Distributed Coordination of Asymmetric Compartmental Systems Under Time-Varying Inputs and Its Applications

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***Abstract*— In this paper, we present new distributed coordination for a set-point regulation of asymmetric compartmental systems under periodically time-varying external inputs. The feature of the proposed coordination is to adjust system parameters, which is directly related to interflow rate among compartments and outflow rate to environment. In contrast to symmetric compartmental systems, there is a difficulty in analysis resulted from a property of compartmental matrix in the asymmetric cases; however, we solve the problem by designing an appropriate coordination range of the system parameters. In order to compensate the time-varying effect in the inputs, we assume that the structure of the inputs is a finite Fourier series, and show that the coordination of the system parameters should have the same structure with the same frequencies. The remainder of the coordination is to determine coefficients of the Fourier series of the system parameters. We develop a distributed update law for obtaining the coefficients. We apply the distributed coordination to water irrigation systems and water distribution systems, and show the effectiveness of the proposed distributed coordination schemes through numerical simulations.**

***Index Terms*— Compartmental systems, distributed coordination, set-point regulation, time varying inputs, water distribution systems, water irrigation systems.**

I. Introduction

COMPARTMENTAL systems feature attribute (or energy) distribution or transportation in complex networks; therefore, these are modeled in various applications such as pharmacokinetics, chemical reaction, ecology [1], [2], epidemics [3], air traffic network [4], contaminant transport systems [5], cancer chemotherapy [6], distillation columns [7], water tank system [8], water irrigation systems [9], water distribution systems [10], and so on.

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Some mathematical properties of the compartmental systems have been studied. For example, all the states of the compartmental systems are nonnegative for all nonnegative initial conditions and input values [11]. Regarding stability property, all eigenvalues of the compartmental matrix of the compartmental systems are nonpositive. Especially, Fife [12] and Foster and Jacquez [13] have shown that the number of zero eigenvalues is equal to that of simple traps. Reachability and observability of the compartmental systems have been studied in [14], and condition of nonoscillatory solution has been presented in [15]. In [16] and [17], positive realization problem, i.e., the existence of realization for the transfer function of the compartmental system, has been analyzed.

There also have been synthesis problems of the compartmental systems. Bastin and Guffens [18] have raised the congestion problem of the compartmental systems, and have suggested a nonlinear output feedback controller preventing the congestion. In [19], a positive observer for the compartmental system has been presented. A model reference tracking via H_∞ control and sliding mode control has been studied in [20] and [21], respectively. As a set-point regulation problem, Bastin and Provost [22] have defined the set point as total state values, and proposed its positive feedback controller. The set point can be defined as the desired state of each compartment. With this definition, a distributed adjustment of external inputs of each compartment for the set-point regulation and its extension to noisy measurements have been studied in [11] and [23], respectively. Alternatively, in [24], the set-point regulation is achieved by the coordination of system parameters, which are related to interflow rate among compartments and outflow rate.

In this paper, we further discuss the distributed coordination scheme of [24]. It is notable that the result of [24] is limited to symmetric linear compartmental systems in which the interconnection graph is undirected and system parameters are symmetric. It is necessary to consider the asymmetrical case in order to extend applicability. Furthermore, one of underlying assumptions on the result of [24] is that the external inputs are fixed as constant. However, the inflow rate can be time-varying in real situations. For example, in water distribution systems [10], the external inputs are water supply from fresh water resources, and the amount of water supply varies seasonally. Thus, it is necessary to extend the result of [24] to the case of time-varying external inputs.

The contributions of this paper are summarized as follows. First, we deal with the distributed coordination for the set-point regulation in the asymmetric linear compartmental systems. Furthermore, we show that distributed coordination is still effective in the case that the coordination of some system parameters is saturated due to physical constraint of the system parameters. In contrast to symmetric compartmental systems, the compartmental matrix of the asymmetric compartmental system may not be negative definite, which makes the analysis of the coordination rule difficult; however, we solve the problem by designing a coordination range of the system parameters, which is closely related to the property of the compartmental matrix.

Second, we deal with the distributed coordination of system parameters in the linear compartmental systems under periodically time-varying external inputs. Assuming that the periodic inputs are represented as a finite Fourier series, we show that the solutions of coordination of the system parameters are also the finite Fourier series with the same frequencies. Then, the coordination problem becomes finding the corresponding coefficients of the Fourier series of the system parameter. We develop a distributed update law for obtaining the coefficients. Actually, it is difficult to analyze the effectiveness of the coordination when the coordination is saturated due to the physical constraint. Alternatively, we find the range of the coefficients of the finite Fourier series, so that it prevents the system parameters from being saturated, and then, we give a constraint to the update law of the coefficients, such that the update is allowed within the range. We analytically show that the set-point regulation problem is still solved despite the constraint of the update law.

Third, we apply the distributed coordination to water irrigation systems [9] and water distribution systems [10]. The set-point regulation of water level is an important issue in the water irrigation systems, because superfluous or deficient water supply to farms can result in the failure of the regulation of water level. By regulating the desired water level, the best operational efficiency can be achieved. Because the water irrigation system is an asymmetric compartmental system, the proposed coordination rule designed for the asymmetric compartmental systems in this paper is applied. In water distribution systems, it is necessary to regulate the appropriate head of the storage reservoir in order to provide hydraulic capacitance and sufficient head of water for the water distribution. In dry season, however, it is difficult to obtain sufficient water from the fresh water source such as a river. In this paper, we use the coordination scheme designed for time-varying external inputs case in order to cope with the variational water supply from the fresh water source.

This paper is organized as follows. In Section II, basics of the compartmental model and its mathematical properties are given. In Section III, the coordination scheme designed for asymmetric linear compartmental systems is presented. In Section IV, the coordination scheme designed for the set-point regulation under the periodically time-varying inputs is presented. In Section V, simulation results for the water irrigation systems and the water distribution systems are presented. In Section VI, we conclude this paper.

II. Preliminaries

The following notations are used. $\bar{\mathbb{R}}_+$ denotes the nonnegative real part. Then, we write $x \in \bar{\mathbb{R}}_+$ to indicate that the scalar x is nonnegative. Similarly, in vector case, $\bar{\mathbb{R}}^n_+$ denotes the nonnegative orthants of $\mathbb{R}^n$, and $y \in \bar{\mathbb{R}}^n_+$ means that the vector y is nonnegative, i.e., every vector component of y is nonnegative, which is denoted by $y \geq\geq 0$. In function case, a real function $z : [0, T] \to \mathbb{R}^n$ is a nonnegative if $z(t) \geq\geq 0$ on the interval $[0, T]$. Given a matrix $A \in \mathbb{R}^{n\times n}$, A^T and $\rho(A)$ denote the transpose and the spectral radius of A, respectively. In addition, we define the symmetric part of A as $A^S \triangleq (1/2)(A + A^T)$.

Given a compartmental system with n individual compartments, the interconnection topology is represented as a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{v_1, \ldots, v_n\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. The adjacency matrix $\mathcal{A} \equiv \{a_{ij}\} \in \mathbb{R}^{n\times n}$ is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in \mathcal{E}, \quad i \neq j \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The adjacency matrix $\mathcal{A}$ is algebraic representation of the graph $\mathcal{G}$.

The state-space model of the ith compartment is

$$\dot{x}_i = -\left(c_{ei}a_{ei} + \sum_{j\neq i} c_{ji}a_{ji}\right)x_i + \sum_{j\neq i} c_{ij}a_{ij}x_j + a_{si}I_i \tag{2}$$

for $i = 1, \ldots, n$, where $x_i \in \mathbb{R}$, $I_i \in \bar{\mathbb{R}}_+$, $c_{ei}, c_{ij} \in \bar{\mathbb{R}}_+$, and $a_{ij} \in \{0, 1\}$ are state, external input, system parameters, and an element of the adjacency matrix $\mathcal{A}$, respectively. In addition, $a_{ei}, a_{si} \in \{0, 1\}$ means the connection of the ith compartment to an environment and external input source, respectively. The physical meaning of the system model (2) is explained as follows. The first term is the outflow rate of the ith compartment, the second term is the inflow rate from neighbor compartments, and the last term is the inflow rate from the external input source. Note that the system (2) is said to be symmetric if the interconnection graph is an undirected graph and the system parameters are symmetric, i.e., $a_{ij} = a_{ji}$ and $c_{ij} = c_{ji}$. In the asymmetric compartmental systems, the symmetrical property is no longer preserved, i.e., $a_{ij} \neq a_{ji}$ or $c_{ij} \neq c_{ji}$. The system model (2) can be compactly expressed as

$$\dot{x} = Mx + BI \tag{3}$$

where $x = [x_1, x_2, \ldots, x_n]^T$, $I = [I_1, I_2, \ldots, I_n]^T$, $B = \mathrm{diag}(a_{s1}, a_{s2}, \ldots, a_{sn})$, and

$$M = \begin{bmatrix} -c_{e1}a_{e1} - \sum_{j\neq 1} c_{j1}a_{j1} & c_{12}a_{12} & \cdots \\ c_{21}a_{21} & -c_{e2}a_{e2} - \sum_{j\neq 2} c_{j2}a_{j2} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \tag{4}$$

which is said to be the compartmental matrix and satisfies the following mathematical properties.

Property 1:

1) M is a Metzler matrix, i.e., nonzero matrix with nonnegative off-diagonal elements.
2) M has nonpositive diagonal elements: $m_{ii} \leq 0$.
3) M is (columnwise) diagonally dominant: $|m_{ii}| - \sum_{j \neq i} m_{ji} \geq 0$.

From the above properties, it can be shown that the compartmental matrix M has all nonpositive eigenvalues. Further results on the stability are summarized as follows.

Definition 1: A trap is a compartment or a set of compartments from which there are no outflows to the environments and to compartments that are not in the trap.

Definition 2: A simple trap is a trap that has no traps inside it.

Proposition 1 [12]: Let M be the compartmental matrix for a linear compartmental system (3). Then, M is singular if and only if the system has a trap.

Proposition 2 [13]: Let M be the compartmental matrix for a linear compartmental system (3). Then, zero is an eigenvalue of M of multiplicity m if and only if the corresponding compartmental system has m simple traps.

Proposition 3 [3]: Let M be the compartmental matrix for a linear compartmental system (3). Then, M is nonsingular if and only if M has all negative eigenvalues.

In addition, the compartmental system (2) is nonnegative with the following definition.

Definition 3: The linear compartmental system given by (3) is nonnegative if for every $x(0) \in \bar{\mathbb{R}}^n_+$ and $I \geq\geq 0$ for $t \geq 0$, the solution $x(t)$ for $t \geq 0$ to (3) is nonnegative.

Proposition 4 [11]: The linear compartmental system given by (3) is nonnegative.

Suppose that the system parameters c_{ei} and c_{ij} are continuously time-varying, then the compartmental system (2) becomes

$$\dot{x}_i = -\left(c_{ei}(t)a_{ei} + \sum_{j \neq i} c_{ji}(t)a_{ji}\right) x_i + \sum_{j \neq i} c_{ij}(t)a_{ij}x_j + a_{si}I_i \tag{5}$$

and its compact form becomes

$$\dot{x} = M(t)x + BI. \tag{6}$$

The following proposition states that the nonnegativeness is also preserved:

Proposition 5 [11]: The linear time-varying compartmental system (6) is nonnegative.

Note that in what follows, for simplicity of notation, we drop the time argument t where there is no ambiguity, e.g., $c_{ei}(t) = c_{ei}$ and $c_{ij}(t) = c_{ij}$.

III. Distributed Coordination of Asymmetric Compartmental Systems

In this section, we formally state the set-point regulation problem of the asymmetric compartmental systems as follows.

Problem 1: Given constant external inputs $I_i \in \bar{\mathbb{R}}_+$, design a distributed coordination rule for the system parameters c_{ei} and c_{ij}, such that the asymmetric compartmental system (5) converges to desired points x_i^d. Here, the system parameters should be adjusted in the ranges of $c_{ei} \in [\underline{c_{ei}}, \overline{c_{ei}}]$ and $c_{ij} \in [\underline{c_{ij}}, \overline{c_{ij}}]$.

There are several comments on the formal problem statement. First, there are two control approaches to achieve the regulation of the desired points x_i^d. One is to adjust the inflow rate of each compartment [11]. The other one, as our control approach, is to adjust interflow rate among compartments and outflow rate, which is implemented by coordinating the system parameters c_{ei} and c_{ij}. The difference between the two approaches can be illustrated by the water level control of the water tank system. The former control approach is that there are direct input channels in each water tank, and water supply through the channels is controlled. In the latter one, controllable valves are installed in each connection channel and adjust water flow rate.

The second comment is that the system parameters c_{ei} and c_{ij} are coordinated in a distributed way. Let $\tilde{c}_{ei}$ and $\tilde{c}_{ij}$ be a solution of the set-point regulation problem, i.e., they satisfy the following constrained linear equations:

$$-\left(\tilde{c}_{ei}a_{ei} + \sum_{j \neq i} \tilde{c}_{ji}a_{ji}\right)x_i + \sum_{j \neq i} \tilde{c}_{ij}a_{ij}x_j + a_{si}I_i = 0 \tag{7}$$

with constraints

$$\tilde{c}_{ei} \in [\underline{c_{ei}}, \overline{c_{ei}}], \tilde{c}_{ij} \in [\underline{c_{ij}}, \overline{c_{ij}}] \quad i, j = 1, \ldots, n, i \neq j \tag{8}$$

where $\underline{c_{ei}} \geq 0$ and $\underline{c_{ij}} \geq 0$. Then, *Problem 1* boils down to finding the solutions $\tilde{c}_{ei}$ and $\tilde{c}_{ij}$. Equation (7) can be directly solved in a centralized way; however, the computation may be burdensome in the case that the compartmental system becomes large-scaled. Furthermore, the equation with constraint (8) is not simple to solve. Motivated by these observations, we design a distributed coordinator for the system parameters, from which parallel computation can be implemented. Furthermore, the coordinators should use only the local information of the compartmental system and the communication graph should be consistent with the interaction graph $\mathcal{G}$.

The third comment is that the designed coordination rule for the set-regulation problem should be effective in teh case that the system parameters are under the physical constraints of $c_{ei} \in [\underline{c_{ei}}, \overline{c_{ei}}]$ and $c_{ij} \in [\underline{c_{ij}}, \overline{c_{ij}}]$. The interflow and outflow rates cannot be negative and increase without bound. We introduce some notations related to the constraints of the system parameters. Let us rewrite the system parameters in the vector form as $\mathbf{c} = [c_{e1}, \ldots, c_{12}, \ldots]^T \in \mathbb{R}^m$, where m is the number of the system parameters, and denote the solution of (7), and the lower and upper bounds in (8) by $\tilde{\mathbf{c}}$, $\underline{\mathbf{c}}$ and $\overline{\mathbf{c}}$, respectively. Then, the constraint (8) becomes $\underline{\mathbf{c}} \leq\leq \tilde{\mathbf{c}} \leq\leq \overline{\mathbf{c}}$. Let $M(\mathbf{c})$ be the compartmental matrix (4) with the system parameters $\mathbf{c}$, and let us define an interval matrix as $M_I \triangleq \{M(\mathbf{c}) | \underline{\mathbf{c}} \leq\leq \mathbf{c} \leq\leq \overline{\mathbf{c}}\}$. Then, constraint (8) becomes $M(\tilde{\mathbf{c}}) \in M_I$. With the interval matrix M_I, the symmetric part of M_I is defined as $M_I^S \triangleq \{(1/2)(M(\mathbf{c}) + M^T(\mathbf{c})) | \underline{\mathbf{c}} \leq\leq \mathbf{c} \leq\leq \overline{\mathbf{c}}\}$.

As the last comment, it is notable that Problem 1 has actually been solved in the case that the compartmental system is symmetric [24]. In this paper, we extend the existing results to the asymmetric compartmental systems. The reason for the extension is that there are several applications of the asymmetric compartmental systems, such as water irrigation system [9], which will be subsequently discussed in Section V.

The following assumptions are necessary for the analysis of the newly designed coordination rule.

Assumption 1: The compartmental system (5) has no trap.

Assumption 2: There exist solutions $\tilde{c}_{ei}$ and $\tilde{c}_{ij}$ in (7) satisfying constraint (8).

Assumption 3: The coordinator of c_{ei} can obtain the information of the ith compartment, and the coordinator of c_{ij} can obtain the information of both the ith and jth compartments.

Assumption 4: All elements of M_I^S are negative definite.

Assumption 1 makes the system (5) be asymptotically stable by *Proposition 1* and *Proposition 3*. When the external input I_i is constant, each state x_i converges to a nonzero value. If we set the solutions as $\tilde{c}_{ei}$ and $\tilde{c}_{ij}$, the states x_i converge to the desired points x_i^d; thus, equation (7) is satisfied. *Assumption 2* is reasonable in practical application, because the coordination range of (8) is designed to be as wide as possible. *Assumption 3* is also practically reasonable, because it is consistent with the physical interconnection topology $\mathcal{G}$. Note that *Assumption 4* is necessary for the completion of mathematical analysis. The condition for *Assumption 4* will be discussed.

Let us define the error of the state variable and parameters, respectively, as

$$\begin{aligned} e_i &= x_i - x_i^d \\ \hat{c}_{ei} &= c_{ei} - \tilde{c}_{ei}, \quad \hat{c}_{ij} = c_{ij} - \tilde{c}_{ij}. \end{aligned}$$

Then, the following error dynamics of (5) can be derived from constraint (7):

$$\begin{aligned} \dot{e}_i = &-\left(c_{ei}a_{ei} + \sum_{j\neq i} c_{ji}a_{ji}\right)e_i + \sum_{j\neq i} c_{ij}a_{ij}e_j \\ &-\left(\hat{c}_{ei}a_{ei} + \sum_{j\neq i} \hat{c}_{ji}a_{ji}\right)x_i^d + \sum_{j\neq i} \hat{c}_{ij}a_{ij}x_j^d. \end{aligned} \quad (9)$$

As our main result, we present the distributed coordination rule that solves *Problem 1*.

Theorem 1: Suppose that *Assumptions 1–4* hold. Then, given a linear asymmetric compartmental system (5) and external input constants $I_i \in \bar{\mathbb{R}}_+$, the error dynamics (9) converges to zero under the following coordination rule:

$$\begin{aligned} \dot{c}_{ei} &= q_{ei}h_{c_{ei}}\left(a_{ei}x_i^d e_i\right), \quad i = 1, \ldots, n \\ \dot{c}_{ij} &= q_{ij}h_{c_{ij}}\left(a_{ij}x_j^d (e_j - e_i)\right), \quad i, j = 1, \ldots, n, i \neq j \end{aligned} \quad (10)$$

where q_{ei} and q_{ij} are positive constants and $h_a(b)$ is the state constraint function [25] defined as

$$h_a(b) = \begin{cases} 0 & \text{if } a = \overline{a} \text{ and } b > 0 \\ 0 & \text{if } a = \underline{a} \text{ and } b < 0 \\ b & \text{otherwise.} \end{cases} \quad (11)$$

Furthermore, $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \bar{\mathbb{R}}_+$.

Proof: See the Appendix. ∎

The analysis of *Theorem 1* is actually based on *Assumption 4*. In symmetric compartmental systems, the interval matrix M_I is symmetric; thus, the symmetric part of the interval matrix M_I becomes itself and is always negative definite. In asymmetric compartmental systems, the interval matrix M_I^S may be indefinite when c_{ij} are sufficiently large but c_{ei} are near to zero, because the dominance of diagonal elements [see *Property 1*.3)] of the interval matrix is no longer preserved. Therefore, it is necessary to reset the lower bound $\underline{\mathbf{c}}$ to be a positive vector if the lower bound $\underline{\mathbf{c}}$ is given as the zero vector. At the same time, the values of $\underline{\mathbf{c}}$ should be as small as possible in order to make a maximally wide coordination range of the system parameters. In this paper, we present the following optimization problem for designing the lower bound $\underline{\mathbf{c}}$:

$$\begin{aligned} &\text{minimize}_{\text{w.r.t. } \underline{\mathbf{c}}} \; \underline{\mathbf{c}}^T W_2 \underline{\mathbf{c}} \\ &\text{s.t. } \lambda_{\max}(M^S(\mathbf{c})) < 0 \quad \forall \mathbf{c} \in \Phi \\ &\qquad 0 \leq\leq \underline{\mathbf{c}} \leq\leq \overline{\mathbf{c}} \end{aligned} \quad (12)$$

where W_2 is a positive-definite diagonal weight matrix, $\lambda_{\max}(\cdot)$ is the maximum eigenvalue of the input matrix, and

$$\Phi \triangleq \{\mathbf{c} | c_{ei} = \underline{c_{ei}} \text{ or } \overline{c_{ei}}, \; c_{ij} = \underline{c_{ij}} \text{ or } \overline{c_{ij}}\}. \quad (13)$$

There are three comments on the above optimization problem. First, the decision variable in the optimization is the lower bound $\underline{\mathbf{c}}$. The constraint $\lambda_{\max}(M^S(\mathbf{c}))$ is actually a function of $\underline{\mathbf{c}}$ because of the definition of Φ ($\overline{c_{ei}}$ and $\overline{c_{ij}}$ are fixed values). Second, the constraint $\lambda_{\max}(M^S(\mathbf{c}))$ is convex with respect to $\underline{\mathbf{c}}$. The function $\lambda_{\max} : \mathbb{S}^n \rightarrow \mathbb{R}$ where $\mathbb{S}^n$ is the space of real symmetric matrix can be redefined as $\lambda_{\max}(X) = \sup\{y^T X y | \|y\|_2 = 1\}$, where $X \in \mathbb{S}^n$. Then, $\lambda_{\max}(M^S(\mathbf{c})) = \sup\{y^T M^S(\mathbf{c}) y | \|y\|_2 = 1\}$. It is easily shown that the term $y^T M^S(\mathbf{c}) y$ is an affine function, that is, convex function, with respect to each element of $\mathbf{c}$, and the supremum of the convex function with respect to y is also a convex function [26]. From the result, the optimization problem is convex optimization. As the last comment, the set of $\mathbf{c}$ such that $\lambda_{\max}(M^S(\mathbf{c})) < 0$ is a convex set. Therefore, we do not need to check *Assumption 4* for the whole region of $\underline{\mathbf{c}} \leq\leq \mathbf{c} \leq\leq \overline{\mathbf{c}}$. It is sufficient to check *Assumption 4* for the vertex set of the region, i.e., the set Φ, which is why we use the above constraint.

IV. Distributed Coordination of Compartmental Systems With Time-Varying External Inputs

In Section III, we dealt with the set-point problem of the asymmetrical systems with the fixed external inputs I_i. In this section, we further extend the problem into the case of time-varying external inputs $I_i(t)$. We first consider symmetrical compartmental systems for simplicity of analysis, and then, we discuss the asymmetrical case. The system model of the symmetrical compartmental system with time-varying inputs is

$$\dot{x}_i = -c_{ei}(t)a_{ei}x_i + \sum_{j\neq i} c_{ij}(t)a_{ij}(x_j - x_i) + a_{si}I_i(t) \quad (14)$$

for $i = 1, \ldots, n$. In this paper, we assume the external inputs $I_i(t)$ to be in the following form:

Assumption 5: The external inputs $I_i(t)$ in (14) is

$$I_i(t) = \gamma_i + \sum_{k=1}^{p} \left[\alpha_i^k \sin(\omega_k t) + \beta_i^k \cos(\omega_k t)\right] \tag{15}$$

where $\gamma_i, \alpha_i^k, \beta_i^k \in \mathbb{R}$ are the Fourier coefficients and $\omega_k \in \bar{\mathbb{R}}_+$ are the corresponding frequencies.

There are two comments on *Assumption* 5. First, the external inputs $I_i(t)$ are periodic functions. The disturbances are classified into diminishing and nondiminishing disturbances. An example of the former one is a temporary system failure or perturbation. The latter one persistently affects the system. The periodic external input is the nondiminishing disturbance with a certain pattern. For example, in water distribution systems [10], the external inputs are water supply from fresh water resources, and the amount of water supply varies according to changing from rainy season to dry season. Therefore, the corresponding flow rate also varies with the periodicity of a year. Second, we approximate the external inputs as finite Fourier series. Although the series cannot perfectly represent the actual inputs, the high accuracy can be achieved if we choose sufficiently large p in (15).

Suppose that there exist $\tilde{c}_{ei}(t)$ and $\tilde{c}_{ij}(t)$ that compensate the time-varying effect of $I_i(t)$ and achieve the regulation of the desired points x_i^d. The solution of $\tilde{c}_{ei}(t)$ and $\tilde{c}_{ij}(t)$ should satisfy the following equation:

$$-\tilde{c}_{ei}(t) a_{ei} x_i^d + \sum_{j \neq i} \tilde{c}_{ij}(t) a_{ij} (x_j^d - x_i^d) + a_{si} I_i(t) = 0 \tag{16}$$

with constraints

$$\underline{c_{ei}} \le \tilde{c}_{ei}(t) \le \overline{c_{ei}}, \ \underline{c_{ij}} \le \tilde{c}_{ij}(t) \le \overline{c_{ij}} \quad i, j = 1, \ldots, n, i \neq j. \tag{17}$$

From (16), we can derive the structure of the solutions $\tilde{c}_{ei}(t)$ and $\tilde{c}_{ij}(t)$. Equation (16) can be rewritten in a matrix form as

$$A\tilde{c}(t) = -BI(t). \tag{18}$$

Premultiplying the inverse of A at both the sides of (18), the solutions $\tilde{c}_{ei}(t)$ and $\tilde{c}_{ij}(t)$ can be written as

$$\tilde{c}_{ei}(t) = \tilde{\gamma}_{ei} + \sum_{k=1}^{p} \left[\tilde{\alpha}_{ei}^k \sin(\omega_k t) + \tilde{\beta}_{ei}^k \cos(\omega_k t)\right] \tag{19a}$$

$$\tilde{c}_{ij}(t) = \tilde{\gamma}_{ij} + \sum_{k=1}^{p} \left[\tilde{\alpha}_{ij}^k \sin(\omega_k t) + \tilde{\beta}_{ij}^k \cos(\omega_k t)\right]. \tag{19b}$$

Therefore, the structure of the solutions $\tilde{c}_{ei}(t)$ and $\tilde{c}_{ij}(t)$ is the same to that of the external inputs $I_i(t)$. Substituting (19) into (16), we get

$$-\tilde{\gamma}_{ei} a_{ei} x_i^d + \sum_{j \neq i} \tilde{\gamma}_{ij} a_{ij} \left(x_j^d - x_i^d\right) + a_{si} \gamma_i = 0 \tag{20a}$$

$$-\tilde{\alpha}_{ei}^k a_{ei} x_i^d + \sum_{j \neq i} \tilde{\alpha}_{ij}^k a_{ij} \left(x_j^d - x_i^d\right) + a_{si} \alpha_{ik} = 0 \tag{20b}$$

$$-\tilde{\beta}_{ei}^k a_{ei} x_i^d + \sum_{j \neq i} \tilde{\beta}_{ij}^k a_{ij} \left(x_j^d - x_i^d\right) + a_{si} \beta_{ik} = 0 \tag{20c}$$

for $k = 1, \ldots, p$. Therefore, solving (16) is equivalent to finding the solution of (20).

In this paper, we design a distributed update law for the coefficients γ_{ei}, γ_{ij}, α_{ei}^k, α_{ij}^k, β_{ei}^k, and β_{ij}^k. Based on the update law, the coordination rules of the system parameters $c_{ei}(t)$ and $c_{ij}(t)$ become

$$c_{ei}(t) = \gamma_{ei}(t) + \sum_{k=1}^{p} \left[\alpha_{ei}^k(t) \sin(\omega_k t) + \beta_{ei}^k(t) \cos(\omega_k t)\right] \tag{21a}$$

$$c_{ij}(t) = \gamma_{ij}(t) + \sum_{k=1}^{p} \left[\alpha_{ij}^k(t) \sin(\omega_k t) + \beta_{ij}^k(t) \cos(\omega_k t)\right]. \tag{21b}$$

For simplicity of the notations, we define the coefficient vectors as

$$z_{ei} = \left[z_{ei}^1 \ z_{ei}^2 \ \cdots \ z_{ei}^l\right] = \left[\gamma_{ei} \ \alpha_{ei}^1 \ \cdots \ \alpha_{ei}^p \ \beta_{ei}^1 \ \cdots \ \beta_{ei}^p\right] \tag{22a}$$

$$z_{ij} = \left[z_{ij}^1 \ z_{ij}^2 \ \cdots \ z_{ij}^l\right] = \left[\gamma_{ij} \ \alpha_{ij}^1 \ \cdots \ \alpha_{ij}^p \ \beta_{ij}^1 \ \cdots \ \beta_{ij}^p\right] \tag{22b}$$

where $l = 2p + 1$. We denote $\tilde{z}_{ei}$ and $\tilde{z}_{ij}$ the solution of (20). The simplest approach to deal with the constraints (17) is applying a saturation function to the coordination rules (21); however, the saturation function causes a problem in analysis. We take an alternative approach to find a region, defined as

$$\Lambda_{ei} = \{z_{ei} | \underline{z_{ei}} \le\le z_{ei} \le\le \overline{z_{ei}}\} \tag{23a}$$

$$\Lambda_{ij} = \{z_{ij} | \underline{z_{ij}} \le\le z_{ij} \le\le \overline{z_{ij}}\} \tag{23b}$$

such that (21) satisfies constraint (17), and then, we give a constraint to the update law of the coefficients, such that the update is allowed within the range. The detailed procedure for finding the region will be discussed later. We summarize the assumption and problem statement as follows.

Assumption 6: There exist solutions $\tilde{\gamma}_{ei}$, $\tilde{\gamma}_{ij}$, $\tilde{\alpha}_{ei}^k$, $\tilde{\alpha}_{ij}^k$, $\tilde{\beta}_{ei}^k$, and $\tilde{\beta}_{ij}^k$ in (19) satisfying constraints (20) and (17).

Problem 2: Design a distributed update law for the coefficients of γ_{ei}, γ_{ij}, α_{ei}^k, α_{ij}^k, β_{ei}^k, and β_{ij}^k in (20), such that system (14) converges to desired points x_i^d. Furthermore, the coefficients should be updated, so that (21) satisfies constraint (17).

Let us define the error of the parameters as

$$\hat{z}_{ei} = z_{ei} - \tilde{z}_{ei}, \quad \hat{z}_{ij} = z_{ij} - \tilde{z}_{ij}.$$

Then, one can easily derive the error dynamics as

$$\begin{aligned} \dot{e}_i &= -c_{ei} a_{ei} e_i + \sum_{j \neq i} c_{ij} a_{ij} (e_j - e_i) - c_{ei} a_{ei} x_i^d \\ &\quad + \sum_{j \neq i} c_{ij} a_{ij} \left(x_j^d - x_i^d\right) + a_{si} I_i \\ &= -c_{ei} a_{ei} e_i + \sum_{j \neq i} c_{ij} a_{ij} (e_j - e_i) - \hat{c}_{ei} a_{ei} x_i^d \\ &\quad + \sum_{j \neq i} \hat{c}_{ij} a_{ij} \left(x_j^d - x_i^d\right), \quad i = 1, \ldots, n \end{aligned} \tag{24}$$

where

$$\hat{c}_{ei}(t) = \hat{\gamma}_{ei}(t) + \sum_{k=1}^{p} \left[\hat{\alpha}_{ei}^k(t) \sin(\omega_k t) + \hat{\beta}_{ei}^k(t) \cos(\omega_k t)\right]$$

$$\hat{c}_{ij}(t) = \hat{\gamma}_{ij}(t) + \sum_{k=1}^{p} \left[\hat{\alpha}_{ij}^k(t) \sin(\omega_k t) + \hat{\beta}_{ij}^k(t) \cos(\omega_k t)\right].$$

The following theorem presents the distributed coordination rule that solves *Problem 2*.

Theorem 2: Suppose that *Assumptions 1, 3, 5, and 6* hold, and $\underline{c_{ei}} > 0$ and $\underline{c_{ij}} > 0$ for $i, j = 1, \ldots, n,\ i \neq j$. Then, given a linear compartmental system (14) and the time-varying external inputs (15), the error dynamics (24) converges to zero under the coordination rule (21) with the following update law:

$$\dot{\gamma}_{ei} = q_{\gamma_{ei}} h_{\gamma_{ei}} \left(a_{ei} x_i^d e_i\right) \tag{25a}$$
$$\dot{\alpha}_{ei}^k = q_{\alpha_{ei}^k} h_{\alpha_{ei}^k} \left(\sin(\omega_k t) a_{ei} x_i^d e_i\right) \tag{25b}$$
$$\dot{\beta}_{ei}^k = q_{\beta_{ei}^k} h_{\beta_{ei}^k} \left(\cos(\omega_k t) a_{ei} x_i^d e_i\right) \tag{25c}$$

and

$$\dot{\gamma}_{ij} = q_{\gamma_{ij}} h_{\gamma_{ij}} \left(a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j)\right) \tag{26a}$$
$$\dot{\alpha}_{ij}^k = q_{\alpha_{ij}^k} h_{\alpha_{ei}^k} \left(\sin(\omega_k t) a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j)\right) \tag{26b}$$
$$\dot{\beta}_{ij}^k = q_{\beta_{ij}^k} h_{\beta_{ei}^k} \left(\cos(\omega_k t) a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j)\right). \tag{26c}$$

Furthermore, $x(t) \geq\geq 0,\ t \geq 0$, for all $x_0 \in \bar{\mathbb{R}}_+$.

Proof: See the Appendix. ■

We can find many candidate of the parameter region of Λ_{ei} and Λ_{ij}, such that (21) satisfies constraint (17). However, a candidate is so narrow that *Assumption 6* may not be satisfied. Therefore, it is necessary to maximize the region of Λ_{ei} and Λ_{ij}, such that the above two conditions are guaranteed simultaneously.

In this paper, we just provide an optimization problem of the region of Λ_{ei}. The same problem can be constructed for the region of Λ_{ij}. Concerning the constraint (17), $c_{ei}(t)$ in (21) can be redefined as $c_{ei}(z_{ei}, t)$, which implies that the trajectory of c_{ei} as well as its maximum and minimum values depend on the parameter values of z_{ei}. For each $t \in [t_0,\ t_0 + T]$ where T is the period of c_{ei}, $c_{ei}(z_{ei}, t)$ is convex in z_{ei}, because $c_{ei}(z_{ei}, t)$ is an affine function with respect to z_{ei} for each t, and then, the functions f_{ei} and g_{ei}, defined, respectively, as

$$f_{ei}(z_{ei}) = \sup_{t \in [t_0, t_0+T]} c_{ei}(z_{ei}, t)$$
$$g_{ei}(z_{ei}) = \inf_{t \in [t_0, t_0+T]} c_{ei}(z_{ei}, t)$$

are also convex and concave functions, respectively [26]. Therefore, sublevel set $\overline{\mathbf{\Omega}}_{ei}$ and superlevel set $\underline{\mathbf{\Omega}}_{ei}$ defined as

$$\overline{\mathbf{\Omega}}_{ei} = \{z_{ei} | f(z_{ei}) \leq \overline{c_{ei}}\}$$
$$\underline{\mathbf{\Omega}}_{ei} = \{z_{ei} | f(z_{ei}) \geq \underline{c_{ei}}\}$$

are convex sets. The parameter set Λ_{ei} is a special kind of polytopes, i.e., hypercube, whose vertex set can be defined as

$$\mathcal{V}_{ei} = \left\{z_{ei} \in \Lambda_{ei} | z_{ei}^k = \underline{z_{ei}^k} \text{ or } z_{ei}^k = \overline{z_{ei}^k},\ k = 1, \ldots, l\right\}. \tag{27}$$

If all the elements of $\mathcal{V}_{ei}$ are contained in $\overline{\mathbf{\Omega}}_{ei} \cap \underline{\mathbf{\Omega}}_{ei}$, the set Λ_{ei} is also included in $\overline{\mathbf{\Omega}}_{ei} \cap \underline{\mathbf{\Omega}}_{ei}$ by convex property. Therefore, constraint (17) becomes

$$f(y) \leq \overline{c_{ei}} \quad \text{and } g(y) \geq \underline{c_{ei}} \quad \forall y \in \mathcal{V}_{ei}. \tag{28}$$

As a cost function, a geometric programming can be used to maximize the volume of the parameter set Λ_{ei}; however, the geometric programming with the constraint (28) cannot be transformed into a convex optimization problem. We alternatively define cost function as the length of a diagonal line of the hypercube Λ_{ei}, which is proportional to the volume. In summary, we present the following convex optimization problem:

$$\begin{aligned} &\text{maximize } \lambda \\ &\text{s.t. } f(y) - \overline{c_{ei}} \leq 0 \quad \forall y \in \mathcal{V}_{ei} \\ &\quad\ \underline{c_{ei}} - g(y) \leq 0 \quad \forall y \in \mathcal{V}_{ei} \\ &\quad\ \lambda \geq 0 \\ &\quad\ \overline{z_{ei}} - \underline{z_{ei}} = \lambda w \end{aligned} \tag{29}$$

with variables $\overline{z_{ei}}$, $\underline{z_{ei}}$, λ, and a unit positive constant vector $w \in \mathbb{R}^l$. The vector w is a direction vector of the diagonal line of the hypercube. The volume of Λ_{ei} is actually dependent on the direction vector. If the solution of the optimization problem is unsatisfactory, we need to change the direction. The direction vector w also implies the weight of the coefficient range. For example, if we assign the higher weight on γ_{ei}, the larger range of $[\underline{\gamma_{ei}},\ \overline{\gamma_{ei}}]$ will be obtained.

Next, we extend *Theorem 2* to the asymmetric case. The system model becomes (5) with the time-varying external inputs $I_i(t)$, which also satisfies *Assumption 5*. Given the desired points x_i^d, the following result shows the coordination rule for the set-point regulation problem.

Theorem 3: Suppose that *Assumptions 1 and 3–6* hold. Then, given a linear compartmental system (5) with the time-varying external inputs (15), the error dynamics converges to zero under the coordination rule (21) with the following update law:

$$\dot{\gamma}_{ei} = q_{\gamma_{ei}} h_{\gamma_{ei}} \left(a_{ei} x_i^d e_i\right) \tag{30a}$$
$$\dot{\alpha}_{ei}^k = q_{\alpha_{ei}^k} h_{\alpha_{ei}^k} \left(\sin(\omega_k t) a_{ei} x_i^d e_i\right) \tag{30b}$$
$$\dot{\beta}_{ei}^k = q_{\beta_{ei}^k} h_{\beta_{ei}^k} \left(\cos(\omega_k t) a_{ei} x_i^d e_i\right) \tag{30c}$$

and

$$\dot{\gamma}_{ij} = q_{\gamma_{ij}} h_{\gamma_{ij}} \left(a_{ij} x_j^d (e_i - e_j)\right) \tag{31a}$$
$$\dot{\alpha}_{ij}^k = q_{\alpha_{ij}^k} h_{\alpha_{ei}^k} \left(\sin(\omega_k t) a_{ij} x_j^d (e_i - e_j)\right) \tag{31b}$$
$$\dot{\beta}_{ij}^k = q_{\beta_{ij}^k} h_{\beta_{ei}^k} \left(\cos(\omega_k t) a_{ij} x_j^d (e_i - e_j)\right). \tag{31c}$$

Furthermore, $x(t) \geq\geq 0,\ t \geq 0$, for all $x_0 \in \bar{\mathbb{R}}_+$.

One can readily prove the result by following the proofs of *Theorem 1* and *2* with the Lyapunov function

$$V = \frac{1}{2}\sum_{i=1} e_i^2 + \frac{1}{2}\sum_{i=1} \hat{z}_{ei}^T Q_{ei} \hat{z}_{ei} + \frac{1}{2}\sum_{i=1}\sum_{j \neq i} \hat{z}_{ij}^T Q_{ij} \hat{z}_{ij}.$$

Similar to the symmetrical case, we can find the coefficient region of Λ_{ei} and Λ_{ij}. At first, we find $\underline{\mathbf{c}}$ satisfying *Assumption 4*, which is obtained by solving the optimization problem (12). Then, we solve the optimization problem (29) in order to obtain the coefficient region of Λ_{ei} and Λ_{ij}, such that (21) satisfies constraint (17).

V. Applications

A. Water Irrigation Systems

The water irrigation system is shown in Fig. 1. There is a descent slope in each open-water channel; thus, water is

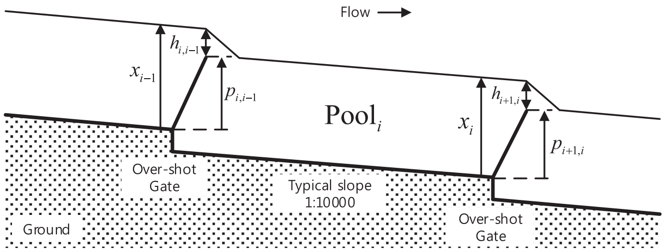

Fig. 1. Diagram for model of a water irrigation system.

transported along the channels by only gravitational force. The overshot gates are installed at the end of each channel, and adjust the flow rate of water through the network.

Model of the open-water channel dynamics has been originated from Saint–Venant equations, which are nonlinear partial differential equations for a mass and momentum balance along the length of each pool [27]. However, the model is somewhat complex to be applied to system identification and closed-loop analysis and synthesis. From the motivation, the following simplified model that is appropriate for control design has been introduced in [9]:

$$\alpha_i \dot{x}_i = \gamma_{i,i-1} h_{i,i-1}^{\frac{3}{2}}(t-\tau_i) - \gamma_{i+1,i} h_{i+1,i}^{\frac{3}{2}}(t) \tag{32}$$

where $\alpha_i \in \bar{\mathbb{R}}_+$ is the pool surface area, $x_i \in \bar{\mathbb{R}}_+$ is the water level of the pool, $\gamma_{i,i-1} \in \bar{\mathbb{R}}_+$ is a constant, which depends on the geometry of the gate which is installed between the $i-1$th and ith pools, $h_{i,i-1} \in \bar{\mathbb{R}}_+$ is the head over the gate, and $\tau_i \in \bar{\mathbb{R}}_+$ is the time delay it takes for the water to travel the length of the ith pool.

In this paper, we modify the previous model as follows:

$$\begin{aligned} \alpha_i \dot{x}_i &= \gamma_{i,i-1}(c_{i,i-1}(t-\tau_i)x_{i-1}(t-\tau_i))^{\frac{3}{2}} \\ &\quad - \gamma_{i+1,i}(c_{i+1,i}(t)x_i(t))^{\frac{3}{2}} \end{aligned} \tag{33}$$

where $c_{i,i-1} = h_{i,i-1}/x_{i-1}$, which means the ratio of the head over the gate h_i to the water level x_{i-1}. The system parameter $c_{i,i-1}$ can be adjusted by controlling height of the gate $p_{i,i-1}$. As described in Fig. 1, it is easily shown that $h_{i,i-1} = x_{i-1} - p_{i,i-1}$; thus, $c_{i,i-1} = 1 - p_{i,i-1}/x_{i-1}$. If we measure the water level of x_{i-1} in real time, we can make a desired value of $c_{i,i-1}$ by controlling the height of the gate $p_{i,i-1}$. The physical constraint of the range of parameter becomes $c_{i,i-1} \in [0, 1]$. Here $c_{i,i-1} = 0$ implies that $h_{i,i-1} = 0$, so there is no water flow along the channel, and $c_{i,i-1} = 1$ implies that the gate is completely open, i.e., $p_{i,i-1} = 0$, so $h_i = x_{i-1}$.

The model of (33) is defined based on a simple path graph. The model can be easily extended to general directed graph $\mathcal{G}$ as follows:

$$\begin{aligned} \dot{x}_i &= \sum_{j \neq i} \frac{\gamma_{ij}}{\alpha_i} a_{ij}(c_{ij}(t-\tau_i)x_j(t-\tau_i))^{\frac{2}{3}} \\ &\quad - \sum_{j \neq i} \frac{\gamma_{ji}}{\alpha_i} a_{ji}(c_{ji}(t)x_i(t))^{\frac{2}{3}} \\ &\quad - \frac{\gamma_{ei}}{\alpha_i} a_{ei}(c_{ei}(t)x_i(t))^{\frac{2}{3}} + a_{si}\frac{I_i}{\alpha_i}. \end{aligned} \tag{34}$$

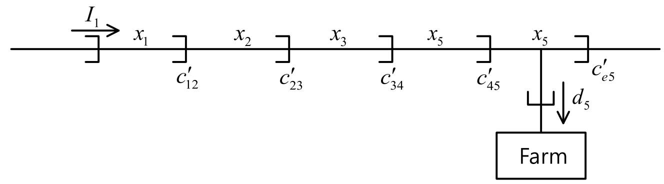

Fig. 2. Simulation setting for water irrigation system.

Note that the graph $\mathcal{G}$ is usually a tree graph and the structure of the model of (34) is consistent with that of the asymmetric compartmental systems (5).

The system model of (34) can be linearized around the desired points x_i^d. Defining $c'_{ei}(t) = (\gamma_{ei}/\alpha_i)(c_{ei}(t))^{(3/2)}$ and $c'_{ij}(t) = (\gamma_{ij}/\alpha_i)(c_{ij}(t))^{(3/2)}$, the model of (34) can be rewritten as

$$\begin{aligned} \dot{x}_i &= -\left(a_{ei}c'_{ei}(t) + \sum_{j\neq i} a_{ji}c'_{ji}(t)\right)(x_i(t))^{\frac{3}{2}} \\ &\quad + \sum_{j\neq i} a_{ij}c'_{ij}(t-\tau_i)(x_j(t-\tau_i))^{\frac{3}{2}} + a_{si}\frac{I_i}{\alpha_i}. \end{aligned}$$

It is easily shown that the linearization of (34) around the desired points x_i^d becomes

$$\begin{aligned} \dot{e}_i &= -\left(a_{ei}s_{ei}c'_{ei}(t) + \sum_{j\neq i} a_{ji}s_{ji}c'_{ji}(t)\right)e_i(t) \\ &\quad + \sum_{j\neq i} a_{ij}s_{ij}c'_{ij}(t-\tau_i)e_j(t-\tau_i) \\ &\quad - \left(a_{ei}\left(c'_{ei}(t) - \tilde{c}'_{ei}\right) + \sum_{j\neq i} a_{ji}\left(c'_{ji}(t) - \tilde{c}'_{ji}\right)\right)\left(x_i^d\right)^{\frac{2}{3}} \\ &\quad + \sum_{j\neq i} a_{ij}\left(c'_{ij}(t-\tau_i) - \tilde{c}'_{ij}\right)\left(x_j^d\right)^{\frac{2}{3}} \end{aligned}$$

where $s_{ei} = (3/2)(x_i^d)^{(1/2)}$ and $s_{ij} = (3/2)(x_j^d)^{(1/2)}$, using the equality

$$\begin{aligned} 0 &= -\left(a_{ei}\tilde{c}'_{ei} + \sum_{j\neq i} a_{ji}\tilde{c}'_{ji}\right)\left(x_i^d\right)^{\frac{3}{2}} \\ &\quad + \sum_{j\neq i} \tilde{c}'_{ij}a_{ij}\left(x_j^d\right)^{\frac{3}{2}} + a_{si}\frac{I_i}{\alpha_i}. \end{aligned}$$

where $\tilde{c}'_{ei}$ and $\tilde{c}'_{ij}$ are the solutions of the set-point regulation problem. Note that the linearized model is similar to the error dynamics (9). The modified coordination rules are

$$\dot{c}'_{ei} = q_{ei}a_{ei}\left(x_i^d\right)^{\frac{3}{2}} e_i, \quad i = 1, \ldots, n \tag{35a}$$

$$\dot{c}'_{ij} = q_{ij}a_{ij}\left(x_j^d\right)^{\frac{3}{2}}(e_j - e_i), \quad i, j = 1, \ldots, n, i \neq j. \tag{35b}$$

Note that the coordination rule (35) is essentially the same as that of (10), because there is no difference except the constants $(x_i^d)^{(3/2)}$ and $(x_j^d)^{(3/2)}$. In addition, the system model of (34) has time delays. The effect of time delay will be discussed in the following simulation results.

We conduct the simulation in order to evaluate the effectiveness of the coordination rule (35). The simulation setting is described in Fig. 2 and its parameter values are based on the irrigation network of [28]. There is an offtake from the fifth pool to the farm, which is regarded as load disturbance d_5. We assume that the first gate is controlled to supply water with fixed flow rate; therefore, the flow can be regarded as the fixed external input I_1. The system model with the parameters becomes

$$\begin{aligned}
\dot{x}_1 &= 0.0201 I_1(t) - 0.0264 c_{21}(t) x_1(t)^{\frac{3}{2}} \\
\dot{x}_2 &= 0.0851 c_{21}(t-2) x_1(t-2)^{\frac{3}{2}} - 0.0847 c_{32}(t) x_2(t)^{\frac{3}{2}} \\
\dot{x}_3 &= 0.0342 c_{32}(t-4) x_2(t-4)^{\frac{3}{2}} - 0.0255 c_{43}(t) x_3(t)^{\frac{3}{2}} \\
\dot{x}_4 &= 0.0277 c_{43}(t-4) x_3(t-4)^{\frac{3}{2}} - 0.0277 c_{54}(t) x_4(t)^{\frac{3}{2}} \\
\dot{x}_5 &= 0.0231 c_{54}(t-6) x_4(t-6)^{\frac{3}{2}} - 0.0231 c_{e5}(t) x_5(t)^{\frac{3}{2}} \\
&\quad - 0.0231 d_5(t).
\end{aligned}$$

The desired water levels for each pool are $x_1^d = 1.45$, $x_2^d = 1.51$, $x_3^d = 1.554$, $x_4^d = 1.5$, and $x_5^d = 1.52$, and the external input is $I_1 = 1$. The load disturbance is

$$d_5(t) = \begin{cases} 0 & \text{for } 0 \le t < 200 \\ 0.1 & \text{for } t \ge 200. \end{cases}$$

The time unit is minutes and unit of the water level is meter Australian Height Datum.

The simulation result of set-point regulation with the coordination rule (35) and its coordination gain of 0.003 is shown in Fig. 3. As shown in Fig. 3(a) and (b), there is a fluctuation of water level at the fifth pool due to the disturbance d_5, and then, it propagates upstream. However, their magnitude is gradually reduced, and the errors eventually converge to zero. Fig. 3(c) shows that the system parameters converge to some constant values, which means that the heights of gates are readjusted after the load disturbance d_5.

Fig. 4 includes the simulation results with its coordination gain of 0.063. As shown in Fig. 4, there are undamped oscillations in each trajectory, which results from the time delay effect. These results show that the coordination rule (35) has limitation on the selection of the coordination gain. However, the coordination rule (35) is still effective in the water irrigation within the limited coordination gain.

B. Water Distribution Systems

In this section, we show that the distributed coordination rule of the compartmental systems can be applied to water distribution systems.

Overall schematic of the water distribution systems is shown in Fig. 5. The urban macro water grid is divided into smaller meso-grids. Each meso-grid consists of a fresh water source, a treatment plant, a storage reservoir, and its own district. The examples of the fresh water source are ground water, water from dams, direct water taken from rivers, and water from sea-water desalination. The water from the fresh water source is transported to the treatment plant for the improvement of water quality. After treatment, the product water is temporally stored in the storage reservoir, and is directly distributed to customers in the district such as residential, industrial, and irrigational area.

The storage reservoir should provide hydraulic capacitance in the meso-grid and sufficient head of water for the water distribution; thus, it is important to maintain the appropriate head of water in the storage reservoir. However, back-up of the product water is necessary for stable regulation of the head of water. Suppose that the fresh water sources 1 and 3 in Fig. 5 are water from a dam and a river, respectively. In drought season, it is difficult to obtain sufficient water from the river, which results in the shortage of water supply to the storage reservoir 3. As a result, the head of water in the storage reservoir 3 goes down, and in worst case, the water supply from the reservoir to district 3 could be cut off.

One solution for the back-up of the storage reservoir 3 is to construct another water supply line from the fresh water source 1 to the storage reservoir 3, because the dam provides relatively stable water supply; however, the construction would be economically infeasible due to geographical constraints. An alternative solution proposed in [10] is that an additional reservoir at the center of the three storage reservoirs is installed and each storage reservoir is connected to the central reservoir. The central reservoir allocates water from one water grid to the other one, e.g., the central reservoir takes water from the

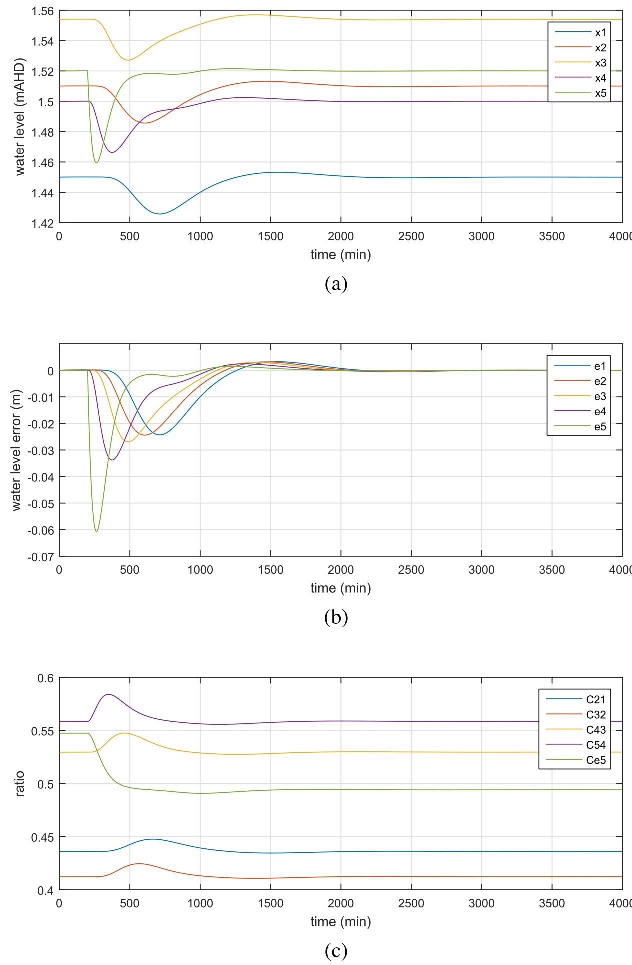

Fig. 3. Simulation results with a coordination gain of 0.003. (a) State trajectories. (b) Error trajectories. (c) System parameter trajectories.

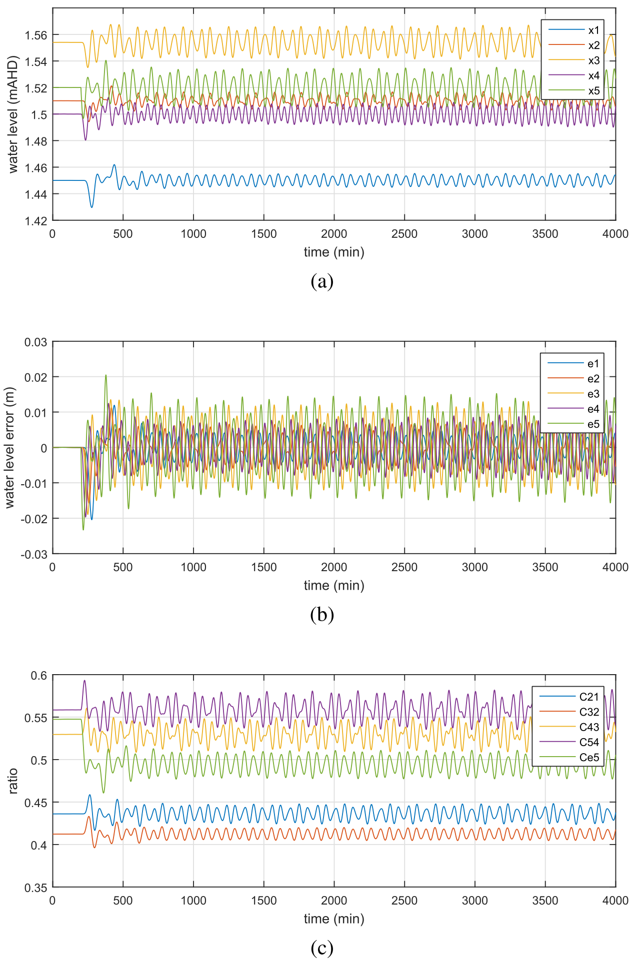

Fig. 4. Simulation results with a coordination gain of 0.063. (a) State trajectories. (b) Error trajectories. (c) System parameter trajectories.

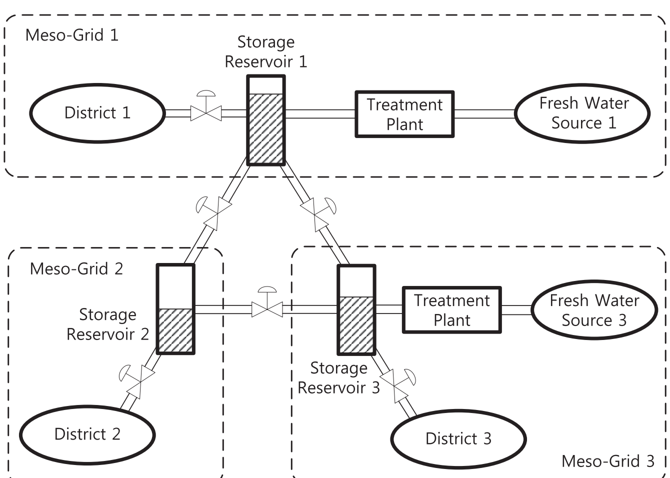

Fig. 5. Illustration of a water distribution system.

storage reservoir 1 and sends it to the storage reservoir 3. Essential idea of the solution is that the whole product water is shared with each storage reservoir through connection lines. In this paper, instead of the construction of the central reservoir, we make direct connection lines among the storage reservoirs for sharing the product water described as Fig. 5. Controllable valves are installed at each connection line and

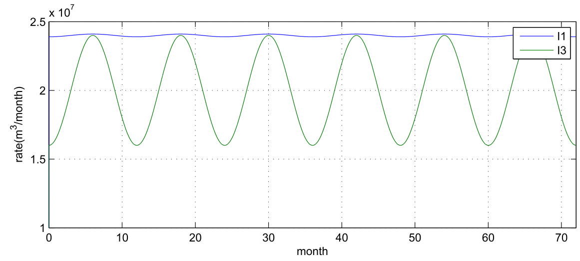

Fig. 6. Water supply rates from fresh water sources 1 and 3.

play role of allocation of the water by adjusting flow rate. In this configuration, the coordination rule of each valve presented in this paper can achieve the regulation of head of water, which will be shown in the following simulation results.

For the simulation, a system model for the storage reservoir is necessary. The storage reservoir can be represented by a water tank; thus, we set the following model of the storage reservoir with slight modification of models of water tank systems [8]:

$$A_i\dot{h}_i = -a_{ei}c_{ei}\sqrt{2gh_i}+\sum_{i\neq j}a_{ji}c_{ji}\operatorname{sgn}(h_j-h_i)\sqrt{2g|h_j-h_i|} + a_{si}I_i(t) \tag{36}$$

where $\operatorname{sgn}(\cdot)$ is the sign function, g is the acceleration of gravity, h_i is the water level, I_i is the rate of water supply from the corresponding fresh water source, A_i is the cross-sectional area of the ith storage reservoir, and c_{ei} is the cross-sectional area of distribution line, and $c_{ji} = c_{ij}$ is the cross-sectional area of connection line between the jth and ith storage reservoirs. The valves change the coefficients c_{ei} and c_{ji} according to the coordination rule (21). With the desired water level h_i^d and the solutions $\tilde{c}_{ei}(t)$ and $\tilde{c}_{ij}(t)$, the system (36) becomes in equilibrium as

$$0 = -a_{ei}\tilde{c}_{ei}(t)\sqrt{2gh_i^d}+\sum_{i\neq j}a_{ji}\tilde{c}_{ji}(t)\operatorname{sgn}\left(h_j^d-h_i^d\right) \times\sqrt{2g\left|h_j^d-h_i^d\right|}+a_{si}I_i(t). \tag{37}$$

Let us assume that $h_i^d > 0$ and $h_i^d \neq h_j^d$ for all $i, j = 1, \ldots, n, i \neq j$. Using (37), the linearization of (36) around the desired points h_i^d is

$$\begin{aligned}A_i\dot{e}_i &= -a_{ei}c_{ei}s_{ei}e_i+\sum_{i\neq j}a_{ij}c_{ij}s_{ji}(e_j-e_i)\\ &\quad - a_{ei}(c_{ei}-\tilde{c}_{ei})\sqrt{2gh_i^d}+\sum_{i\neq j}a_{ij}(c_{ij}-\tilde{c}_{ij})\\ &\quad \times\operatorname{sgn}\left(h_j^d-h_i^d\right)\sqrt{2g\left|h_j^d-h_i^d\right|}\end{aligned} \tag{38}$$

where $s_{ei} = g/(2gh_i^d)^{1/2} > 0$ and $s_{ij} = s_{ji} = g/(2g|h_j^d-h_i^d|)^{1/2} > 0$.

When all the states are sufficiently close to the corresponding desired points, the linearized model (38) becomes effective, and the following modified update laws can be used

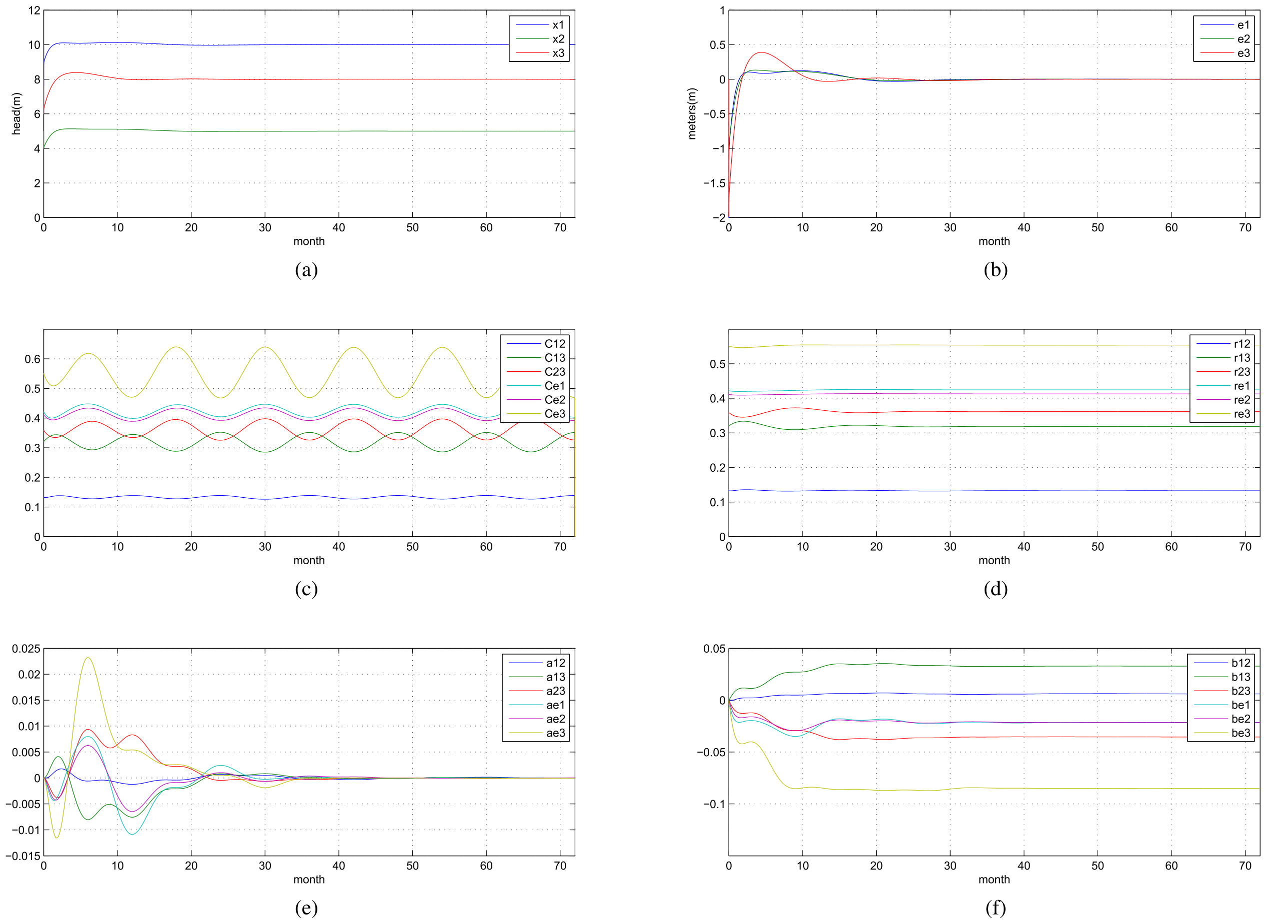

Fig. 7. Simulation results of water distribution systems. (a) State trajectories. (b) Error trajectories. (c) System parameter trajectories. (d) γ trajectories. (e) α trajectories. (f) β trajectories.

for the set-point regulation problem:

$$\begin{aligned}\dot{\gamma}_{ei} &= q_{\gamma_{ei}} h_{\gamma_{ei}} \left(a_{ei}\sqrt{2gx_i^d}e_i\right)\\ \dot{\alpha}_{ei}^k &= q_{\alpha_{ei}^k} h_{\alpha_{ei}^k} \left(\sin(\omega_k t)a_{ei}\sqrt{2gx_i^d}e_i\right)\\ \dot{\beta}_{ei}^k &= q_{\beta_{ei}^k} h_{\beta_{ei}^k} \left(\cos(\omega_k t)a_{ei}\sqrt{2gx_i^d}e_i\right)\end{aligned} \tag{39}$$

and

$$\begin{aligned}\dot{\gamma}_{ij} &= q_{\gamma_{ij}} h_{\gamma_{ij}} \left(a_{ij}\,\mathrm{sgn}(e_i-e_j)\sqrt{2g\left|x_i^d-x_j^d\right|}\right)\\ \dot{\alpha}_{ij}^k &= q_{\alpha_{ij}^k} h_{\alpha_{ei}^k} \left(\sin(\omega_k t)a_{ij}\,\mathrm{sgn}(e_i-e_j)\sqrt{2g\left|x_i^d-x_j^d\right|}\right)\\ \dot{\beta}_{ij}^k &= q_{\beta_{ij}^k} h_{\beta_{ei}^k} \left(\cos(\omega_k t)a_{ij}\,\mathrm{sgn}(e_i-e_j)\sqrt{2g\left|x_i^d-x_j^d\right|}\right).\end{aligned} \tag{40}$$

The details of the simulation setting are shown in Fig. 5. Note that the meso-grid 2 has no direct water supply from a fresh water source; instead, it receives the water from the storage reservoir 1 and 3. We will show through simulation that the head of the storage reservoir 2 can be regulated in the configuration. Fig. 6 shows water supply from the fresh water sources 1 and 3, i.e., I_1 and I_3. Generally speaking, precipitation is high in summer season and low in winter season; thus, we approximate the shapes of I_1 and I_3 as a cosine function with a one-year period. As mentioned above, the fresh water sources 1 and 3 are the dam and river, respectively; thus, the amplitude of I_1 is far smaller than that of I_3. As simulation parameters, $A_1 = A_2 = A_3 = 10^4$ m^2, and the desired heads of each storage reservoir are $h_1^d = 10$ m, $h_2^d = 5$ m, and $h_3^d = 8$ m. The time unit is months.

As the simulation results, the coordination performance is dependent on the gains of (30) and (31). The optimal results shown in Fig. 7 are obtained from the following gains: $q_{\gamma_{12}} = q_{\gamma_{13}} = q_{\gamma_{23}} = 0.003$, $q_{\gamma_{e1}} = q_{\gamma_{e2}} = q_{\gamma_{e3}} = 0.0003$, $q_{\alpha_{12}} = q_{\alpha_{13}} = q_{\alpha_{23}} = q_{\alpha_{e1}} = q_{\alpha_{e2}} = q_{\alpha_{e3}} = 0.003$, and $q_{\beta_{12}} = q_{\beta_{13}} = q_{\beta_{23}} = q_{\beta_{e1}} = q_{\beta_{e2}} = q_{\beta_{e3}} = 0.003$. (We abbreviate $q_{\gamma_{ij}^k}$ as $q_{\gamma_{ij}}$, because there is only one case with $k = 1$ in this simulation.) As shown in Fig. 7(a), the water heads of each storage reservoir are regulated at the desired value, and the initial errors converge to zero as described in Fig. 7(b). Therefore, the water heads are eventually regulated at the desired value in spite of the variation of the water supply shown in Fig. 6. The coefficient trajectories shown in Fig. 7(c) are well consistent with our intuition. For example, c_{e1}, c_{e2}, and c_{e3} in winter season are minimum, because the direct water supplies of I_1 and I_3 are low. The degree of variation of c_{e3} is relatively large, because I_3 is also large. It is notable that c_{12}, c_{13} in winter season are maximum, because the storage reservoirs 2 and 3 should maximally receive water from the storage reservoir 1. On the other hand, c_{23} in winter season is minimum, because the storage reservoir 3 has little spare water to feed to the storage reservoir 2. Fig. 7(d)–(f) show that the evolutions of parameter under the update law of (30) and (31) eventually converge to their solution of (20).

VI. Conclusion

In this paper, we extended the results of [24] in two ways. First, we designed the distributed coordination scheme for the asymmetric compartmental systems. In contrast to the symmetric cases, Assumption 4 is necessary for a completion of mathematical analysis, because there are some exceptional cases where the matrix M_I^S is not negative definite. In order to overcome the difficulty in analysis, we present a convex optimization problem, which determines the coordination range of the system parameters to satisfy Assumption 4. Second, we designed the distributed coordination scheme for periodically time-varying external inputs.

We applied our theoretical results to the water irrigation systems and the water distribution systems. In the water irrigation systems, the distributed coordination for the asymmetric case is used, and its effectiveness is shown despite the presence of time delay effects. In the water distribution systems, it is shown that the heads of water in the storage reservoirs are maintained despite the time-varying water supply from the fresh water sources.

Appendix

Proof of Theorem 1: Consider the following Lyapunov candidate as

$$V = \frac{1}{2}\sum_{i=1} e_i^2 + \frac{1}{2}\sum_{i=1}\frac{1}{q_{ei}}(c_{ei}-\tilde{c}_{ei})^2 + \frac{1}{2}\sum_{i=1}\sum_{j\neq i}\frac{1}{q_{ij}}(c_{ij}-\tilde{c}_{ij})^2. \tag{41}$$

The function is positive definite and radially unbounded. The derivative of the Lyapunov function is

$$\begin{aligned}\dot{V} &= \sum_{i=1} e_i\left\{-\left(c_{ei}a_{ei}+\sum_{j\neq i}c_{ji}a_{ji}\right)e_i + \sum_{j\neq i}c_{ij}a_{ij}e_j\right\}\\ &\quad + \sum_{i=1} e_i\left\{-\left((c_{ei}-\tilde{c}_{ei})a_{ei}+\sum_{j\neq i}(c_{ji}-\tilde{c}_{ji})a_{ji}\right)x_i^d + \sum_{j\neq i}(c_{ij}-\tilde{c}_{ij})a_{ij}x_j^d\right\}\\ &\quad + \sum_{i=1}\frac{1}{q_{ei}}\dot{c}_{ei}(c_{ei}-\tilde{c}_{ei}) + \sum_{i=1}\sum_{j\neq i}\frac{1}{q_{ij}}\dot{c}_{ij}(c_{ij}-\tilde{c}_{ij}).\end{aligned} \tag{42}$$

Because the coordination rule (10) is constrained from (11), we need to consider the following three cases.

Case 1: The system parameters are within the ranges: $c_{ei} \in [\underline{c_{ei}}, \overline{c_{ei}}]$, $c_{ij} \in [\underline{c_{ij}}, \overline{c_{ij}}]$.

The coordination rule (10) becomes

$$\begin{aligned}\dot{c}_{ei} &= q_{ei}a_{ei}x_i^d e_i, \quad i=1,\ldots,n\\ \dot{c}_{ij} &= q_{ij}a_{ij}x_j^d(e_j-e_i), \quad i,j=1,\ldots,n, i\neq j.\end{aligned} \tag{43}$$

Substituting (43) into (42)

$$\begin{aligned}\dot{V} &= \sum_{i=1} e_i\left\{-c_{ei}a_{ei}e_i + \sum_{j\neq i}c_{ij}a_{ij}(e_j-e_i)\right\}\\ &= e^T M(t)e\\ &= \frac{1}{2}e^T(M(t)^T+M(t))e.\end{aligned} \tag{44}$$

Because $M(t) \in M_I$ for all $t \geq 0$, $M(t)^T + M(t) \in M_I^S$ for all $t \geq 0$.

Case 2: One of the values of c_{ei} is under the state constraint. Let c_{ep} be the variable. When c_{ep} is constrained at upper limit, i.e., $c_{\text{ep}} = \overline{c_{\text{ep}}}$, its time derivative $\dot{c}_{\text{ep}}$ becomes zero by (11), and (42) becomes

$$\begin{aligned}\dot{V} &= -\sum_{i\neq p} c_{ei}a_{ei}e_i^2 - \overline{c_{ep}}a_{ep}e_p^2\\ &\quad + \sum_{i=1}\sum_{j\neq i}c_{ij}a_{ij}e_i(e_j-e_i) - (\overline{c_{ep}}-\tilde{c}_{ep})a_{ep}x_p^d e_p.\end{aligned} \tag{45}$$

Because the solution $\tilde{c}_{ep}$ is within the range of $[\underline{c_{ep}}, \overline{c_{ep}}]$, $\overline{c_{ep}} - \tilde{c}_{ep} \geq 0$. By definition of (11), $a_{ep}x_p^d e_p > 0$. Therefore, $-(\overline{c_{ep}} - \tilde{c}_{ep})a_{ep}x_p^d e_p$ is nonpositive, and $\dot{V}$ becomes

$$\begin{aligned}\dot{V} &\leq -\sum_{i\neq p} c_{ei}a_{ei}e_i^2 - \overline{c_{ep}}a_{ep}e_p^2 + \sum_{i=1}\sum_{j\neq i}c_{ij}a_{ij}e_i(e_j-e_i)\\ &= e^T M_{\overline{c_{ep}}}e\\ &= \frac{1}{2}e^T\left(M_{\overline{c_{ep}}}^T + M_{\overline{c_{ep}}}\right)e\end{aligned}$$

where $M_{\overline{c_{ep}}}$ is the matrix $M(t)$, which satisfies $c_{ep} = \overline{c_{ep}}$.

When c_{ep} is constrained at lower limit, i.e., $c_{ep} = \underline{c_{ep}}$, its time derivative $\dot{c}_{ep}$ becomes zero, and (42) becomes

$$\begin{aligned}\dot{V} &= -\sum_{i\neq p} c_{ei}a_{ei}e_i^2 - \underline{c_{ep}}a_{ep}e_p^2\\ &\quad + \sum_{i=1}\sum_{j\neq i}c_{ij}a_{ij}e_i(e_j-e_i) - (\underline{c_{ep}}-\tilde{c}_{ep})a_{ep}x_p^d e_p.\end{aligned} \tag{46}$$

With the similar analysis, $-(\underline{c_{ep}} - \tilde{c}_{ep})a_{ep}x_p^d e_p$ is nonpositive $\dot{V} \leq \frac{1}{2}e^T(M_{\underline{c_{ep}}}^T + M_{\underline{c_{ep}}})e$, where the matrix $M_{\underline{c_{ep}}}$ is the matrix $M(t)$, which satisfies $c_{ep} = \underline{c_{ep}}$.

Case 3: One of the values of c_{ij} is under the state constraint. Let c_{pq} be the variable. When c_{pq} is constrained at upper limit, i.e., $c_{pq} = \overline{c_{pq}}$, then the time derivative $\dot{c}_{pq}$ becomes zero, and (42) becomes

$$\begin{aligned}\dot{V} &= -\sum_{i=1} c_{ei}a_{ei}e_i^2 + \sum_{i\neq p,q}\sum_{j\neq i}c_{ij}a_{ij}e_i(e_j-e_i)\\ &\quad -\overline{c_{pq}}a_{pq}(e_p^2 - e_pe_q) - (\overline{c_{pq}}-\tilde{c}_{pq})a_{pq}x_q^d(e_p-e_q).\end{aligned} \tag{47}$$

Because the solution $\tilde{c}_{\mathrm{pq}}$ is within the range of $[\underline{c_{\mathrm{pq}}}, \overline{c_{\mathrm{pq}}}]$, $\overline{c_{\mathrm{pq}}} - \tilde{c}_{\mathrm{pq}} \geq 0$. By definition of (11), $a_{pq}(x_p^d - x_q^d)(e_p - e_q) > 0$. Therefore, $-(\overline{c_{pq}} - \tilde{c}_{pq})a_{pq}x_q^d(e_p - e_q)$ is nonpositive, and $\dot{V}$ becomes

$$\begin{aligned}\dot{V} &\leq -\sum_{i=1} c_{ei}a_{ei}e_i^2 + \sum_{i\neq p,q}\sum_{j\neq i} c_{ij}a_{ij}e_i(e_j - e_i) \\ &\quad - \overline{c_{pq}}a_{pq}\left(e_p^2 - e_pe_q\right) \\ &= e^T M_{\overline{c_{pq}}} e = \frac{1}{2}e^T\left(M_{\overline{c_{pq}}}^T + M_{\overline{c_{pq}}}\right)e.\end{aligned}$$

When c_{pq} is constrained at lower limit, i.e., $c_{pq} = \underline{c_{pq}}$, $\dot{c}_{pq}$ is zero, and (42) becomes

$$\begin{aligned}\dot{V} &= -\sum_{i=1} c_{ei}a_{ei}e_i^2 + \sum_{i\neq p,q}\sum_{j\neq i} c_{ij}a_{ij}e_i(e_j - e_i) \\ &\quad - \underline{c_{pq}}a_{pq}\left(e_p^2 - e_pe_q\right) - (\underline{c_{pq}} - \tilde{c}_{pq})a_{pq}x_q^d(e_p - e_q).\end{aligned} \tag{48}$$

With the similar analysis, $-(\underline{c_{pq}} - \tilde{c}_{pq})a_{pq}x_q^d(e_p - e_q)$ is nonpositive and $\dot{V} \leq \frac{1}{2}e^T(M_{\underline{c_{pq}}}^T + M_{\underline{c_{pq}}})e$.

We can extend the analysis of Cases 2 and 3 to the case of the coefficient constraint of multiple values of c_{ei} and c_{ij}. Because all corresponding matrices $M(t)$ under the constraint are in M_I, $M(t)^T + M(t) \in M_I^S$. By Assumption 4, the interval matrix M_I^S is negative definite. Therefore, for the all cases, $\dot{V}$ satisfies

$$\dot{V} \leq \frac{1}{2}\alpha e^T e$$

where $\alpha \triangleq \max_{M\in M_I}\rho(M^T + M)$. Because $\dot{V}$ is negative semidefinite for variables e, c_{ei}, and c_{ij}, V is bounded, and $e_i, \dot{e}_i, c_{ij}, \dot{c}_{ij}$ are also bounded. $\ddot{V}$ is bounded from (9) and (10); thus, $\dot{V}$ is uniformly continuous. Applying Barbalat's lemma, e_i converges to zero. In addition, c_{ei} and c_{ij} converge to constant values from (10). Because the coefficients c_{ei} and c_{ij} remain in the range of $[\underline{c_{ei}}, \overline{c_{ei}}]$ and $[\underline{c_{ij}}, \overline{c_{ij}}]$, respectively, all the properties of the compartmental matrix in the time-varying matrix $M(t)$ is preserved, and by *Proposition 5*, $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \bar{\mathbb{R}}_+$. □

Proof of Theorem 2: Let us set the Lyapunov function as

$$V = \frac{1}{2}\sum_{i=1} e_i^2 + \frac{1}{2}\sum_{i=1}\hat{z}_{ei}^T Q_{ei}\hat{z}_{ei} + \frac{1}{2}\sum_{i=1}\sum_{j>i}\hat{z}_{ij}^T Q_{ij}\hat{z}_{ij} \tag{49}$$

where

$$\begin{aligned}Q_{ei} &= \mathrm{diag}\left(\frac{1}{q_{\gamma_{ei}}}\ \frac{1}{q_{\alpha_{ei}^1}}\ \cdots\frac{1}{q_{\alpha_{ei}^p}}\ \frac{1}{q_{\beta_{ei}^1}}\cdots\ \frac{1}{q_{\beta_{ei}^p}}\right) \\ Q_{ij} &= \mathrm{diag}\left(\frac{1}{q_{\gamma_{ij}}}\ \frac{1}{q_{\alpha_{ij}^1}}\ \cdots\frac{1}{q_{\alpha_{ij}^p}}\ \frac{1}{q_{\beta_{ij}^1}}\cdots\ \frac{1}{q_{\beta_{ij}^p}}\right).\end{aligned}$$

Then, its time derivative becomes

$$\begin{aligned}\dot{V} &= \sum_{i=1} e_i\left\{-c_{ei}a_{ei}e_i + \sum_{j\neq i} c_{ij}a_{ij}(e_j - e_i)\right\} \\ &\quad + \sum_{i=1} e_i\left\{-\hat{c}_{ei}a_{ei}x_i^d + \sum_{j\neq i}\hat{c}_{ij}a_{ij}(x_j^d - x_i^d)\right\} \\ &\quad + \sum_{i=1}\dot{\hat{z}}_{ei}^T Q_{ei}\hat{z}_{ei} + \sum_{i=1}\sum_{j>i}\dot{\hat{z}}_{ij}^T Q_{ij}\hat{z}_{ij} \\ &= e^T M(t)e - \sum_{i=1}\hat{c}_{ei}a_{ei}x_i^d e_i \\ &\quad - \sum_{i=1}\sum_{j>i}\hat{c}_{ij}a_{ij}(x_j^d - x_i^d)(e_i - e_j) \\ &\quad + \sum_{i=1}\dot{\hat{z}}_{ei}^T Q_{ei}\hat{z}_{ei} + \sum_{i=1}\sum_{j>i}\dot{\hat{z}}_{ij}^T Q_{ij}\hat{z}_{ij}.\end{aligned}$$

From (21), the term $-\sum_{i=1}\hat{c}_{ei}a_{ei}x_i^d e_i$ becomes

$$\begin{aligned}&-\sum_{i=1}\hat{c}_{ei}a_{ei}x_i^d e_i \\ &\quad = -\sum_{i=1}\hat{\gamma}_{ei}a_{ei}x_i^d e_i - \sum_{i=1}\sum_{k=1}^{p}\hat{\alpha}_{ei}^k\sin(\omega_k t)a_{ei}x_i^d e_i \\ &\qquad - \sum_{i=1}\sum_{k=1}^{p}\hat{\beta}_{ei}^k\cos(\omega_k t)a_{ei}x_i^d e_i\end{aligned}$$

and the term $-\sum\limits_{i=1}\sum\limits_{j>i}\hat{c}_{ij}a_{ij}(x_j^d - x_i^d)(e_i - e_j)$ becomes

$$\begin{aligned}&-\sum_{i=1}\sum_{j>i}\hat{c}_{ij}a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j) \\ &\quad = -\sum_{i=1}\sum_{j>i}\hat{\gamma}_{ij}a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j) \\ &\qquad - \sum_{i=1}\sum_{j>i}\sum_{k=1}^{p}\hat{\alpha}_{ij}^k\sin(\omega_k t)a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j) \\ &\qquad - \sum_{i=1}\sum_{j>i}\sum_{k=1}^{p}\hat{\beta}_{ij}^k\cos(\omega_k t)a_{ij}\left(x_j^d - x_i^d\right)(e_i - e_j).\end{aligned}$$

From (22), the terms $\sum_{i=1}\dot{\hat{z}}_{ei}^T Q_{ei}\hat{z}_{ei}$ and $\sum\limits_{i=1}\sum_{j>i}\dot{\hat{z}}_{ij}^T Q_{ij}\hat{z}_{ij}$ become

$$\begin{aligned}\sum_{i=1}\dot{\hat{z}}_{ei}^T Q_{ei}\hat{z}_{ei} &= \sum_{i=1}\frac{1}{q_{\gamma_{ei}}}\hat{\gamma}_{ei}\dot{\gamma}_{ei} + \sum_{i=1}\sum_{k=1}^{p}\frac{1}{q_{\alpha_{ei}^k}}\hat{\alpha}_{ei}^k\dot{\alpha}_{ei}^k \\ &\quad + \sum_{i=1}\sum_{k=1}^{p}\frac{1}{q_{\beta_{ei}^k}}\hat{\beta}_{ei}^k\dot{\beta}_{ei}^k\end{aligned}$$

$$\begin{aligned}\sum_{i=1}\sum_{j>i}\dot{\hat{z}}_{ij}^T Q_{ij}\hat{z}_{ij} &= \sum_{i=1}\sum_{j>i}\frac{1}{q_{\gamma_{ij}}}\hat{\gamma}_{ij}\dot{\gamma}_{ij} + \sum_{i=1}\sum_{j>i}\sum_{k=1}^{p}\frac{1}{q_{\alpha_{ij}^k}}\hat{\alpha}_{ij}^k\dot{\alpha}_{ij}^k \\ &\quad + \sum_{i=1}\sum_{j>i}\sum_{k=1}^{p}\frac{1}{q_{\beta_{ij}^k}}\hat{\beta}_{ij}^k\dot{\beta}_{ij}^k.\end{aligned}$$

Because (30) and (31) are determined from the constraint function (11), it is necessary to analyze the three cases.

Case 1): The parameters are within the range (23). It is easily shown that

$$\dot{V} = e^T M(t) e. \tag{50}$$

Let M_I denote an interval matrix defined as

$$M_I \triangleq \{M(t) | z_{ei} \in \Lambda_{ei},\ z_{ij} \in \Lambda_{ij} \text{ for } i, j = 1, \ldots, n,\ i \neq j\}.$$

The matrix $M(t)$ is time-varying but $M(t) \in M_I$ is pointwise-in-time.

Case 2): Suppose that $\gamma_{ei} = \overline{\gamma_{ei}}$ and the other parameters are within the range of (23). The solution $\tilde{\gamma}_{ei}$ is within the range of $[\underline{\gamma_{ei}}, \overline{\gamma_{ei}}]$; thus, $\hat{\gamma}_{ei} = \overline{\gamma_{ei}} - \tilde{\gamma}_{ei} \geq 0$. By definition of (11), $\overline{a_{ei} x_i^d e_i} > 0$. Consequently, $-\hat{\gamma}_{ei} a_{ei} x_i^d e_i$ is nonpositive; therefore

$$\begin{aligned}\dot{V} &= -e^T M_{\overline{\gamma_{ei}}}(t) e - \hat{\gamma}_{ei} a_{ei} x_i^d e_i \\ &\leq e^T M_{\overline{\gamma_{ei}}}(t) e\end{aligned}$$

where $M_{\overline{\gamma_{ei}}}$ is the matrix $M(t)$ with $\gamma_{ei} = \overline{\gamma_{ei}}$.

Case 3): Suppose that $\gamma_{ei} = \underline{\gamma_{ei}}$ and the other parameters are within the range of (23). The solution $\tilde{\gamma}_{ei}$ is within the range of $[\underline{\gamma_{ei}}, \overline{\gamma_{ei}}]$; thus, $\hat{\gamma}_{ei} = \underline{\gamma_{ei}} - \tilde{\gamma}_{ei} \leq 0$. By definition of (11), $\overline{a_{ei} x_i^d e_i} < 0$. Consequently, $-\hat{\gamma}_{ei} a_{ei} x_i^d e_i$ is nonpositive; therefore

$$\begin{aligned}\dot{V} &= -e^T M_{\underline{\gamma_{ei}}}(t) e - \hat{\gamma}_{ei} a_{ei} x_i^d e_i \\ &\leq e^T M_{\underline{\gamma_{ei}}}(t)\end{aligned}$$

where $M_{\underline{\gamma_{ei}}}$ is the matrix $M(t)$ with $\gamma_{ei} = \underline{\gamma_{ei}}$.

The same analyses of Cases 2 and 3 can be readily applied to the other parameters of γ_{ij}, α_{ei}^k,α_{ij}^k, β_{ei}^k, and β_{ij}^k. Furthermore, the same result can be derived in the case that multiple coefficients are under the constraint (11) at upper (Case 2) or lower (Case 3) limit. For the all cases, it is shown that $M(t) \in M_I$, $\forall t \geq 0$. Because of *Assumption 1* and $\underline{c_{ei}}$, $\underline{c_{ij}} > 0$, *Proposition 1* and *Proposition 3* are satisfied, i.e., all the elements of M_I are nonsingular and negative definite. Therefore

$$\dot{V} \leq \alpha e^T e$$

where $\alpha \triangleq \max_{M \in M_I} \rho(M)$. V is bounded, because $\dot{V}$ negative semidefinite for variables e, z_{ei}, and z_{ij}. Thus, $e_i, \dot{e}_i, z_{ij}, \dot{z}_{ij}$ are also bounded. $\ddot{V}$ is also bounded from (24), (30), and (31); thus, $\dot{V}$ is uniformly continuous. Therefore, it is shown by Barbalat's lemma that $\dot{V}$ converges to zero, i.e, e_i converges to zero. From (30) and (31), z_{ei} and z_{ij} also converge to constant vectors, which are the solution of (20). Because $M(t) \in M_I$, and by *Proposition 5*, $x(t) \geq\geq 0$, $t \geq 0$, for all $x_0 \in \bar{\mathbb{R}}_+$. □

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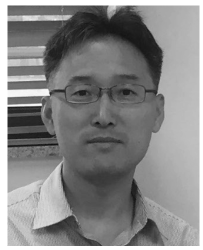

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