}i}} \right| \tag{20}$$

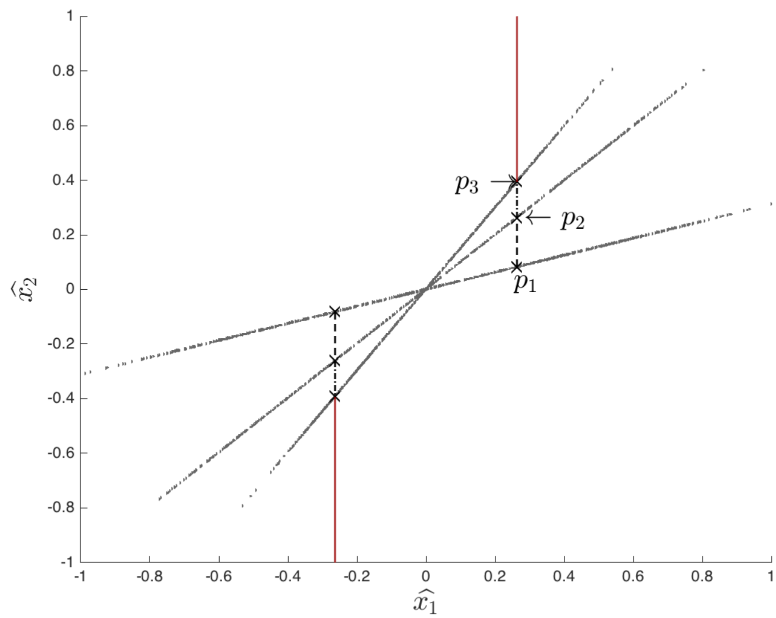

Figure 5: Quantisation of mixtures repaired by $\ell 1$ mimimisation.

$$\hat{x}_{1,n} = \frac{x_{2,n}}{\nabla_{\mathbf{s}l_n}} \tag{21}$$

The results of quantising in this manner for our example are shown in Figure 5.

2.5 Source estimation

In the underdetermined case, we no longer have a square mixing matrix $\hat{\mathbf{A}}$ and can not obtain estimated sources $\hat{\mathbf{s}}$ by inverting this matrix. An alternative source recovery method is needed. We estimate each source coefficient $\hat{s}_{i,n}$ as a weighted sum of the mixture coefficients, $\hat{x}_{1,n} + \nabla_{\mathbf{s}i}\hat{x}_{2,n}$. Note that, ignoring scaling factors, this is equivalent to generating a rotation matrix from the slope $\nabla_{\mathbf{s}i}$ and using this matrix to rotate the mixtures such that the source line i is parallel with x_1. We use this value as the source coefficient $\hat{s}_{i,n}$ if the known source label l_n from Equation (39) is i. If the label does not match, we set the source coefficient to zero because the sources are disjoint in time. The equation below shows the described source recovery method.

$$\hat{s}_{i,n} = \begin{cases} \hat{x}_{1,n} + \nabla_{\mathbf{s}l_n}\hat{x}_{2,n} & \text{if } l_n = i, \\ 0 & \text{otherwise.} \end{cases} \tag{22}$$

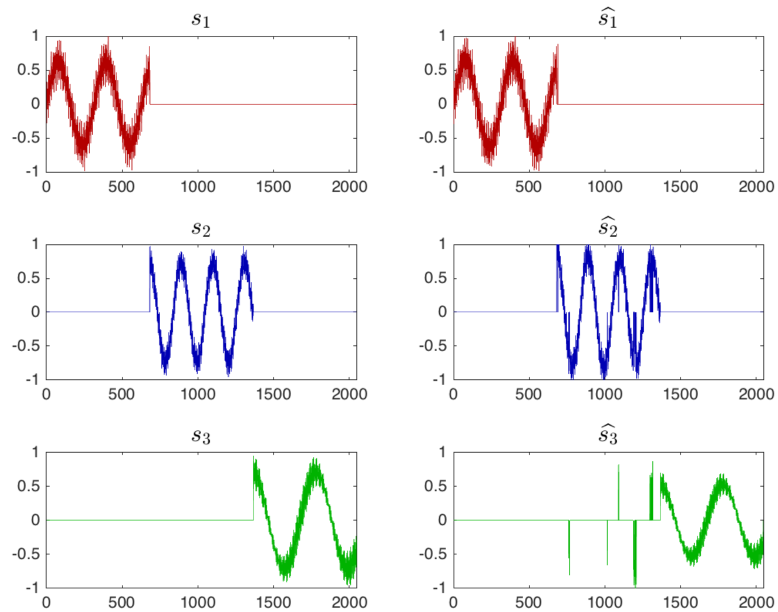

Figure 6: Sources estimated from clipped mixtures by rotation.

Figure 6 shows the sources estimated for our example.

3 Two clipped mixtures

We can also handle the more challenging case where both mixtures are clipped. For this case, the quantisation of trivial points step and the optimisation step are altered.

3.1 Mixing matrix estimation

Mixing matrix estimation is performed in the same way described in Section 2.1.

3.2 Quantisation of repairable time points

Points clipped in both mixtures cannot be trivially repaired. Points clipped in only one mixture can be quantised using information from the other mixture. We split the set of entries of $\nabla_{\mathbf{s}}$ into four subsets, denoted $\nabla_{\mathbf{z1}}$, $\nabla_{\mathbf{z2}}$, $\nabla_{\mathbf{z3}}$ and $\nabla_{\mathbf{z4}}$, each

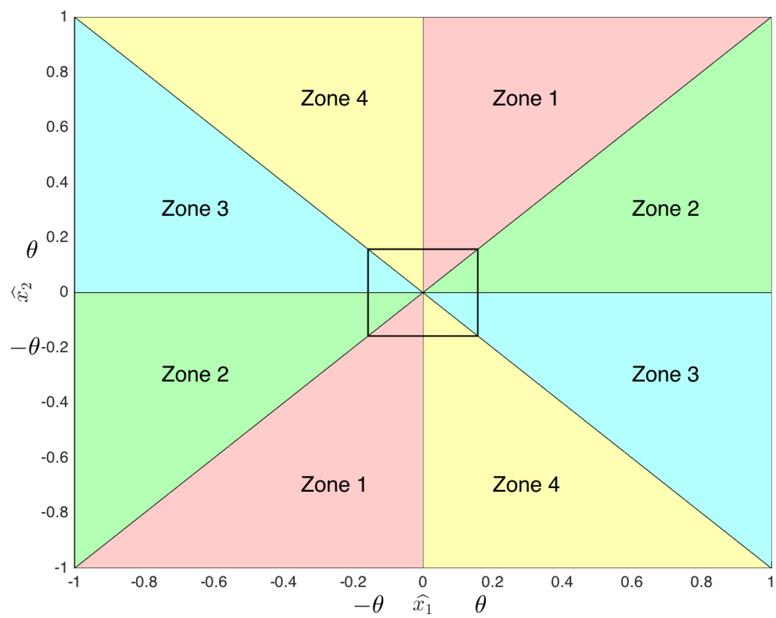

Figure 7: Source lines are grouped into Zones 1 to 4.

of which corresponds to one of four geometric *zones*, shown in Figure 7. The subsets of slopes split into zones are defined by the equations below.

$$\nabla_{\mathbf{z1}} = \left\{ \frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{s}} : \left|x'_{2,i}\right| > \left|x'_{1,i}\right|, \frac{x'_{2,i}}{x'_{1,i}} > 0 \right\} \tag{23}$$

$$\nabla_{\mathbf{z2}} = \left\{ \frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{s}} : \left|x'_{2,i}\right| < \left|x'_{1,i}\right|, \frac{x'_{2,i}}{x'_{1,i}} > 0 \right\} \tag{24}$$

$$\nabla_{\mathbf{z3}} = \left\{ \frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{s}} : \left|x'_{2,i}\right| < \left|x'_{1,i}\right|, \frac{x'_{2,i}}{x'_{1,i}} < 0 \right\} \tag{25}$$

$$\nabla_{\mathbf{z4}} = \left\{ \frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{s}} : \left|x'_{2,i}\right| > \left|x'_{1,i}\right|, \frac{x'_{2,i}}{x'_{1,i}} < 0 \right\} \tag{26}$$

Denote by $\nabla_{\mathbf{z}}^{\mathbf{1}}$ the entries of $\nabla_{\mathbf{z1}}$ and by $\nabla_{\mathbf{z}}^{\mathbf{3}}$ the entries of $\nabla_{\mathbf{z3}}$ in descending order. Denote by $\nabla_{\mathbf{z}}^{\mathbf{2}}$ the entries of $\nabla_{\mathbf{z2}}$ and by $\nabla_{\mathbf{z}}^{\mathbf{4}}$ the entries of $\nabla_{\mathbf{z4}}$ in ascending order. These lists of slopes are expressed formally by the equations below.

$$\nabla_{\mathbf{z}}^{\mathbf{1}} = \left[\frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{z1}} : \frac{x'_{2,i}}{x'_{1,i}} \geq \frac{x'_{2,i+1}}{x'_{1,i+1}} \forall i = 1...|\nabla_{\mathbf{z1}}| \right] \tag{27}$$

$$\nabla_{\mathbf{z}}^{\mathbf{2}} = \left[\frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{z2}} : \frac{x'_{2,i}}{x'_{1,i}} \leq \frac{x'_{2,i+1}}{x'_{1,i+1}} \forall i = 1...|\nabla_{\mathbf{z2}}| \right] \tag{28}$$

$$\nabla_{\mathbf{z}}^{\mathbf{3}} = \left[\frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{z3}} : \frac{x'_{2,i}}{x'_{1,i}} \geq \frac{x'_{2,i+1}}{x'_{1,i+1}} \forall i = 1...|\nabla_{\mathbf{z3}}| \right] \tag{29}$$

$$\nabla_{\mathbf{z}}^{\mathbf{4}} = \left[\frac{x'_{2,i}}{x'_{1,i}} \in \nabla_{\mathbf{z4}} : \frac{x'_{2,i}}{x'_{1,i}} \leq \frac{x'_{2,i+1}}{x'_{1,i+1}} \forall i = 1...|\nabla_{\mathbf{z4}}| \right] \tag{30}$$

We define $[\nabla_{\mathbf{z}}^{\mathbf{1}}]_i$ as the ith element of $\nabla_{\mathbf{z}}^{\mathbf{1}}$, the list of Zone 1 slopes in descending order, and $[\nabla_{\mathbf{z}}^{\mathbf{2}}]_i$ as the ith element of $\nabla_{\mathbf{z}}^{\mathbf{2}}$, the list of Zone 2 slopes in ascending order. We define $[\nabla_{\mathbf{z}}^{\mathbf{3}}]_i$ as the ith element of $\nabla_{\mathbf{z}}^{\mathbf{3}}$, the list of Zone 3 slopes in descending order, and $[\nabla_{\mathbf{z}}^{\mathbf{4}}]_i$ as the ith element of $\nabla_{\mathbf{z}}^{\mathbf{4}}$, the list of Zone 4 slopes in ascending order.

Denote by $z1_i$ the value of x'_1 when the Zone 1 source line with slope $[\nabla_{\mathbf{z}}^{\mathbf{1}}]_i$ intersects the clipping threshold in the other mixture ($x'_2 = \theta$). Denote by $z4_i$ the same for the Zone 4 source line $[\nabla_{\mathbf{z}}^{\mathbf{4}}]_i$. Denote by $z2_i$ the value of x'_2 when the Zone 2 source line with slope $[\nabla_{\mathbf{z}}^{\mathbf{2}}]_i$ intersects the clipping threshold in the other mixture ($x'_1 = \theta$). Denote by $z3_i$ the same for the Zone 3 source line $[\nabla_{\mathbf{z}}^{\mathbf{3}}]_i$.

$$z1_i = \frac{\theta}{[\nabla_{\mathbf{z}}^{\mathbf{1}}]_i} \tag{31}$$

$$z2_i = [\nabla_{\mathbf{z}}^{\mathbf{2}}]_i \theta \tag{32}$$

$$z3_i = [\nabla_{\mathbf{z}}^{\mathbf{3}}]_i \theta \tag{33}$$

$$z4_i = \frac{\theta}{[\nabla_{\mathbf{z}}^{\mathbf{4}}]_i} \tag{34}$$

A time point with index n can be repaired if it satisfies two conditions:

1. It is clipped in x'_1, the first mixture $\left(\left|x'_{1,n}\right| = \theta\right)$, or x'_2, the second mixture $\left(\left|x'_{2,n}\right| = \theta\right)$, but not both.

2. If clipped in x'_1, the x'_2 coefficient lies between the intersection points $z2_1$ and $z2_2$ or $z3_1$ and $z3_2$ (the two least steep Zone 2 slopes and Zone 3 slopes, respectively). If clipped in x'_2, the x'_1 coefficient lies between the intersection points $z1_1$ and $z1_2$ or $z4_1$ and $z4_2$ (the two most steep Zone 1 slopes and Zone 4 slopes, respectively).

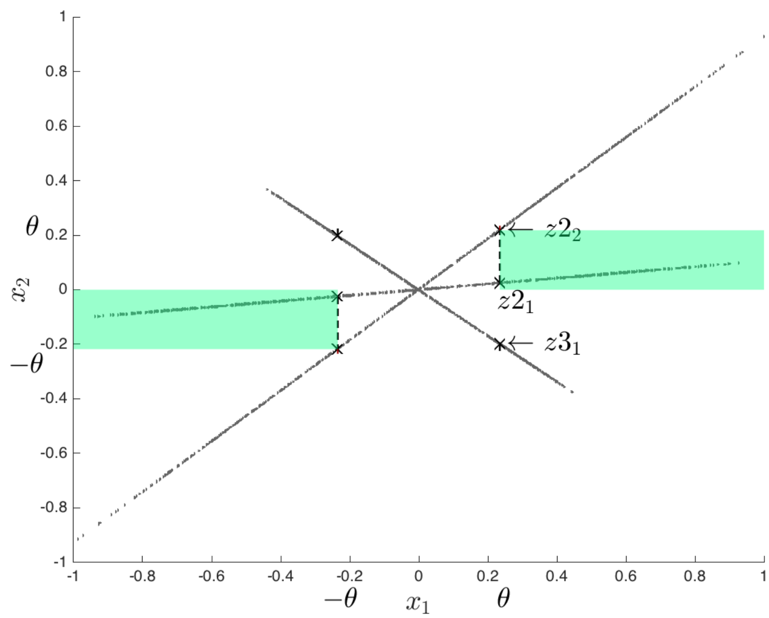

Figure 8: Possible x_1 and x_2 mixture values for the first source at the clipped points.

If a time point satisfies both conditions, it must belong to the source line nearest to whichever axis forms part of the zone boundary. For example, a repairable time point clipped in x'_1 must belong to the least steep source line. This is the source line that is nearest to the x1 axis (Figure 8), which forms part of the zone 2 boundary (Figure 7). When we observe clipped data, we know that a point $(x'_{1,n}, x'_{2,n})^T$ on the dashed line between $z2_1$ and $z2_2$ (Figure 8) is repairable. We also know that the original point $(x_{1,n}, x_{2,n})^T$ was located on a the horizontal straight line $x_{2,n} = x'_{2,n}$, because only the first mixture coordinate $x_{1,n}$ is unknown. The only source line that intersects the partially bounded region $\left|x_{1,n}\right| \geq \theta \wedge z2_1 \leq \left|x_{2,n}\right| < z2_2$ is the least steep, having slope $[\nabla^{\mathbf{2}}_{\mathbf{z}}]_1$. This region is shaded in Figure 8.

With knowledge of one coordinate $x'_{2,n}$, the slope $[\nabla^{\mathbf{2}}_{\mathbf{z}}]_1$ and that all source lines pass through the origin, we can calculate $x'_{1,n}$ geometrically for points in the region $\left|x_{1,n}\right| \geq \theta \wedge z2_1 \leq \left|x_{2,n}\right| < z2_2$. For reasons of symmetry, the same applies to Zone

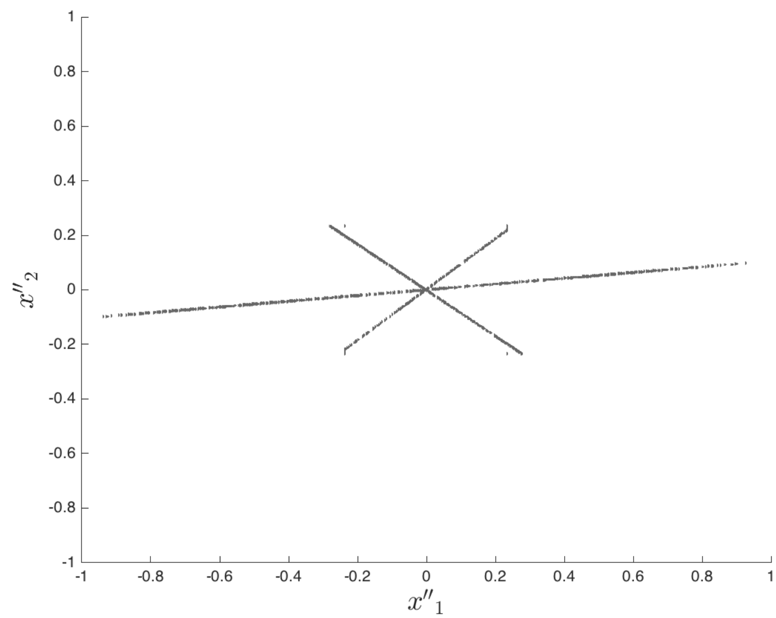

Figure 9: Clipped mixtures after repairing repairable points.

3 points in the region $\left|x_{1,n}\right| \geq \theta \wedge z3_1 \geq \left|x_{2,n}\right| > z3_2$ using the slope $[\nabla_{\mathbf{z}}^{\mathbf{3}}]_1$:

$$x''_{1,n} = \begin{cases} x'_{2,n}/[\nabla_{\mathbf{z}}^{\mathbf{2}}]_1 & \text{if } \left|x'_{1,n}\right| \geq \theta \wedge z2_1 \leq \left|x'_{2,n}\right| < z2_2, \\ x'_{2,n}/[\nabla_{\mathbf{z}}^{\mathbf{3}}]_1 & \text{if } \left|x'_{1,n}\right| \geq \theta \wedge z3_1 \geq \left|x'_{2,n}\right| > z3_2, \\ x'_{1,n} & \text{otherwise.} \end{cases} \tag{35}$$

The same idea applies to the remaining zones 1 and 4 and, if one were to swap the axes, the symmetry of the zones becomes apparent. With knowledge of one coordinate $x'_{1,n}$ and the slope $[\nabla_{\mathbf{z}}^{\mathbf{1}}]_1$, we can calculate $x'_{2,n}$ geometrically for points in the region $\left|x_{2,n}\right| \geq \theta \wedge z1_1 \leq \left|x_{1,n}\right| < z1_2$. For reasons of symmetry, the same applies to Zone 4 points in the region $\left|x_{2,n}\right| \geq \theta \wedge z4_1 \geq \left|x_{1,n}\right| > z4_2$ using the slope $[\nabla_{\mathbf{z}}^{\mathbf{4}}]_1$:

$$x''_{2,n} = \begin{cases} x'_{1,n}[\nabla_{\mathbf{z}}^{\mathbf{1}}]_1 & \text{if } \left|x'_{2,n}\right| \geq \theta \wedge z1_1 \leq \left|x'_{1,n}\right| < z1_2, \\ x'_{1,n}[\nabla_{\mathbf{z}}^{\mathbf{4}}]_1 & \text{if } \left|x'_{2,n}\right| \geq \theta \wedge z4_1 \geq \left|x'_{1,n}\right| > z4_2, \\ x'_{2,n} & \text{otherwise.} \end{cases} \tag{36}$$

Figure 9 shows the result of this quantisation for our example.

3.3 Declipping by $\ell 1$ minimisation

We constrain the optimisation problem by the mixture model $\mathbf{x}'' = \hat{\mathbf{A}}\hat{\mathbf{s}} = \hat{\mathbf{A}}\mathbf{r}^T\mathbf{\Psi}^T$. We know, however, that some time points in x_1'' and x_2'' were unrepairable and should be ignored in the constraints. We formulate a constraint each for x_1'' and x_2''. A pair of masking matrices $\mathbf{C_{u1}}$ and $\mathbf{C_{u2}}$ are constructed, consisting of the $\mathbf{I}_N$ rows corresponding to indices of all unclipped and repairable points in x_1'' and x_2'' respectively, which can be used to isolate these points. These matrices are used to form the first pair of constraints in Equation (37).

We also add constraints that ensure the sparse sources $\mathbf{r}$ obey the clipping model in Equation (2), similar to those developed in [1]. Masking matrices $\mathbf{C_{+1}}$ and $\mathbf{C_{-1}}$ contain $\mathbf{I}_N$ rows that correspond, respectively, to the indices of points positively clipped ($x'_{1,n} \geq \theta$) and negatively clipped ($x'_{1,n} \leq \theta$) only in x_1''. For those points only clipped in the second mixture, x_2'', the equivalent matrices $\mathbf{C_{+2}}$ and $\mathbf{C_{-2}}$ are constructed. These four matrices enable points clipped in only one mixture to be isolated and constrained, forming the second and third pairs of constraints in Equation (37).

We can also constrain those points which are clipped in both mixtures. Since we assume noiseless mixing and time disjoint sources, we know that the sign of the slope of a point will not be altered by clipping. Each estimated source $\hat{s}_i$ which has a positive slope when mixed ($\hat{\mathbf{A}}_{2,i}/\hat{\mathbf{A}}_{1,i} > 0$) must be zero at those indices n where the clipped mixtures $\mathbf{x}''$ yield a negative slope ($x''_{2,n}/x''_{1,n} < 0$). At these indices, we know the slope is negative from the mixtures, so the sources whose lines have a positive slope must be zero because of the time disjointness.

We denote by $\mathbf{B_p}$ a masking matrix consisting of the $\mathbf{I}_D$ rows that correspond to the indices of the rows of $\hat{\mathbf{A}}$ that yield a source line with a positive slope ($\hat{\mathbf{A}}_{2,i}/\hat{\mathbf{A}}_{1,i} > 0$). An equivalent masking matrix $\mathbf{B_q}$ is constructed to isolate source lines with a negative slope ($\hat{\mathbf{A}}_{2,i}/\hat{\mathbf{A}}_{1,i} < 0$). We denote by $\mathbf{C_{bp}}$ a masking matrix consisting of $\mathbf{I}_N$ rows that correspond to the indices of points clipped in both x_1'' and x_2'' that have a positive slope ($x''_{2,n}/x''_{1,n} > 0$). An equivalent masking matrix $\mathbf{C_{bq}}$ is constructed to isolate points clipped in both mixtures that have a negative slope ($x''_{2,n}/x''_{1,n} < 0$). These four masking matrices are used to form the final pair of constraints in Equation (37). The final form of the optimisation problem, where $\hat{\mathbf{A}}_1$ and $\hat{\mathbf{A}}_2$ refer to the first and second rows of the matrix $\hat{\mathbf{A}}$ respectively, is as follows:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{D} \|\mathbf{r}_i\|_1 \\
\text{subject to} \quad & \hat{\mathbf{A}}_1 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{u1}}^T = x_1'' \mathbf{C_{u1}}^T, \\
& \hat{\mathbf{A}}_2 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{u2}}^T = x_2'' \mathbf{C_{u2}}^T, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{+1}}^T \geq \widehat{\theta}, \\
& \hat{\mathbf{A}}_1 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{-1}}^T \leq -\widehat{\theta}, \\
& \hat{\mathbf{A}}_2 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{+2}}^T \geq \widehat{\theta}, \\
& \hat{\mathbf{A}}_2 \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{-2}}^T \leq -\widehat{\theta}, \\
& \mathbf{B_p} \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{bq}}^T = 0, \\
& \mathbf{B_q} \mathbf{r}^T \boldsymbol{\Psi}^T \mathbf{C_{bp}}^T = 0.
\end{aligned}
\tag{37}
$$

After solving the optimisation problem, the following equation yields an estimate of the declipped mixtures $\hat{\mathbf{x}}$:

$$\hat{\mathbf{x}} = \hat{\mathbf{A}} \mathbf{r}^T \boldsymbol{\Psi}^T \tag{38}$$

Figure 10 shows the result of this repair for our example.

3.4 Quantisation of declipped points

The mixture points must lie on known source lines, which is not a guaranteed outcome of the optimisation problem. For this reason, we need to quantise the points so they they lie on the source lines. Since the second mixture x_2 was not clipped, we should only adjust $\hat{x}_1$. In the scattergram, this quantisation translates to horizontal adjustments of the repaired points. Each repaired point is quantised to the closest source line i using the following pair of equations:

$$l_n = \arg\min_i \left| \hat{x}_{1,n} - \frac{x_{2,n}}{\nabla_{\mathbf{s}i}} \right| \tag{39}$$

$$\hat{x}_{1,n} = \frac{x_{2,n}}{\nabla_{\mathbf{s}l_n}} \tag{40}$$

The results of quantising in this manner for our example are shown in Figure 11.

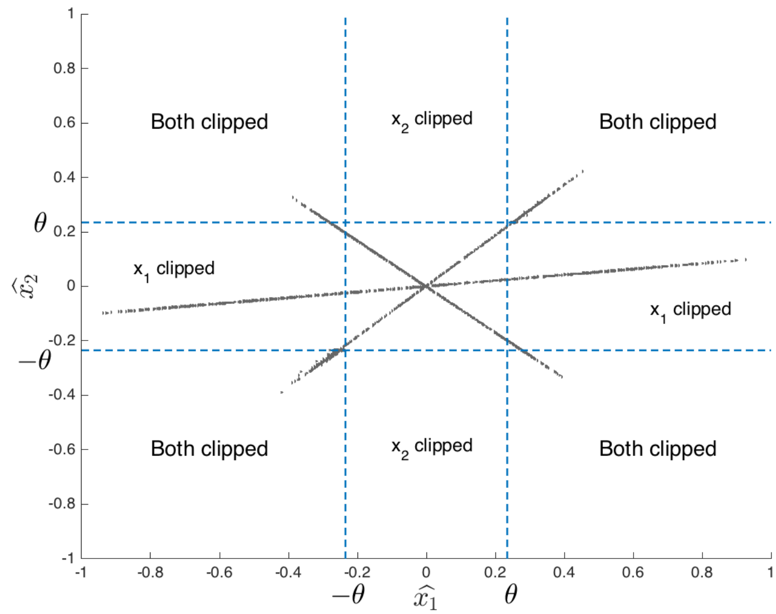

Figure 10: Clipped mixtures after clipped points have been repaired by $\ell 1$ minimisation.

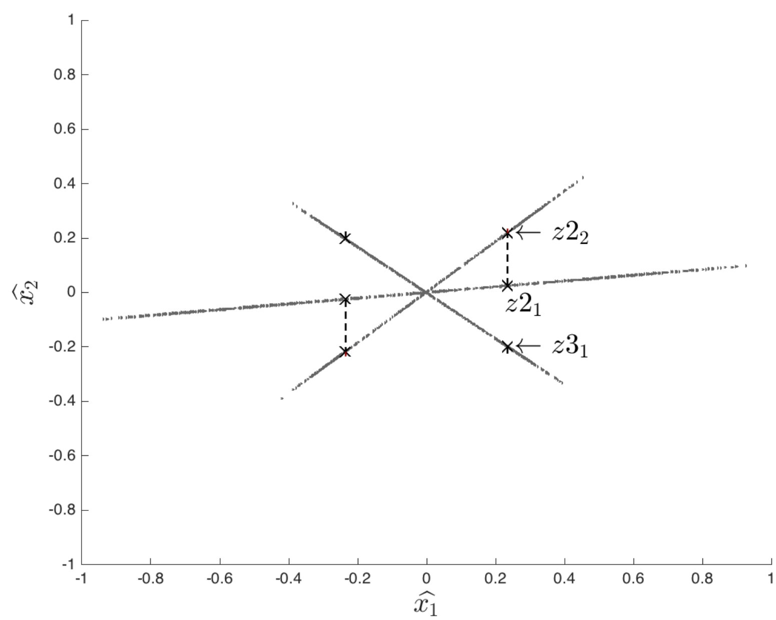

Figure 11: Quantisation of mixtures repaired by $\ell 1$ mimimisation.

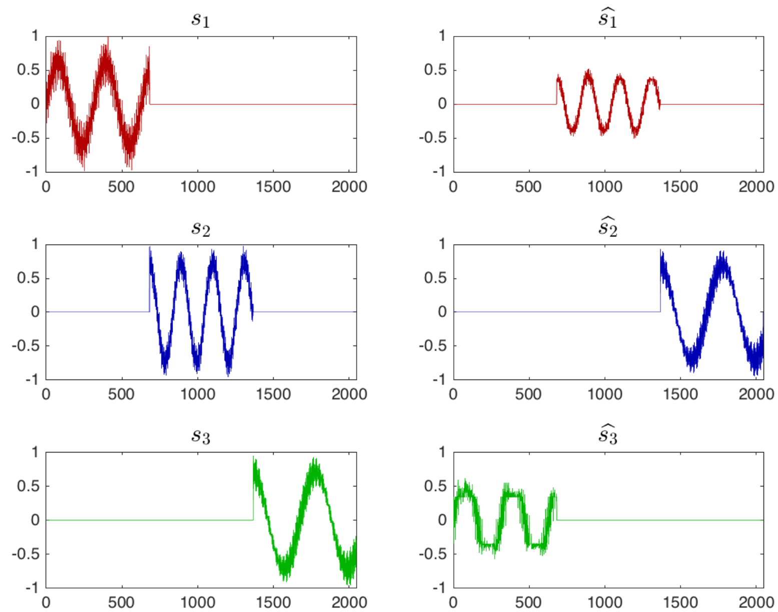

Figure 12: Sources estimated from both clipped mixtures.

3.5 Source estimation

Source estimation is performed in the same way described in Section 2.5. Figure 12 shows the sources estimated for our example.

References

[1] B. Defraene, N. Mansour, S. De Hertogh, T. van Waterschoot, M. Diehl, and M. Moonen, "Declipping of Audio Signals Using Perceptual Compressed Sensing," *IEEE Transactions on Audio, Speech, and Language Processing*, vol. 21, no. 12, pp. 2627–2637, 2013.

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