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Measuring the tail risk: An asymptotic approach

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ABSTRACT

Given a vector of non-negative random variables (X, Y) with marginal survival functions $\overline{F}$ and $\overline{G}$, this paper obtains asymptotic approximations for the quantity

$$\phi_{X,Y}(p) := \mathbb{E}\left[X|\overline{F}(X)\overline{G}(Y) \le p\right]$$

as $p \downarrow 0$. Interpreting X and Y as two risks, $\phi_{X,Y}(p)$ defines a novel risk measure where the extreme region as $p \downarrow 0$ is attained when at least one of the two risks, rather than only one of them as in common cases, is large. We conduct our study under wide dependence structures between X and Y, which cover both asymptotic independence and asymptotic dependence cases. Some numerical examples are also shown to analyze the sensitivity of our obtained results with respect to dependence parameters.

1. Introduction and background

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and denote by $L_+(\mathbb{P})$ the set of non-negative random variables. Consider $X \in L_+(\mathbb{P})$ and $Y \in L_+(\mathbb{P})$ as two random insurance risks possessing *distribution functions* (df's) F and G, respectively that are assumed to have ultimate right tails, i.e. $\inf\{x \in \Re : F(x) = 1\} = \infty$ and $\inf\{x \in \Re : G(x) = 1\} = \infty$. The corresponding survival functions are $\overline{F} := 1 - F$ and $\overline{G} := 1 - G$.

This paper considers the following quantity defined as a conditional expectation:

$$\phi_{X,Y}(p) := \mathbb{E}\left[X|\overline{F}(X)\overline{G}(Y) \le p\right], \qquad p \in (0, 1]. \tag{1.1}$$

We aim to find asymptotic approximations for $\phi_{X,Y}(p)$ as $p \downarrow 0$ under wide dependence structures between X and Y, which cover both asymptotic independence and asymptotic dependence cases.

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Our main motivation for studying $\phi_{X,Y}(p)$ is that it owns a strong application background in risk theory. Specifically, assume that X is a risk from an insurer/investor portfolio of risks and Y is the common risk or the reference risk of the portfolio that specifies the adverse scenario for which the exercise is performed. Note that, in the context of capital allocation, Y usually represents the total risk portfolio (for example, see [16]). A key task of risk theory is to evaluate the risk exposure of X under certain extreme region related to X and/or Y. This purpose results in the emergence of many risk measures in terms of conditional expectations. A commonly-used risk measure is the popular *Marginal Expected Shortfall* (MES), which is mathematically formulated as $\mathbb{E}\left[X|\overline{G}(Y) \leq p\right]$ (see [5], [8] and [14]). Clearly, the extreme region of the MES as $p \downarrow 0$ is only related to the common/reference risk Y and it is attained only when Y becomes large. Now, observing (1.1), our $\phi_{X,Y}(p)$ actually defines a novel risk measure where the extreme region is related to both the common/reference risk Y and the risk X itself and it is attained when at least one of X and Y is large. This synthetic consideration is the crucial difference with the MES and it may not affect the results much if the common/reference risk acts as the main driving risk, while the case will be changed if for example some dominant risks are present in the portfolio where "medium" and "small" type risks are ignored if only the common/reference risk is considered. Our numerical examples below have shown that our $\phi_{X,Y}(p)$ as a risk measure outperforms the MES in the sense that it is always less sensitive to the chosen model (dependence and marginal distributions) and always leads to non-trivial results, which provides clear evidence to support our approach.

The rest of this paper consists of four sections. Section 2 introduces various concepts and notations. Sections 3 and 4 show our main asymptotic results for $\phi_{X,Y}(p)$ under the asymptotic independence and asymptotic dependence cases, respectively. The paper is concluded with some numerical discussions included in Section 5.

2. Preliminaries

Let $\{X_i; i \geq 1\}$ be a sequence of independent and identically distributed random variables with common df F. *Extreme Value Theory* (EVT) assumes that there are constants $a_n > 0$ and $b_n \in \Re$ such that

$$\lim_{n\to\infty} \mathbb{P}\left(a_n\left(\max_{1\leq i\leq n} X_i - b_n\right) \leq x\right) = Q(x), \quad x \in \Re.$$

In this case, Q is called an *Extreme Value Distribution* and F is said to belong to the *max-domain of attraction of* Q, denoted by $F \in \mathrm{MDA}(Q)$. If Q is non-degenerate and is required to have an ultimate right tail, then the Fisher–Tippett Theorem (see [12]) implies that Q is of one of the following two types: $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}$ for all $x > 0$ with $\alpha > 0$ or $\Lambda(x) = \exp\{-\mathrm{e}^{-x}\}$ for all $x \in \Re$. The first scenario makes X to have a *Fréchet* tail or in other words, *regularly varying* at ∞ with index $-\alpha$, i.e.

$$\lim_{t\to\infty} \frac{\overline{F}(tz)}{\overline{F}(t)} = z^{-\alpha}, \quad z > 0. \tag{2.1}$$

We signify the above by $F \in \mathcal{R}_{-\alpha}$. The second scenario makes X to have a Gumbel tail and it is well-known (for example, see [11]) that there exists a positive measurable function a such that

$$\lim_{t\to\infty} \frac{\overline{F}\big(t + za(t)\big)}{\overline{F}(t)} = \mathrm{e}^{-z}, \quad z \in \Re. \tag{2.2}$$

Relation (2.2) implies that X has a rapidly varying tail, written as $F \in \mathcal{R}_{-\infty}$, i.e.

$$\lim_{t\to\infty} \frac{\overline{F}(tz)}{\overline{F}(t)} = 0, \quad z > 1. \tag{2.3}$$

For further details of regular variation and rapid variation, we refer the reader to [7] or [11].

It is necessary to recall the important concept of *copula*, which is a commonly-used tool for measuring dependence amongst random variables. Let Z_1 and Z_2 be two random variables with df's V_1 and V_2, respectively. It is well-known that the dependence structure associated with a random vector can be characterized in terms of its copula. By definition, a bivariate copula is a two-dimensional df defined on $[0,1]^2$ with uniform marginal distributions. Due to Sklar's Theorem (see [23]), if V_1 and V_2 are continuous, then there exists a unique copula C such that $\mathbb{P}\big(Z_1 \le x, Z_2 \le y\big) = C\big(V_1(x), V_2(y)\big)$. The *survival copula* $\widehat{C}$ is defined as the copula corresponding to the joint survival function, i.e. $\mathbb{P}\big(Z_1 > x, Z_2 > y\big) = \widehat{C}\big(\overline{V}_1(x), \overline{V}_2(y)\big)$ and thus, we have

$$\widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v), \quad (u,v) \in [0,1]^2.$$

The *generalized inverse function* is another concept heavily used in this paper, which is given by $f^{\leftarrow}(y) := \inf\big\{x \in \Re : f(x) \ge y\big\}$ if f is a non-decreasing function with the convention $\inf\emptyset = \infty$. If f is a non-increasing function, then $f^{\leftarrow}(y) := \inf\big\{x \in \Re : f(x) \le y\big\}$.

By definition, Z_1 and Z_2 are said to be *asymptotically independent* if

$$\lim_{q\uparrow 1} \mathbb{P}\big(\, Z_2 > V_2^{\leftarrow}(q) \big|\, Z_1 > V_1^{\leftarrow}(q)\big) = 0. \tag{2.4}$$

Moreover, Z_1 and Z_2 are *asymptotically dependent* if

$$\liminf_{q\uparrow 1} \mathbb{P}\big(\, Z_2 > V_2^{\leftarrow}(q) \big|\, Z_1 > V_1^{\leftarrow}(q)\big) > 0. \tag{2.5}$$

Recall that the concept of asymptotic independence stems from Definition 5.30 of [18] and not only, while the asymptotic dependence is related to equation (1.2) of [2]. It is not difficult to find that, if Z_1 and Z_2 are continuous random variables with copula C, then (2.4) and (2.5) can be respectively rewritten as

$$\lim_{u\downarrow 0} \frac{\widehat{C}(u,u)}{u} = 0 \quad \text{and} \quad \liminf_{u\downarrow 0} \frac{\widehat{C}(u,u)}{u} > 0. \tag{2.6}$$

We now introduce the concept of *vague convergence* prior to defining the *multivariate regular variation*, which is a key ingredient for proving our main results under the asymptotic dependence case. Consider an d-dimensional cone $[0,\infty]^d \backslash \{\mathbf{0}\}$ equipped with a Borel sigma-field $\mathcal{B}$. A measure on the cone is called Radon if its value is finite for every compact set in $\mathcal{B}$. For a sequence of Radon measures $\big\{\nu, \nu_n, n = 1, 2, \dots\big\}$ on $[0,\infty]^d \backslash \{\mathbf{0}\}$, we say that ν_n vaguely converges to ν as $n \to \infty$, written as $\nu_n \xrightarrow{v} \nu$, if

$$\lim_{n\to\infty} \int_{[0,\infty]^d\backslash\{\mathbf{0}\}} f\big(\mathbf{z}\big)\nu_n\big(\mathrm{d}\mathbf{z}\big) = \int_{[0,\infty]^d\backslash\{\mathbf{0}\}} f\big(\mathbf{z}\big)\nu\big(\mathrm{d}\mathbf{z}\big)$$

holds for every non-negative continuous function f with compact support. It is known that $\nu_n \xrightarrow{v} \nu$ on $[0,\infty]^d \backslash \{\mathbf{0}\}$ if and only if

$$\lim_{n\to\infty} \nu_n\big[\mathbf{0}, \mathbf{x}\big]^c = \nu\big[\mathbf{0}, \mathbf{x}\big]^c$$

is true for every continuity point $\mathbf{x} \in [0,\infty]^d \backslash \{\mathbf{0}\}$ of $\nu\big[\mathbf{0}, \mathbf{x}\big]^c$. For more details and related discussions, we refer the reader to Section 3.3.5 and Lemma 6.1 of [22].

A d-dimensional random vector $\mathbf{X} = (X_1, \ldots, X_d)$ follows a *multivariate regular variation* (*MRV*) structure if there exist a positive normalizing function $b(t) \uparrow \infty$ as $t \to \infty$ and a Radon measure ν on $[0,\infty]^d \backslash \{\mathbf{0}\}$, which is not identically 0, such that

$$t\mathbb{P}\left(\frac{\mathbf{X}}{b(t)} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \qquad \text{on } [0,\infty]^d \backslash \{\mathbf{0}\}. \tag{2.7}$$

The function b may not be unique and different choices are likely to generate limiting measures that differ only by a constant factor. In the case that the marginal distributions of $\mathbf{X}$ are tail equivalent to some df F_* (i.e. $\lim_{t\to\infty} \mathbb{P}(X_i > t)/\overline{F}_*(t) = c_i$ with some $c_i > 0$ for all $i \in \{1,\ldots d\}$), a possible choice is $b(t) = F_*^{\leftarrow}(1 - 1/t)$, which leads to

$$\frac{1}{\overline{F}_*(t)}\mathbb{P}\left(\frac{\mathbf{X}}{t} \in \cdot\right) \xrightarrow{v} \nu(\cdot) \qquad \text{on } [0,\infty]^d \backslash \{\mathbf{0}\},$$

where $\overline{F}_* = 1 - F_*$. A by-product of relation (2.7) is that the limit measure ν is homogeneous, i.e. there exists some index $0 < \alpha < \infty$ such that $\nu(xB) = x^{-\alpha}\nu(B)$ for all $B \in \mathcal{B}$ (for details, see page 178 of [22]) and hence, we write $\mathbf{X} \in \mathrm{MRV}_{-\alpha}$. The homogeneity property of ν implies that $\nu\big[\mathbf{0}, \mathbf{x}\big]^c$ is continuous in $\mathbf{x}$ for every $\mathbf{x} > \mathbf{0}$. Moreover, $\nu\big(\mathbf{x}, \boldsymbol{\infty}\big] > 0$ is true for some $\mathbf{x} > \mathbf{0}$ if and only if it holds for every $\mathbf{x} > \mathbf{0}$. For more discussions on this concept, we refer the reader to Section 5.4.2 of [21] or Section 6.1.4 of [22].

We end this section with a summary of notations used in this paper. Unless otherwise stated, all limit relationships hold as $p \downarrow 0$. For two real-valued functions f_1 and f_2 that are not 0 in the right neighbourhood of 0, we write $f_1(p) \sim f_2(p)$ if $\lim_{p\downarrow 0} f_1(p)/f_2(p) = 1$, $f_1(p) = O\big(f_2(p)\big)$ if $\limsup_{p\downarrow 0} |f_1(p)/f_2(p)| < \infty$ and $f_1(p) = o\big(f_2(p)\big)$ if $\lim_{p\downarrow 0} f_1(p)/f_2(p) = 0$. Finally, $\mathbf{1}_{\{\cdot\}}$ represents the indicator function.

3. Main results under asymptotic independence

This section establishes asymptotic approximations of the expected loss $\phi_{X,Y}(p)$ defined in (1.1), where the extreme region is given by $\overline{F}(X)\overline{G}(Y) \le p$ for small values of p. This means that at least one of $\overline{F}(X)$ and $\overline{G}(Y)$ is small, which implies that X or Y is in an extreme region. Therefore, the level of risk exhibited by X in an extreme region defined in tandem by both risks is expected to be very sensitive to the specific dependence structure between X and Y.

Consider first a general dependence structure between X and Y given in Assumption 3.1, whose initial version is proposed in [4]. This assumption describes a popular dependence structure possessing the asymptotic independence property as detailed in Remark 3.1 and it has been widely applied in various fields (for example, see [1], [2], [10], [17] and [24]).

Assumption 3.1. There is some non-negative function $g : [0,\infty) \mapsto [0,\infty)$ such that

$$\lim_{t\to\infty} \frac{\mathbb{P}(Y > t | X = x)}{\overline{G}(t)} = g(x) \tag{3.1}$$

holds uniformly for $x \in [0,\infty)$ with $\lim\limits_{x\to\infty} g(x) = g^* > 0$.

Remark 3.1. If relation (3.1) holds uniformly for $x \in [0,\infty)$, then X and Y are asymptotically independent. One finds this result by integrating both sides of (3.1) with respect to $\mathbb{P}(X \in \mathrm{d}x)$ over the range $[0,\infty)$, which leads to

$$\int_{0-}^{\infty} g(x)\mathbb{P}(X \in \mathrm{d}x) = 1 < \infty.$$

Moreover,

$$\begin{aligned}\mathbb{P}(X > F^{\leftarrow}(q)|\,Y > G^{\leftarrow}(q)) &= \frac{1}{\mathbb{P}(Y > G^{\leftarrow}(q))}\int_{F^{\leftarrow}(q)}^{\infty} \mathbb{P}(Y > G^{\leftarrow}(q)|\,X = x)\mathbb{P}(X \in \mathrm{d}x)\\ &\sim \int_{F^{\leftarrow}(q)}^{\infty} g(x)\mathbb{P}(X \in \mathrm{d}x), \quad q \uparrow 1\\ &= 0,\end{aligned}$$

which concludes our claim.

In the remaining part of the paper, we write $\xi = \overline{F}(X)$ and $\eta = \overline{G}(Y)$. Therefore, if F and G are continuous, then ξ and η are uniformly distributed on $[0, 1]$. Relation (3.1) may be rewritten in terms of ξ and η, i.e.

$$\lim_{v\downarrow 0} \frac{\mathbb{P}\left(\eta \le v|\xi = u\right)}{v} = \widetilde{g}(u) \tag{3.2}$$

holds uniformly for $u \in (0, 1]$ with $\widetilde{g} := g \circ \overline{F}^{\leftarrow}$. Note also that, if (X, Y) follows a copula C, then the copula of (ξ, η) is just $\widehat{C}$ and we have

$$\mathbb{P}\left(\eta \le v|\xi = u\right) = \frac{\partial \widehat{C}(u, v)}{\partial u}. \tag{3.3}$$

In view of the above, we may restate Assumption 3.1 in terms of the copula of (X, Y), i.e. there is some positive function $\widetilde{g} : (0, 1] \mapsto [0, \infty)$ such that $\lim_{u\downarrow 0} \widetilde{g}(u) = g^* > 0$ and the relation

$$\lim_{v\downarrow 0} \frac{\partial \widehat{C}(u, v)/\partial u}{v} = \widetilde{g}(u) \tag{3.4}$$

holds uniformly for $u \in (0, 1]$.

The asymptotic property displayed in (3.4) is satisfied by many commonly-used bivariate copulae. We further provide with three specific examples related to our subsequent discussions and all calculations are omitted. Many more other examples can be found in Section 3 of [17] or [24].

Example 3.1. The Johnson–Kotz iterated FGM copula (see [6] or [15]) is given by

$$C(u, v) = uv + (\theta + \lambda uv)\, uv(1 - u)(1 - v), \quad \theta \in [-1, 1]$$

and $-1 - \theta < \lambda < \left(3 - \theta + \sqrt{9 - 6\theta - 3\theta^2}\right)/2$. Particularly, if $\lambda = 0$ then the above reduces to the Farlie–Gumbel–Morgenstern (FGM) copula with $\theta \in (-1, 1]$. Assumption 3.1 holds with

$$\widetilde{g}(u) = 1 + \theta + \lambda - 2\,(\theta + 2\lambda)\, u + 3\lambda u^2 \quad \text{and} \quad g^* = 1 + \theta + \lambda.$$

Example 3.2. The Ali–Mikhail–Haq copula is defined as follows

$$C(u, v) = \frac{uv}{1 - \theta(1 - u)(1 - v)}, \quad \theta \in (-1, 1)$$

and satisfies Assumption 3.1 with $\widetilde{g}(u) = 1 + \theta - 2\theta u$ and $g^* = 1 + \theta$.

Example 3.3. The following copula appears in [20]

$$C(u,v) = uv + \frac{\theta}{\pi} v(1-v)\sin(\pi u), \quad \theta \in (-1,1]$$

and satisfies Assumption 3.1 with $\widetilde{g}(u) = 1 + \theta\cos(\pi u)$ and $g^* = 1 + \theta$.

The next lemma is crucial in deriving the asymptotic approximations for $\phi_{X,Y}(p)$.

Lemma 3.1. *Let Assumption 3.1 hold. If F and G are continuous, then*

$$\mathbb{P}\big(\overline{F}(X)\overline{G}(Y) \le p\big) = \mathbb{P}(\xi\eta \le p) \sim g^* p \log \frac{1}{p}. \tag{3.5}$$

Proof. We write

$$\mathbb{P}\left(\xi\eta \le p\right) = \left(\int_0^{p\log\log(1/p)} + \int_{p\log\log(1/p)}^{1} \right) \mathbb{P}\left(\eta \le \frac{p}{u} \Big| \, \xi = u\right) \mathrm{d}u =: I_1(p) + I_2(p). \tag{3.6}$$

It is clear that

$$I_1(p) \le p \log\log \frac{1}{p} = o(1) p \log \frac{1}{p}. \tag{3.7}$$

By Assumption 3.1 or (3.2), we have

$$I_2(p) \sim p \int_{p\log\log(1/p)}^{1} \frac{\widetilde{g}(u)}{u} \mathrm{d}u. \tag{3.8}$$

For every $\varepsilon > 0$, since $\lim_{u \downarrow 0} \widetilde{g}(u) = g^*$, there is some small $\delta > 0$ such that the relation

$$(1-\varepsilon)\, g^* \le \widetilde{g}(u) \le (1+\varepsilon)\, g^* \tag{3.9}$$

holds for all $u \in (0,\delta]$. Choose $p > 0$ small enough such that $p \log\log(1/p) < \delta$. We further write

$$\int_{p\log\log(1/p)}^{1} \frac{\widetilde{g}(u)}{u} \mathrm{d}u = \left(\int_{p\log\log(1/p)}^{\delta} + \int_{\delta}^{1} \right) \frac{\widetilde{g}(u)}{u} \mathrm{d}u. \tag{3.10}$$

Now, integrating both sides of (3.2) with respect to $\mathbb{P}(\xi \in \mathrm{d}u) = \mathrm{d}u$ over the range $(0,1]$ leads to $\int_0^1 \widetilde{g}(u)\mathrm{d}u = 1 < \infty$, i.e. $\widetilde{g}$ is integrable over $(0,1]$ and hence, $\widetilde{g}(\cdot)/\cdot$ is integrable over $(\delta,1]$. Thus, the second term of (3.10) is finite and hence, is negligible compared to $\log(1/p)$ as $p \downarrow 0$. In the light of (3.9), the first term of (3.10) satisfies

$$\int_{p\log\log(1/p)}^{\delta} \frac{\widetilde{g}(u)}{u} \mathrm{d}u \ge (1-\varepsilon) g^* \left(\log\delta + \log\frac{1}{p} - \log\log\log\frac{1}{p} \right) \sim (1-\varepsilon) g^* \log\frac{1}{p},$$

and

$$\int_{p\log\log(1/p)}^{\delta} \frac{\widetilde{g}(u)}{u}\mathrm{d}u \le (1+\varepsilon)g^* \left(\log\delta + \log\frac{1}{p} - \log\log\log\frac{1}{p}\right) \sim (1+\varepsilon)g^*\log\frac{1}{p}.$$

Plugging the above estimates into (3.10) and noting the arbitrariness of ε, we have

$$\int_{p\log\log(1/p)}^{1} \frac{\widetilde{g}(u)}{u}\mathrm{d}u \sim g^*\log\frac{1}{p},$$

which combined with (3.8) imply that $I_2(p) \sim g^* p\log(1/p)$. The latter, equations (3.6) and (3.7) conclude (3.5). The proof is now complete. $\square$

We now go back to our ultimate aim, which is to estimate $\phi_{X,Y}(p)$. It is not difficult to see that

$$\phi_{X,Y}(p) = \int_0^\infty \mathbb{P}\big(X > x|\xi\eta \le p\big)\mathrm{d}x = \frac{1}{\mathbb{P}(\xi\eta \le p)}\int_0^\infty \mathbb{P}\big(\xi \le \overline{F}(x), \xi\eta \le p\big)\mathrm{d}x. \tag{3.11}$$

By noting Lemma 3.1, we may find that the integral term of (3.11) is the only estimate we have to deal with. Now,

$$\begin{aligned}
&\int_0^\infty \mathbb{P}\left(\xi \le \overline{F}(x), \xi\eta \le p\right)\mathrm{d}x \\
&= \left(\int_0^{\overline{F}^{\leftarrow}(p)} + \int_{\overline{F}^{\leftarrow}(p)}^{\infty}\right) \mathbb{P}\left(\xi \le \overline{F}(x), \xi\eta \le p\right)\mathrm{d}x \\
&= \int_0^{\overline{F}^{\leftarrow}(p)} \left(\int_0^{p} + \int_p^{\overline{F}(x)}\right) \mathbb{P}\left(\eta \le \frac{p}{u}\,\Big|\, \xi = u\right)\mathrm{d}u\mathrm{d}x + \int_{\overline{F}^{\leftarrow}(p)}^{\infty} \mathbb{P}\left(\xi \le \overline{F}(x)\right)\mathrm{d}x \qquad (3.12)\\
&= p\overline{F}^{\leftarrow}(p) + \int_p^1 \overline{F}^{\leftarrow}(u)\mathbb{P}\left(\eta \le \frac{p}{u}\,\Big|\, \xi = u\right)\mathrm{d}u + \int_{\overline{F}^{\leftarrow}(p)}^{\infty} \overline{F}(x)\mathrm{d}x \\
&=: p\overline{F}^{\leftarrow}(p) + I(p) + J(p),
\end{aligned}$$

where an obvious exchange of integrals is made to get $I(p)$. It is clear that only $I(p)$ and $J(p)$ need further work, while only $I(p)$ is sensitive to the dependence between ξ and η. Hence, the main challenge to study the asymptotic behaviour of $\phi_{X,Y}$ is to estimate $I(p)$ under specific dependence structures. Unfortunately, the general dependence structure given in Assumption 3.1 does not allow us to obtain precise asymptotic approximations for $I(p)$. The main reason lies in that (3.2) provides us with the first order approximation of $\mathbb{P}\left(\eta \le v|\, \xi = u\right)$ as $v \downarrow 0$, which is not sufficient.

Despite the above disappointing conclusion, an interesting specific scenario can be investigated. Namely, if X has a regularly varying tail and C satisfies Assumption 3.2, which is a refinement of Assumption 3.1 (as explained in Remark 3.3), then the precise asymptotic result for $\phi_{X,Y}(p)$ as $p \downarrow 0$ is possible.

Assumption 3.2. There exists a positive integer n such that the copula of (X, Y) satisfies

$$\frac{\partial \widehat{C}(u,v)}{\partial u} = \sum_{i=1}^{n} v^i l_i(u,v), \tag{3.13}$$

where $l_1, \ldots, l_n$ are some continuous functions on $[0,1]^2$. For each $i \in \{1, \ldots, n\}$, assume also that there is some constant l_i^* such that $l_i(0,v) = l_i^*$ for all $v \in [0,1]$ with $l_1^* > 0$.

Remark 3.2. In view of (3.3), $\partial \widehat{C}(u,1)/\partial u = 1$ for all $u \in [0,1]$. Thus, putting $v = 1$ on both sides of (3.13) leads to $\sum_{i=1}^{n} l_i(u,1) = 1$ for all $u \in [0,1]$ and hence, $\sum_{i=1}^{n} l_i^* = 1$.

Remark 3.3. It is not difficult to check that Assumption 3.2 is a special case of Assumption 3.1. In fact, Assumption 3.2 implies that

$$\frac{\partial \widehat{C}(u,v)/\partial u}{v} = l_1(u,v) + \sum_{i=2}^{n} v^{i-1} l_i(u,v),$$

where the sum is understood as 0 if $n = 1$. Note that l_i is continuous on $[0,1]^2$, which implies that l_i is uniformly continuous and bounded on $[0,1]^2$ for each $i \in \{1, \ldots, n\}$. Hence, we have

$$\lim_{v \downarrow 0} \sup_{u \in [0,1]} |l_1(u,v) - l_1(u,0)| \le \lim_{v \downarrow 0} \sup_{|u_1 - u_2| + |v_1 - v_2| \le v} |l_1(u_1,v_1) - l_1(u_2,v_2)| = 0,$$

and

$$\lim_{v \downarrow 0} \sup_{u \in [0,1]} \left| \sum_{i=2}^{n} v^{i-1} l_i(u,v) \right| \le \sum_{i=2}^{n} \lim_{v \downarrow 0} v^{i-1} \sup_{(u,v) \in [0,1]^2} |l_i(u,v)| = 0.$$

These indicate that (3.4) holds uniformly for $u \in [0,1]$ with $\widetilde{g}(u) = l_1(u,0)$. Moreover, Assumption 3.2 implies that

$$\lim_{u \downarrow 0} \widetilde{g}(u) = \lim_{u \downarrow 0} l_1(u,0) = l_1(0,0) = l_1^* > 0.$$

Therefore, Assumption 3.1 holds with $\widetilde{g}(u) = l_1(u,0)$ and $g^* = l_1^*$.

Remark 3.4. Following the same arguments given in Remark 3.3, for each $i \in \{1, \ldots, n\}$, the uniform continuity of l_i on $[0,1]^2$ implies that

$$\lim_{u \downarrow 0} l_i(u,v) = l_i(0,v) = l_i^*$$

holds uniformly for $v \in [0,1]$ under Assumption 3.2.

Remark 3.5. It is not difficult to verify that all Examples 3.1–3.3 satisfy Assumption 3.2. Specifically, for Example 3.1, we have $n = 3$ and

$$\begin{cases} l_1(u,v) = l_1(u) = 1 + \theta + \lambda - 2(\theta + 2\lambda)u + 3\lambda u^2, \\ l_2(u,v) = l_2(u) = -\theta - 2\lambda + 2(\theta + 4\lambda)u - 6\lambda u^2, \\ l_3(u,v) = l_3(u) = \lambda - 4\lambda u + 3\lambda u^2, \end{cases}$$

with $l_1^* = 1 + \theta + \lambda$, $l_2^* = -\theta - 2\lambda$ and $l_3^* = \lambda$. For Example 3.2, we have $n = 2$ and

$$l_1(u,v) = \frac{1+\theta-2\theta u}{(1-\theta uv)^2} \quad \text{and} \quad l_2(u,v) = -\frac{\theta\left(1-\theta u^2\right)}{(1-\theta uv)^2},$$

with $l_1^* = 1+\theta$ and $l_2^* = -\theta$. For Example 3.3, we have $n=2$ and

$$l_1(u,v) = l_1(u) = 1+\theta\cos(\pi u) \quad \text{and} \quad l_2(u,v) = l_2(u) = -\theta\cos(\pi u),$$

with $l_1^* = 1+\theta$ and $l_2^* = -\theta$.

Before proceeding further discussions, we summarize some well-known Karamata-type results for regularly or rapidly varying functions for later use. We refer the reader to Theorems A3.6 and A3.12(a) of [11] for further details.

Lemma 3.2. *Let h be a positive function from the class $\mathcal{R}_\beta$ for some $-\infty \le \beta < \infty$ such that h is locally bounded in $[t_0,\infty)$ for some $t_0 \ge 0$.*

(i) *If $-1 < \beta < \infty$ then*

$$\lim_{t\to\infty} \frac{\int_{t_0}^t h(x)\mathrm{d}x}{th(t)} = \frac{1}{\beta+1}. \tag{3.14}$$

(ii) *If $-\infty < \beta < -1$ then*

$$\lim_{t\to\infty} \frac{\int_t^\infty h(x)\mathrm{d}x}{th(t)} = -\frac{1}{\beta+1}. \tag{3.15}$$

(iii) *If $\beta = -1$ then (3.14) remains true with $1/(\beta+1)$ understood as ∞. If $\beta = -1$ and $\int_{t_0}^\infty h(x)\mathrm{d}x < \infty$ then (3.15) remains true with $-1/(\beta+1)$ is understood as ∞.*

(iv) *If $\beta = -\infty$ and h is non-increasing, then it holds for every $-\infty < s < \infty$ that*

$$\lim_{t\to\infty} \frac{\int_t^\infty x^s h(x)\mathrm{d}x}{t^{s+1}h(t)} = 0.$$

Now, we are ready to state our first main result for the asymptotic behaviour of $\phi_{X,Y}(p)$.

Theorem 3.1. *Assume that F and G are continuous and let C be the copula of (X,Y) such that Assumption 3.2 is satisfied. If $F \in \mathcal{R}_{-\alpha}$ for some $1 < \alpha < \infty$, then*

$$\phi_{X,Y}(p) \sim \left[\frac{\alpha}{(\alpha-1)l_1^*} + \sum_{i=1}^n \frac{l_i^*}{(i-1+1/\alpha)l_1^*}\right] \frac{\overline{F}^{\leftarrow}(p)}{\log(1/p)}. \tag{3.16}$$

Proof. In view of the analysis immediately after Lemma 3.1, we only need to estimate $I(p)$ and $J(p)$ from (3.12). Now, since $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$, Lemma 3.2(ii) leads to

$$J(p) \sim \frac{1}{\alpha-1} p\overline{F}^{\leftarrow}(p). \tag{3.17}$$

We next focus on $I(p)$ under Assumption 3.2. A combination of (3.3), (3.12) and (3.13) gives that

$$I(p) = \sum_{i=1}^n \int_p^1 \overline{F}^{\leftarrow}(u) \frac{p^i}{u^i} l_i\left(u, \frac{p}{u}\right) \mathrm{d}u =: \sum_{i=1}^n I_i(p). \tag{3.18}$$

For each $i \in \{1, \dots, n\}$, an obvious variable substitution leads to

$$I_i(p) = p^i \int_1^{1/p} x^{i-2} \overline{F}^{\leftarrow}\left(x^{-1}\right) l_i\left(x^{-1}, px\right) \mathrm{d}x. \tag{3.19}$$

Due to Remark 3.4, for every $\varepsilon > 0$, there is some M large enough such that $l_i^* - \varepsilon \le l_i\left(x^{-1}, v\right) \le l_i^* + \varepsilon$ holds for all $x > M$ and $v \in [0, 1]$. Hence, for p being in the right neighbourhood of 0 such that $1/p > M$, the relation $l_i^* - \varepsilon \le l_i\left(x^{-1}, px\right) \le l_i^* + \varepsilon$ holds for all $x \in (M, 1/p]$. Thus,

$$I_i(p) = p^i \left(\int_1^M + \int_M^{1/p} \right) x^{i-2} \overline{F}^{\leftarrow}\left(x^{-1}\right) l_i\left(x^{-1}, px\right) \mathrm{d}x =: I_{i1}(p) + I_{i2}(p). \tag{3.20}$$

Noting that l_i is bounded and $\overline{F}^{\leftarrow}(p) \to \infty$ as $p \downarrow 0$, we have

$$|I_{i1}(p)| \le D_{i,M} \cdot p^i = o(1) p \overline{F}^{\leftarrow}(p), \tag{3.21}$$

where $D_{i,M}$ is a positive constant that only depends upon i and M. For I_{i2}, it follows that

$$(l_i^* - \varepsilon)\, p^i \int_M^{1/p} x^{i-2} \overline{F}^{\leftarrow}\left(x^{-1}\right) \mathrm{d}x \le I_{i2}(p) \le (l_i^* + \varepsilon)\, p^i \int_M^{1/p} x^{i-2} \overline{F}^{\leftarrow}\left(x^{-1}\right) \mathrm{d}x. \tag{3.22}$$

Since $F \in \mathcal{R}_{-\alpha}$, we have $1/\overline{F} \in \mathcal{R}_\alpha$ and hence, $\overline{F}^{\leftarrow}(1/\cdot) = (1/\overline{F})^{\leftarrow}(\cdot) \in \mathcal{R}_{1/\alpha}$ (see, e.g. Proposition 2.6(v) of [22]). Thus, $(\cdot)^{i-2}\overline{F}^{\leftarrow}(1/\cdot) \in \mathcal{R}_{1/\alpha+i-2}$ with $1/\alpha + i - 2 > -1$. Then, applying Lemma 3.2(i) to (3.22) yields that

$$\frac{l_i^* - \varepsilon}{i - 1 + 1/\alpha} \le \liminf_{p \downarrow 0} \frac{I_{i2}(p)}{p \overline{F}^{\leftarrow}(p)} \le \limsup_{p \downarrow 0} \frac{I_{i2}(p)}{p \overline{F}^{\leftarrow}(p)} \le \frac{l_i^* + \varepsilon}{i - 1 + 1/\alpha},$$

which together with (3.20), (3.21) and the arbitrariness of ε give

$$\lim_{p \downarrow 0} \frac{I_i(p)}{p \overline{F}^{\leftarrow}(p)} = \frac{l_i^*}{i - 1 + 1/\alpha}. \tag{3.23}$$

The latter and (3.18) imply that

$$I(p) \sim \sum_{i=1}^n \frac{l_i^*}{i - 1 + 1/\alpha} p \overline{F}^{\leftarrow}(p),$$

which together with (3.12) and (3.17) lead to

$$\int_0^\infty \mathbb{P}\left(\xi \le \overline{F}(x), \xi\eta \le p\right) \mathrm{d}x \sim \left(\frac{\alpha}{\alpha - 1} + \sum_{i=1}^n \frac{l_i^*}{i - 1 + 1/\alpha} \right) p \overline{F}^{\leftarrow}(p).$$

Recalling Lemma 3.1 and $g^* = l_1^*$ (see Remark 3.3), it holds that $\mathbb{P}\left(\xi\eta \le p\right) \sim l_1^* p \log(1/p)$, which together with (3.11) give (3.16). This completes the proof. □

We now show some further discussions on Theorem 3.1 in the rapid variation case. Observing the proof of Theorem 3.1, one may find that we are not able to obtain precise approximations for $\phi_{X,Y}(p)$ under the framework of Theorem 3.1 when X has a rapidly varying tail. The concrete reason lies in that $\overline{F}^{\leftarrow}(1/\cdot) \in \mathcal{R}_0$ and the argument for estimating $I_1(p)$ defined in (3.19) involves the critical case of Karamata's Theorem, i.e. Lemma 3.2(iii) with $\beta = -1$, for which no precise approximation is available. If the conditions of Theorem 3.1 hold with $F \in \mathcal{R}_{-\infty}$ then, following the same logic displayed in equations (3.19)–(3.23) and applying Lemma 3.2(iii), we get that $p\overline{F}^{\leftarrow}(p) = o\big(I_1(p)\big)$, where we also used the fact that $l_1^* > 0$.

On the other hand, relation (3.23) still holds with $1/\alpha$ understood as 0, which indicates that $I_i(p) = O(1)p\overline{F}^{\leftarrow}(p)$ for all $i \in \{2, \ldots, n\}$. Additionally, by Lemma 3.2(iv), relation (3.17) remains true for $F \in \mathcal{R}_{-\infty}$ if we understand $1/(\alpha - 1)$ as 0 and read the right-hand side of (3.17) as $o(1)p\overline{F}^{\leftarrow}(p)$. Combining all of these with (3.12) and (3.18), we get

$$\int_0^\infty \mathbb{P}\left(\xi \leq \overline{F}(x), \xi\eta \leq p\right) \mathrm{d}x \sim I_1(p),$$

which together with (3.11) and the fact that $\mathbb{P}\left(\xi\eta \leq p\right) \sim l_1^* p \log(1/p)$ (concluded at the end of the proof of Theorem 3.1) imply that

$$\phi_{X,Y}(p) \sim \frac{I_1(p)}{l_1^* p \log\left(1/p\right)}. \tag{3.24}$$

Hence, the key point to derive precise approximations for $\phi_{X,Y}(p)$ when $F \in \mathcal{R}_{-\infty}$ is to further estimate $I_1(p)$. This depends on the specific form of $\overline{F}$ and we show below two specific examples.

Example 3.4. Assume that the conditions of Theorem 3.1 are satisfied, where F is now chosen to be exponentially distributed as $\overline{F}(x) = \mathbf{1}_{\{x\leq 0\}} + \mathrm{e}^{-\sigma x}\mathbf{1}_{\{x>0\}}$ with $\sigma > 0$. In this case F is light-tailed in the sense that the moment generating function $\hat{F}(z) = \int_0^\infty \mathrm{e}^{zx}\mathrm{d}F(x)$ is finite for all $0 < z < \sigma$. It is not difficult to check that $\overline{F}^{\leftarrow}\left(x^{-1}\right) = \frac{1}{\sigma}\log x$ holds for all $x > 1$. For every $\varepsilon > 0$, one may choose a large enough M such that (3.20) holds. It is clear that (3.21) holds for $i = 1$ with $I_{11}(p) = o(1)p\log(1/p)$. For $I_{12}(p)$, note that

$$\int_M^{1/p} x^{-1}\overline{F}^{\leftarrow}\left(x^{-1}\right)\mathrm{d}x = \frac{1}{\sigma}\int_M^{1/p} \frac{\log x}{x}\mathrm{d}x \sim \frac{1}{2\sigma}\left(\log p\right)^2.$$

The latter and relation (3.22) yield that

$$\frac{l_1^* - \varepsilon}{2\sigma} \leq \liminf_{p\downarrow 0} \frac{I_{12}(p)}{p\left(\log p\right)^2} \leq \limsup_{p\downarrow 0} \frac{I_{12}(p)}{p(\log p)^2} \leq \frac{l_1^* + \varepsilon}{2\sigma}.$$

Thus, due to the arbitrariness of ε and equation (3.20), we obtain that

$$I_1(p) \sim \frac{l_1^*}{2\sigma}p(\log p)^2,$$

which together with (3.24) give that

$$\phi_{X,Y}(p) \sim \frac{1}{2\sigma}\log\frac{1}{p}.$$

Example 3.5. Assume that the conditions of Theorem 3.1 are satisfied, where F is now chosen such that $\overline{F}(x) = \mathbf{1}_{\{x\leq 1\}} + \mathrm{e}^{-(\log x)^{\gamma}}\mathbf{1}_{\{x>1\}}$ with $\gamma > 1$. In this case F is heavy-tailed, since its moment generating function $\hat{F}(z) = \infty$ for all $z > 0$. Clearly, $\overline{F}^{\leftarrow}\left(x^{-1}\right) = \mathrm{e}^{(\log x)^{1/\gamma}}$ for all $x > 1$. Using the same reasoning as in Example 3.4, we get that $I_{11}(p) = o(1)p\mathrm{e}^{(\log(1/p))^{1/\gamma}}$. Further,

$$\lim_{t\to\infty} \frac{\int_M^t \mathrm{e}^{(\log x)^{1/\gamma}}/x\mathrm{d}x}{\gamma\left(\log t\right)^{1-1/\gamma}\mathrm{e}^{(\log t)^{1/\gamma}}} = 1,$$

due to L'Hôspital's rule, which together with (3.22) imply that

$$\begin{aligned}\gamma\left(l_1^* - \varepsilon\right) &\leq \liminf_{p\downarrow 0} \frac{I_{12}(p)}{p\big(\log\left(1/p\right)\big)^{1-1/\gamma}\mathrm{e}^{\left(\log(1/p)\right)^{1/\gamma}}} \\ &\leq \limsup_{p\downarrow 0} \frac{I_{12}(p)}{p\big(\log\left(1/p\right)\big)^{1-1/\gamma}\mathrm{e}^{\left(\log(1/p)\right)^{1/\gamma}}} \leq \gamma\left(l_1^* + \varepsilon\right).\end{aligned}$$

Thus, due to the arbitrariness of ε and equation (3.20), we obtain that

$$I_1(p) \sim \gamma l_1^* p\left(\log\frac{1}{p}\right)^{1-1/\gamma}\mathrm{e}^{\left(\log(1/p)\right)^{1/\gamma}},$$

which together with (3.24) yield that

$$\phi_{X,Y}(p) \sim \gamma\mathrm{e}^{\left(\log(1/p)\right)^{1/\gamma}}\big(\log(1/p)\big)^{-1/\gamma}.$$

We next explore another important asymptotic independence structure beyond the scope of Assumption 3.1. Consider now the well-known Fréchet–Hoeffding lower bound copula defined as $W(u,v) := \max\{u+v-1, 0\}$. This copula has the asymptotic independence property defined in (2.6), but it does not satisfy Assumption 3.1, since its corresponding function $\widetilde{g}$ from (3.4) satisfies $\widetilde{g} \equiv 0$ and hence, $g^* = 0$. This dependence structure is analyzed in Proposition 3.1 and its asymptotic approximation for $\phi_{X,Y}(p)$ is shown to be totally different with that shown in Theorem 3.1, confirming one more time how sensitive the asymptotic behaviour of $\phi_{X,Y}(p)$ is with respect to the dependence between X and Y.

Proposition 3.1. *Assume that F and G are continuous, the copula of (X,Y) is given by W and $F \in \mathcal{R}_{-\alpha}$ for some $1 < \alpha \leq \infty$. Then,*

$$\phi_{X,Y}(p) \sim \frac{\alpha}{2\left(\alpha - 1\right)}\overline{F}^{\leftarrow}(p),$$

where $\alpha/(\alpha-1)$ is understood as 1 if $\alpha = \infty$.

Proof. Note first that

$$\mathbb{P}\left(\eta \leq v | \xi = u\right) = \frac{\partial\widehat{W}\left(u,v\right)}{\partial u} = \mathbf{1}_{\{1\geq v\geq 1-u\geq 0\}} + 0\cdot\mathbf{1}_{\{0\leq v<1-u\leq 1\}}.$$

Plugging this into $I(p)$ defined in (3.12), we have for any $0 < p < 1/4$ that

$$\begin{aligned}I(p) &= \int\limits_{p}^{(1-\sqrt{1-4p})/2} \overline{F}^{\leftarrow}(u)\mathrm{d}u + \int\limits_{(1+\sqrt{1-4p})/2}^{1} \overline{F}^{\leftarrow}(u)\mathrm{d}u \\ &\leq \left(\frac{1-\sqrt{1-4p}}{2} - p\right)\overline{F}^{\leftarrow}(p) + \frac{1-\sqrt{1-4p}}{2}\overline{F}^{\leftarrow}\left(\frac{1}{2}\right) \\ &= o(1)p\overline{F}^{\leftarrow}(p).\end{aligned}$$

As mentioned before Example 3.4, if $\overline{F} \in \mathcal{R}_{-\infty}$ then relation (3.17) still holds and should be read as $J(p) = o(1)p\overline{F}^{\leftarrow}(p)$. Hence, the above equation, (3.12) and (3.17) imply that

$$\int\limits_{0}^{\infty} \mathbb{P}\left(\xi \leq \overline{F}(x), \xi\eta \leq p\right)\mathrm{d}x \sim \frac{\alpha}{\alpha - 1} p\overline{F}^{\leftarrow}(p).$$

Thus, one may conclude our claim by recalling equation (3.11) and the fact that

$$\mathbb{P}\left(\xi\eta \leq p\right) = \int\limits_{0}^{1} \mathbb{P}\left(\eta \leq \frac{p}{u}\middle|\, \xi = u\right)\mathrm{d}u = \int\limits_{0}^{(1-\sqrt{1-4p})/2} \mathrm{d}u + \int\limits_{(1+\sqrt{1-4p})/2}^{1} \mathrm{d}u \sim 2p.$$

The proof is now complete. □

4. Main results under asymptotic dependence

This section investigates the extreme behaviour of the quantity defined in (1.1) under the asymptotic dependence assumption between X and Y. The following set of assumptions allows us to deliver explicit results.

Assumption 4.1. There exists a non-degenerate function $H : [0, \infty)^2 \to [0, \infty)$ such that the copula C of (X, Y) satisfies

$$\lim_{u\to 0} \frac{\widehat{C}(ux, uy)}{u} = H(x, y) \tag{4.1}$$

for every $(x, y) \in [0, \infty)^2$.

Note that the function H is homogeneous of order 1 and $\big(1/\overline{F}(X), 1/\overline{G}(Y)\big) = \big(1/\xi, 1/\eta\big)$ belongs to MRV_{-1} such that

$$\frac{1}{\overline{F}(t)}\mathbb{P}\left(\overline{F}(t)\left(\frac{1}{\xi}, \frac{1}{\eta}\right) \in \cdot\right) \xrightarrow{v} \nu(\cdot) \quad \text{as } t \to \infty \quad \text{on } [0, \infty]^2 \backslash \{\mathbf{0}\}, \tag{4.2}$$

where $\nu\big((x, \infty] \times (y, \infty]\big) := H\big(1/x, 1/y\big)$ for all $(x, y) \in [0, \infty]^2\backslash\{\mathbf{0}\}$ (see [3]).

Some typical copula examples satisfying Assumption 4.1 are given in the following Examples 4.1–4.3. Actually, one can verify that Assumption 4.1 is satisfied by many Archimedean copulas implying the asymptotic dependence (see [9] or [19]).

Example 4.1. The Fréchet–Hoeffding upper bound copula

$$C(u,v) = M(u,v) := \min\{u,v\}$$

satisfies Assumption 4.1 with $H(x,y) = \min\{x,y\}$.

Example 4.2. Consider the copula $C(u,v)$ such that

$$\widehat{C}(u,v) = \left(\max\left\{u^{-\theta} + v^{-\theta} - 1, 0\right\}\right)^{-1/\theta}, \quad \theta > 0,$$

i.e. the survival copula is the Clayton copula. Then $C(u,v)$ satisfies Assumption 4.1 with $H(x,y) = \left(x^{-\theta} + y^{-\theta}\right)^{-1/\theta}$.

Example 4.3. The Archimedean copula (for example, see (4.2.6) in Table 4.1 of [19])

$$C(u,v) = 1 - \left[(1-u)^{\theta} + (1-v)^{\theta} - (1-u)^{\theta}(1-v)^{\theta}\right]^{1/\theta}, \quad \theta > 1,$$

satisfies Assumption 4.1 with $H(x,y) = x + y - \left(x^{\theta} + y^{\theta}\right)^{1/\theta}$.

We are now ready to provide the main results of this section, which are given as Theorem 4.1.

Theorem 4.1. *If Assumption 4.1 holds with continuous F and G, then*

$$\lim_{p\downarrow 0} \frac{\phi_{X,Y}(p)}{\overline{F}^{\leftarrow}(\sqrt{p})} = \begin{cases} \int_0^{\infty} \frac{\nu((x,y):\ xy>1,\ x>z^{\alpha})}{\nu((x,y):\ xy>1)}\mathrm{d}z, & \text{if } F \in \mathcal{R}_{-\alpha} \text{ with } \alpha > 1, \\ 1, & \text{if } F \in \mathrm{MDA}(\Lambda). \end{cases}$$

Proof. Note first that our limit is the same as $\lim_{p\downarrow 0} \phi_{X,Y}\left(p^2\right)/\overline{F}^{\leftarrow}(p)$. Assume first that $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$. Clearly,

$$\begin{aligned} \phi_{X,Y}\left(p^2\right) &= \int_0^{\infty} \mathbb{P}\left(X > x|\xi\eta \le p^2\right)\mathrm{d}x \\ &= \int_0^{\infty} \mathbb{P}\left(\xi \le \overline{F}(x)|\xi\eta \le p^2\right)\mathrm{d}x \qquad (4.3) \\ &= \overline{F}^{\leftarrow}(p)\int_0^{\infty} \mathbb{P}\Big(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right)|\xi\eta \le p^2\Big)\mathrm{d}z, \end{aligned}$$

where the last step is due to an obvious change of variables. Now,

$$\mathbb{P}\Big(\xi\eta \le p^2\Big) \sim \nu((x,y):\ xy>1)p \qquad (4.4)$$

and

$$\mathbb{P}\Big(\xi \le pz, \xi\eta \le p^2\Big) \sim \nu\big((x,y):\ xy>1,\ x>z^{-1}\big)p, \quad z>0, \qquad (4.5)$$

hold due to (4.2) and Proposition A2.12 of [11], which are applied to the following two sets:

$$S_1 := \{(x,y):\ xy > 1\} \quad \text{and} \quad S_2 := \{(x,y):\ xy > 1,\ x > z^{-1}\}.$$

Note that the latter proposition could be applied since $\nu(\partial S_1) = \nu(\partial S_2) = 0$ holds. Note also that $\nu(\partial S_1) = 0$ is justified in the proof of Theorem 4.1(ii) of [3], while $\nu(\partial S_2) = 0$ is true because of $\nu(\partial S_1) = 0$ and the fact that $\nu\big(x = z^{-1}\big) = 0$ due to the uniform convergence of (2.1) on $[c, \infty)$ for any $c > 0$ (see Theorem 1.5.2 of [7]). In addition, for every $z > 0$, it follows from (2.1) that $\overline{F}\left(z\overline{F}^{\leftarrow}(p)\right) \sim pz^{-\alpha}$ and in turn, $(1-\varepsilon)pz^{-\alpha} \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right) \le (1+\varepsilon)pz^{-\alpha}$ holds for p in the right neighbourhood of 0 and any $0 < \varepsilon < 1$. Hence,

$$\begin{aligned}\mathbb{P}\left(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right), \xi\eta \le p^2\right) &\le \mathbb{P}\left(\xi \le (1+\varepsilon)pz^{-\alpha}, \xi\eta \le (1+\varepsilon)^2 p^2\right)\\ &\sim (1+\varepsilon)\nu\big((x,y):\ xy > 1,\ x > z^{\alpha}\big)p\end{aligned}$$

and

$$\begin{aligned}\mathbb{P}\left(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right), \xi\eta \le p^2\right) &\ge \mathbb{P}\left(\xi \le (1-\varepsilon)pz^{-\alpha}, \xi\eta \le (1-\varepsilon)^2 p^2\right)\\ &\sim (1-\varepsilon)\nu\big((x,y):\ xy > 1,\ x > z^{\alpha}\big)p,\end{aligned}$$

by keeping in mind (4.5). Thus, the arbitrariness of ε indicates that for every $z > 0$ we have

$$\mathbb{P}\left(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right), \xi\eta \le p^2\right) \sim \nu\big((x,y):\ xy > 1,\ x > z^{\alpha}\big)p. \tag{4.6}$$

Recall that $F \in \mathcal{R}_{-\alpha}$ and thus, one may apply the well-known Potter's bound (see Proposition 2.2.3 of [7]), which gives that

$$\frac{\mathbb{P}\left(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right), \xi\eta \le p^2\right)}{p} \le \frac{\overline{F}\left(z\overline{F}^{\leftarrow}(p)\right)}{p} \le 2z^{-\alpha'}$$

for every $1 < \alpha' < \alpha$, any p in the right neighbourhood of 0 and all $z > 1$. The latter and equation (4.4) imply that

$$\mathbb{P}\Big(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right) | \xi\eta \le p^2\Big) \le \mathbf{1}_{\{0<z\le 1\}} + \frac{2z^{-\alpha'}}{\nu((x,y):\ xy > 1)/2}\mathbf{1}_{\{z>1\}}.$$

The right-hand side of the above is integrable with respect to z over $(0, \infty)$ and therefore, one may apply the Dominated Convergence Theorem in (4.3). The latter, equations (4.4) and (4.6) lead to

$$\begin{aligned}\lim_{p\downarrow 0}\frac{\phi_{X,Y}\left(p^2\right)}{\overline{F}^{\leftarrow}(p)} &= \lim_{p\downarrow 0}\int_0^\infty \frac{\mathbb{P}\Big(\xi \le \overline{F}\left(z\overline{F}^{\leftarrow}(p)\right), \xi\eta \le p^2\Big)}{\mathbb{P}\Big(\xi\eta \le p^2\Big)}\mathrm{d}z\\ &= \int_0^\infty \frac{\nu\big((x,y):\ xy > 1,\ x > z^{\alpha}\big)}{\nu\big((x,y):\ xy > 1\big)}\mathrm{d}z.\end{aligned}$$

This justifies our first claim for $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$.

It remains to prove the second case where $F \in \mathrm{MDA}(\Lambda)$. Let a be the corresponding scaling function defined in (2.2). Clearly,

$$\phi_{X,Y}\left(p^2\right) = \left(\int\limits_0^{\overline{F}^{\leftarrow}(p)} + \int\limits_{\overline{F}^{\leftarrow}(p)}^{\infty} \right) \mathbb{P}\left(\xi \le \overline{F}(x) | \xi\eta \le p^2\right) \mathrm{d}x =: K_1(p) + K_2(p). \tag{4.7}$$

A straightforward change of variables shows that

$$\begin{aligned} K_2(p) &= a\left(\overline{F}^{\leftarrow}(p)\right) \int\limits_0^{\infty} \mathbb{P}\left(\xi \le \overline{F}\left(\overline{F}^{\leftarrow}(p) + a\left(\overline{F}^{\leftarrow}(p)\right) z\right) \Big| \xi\eta \le p^2\right) \mathrm{d}z \\ &\le a\left(\overline{F}^{\leftarrow}(p)\right) \int\limits_0^{\infty} \frac{\mathbb{P}\left(\xi \le \overline{F}\left(\overline{F}^{\leftarrow}(p) + a\left(\overline{F}^{\leftarrow}(p)\right) z\right)\right)}{\mathbb{P}\left(\xi\eta \le p^2\right)} \mathrm{d}z \\ &= a\left(\overline{F}^{\leftarrow}(p)\right) \frac{p}{\mathbb{P}\left(\xi\eta \le p^2\right)} \int\limits_0^{\infty} \frac{\overline{F}\left(\overline{F}^{\leftarrow}(p) + a\left(\overline{F}^{\leftarrow}(p)\right) z\right)}{p} \mathrm{d}z. \end{aligned} \tag{4.8}$$

Lemma 2.2 of [21] or relation (5.7) of [13] implies that

$$\frac{\overline{F}\left(\overline{F}^{\leftarrow}(p) + a\left(\overline{F}^{\leftarrow}(p)\right) z\right)}{p} \le (1+\epsilon)(1+\epsilon z)^{-1/\epsilon},$$

for every $0 < \epsilon < 1$, all p in the right neighbourhood of 0 and all $z > 0$. The right-hand side of the above is integrable with respect to z over $(0, \infty)$. Thus, the Dominated Convergence Theorem could be applied in (4.8), which together with relations (2.2) and (4.4) lead to

$$\limsup_{p \downarrow 0} \frac{K_2(p)}{\overline{F}^{\leftarrow}(p)} \le \lim_{p \downarrow 0} \frac{a\left(\overline{F}^{\leftarrow}(p)\right)}{\overline{F}^{\leftarrow}(p)} \Big(\nu\big((x,y):\ xy > 1\big)\Big)^{-1} \int\limits_0^{\infty} \mathrm{e}^{-z} \mathrm{d}z = 0, \tag{4.9}$$

since $a(t) = o(t)$ as $t \to \infty$ (see [11]).

We next focus on $K_1(p)$ and for every $s > 0$, we may write that

$$\begin{aligned} K_1(p) &= \overline{F}^{\leftarrow}(p) - \left(\int\limits_0^{\overline{F}^{\leftarrow}(p) - sa\left(\overline{F}^{\leftarrow}(p)\right)} + \int\limits_{\overline{F}^{\leftarrow}(p) - sa\left(\overline{F}^{\leftarrow}(p)\right)}^{\overline{F}^{\leftarrow}(p)} \right) \mathbb{P}\left(\xi > \overline{F}(x) | \xi\eta \le p^2\right) \mathrm{d}x \\ &=: \overline{F}^{\leftarrow}(p) - K_{11}(p,s) - K_{12}(p,s). \end{aligned} \tag{4.10}$$

Clearly,

$$\begin{aligned} \frac{K_{11}(p,s)}{\overline{F}^{\leftarrow}(p)} &\le \frac{\overline{F}^{\leftarrow}(p) - sa\left(\overline{F}^{\leftarrow}(p)\right)}{\overline{F}^{\leftarrow}(p)} \times \frac{\mathbb{P}\left(\eta \le p^2 / \overline{F}\left(\overline{F}^{\leftarrow}(p) - sa\left(\overline{F}^{\leftarrow}(p)\right)\right)\right)}{\mathbb{P}\left(\xi\eta \le p^2\right)} \\ &\le \frac{p}{\mathbb{P}\left(\xi\eta \le p^2\right)} \times \frac{p}{\overline{F}\left(\overline{F}^{\leftarrow}(p) - sa\left(\overline{F}^{\leftarrow}(p)\right)\right)}. \end{aligned}$$

Equations (2.2) and (4.4) suggest that

$$\lim_{s \to \infty} \limsup_{p \downarrow 0} \frac{K_{11}(p,s)}{\overline{F}^{\leftarrow}(p)} \le \lim_{s \to \infty} \Big(\nu\big((x,y):\ xy > 1\big)\Big)^{-1} \mathrm{e}^{-s} = 0.$$

Further,

$$\lim_{s\to\infty}\limsup_{p\downarrow 0}\frac{K_{12}(p,s)}{\overline{F}^{\leftarrow}(p)} \le \lim_{s\to\infty}\limsup_{p\downarrow 0}\frac{sa\left(\overline{F}^{\leftarrow}(p)\right)}{\overline{F}^{\leftarrow}(p)} = 0,$$

since $a(t)=o(t)$ as $t\to\infty$. Plugging the last two equations into (4.10) gives $K_1(p)\sim\overline{F}^{\leftarrow}(p)$, which together with (4.7) and (4.9) yield our second claim, i.e. $\phi_{X,Y}\left(p^2\right)\sim\overline{F}^{\leftarrow}(p)$ whenever $F\in\mathrm{MDA}(\Lambda)$. The proof is now complete. □

One may check that Theorem 4.1 provides transparent results for $F\in\mathcal{R}_{-\alpha}$ with $\alpha>1$ under the frameworks given by Examples 4.1–4.3. We just specify the details about Examples 4.1 and 4.2 in the following remark for later use.

Remark 4.1. Recalling Example 4.1, if (X,Y) follows the Fréchet–Hoeffding upper bound copula, then Assumption 4.1 holds with $H(x,y)=\min\{x,y\}$. Thus,

$$\frac{\nu\left(\mathrm{d}x\times(y,\infty]\right)}{\mathrm{d}x} = -\frac{\partial H(1/x,1/y)}{\partial x} = x^{-2}\frac{\partial H(1/x,1/y)}{\partial\left(x^{-1}\right)} = x^{-2}\mathbf{1}_{\{x\ge y>0\}}+0\cdot\mathbf{1}_{\{0<x<y\}}.$$

Hence,

$$\nu((x,y):\ xy>1) = \int_0^\infty \nu\left(\mathrm{d}x\times(1/x,\infty]\right) = \int_1^\infty \frac{1}{x^2}\mathrm{d}x = 1.$$

Additionally, for $\alpha>1$,

$$\begin{aligned}\int_0^\infty \nu\left((x,y):\ xy>1,\ x>z^\alpha\right)\mathrm{d}z &= \int_0^\infty\int_{z^\alpha}^\infty \nu\left(\mathrm{d}x\times(1/x,\infty]\right)\mathrm{d}z\\ &= \int_0^1\int_1^\infty \frac{1}{x^2}\mathrm{d}x\mathrm{d}z + \int_1^\infty\int_{z^\alpha}^\infty \frac{1}{x^2}\mathrm{d}x\mathrm{d}z\\ &= \frac{\alpha}{\alpha-1}.\end{aligned}$$

Consequently, Theorem 4.1 tells us that

$$\lim_{p\downarrow 0}\frac{\phi_{X,Y}(p)}{\overline{F}^{\leftarrow}\left(\sqrt{p}\right)} = \begin{cases}\dfrac{\alpha}{\alpha-1}, & \text{if } F\in\mathcal{R}_{-\alpha} \text{ with } \alpha>1,\\ 1, & \text{if } F\in\mathrm{MDA}(\Lambda).\end{cases} \tag{4.11}$$

Moreover, if (X,Y) follows the copula specified in Example 4.2, then Assumption 4.1 holds with $H(x,y)=\left(x^{-\theta}+y^{-\theta}\right)^{-1/\theta}$. By the same idea as above and more cumbersome computations, we can obtain from Theorem 4.1 that

$$\lim_{p\downarrow 0}\frac{\phi_{X,Y}(p)}{\overline{F}^{\leftarrow}\left(\sqrt{p}\right)} = \begin{cases}\dfrac{\Gamma\left(\frac{\alpha+1}{2\alpha\theta}+1\right)\Gamma\left(\frac{\alpha-1}{2\alpha\theta}\right)}{\Gamma\left(\frac{1}{2\theta}+1\right)\Gamma\left(\frac{1}{2\theta}\right)}, & \text{if } F\in\mathcal{R}_{-\alpha} \text{ with } \alpha>1,\\ 1, & \text{if } F\in\mathrm{MDA}(\Lambda).\end{cases}$$

It is interesting to note that, within the structure of the copula M, relation (4.11) is valid for all $F\in\mathcal{R}_{-\infty}$, which is a weaker condition than $F\in\mathrm{MDA}(\Lambda)$. We summarize this finding in the next proposition.

Proposition 4.1. *Assume that F and G are continuous functions, $F \in \mathcal{R}_{-\alpha}$ for some $1 < \alpha \le \infty$ and the copula of (X, Y) is given by M. Then,*

$$\phi_{X,Y}(p) \sim \frac{\alpha}{\alpha - 1} \overline{F}^{\leftarrow}(\sqrt{p}),$$

where $\alpha/(\alpha - 1)$ is understood as 1 in case $\alpha = \infty$.

Proof. Our main reasoning is based on relation (3.12) and the Karamata-type results displayed in Lemma 3.2. Clearly,

$$\mathbb{P}(\eta \le v | \xi = u) = \frac{\partial \widehat{M}(u,v)}{\partial u} = \mathbf{1}_{\{1 \ge v \ge u \ge 0\}} + 0 \cdot \mathbf{1}_{\{0 \le v < u \le 1\}},$$

which in turn gives that

$$I(p) = \int_p^{\sqrt{p}} \overline{F}^{\leftarrow}(u)\mathrm{d}u = \int_{1/\sqrt{p}}^{\infty} x^{-2}\overline{F}^{\leftarrow}\left(x^{-1}\right)\mathrm{d}x - \int_{1/p}^{\infty} x^{-2}\overline{F}^{\leftarrow}\left(x^{-1}\right)\mathrm{d}x. \tag{4.12}$$

Since $(\cdot)^{-2}\overline{F}^{\leftarrow}(1/\cdot) \in \mathcal{R}_{1/\alpha - 2}$ with $1/\alpha - 2 < -1$, Lemma 3.2(ii) yields

$$\int_{1/\sqrt{p}}^{\infty} x^{-2}\overline{F}^{\leftarrow}\left(x^{-1}\right)\mathrm{d}x \sim \frac{\alpha\sqrt{p}}{\alpha - 1}\overline{F}^{\leftarrow}(\sqrt{p}) \quad \text{and} \quad \int_{1/p}^{\infty} x^{-2}\overline{F}^{\leftarrow}\left(x^{-1}\right)\mathrm{d}x \sim \frac{\alpha p}{\alpha - 1}\overline{F}^{\leftarrow}(p) = o(1)\sqrt{p}\overline{F}^{\leftarrow}(\sqrt{p}).$$

Plugging these estimates into (4.12) leads to

$$I(p) \sim \frac{\alpha}{\alpha - 1}\sqrt{p}\overline{F}^{\leftarrow}(\sqrt{p}).$$

Recall that (3.17) holds if $1 < \alpha < \infty$ and $J(p) = o(1)p\overline{F}^{\leftarrow}(p)$ if $\alpha = \infty$ due to the arguments given before Example 3.4. Thus, (3.12) and the above equation give that

$$\int_0^{\infty} \mathbb{P}\left(\xi \le \overline{F}(x), \xi\eta \le p\right)\mathrm{d}x \sim \frac{\alpha}{\alpha - 1}\sqrt{p}\overline{F}^{\leftarrow}(\sqrt{p}).$$

Finally,

$$\mathbb{P}(\xi\eta \le p) = \int_0^1 \mathbb{P}\left(\eta \le \frac{p}{u} \middle| \xi = u\right)\mathrm{d}u = \int_0^{\sqrt{p}} \mathrm{d}u = \sqrt{p}.$$

Equation (3.11) and the very last two relations confirm our claim. □

5. Numerical discussions

The previous two sections have investigated the limiting behaviour of $\phi_{X,Y}(p)$ under various assumptions. The general result could be stated as follows:

$$\phi_{X,Y}(p) = \mathbb{E}\left[X | \overline{F}(X)\overline{G}(Y) \le p\right] \sim A \times r(p),$$

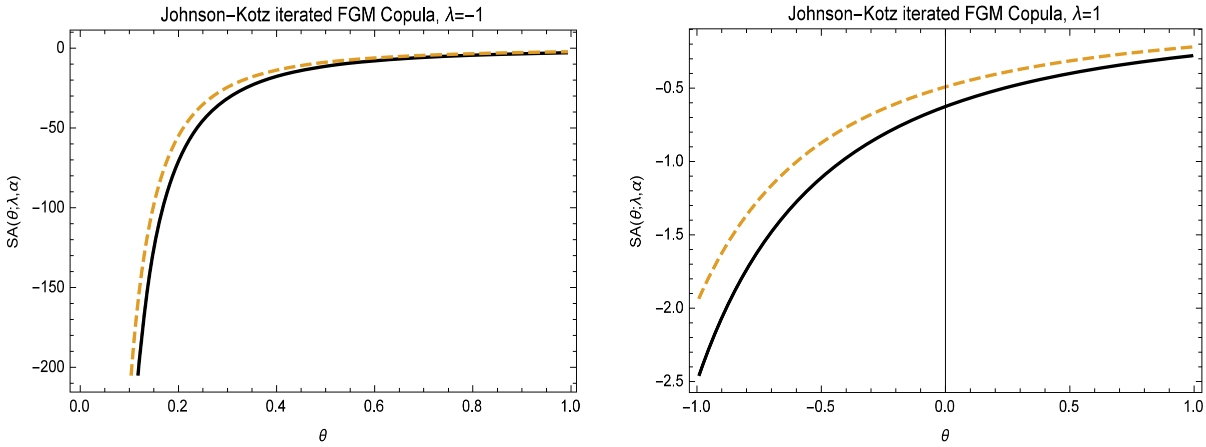

Fig. 1. SA for Example 3.1 with $\alpha = 2$ (solid line) and $\alpha = 5$ (dashed line).

where r and A are the rate of convergence and its corresponding asymptotic constant that both depend on the tail behaviour of copula C and marginal risk X. Our aim is now to understand the stability of our asymptotic results and discuss the pros and cons of the available estimates. While Monte-Carlo simulations may identify the speed of convergence for some specific dependence models, we choose to interpret our results from a different perspective. That is, we aim to understand the parameter risk or in other words, how sensitive the results are with respect to the choice model parameters, which could be estimated or obtained via expert-opinion. This exercise is also known as *sensitivity analysis* (SA). Our numerical illustrations consider the SA with respect to the dependence model parameters, since the choice of the dependence model is of crucial importance, as we noticed in Sections 3 and 4.

The case in which A is a positive constant would be considered as a safeguard, since the choice of the dependence model does not have a huge impact over the asymptotic approximation and accurate marginal models would become the primary interest. If A depends upon the dependence model, then it is imperative to perform a SA in order to understand the priorities for the model validation process.

If $F \in \mathcal{R}_{-\alpha}$ with $1 < \alpha < \infty$ and C is as in Example 3.1, then Theorem 3.1 tells us that

$$A_1(\theta, \lambda; \alpha) := \alpha \times \frac{\alpha^3(\theta+\lambda+1) + \alpha^2(\theta+\lambda+3) + \alpha(2-2\theta-3\lambda) + \lambda}{(\alpha-1)(\alpha+1)(\alpha+2)(\theta+\lambda+1)}.$$

The SA is just the derivative of $A_1(\theta, \lambda; \alpha)$ with respect to the parameter of interest, i.e. θ and λ, respectively. Figs. 1 and 2 illustrate the SA for the two parameters. Fig. 2 tells us that one should be careful when estimating the parameter λ, irrespective of the estimate for θ. Fig. 1 is even more suggestive and shows that a low estimated value for λ increases the estimation error for our asymptotic approximations; the SA results when $\lambda = -1$ illustrate a huge change in value of our estimates. Examples 3.2 and 3.3 lead to the same asymptotic constants and we have

$$A_2(\theta; \alpha) := \alpha^2 \times \frac{\alpha\theta + \alpha - \theta + 1}{(\alpha-1)(\alpha+1)(\theta+1)}.$$

Fig. 3 shows that our asymptotic estimates are very sensitive to the change in θ estimate.

As mentioned in Section 1, the MES is an alternative tail risk measure that has been discussed in the literature, namely $\mathbb{E}\left[X|\overline{G}(Y) \le p\right]$ (for details, see [5] and references therein). If $\overline{F}(t) = O\big(\overline{G}(t)\big)$ as $t \to \infty$ and the limit

$$\lim_{t\to\infty} \mathbb{P}(X > tx | Y > t) =: h(x) \in [0, 1]$$

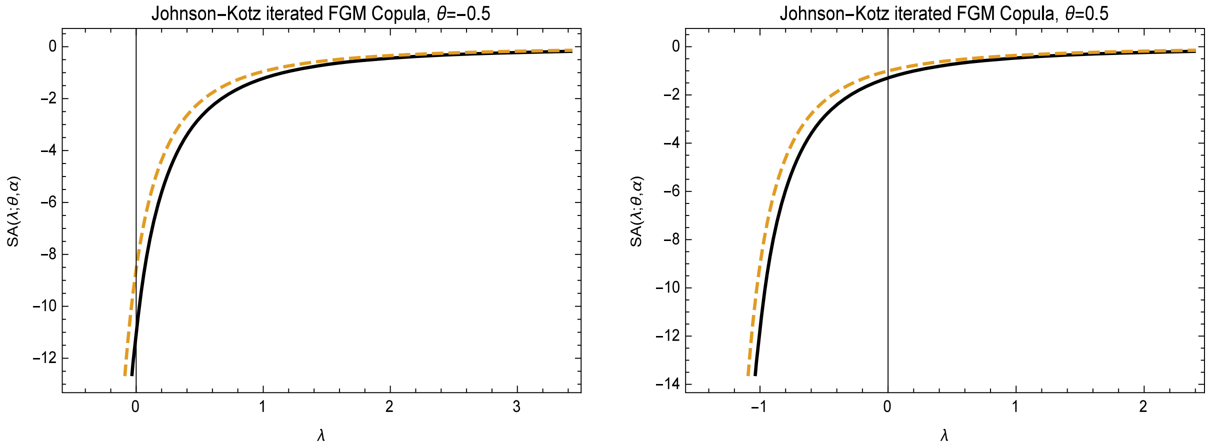

Fig. 2. SA for Example 3.1 with $\alpha = 2$ (solid line) and $\alpha = 5$ (dashed line).

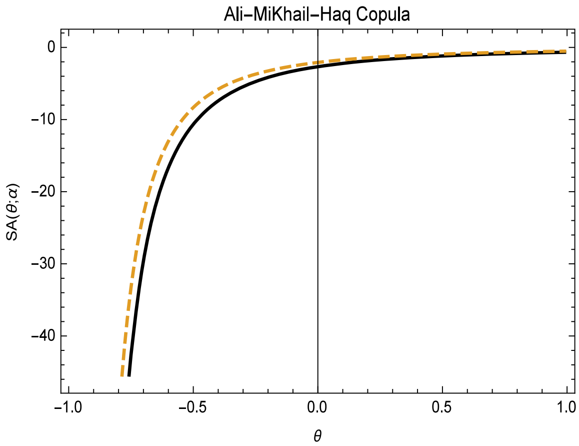

Fig. 3. SA for Example 3.2 with $\alpha = 2$ (solid line) and $\alpha = 5$ (dashed line).

exists almost everywhere for $x > 0$, then one may use Theorem 3.1 of [5] to find that

$$\lim_{p\downarrow 0} \frac{1}{\overline{G}^{\leftarrow}(p)} \mathbb{E}\left[X|\overline{G}(Y) \le p\right] = \int_0^{\infty} h(x)\,\mathrm{d}x. \tag{5.1}$$

By Lemma 3.1(ii) of [5], if the asymptotic independence occurs between X and Y, then in most cases $h(x) = 0$ for all $x > 0$, which is not fit for the estimation purpose. This shows the advantage of using our $\phi_{X,Y}(p) = \mathbb{E}\left[X|\overline{F}(X)\overline{G}(Y) \le p\right]$ as the tail risk measure over the well-known tail risk measure $\mathbb{E}\left[X|\overline{G}(Y) \le p\right]$.

The asymptotic independence has been assumed in the previous examples and therefore, we turn our attention towards the asymptotic dependence case as discussed in Theorem 4.1. Recall that $A = 1$ if $F \in MDA(\Lambda)$, which makes the SA superfluous and thus, we further assume that $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$. Assume that (X, Y) follows the copula specified in Example 4.2, i.e. $\widehat{C}(u, v) = \left(\max\left\{u^{-\theta} + v^{-\theta} - 1, 0\right\}\right)^{-1/\theta}$ for some $\theta > 0$. Then, Assumption 4.1 holds with $H_{Cl}(x, y; \theta) = \left(x^{-\theta} + y^{-\theta}\right)^{-1/\theta}$ and Remark 4.1 tells us that

$$A_3(\theta; \alpha) := \frac{\Gamma\left(\frac{\alpha+1}{2\alpha\theta} + 1\right)\Gamma\left(\frac{\alpha-1}{2\alpha\theta}\right)}{\Gamma\left(\frac{1}{2\theta} + 1\right)\Gamma\left(\frac{1}{2\theta}\right)}.$$

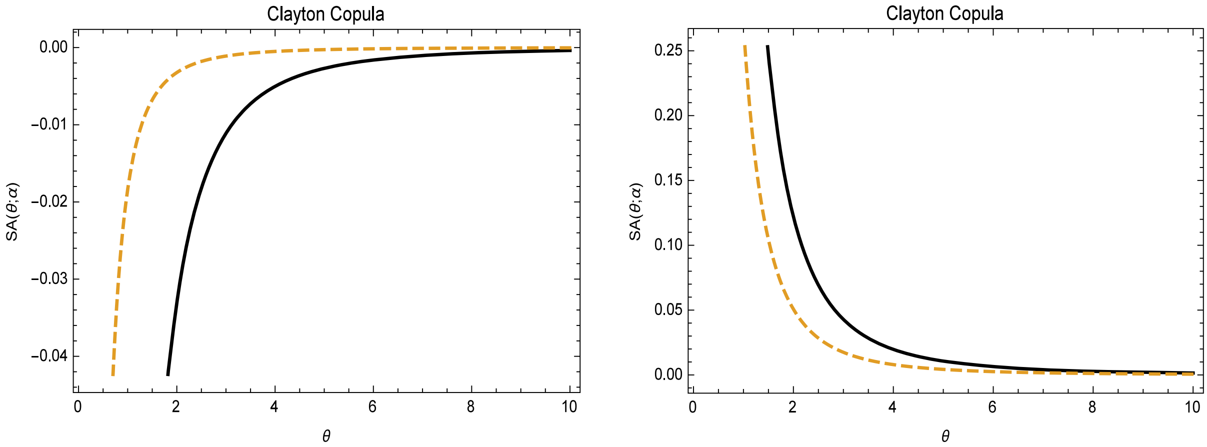

Fig. 4. SA for the Clayton copula for Theorem 4.1 (left) and (5.2) (right) with $\alpha = 2$ (solid line) and $\alpha = 5$ (dashed line).

Now, if the asymptotic dependence follows as in Assumption 4.1, then one may use relations (2.1), (2.3) and (4.1) to conclude that the asymptotic constant from (5.1) is given by

$$A_3^*(\theta;\alpha) := \begin{cases} \int_0^\infty H\big(x^{-\alpha}, 1\big)\,\mathrm{d}x, & \text{if } \ F \in \mathcal{R}_{-\alpha} \ \text{ with } \alpha > 1, \\ 1, & \text{if } \ F \in \mathrm{MDA}(\Lambda), \end{cases}$$

provided that $\overline{F}(t) \sim \overline{G}(t)$ as $t \to \infty$. Clearly, the above is reduced to

$$A_3^*(\theta;\alpha) = \int_0^\infty H_{Cl}\big(x^{-\alpha}, 1; \theta\big)\,\mathrm{d}x = \frac{\Gamma\left(1 + \frac{1}{\alpha\theta}\right)\Gamma\left(\frac{\alpha-1}{\alpha\theta}\right)}{\Gamma\left(\frac{1}{\theta}\right)}, \tag{5.2}$$

if $F \in \mathcal{R}_{-\alpha}$ with $\alpha > 1$. Fig. 4 shows a low sensitivity for the MES, while the SA for our proposed tail risk measure illustrates that the estimation error of parameter θ has very little impact over the asymptotic estimates. Once again, our proposed tail risk measure, i.e. $\mathbb{E}\left[X|\overline{F}(X)\overline{G}(Y) \le p\right]$ exhibits a lower sensitivity to the risk parameter as compared to the MES, i.e. $\mathbb{E}\left[X|\overline{G}(Y) \le p\right]$.

In a nutshell, we believe that the new tail risk measure has a great potential and our numerical illustrations have shown clear evidence of why one should consider (1.1) to compare the risk exposure of various individual risks.

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