Combat Systems Engineering - Radar Systems

Class 3 – Search and Detection Overview

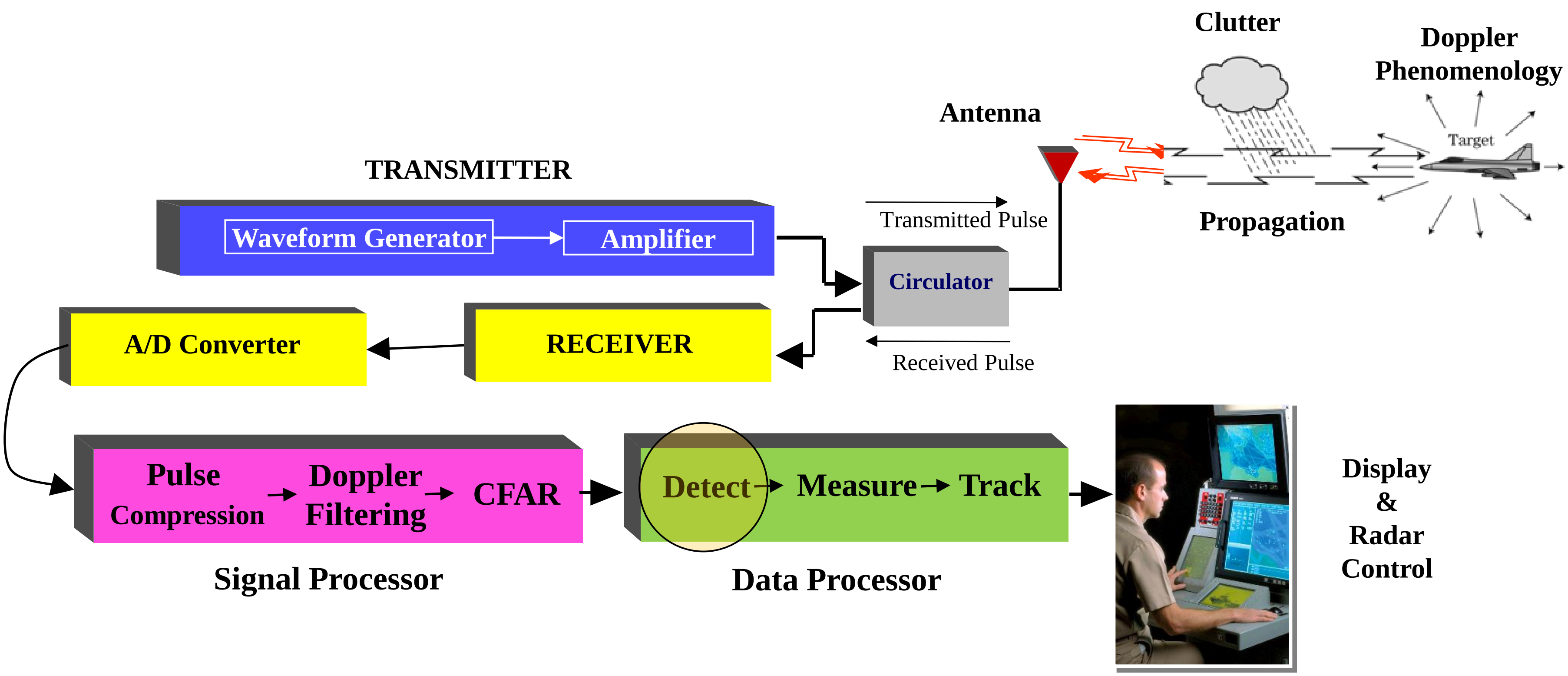
POMR Chapter #3 Lecture

Instructor – Ingar Blosfelds

Class Schedule

Class		Subject	Date
1	**Overview**	*Introduction*	*9/5/2018*
2		*Radar Equation*	*9/12/2018*
3		Detection / Probability	9/19/2018
4	**External Factors**	Propagation Effects	9/26/2018
5		Clutter Characteristics	10/3/2018
6		Target Reflectivity / Fluctuation Models	10/10/2018
7	-Midterm distributed-	Doppler Phenomenology / Fourier Transform	10/17/2018
8	**Subsystems**	Antennas	10/24/2018
9	- Midterm due -	Transmitters / Solid State Antennas	10/31/2018
10		Receivers / Exciters	11/7/2018
11	**Signal/Data Processing**	Signal Processing	11/14/2018
12	- Thanksgiving -	Pulse Compression Waveforms	11/21/2018
13		Doppler Waveforms	11/28/2018
14	- Final distributed -	CFAR	12/5/2018
15		Radar Tracking	12/12/2018
	FINAL EXAM	Final Exam Due at 9 pm	12/19/2018

Basic Radar Block Diagram



Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Random Variables - Introduction

- If each outcome of a probability experiment is assigned a numerical value, then as we observe the results of the experiment we are observing a random variable
- Random variable: A variable that assumes a unique numerical value for each of the outcomes of the sample space of a probability experiment.
  - It is called "random" because the value it assumes is the result of a "by chance", or random, experiment
- Let X represent a random variable having values x. Then X(s) is a function of the elements s of sample space S
  - The domain of the function X(s) is S, and the range of X is some portion of the real line ($X : S \rightarrow R$)
    - Note that the real line spans from $-\infty$ to $+\infty$

Random Variables - Introduction

- <u>Discrete Random Variable</u>: A quantitative random variable that can assume a countable number of values. Intuitively, there is a gap between any two values.
  - Example: The number of heads observed on 5 coin tosses {0, 1, 2, 3, 4, 5}
- <u>Continuous Random Variable</u>: A quantitative random variable that can assume an uncountable number of values. Intuitively, the continuous random variable can assume any value along a line interval, including every possible value between any two values.
  - Example: qualifying speeds for a race in mi/hr [150, 250]

Random Variables – continuous RV

- **<u>Cumulative Distribution Function (CDF)</u>: defined for random variable X as:**

$$F_X(x) = P[X \le x]$$

- **Properties of the CDF:**
  1. **$F_X(\infty) = 1$**
  2. **If $x_1 \le x_2$, then $F_X(x_1) \le F_X(x_2)$ (i.e., $F_X(x)$ is a non-decreasing function)**
  3. **$F_X(x)$ is continuous:**

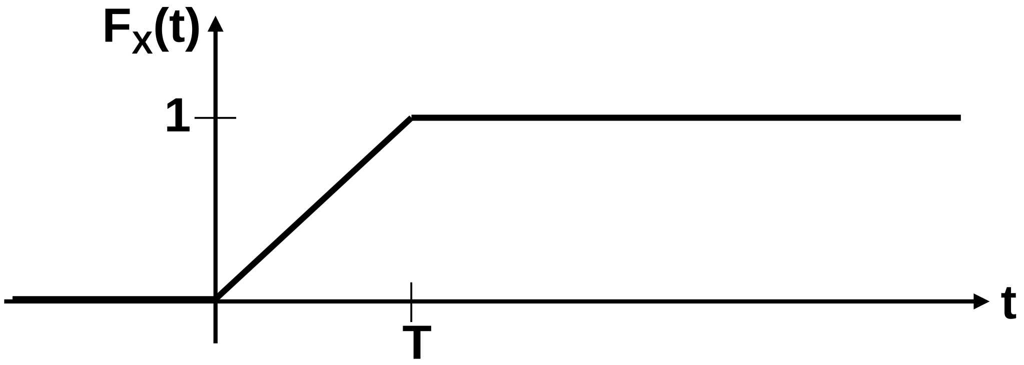
$$F_X(x) = \lim_{\varepsilon \to 0} F_X(x + \varepsilon), \ \varepsilon > 0$$

  - **An outcome of these properties is the useful relationship: $P[x_1 < X \le x_2] = F_x(x_2) - F_x(x_1)$**

Random Variables – Continuous RV

- **<u>CDF example</u>: Example: a bus arrives at random in (0,T]. RV X denotes the time of arrival, and assume the bus is equally likely to arrive within the interval (0,T]**

$$F_X(t) = \begin{cases} 0, & t \le 0 \\ t/T, & 0 < t \le T \\ 1, & t > T \end{cases}$$



Random Variables – continuous RV

- Probability density function (pdf): defined for random variable X as:

$$f_X(x) = \frac{dF_X(x)}{dx}$$

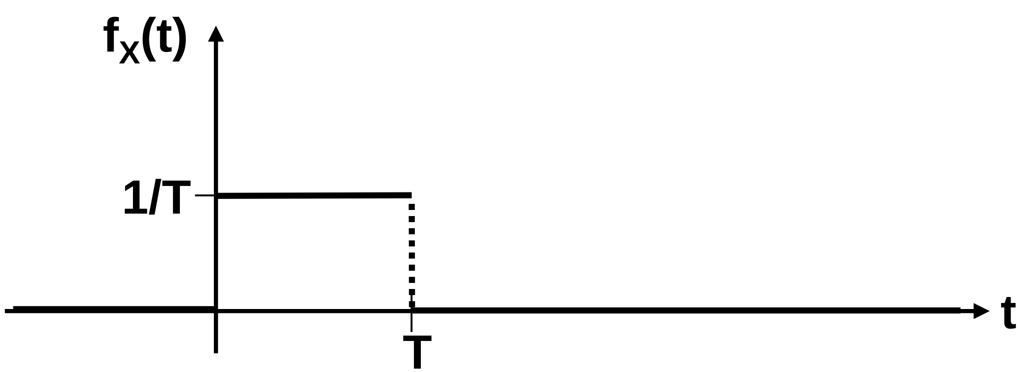
- Properties of the pdf:

1. $\int_{-\infty}^{\infty} f_X(u)du = F_X(\infty) - F_X(-\infty) = 1$

2. $F_X(x) = \int_{-\infty}^{x} f_X(u)du = P[X \le x]$

3. $F_X(x_2) - F_X(x_1) = \int_{-\infty}^{x_2} f_X(u)du - \int_{-\infty}^{x_1} f_X(u)du = P[x_1 < x \le x_2]$

- Example: pdf of RV from previous slide:



Probability Density Functions

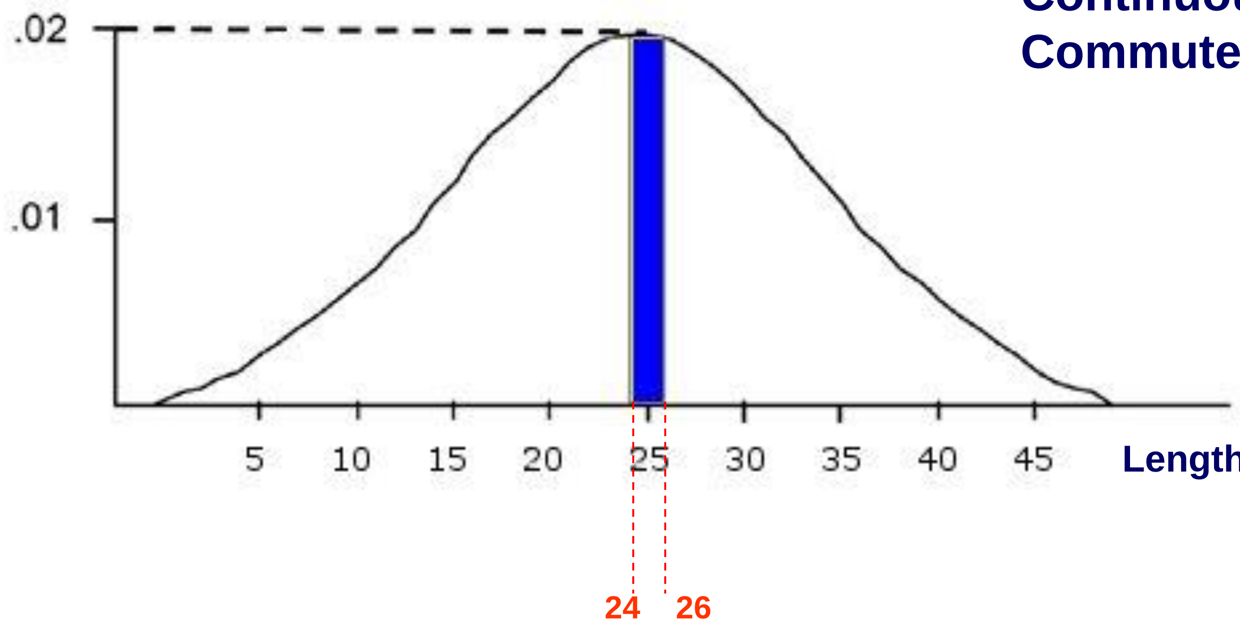
- For a continuous random variable X, a probability density function is a function such that

(1) $$f(x) \geq 0$$

(2) $$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

(3) $$P(a \leq x \leq b) = \int_{a}^{b} f(x)\,dx$$

Probability Density Functions (cont'd)

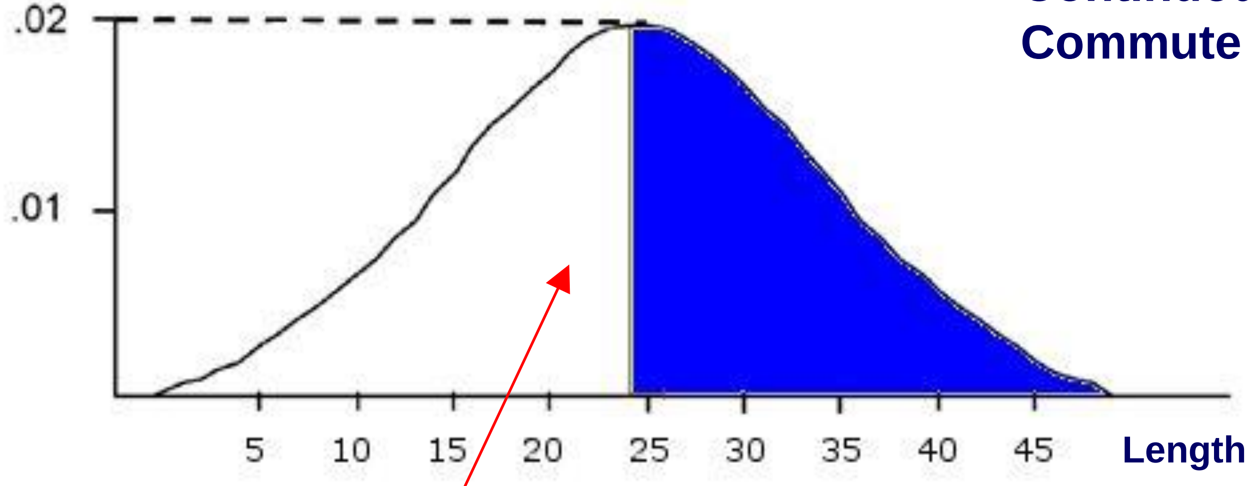


Continuous Distribution of Commute Times to Work

Length of commute (minutes)

- **Probability that someone picked at random has a commute between 24 and 26 minutes long is 0.02 x 2 = 0.04 → 4%**
  - **Equal to the area under PDF curve between 24 and 26**

Probability Density Functions (cont'd)



Continuous Distribution of Commute Times to Work

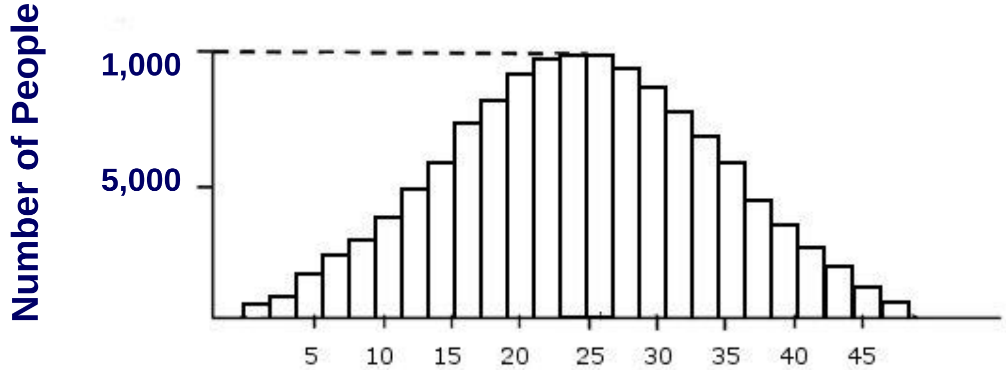
Length of commute (minutes)

Area under PDF curve is one

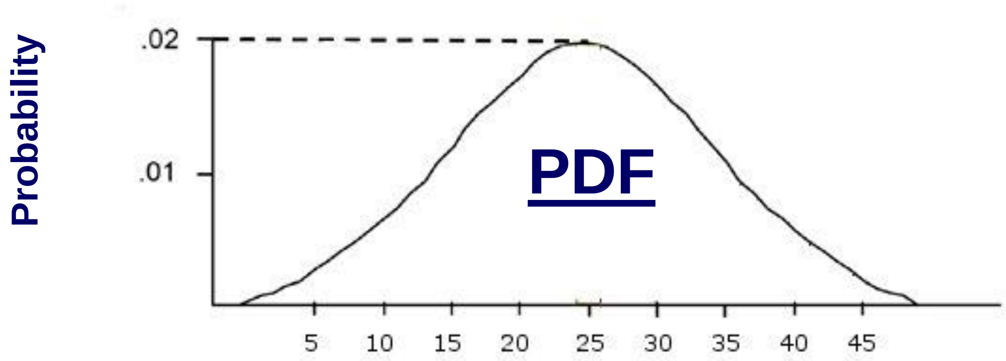
- **Probability that someone picked at random has a commute longer than 25 minutes is 0.5 → 50%**
  - **Equal to the area under PDF curve greater than 25**

Histograms and Probability Density Functions

Histogram



Discrete Increments



Continuous

PDF for a Continuous Random Variable

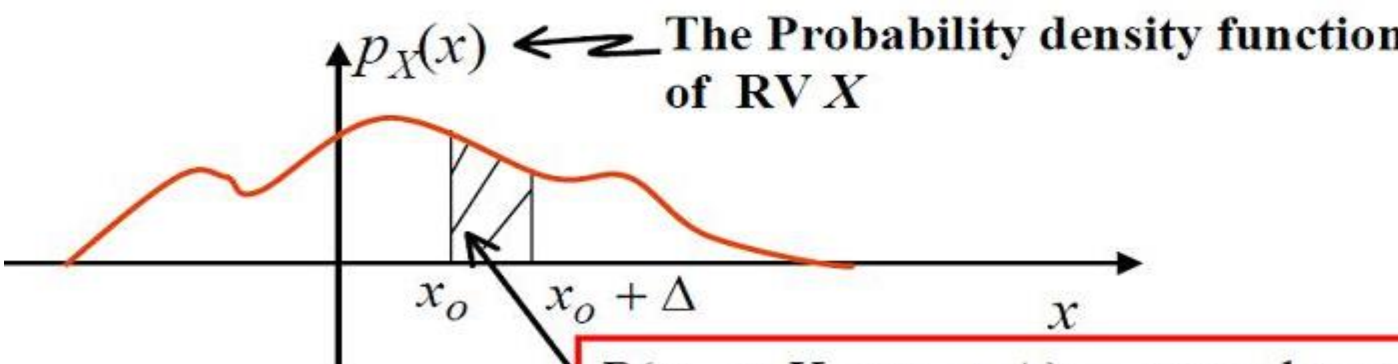
Given Continuous RV X…

What is the probability that $X = x_0$?

- Oddity : $P(X = x_0) = 0$

  Otherwise the Prob. "Sums" to infinity

- Need to think of Prob. *Density* Function (PDF)



$$P(x_0 < X < x_0 + \Delta) = \text{ area shown}$$

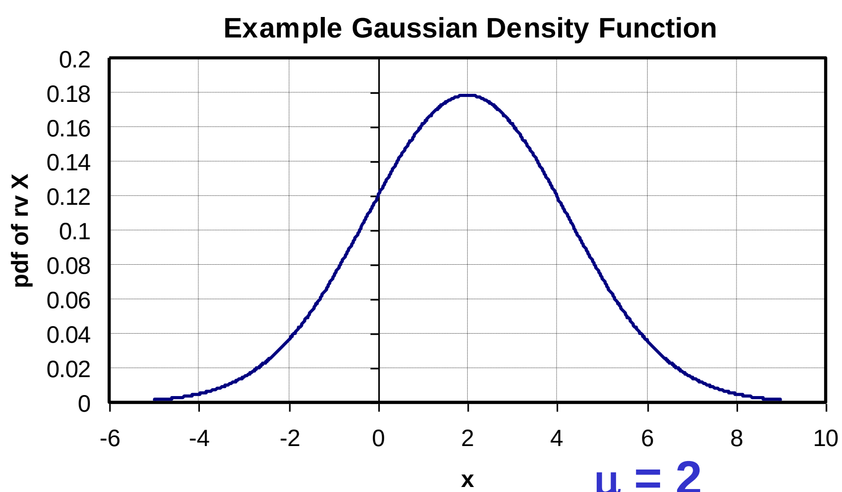
$$= \int_{x_o}^{x_o+\Delta} p_X(x)dx$$

Continuous RV

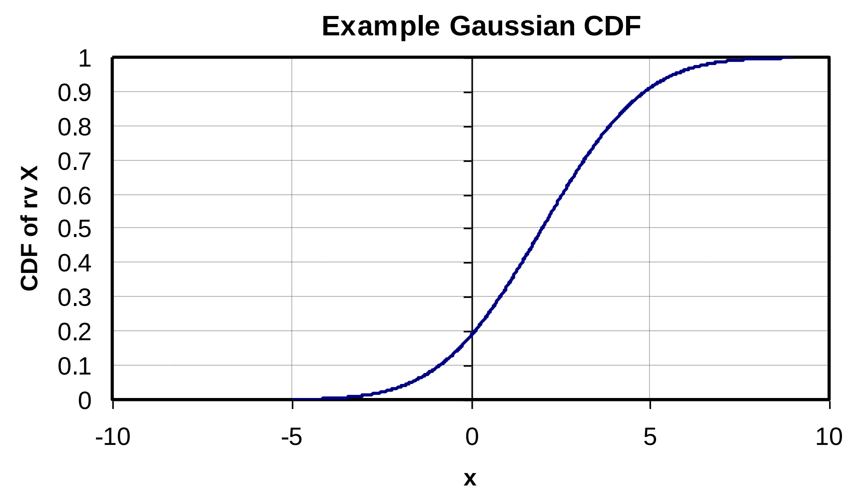
- **pdf example**: Gaussian pdf is defined as:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}, \quad -\infty < x < \infty$$

μ = mean of X
σ = standard deviation of X



$\mu = 2$
$\sigma = \text{sqrt}(5)$



Continuous RV

- **Gaussian random variable:**
  - **Many physical processes can be modeled as Gaussian, and it is thus a good tool for characterizing & analyzing the statistical properties of such processes. Example: thermal noise in a radar system**
  - **By law of large numbers, the sum of many RV's is Gaussian:**
    - **Thermal noise, multipath fading, radar cross section (In-phase/quadrature components)**
  - **The CDF for the Gaussian RV is:**

$$F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left[\frac{u-\mu}{\sigma}\right]^2} du$$

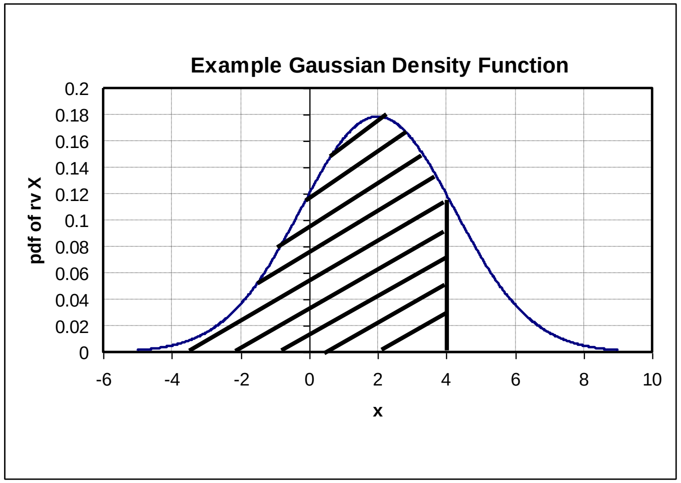
  - **The above integral has no closed form expression; however, this expression is well tabulated via the "error function", or erf(x)**
    - **Transformation $z = (x - \mu)/\sigma$**

Continuous RV

- **Gaussian random variable:**
  - **Use of the error function for the previous example:**

$$P[X \le 4] = F_X(4) = \int_{-\infty}^{4} \frac{1}{\sqrt{2 \cdot \pi \cdot 5}} e^{-\frac{1}{2}\left[\frac{u-2}{\sqrt{5}}\right]^2} du$$

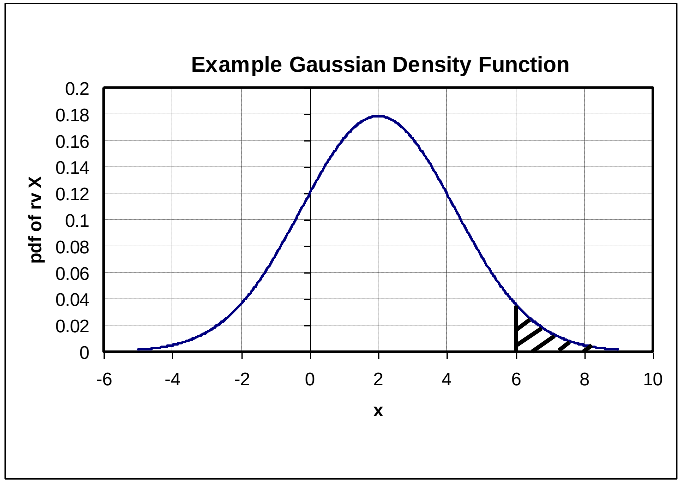
$$= erf\left(\frac{4-2}{\sqrt{5}}\right) = 0.8145$$



  - **Use of the error function to find P[X > x]**

$$P[X > 6] = 1 - F_X(6) = \int_{6}^{\infty} \frac{1}{\sqrt{2 \cdot \pi \cdot 5}} e^{-\frac{1}{2}\left[\frac{u-2}{\sqrt{5}}\right]^2} du$$

$$= 1 - erf\left(\frac{6-2}{\sqrt{5}}\right) = 0.0368$$



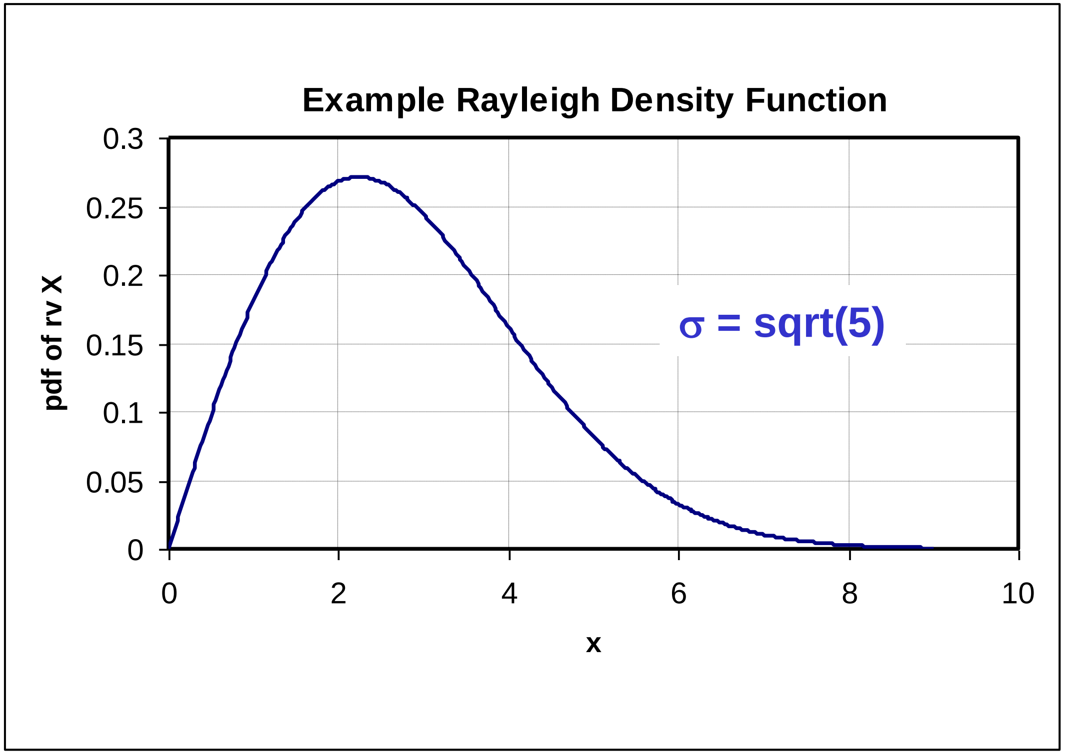
Common Continuous pdfs

- Gaussian: Limit in sum of RV’s
  - Thermal noise, multipath, RCS- I and Q
- Rayleigh: Square root of the sum of squares of two Gaussian RVs
  - Amplitude of complex noise, multipath, RCS
- Rician: Amplitude with two Gaussians and dominant component
  - Amplitude with multipath with line-of-sight component, RCS with dominant scatterer
- Exponential: Sum of squares of two Gaussian RVs
  - Power of noise, multipath and RCS with square law receiver
- Chi-square with n degrees of freedom: Sum of n Gaussian RVs
  - Non-coherent detection of n samples

Rayleigh

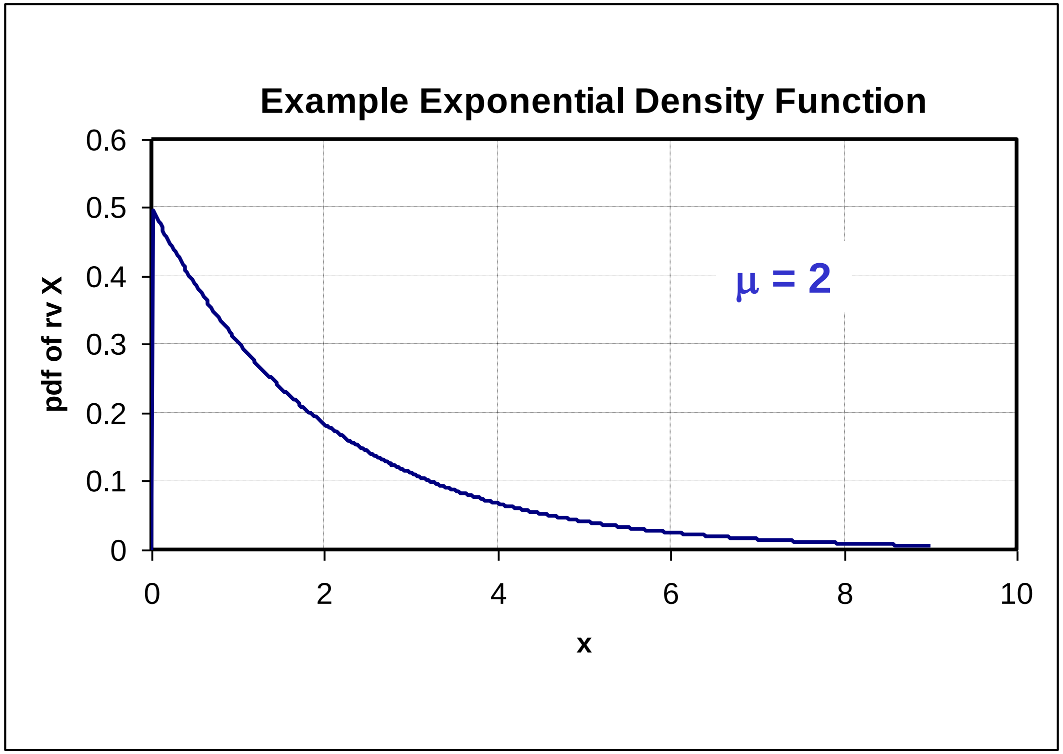
$$f_X(x) = \frac{x}{\sigma^2} \cdot e^{-x^2/2\cdot\sigma^2} \cdot u(x)$$

u(x) is unit step function



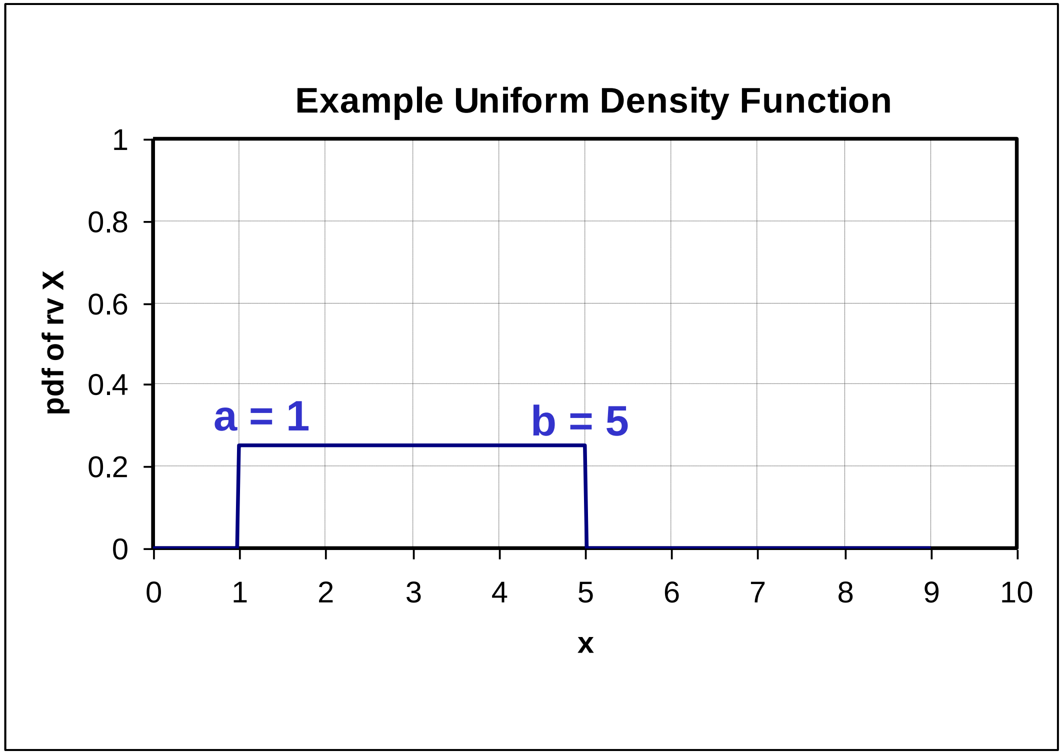
Exponential

$$f_X(x) = \frac{1}{\mu} \cdot e^{-x/\mu} \cdot u(x)$$



Uniform

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a < x \le b \\ 0, & \text{else} \end{cases}$$



Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Expected Values

- Expectations of random variables are used to summarize key properties of the random variables by just a few numbers (such as average value, standard deviation, etc.)
- For sampled data sets, where the underlying probabilistic model may or may not be known, the following common sampled statistics are used for a set of measured values $x_1, x_2, \ldots, x_N$
  - <u>Mean (arithmetic)</u>: represents the "most likely" value of the data set:

$$\mu_x = \frac{1}{N} \cdot \sum_{i=1}^{N} x_i$$

  - <u>Mode</u>: most commonly occurring value of the data set (may not exist, or may not be unique)
  - <u>Median</u>: 50% value of data set – half the samples are less than the median, half are greater; for N odd, median is the middle value of the ordered data set; for N even, median is the mean of the two middle values

Expected Values

- **Sampled statistical measures cont.**
  - **<u>Standard deviation</u>: tells how much the numbers of the measured set spread or deviate from the mean value**

$$\sigma_X = \left[\frac{1}{N} \cdot \sum_{i=1}^{N} (x_i - \mu_x)^2 \right]^{1/2}$$

  - **<u>Variance</u>: square of the standard deviation**
  - **<u>Root mean square (RMS)</u>: a measure of the variation of the magnitude of the data, useful when data set has positive & negative values (such as sinusoids):**

$$RMS = \left[\frac{1}{N} \cdot \sum_{i=1}^{N} x_i^{\,2} \right]^{1/2}$$

Expected Values

- Expected values for random variables are denoted E[·]
- The expected value (or mean value, μ) of a random variable is computed as:

$$E[X] = \sum_{i=1}^{N} x_i \cdot P_X(x_i), \quad \text{discrete RV}$$

$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x)dx, \quad \text{continuous RV}$$

- The r^{th} "moment" for a random variable is computed as:

$$E[X^r] = \sum_{i=1}^{N} x_i^r \cdot P_X(x_i), \quad \text{discrete RV}, r = 0, 1, 2, \ldots$$

$$E[X^r] = \int_{-\infty}^{\infty} x^r \cdot f_X(x)dx, \quad \text{continuous RV}, r = 0, 1, 2, \ldots$$

(Note that for r = 1, it is the mean value; for r = 2, it is the mean squared value)

Expected Values

- The r^{th} central moment of a random variable is computed as:

$$E\left[(X-\mu)^r\right]=\sum_{i=1}^{N}(x_i-\mu)^r\cdot P_X(x_i),\quad \text{discrete RV}, r=0,1,2,\ldots$$

$$E\left[(X-\mu)^r\right]=\int_{-\infty}^{\infty}x\cdot f_X(x)dx,\quad \text{continuous RV}, r=0,1,2,\ldots$$

  - Particularly important is the 2^{nd} moment, called the <u>variance</u>, often symbolized as σ^2

$$\sigma^2=E\left[(X-\mu)^2\right]=E\left[X^2\right]-E\left[2\cdot\mu\cdot X\right]+E\left[\mu^2\right]$$

And noting that μ is a constant, and in general the expected value of a constant is the constant:

$$\begin{aligned}\sigma^2&=E\left[X^2\right]-E\left[2\cdot\mu\cdot X\right]+E\left[\mu^2\right]\\&=E\left[X^2\right]-2\cdot\mu\cdot E\left[X\right]+\mu^2\\&=E\left[X^2\right]-\mu^2\end{aligned}$$

Expected Values

- Expected values for the distributions discussed:

Distribution	RV Type	Mean	Variance
Gaussian	Continuous	μ	σ^2
Rayleigh	Continuous	$\sigma \cdot \sqrt{\frac{\pi}{2}}$	$\frac{4-\pi}{2} \cdot \sigma^2$
Exponential	Continuous	μ	μ^2
Uniform	Continuous	(a + b)/2	$\frac{(b-a)^2}{12}$

Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Functions of Random Variables

- Find the density function of $Y = g(X)$ where X is a continuous random variable with cumulative distribution function $F_X(x)$ and $g(X)$ is a function of X
  - Define $g(X) = aX + b$
  - Define $F_Y(y)$ to be the cumulative distribution function of Y: $F_Y(y) = P(Y \leq y) = P(aX+b \leq y)$

$$P(aX + b \leq y) = P\left(X \leq \frac{y-b}{a}\right) = F_X\left(\frac{y-b}{a}\right)$$

  - To find the probability density function of Y:

$$f_Y(y) = \frac{d}{dy} F_X\left(\frac{y-b}{a}\right) = \frac{1}{a} f_X\left(\frac{y-b}{a}\right)$$

Functions of Random Variables

- **<u>Central Limit Theorem</u>: Very important and useful finding that the sum of many independent, random variables with means μ_1, μ_2, … μ_N, and variances σ_1^2, σ_2^2, …, σ_N^2 tends to a Gaussian distribution:**

$$Y = \sum_{i=1}^{N} X_i$$

approaches a Gaussian pdf as N becomes large with mean:

$$\mu_Y = \sum_{i=1}^{N} \mu_i$$

and variance:

$$\sigma_Y^2 = \sum_{i=1}^{N} \sigma_i^2$$

  - **Note that it is not necessary for the pdfs of each component random variable to be identical, although this is often assumed**

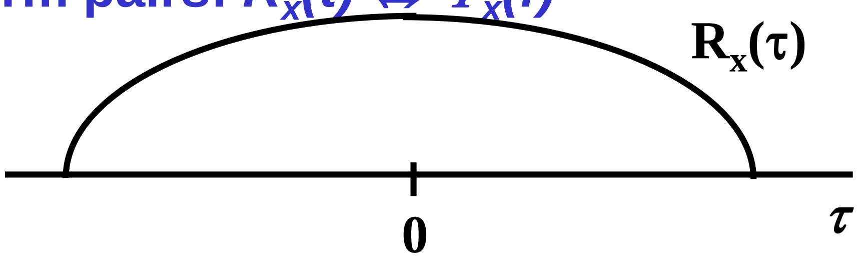
Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Autocorrelation (Energy Signals)

- Defined for real signals as $R_x(t) = x(t)*x(-t)$
  - ** represents convolution*
- Measures signal self-similarity at t
- Useful for synchronization: $|R_x(t)| \leq R_x(0)$

- Energy spectral density and autocorrelation are Fourier transform pairs: $R_x(t) \Leftrightarrow \Psi_x(f)$



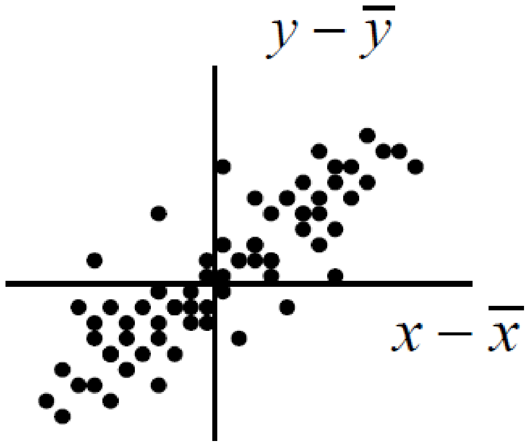
$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = x(\tau) * x(-\tau) \Leftrightarrow X(f)X^*(f) = |X(f)|^2 = \Psi_x(f)$$

Properties Of Autocorrelation Function

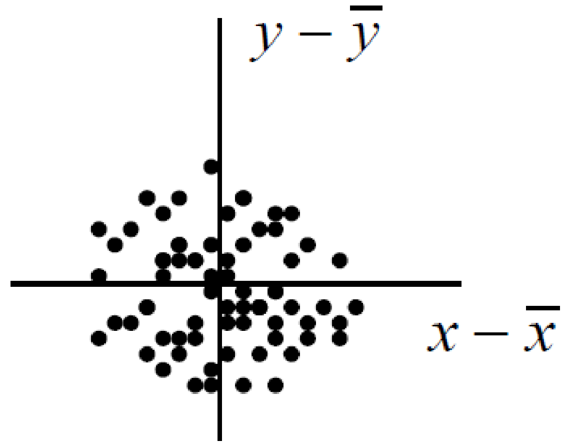
- For real-valued (and wide sense stationary in the case of random signals):
  - Autocorrelation and spectral density form a Fourier transform pair.
  - Autocorrelation is symmetric around zero.
  - Its maximum value occurs at the origin.
  - Its value at the origin is equal to the average power or energy.

Illustrating 3 Main Types of Correlation of Two RVs

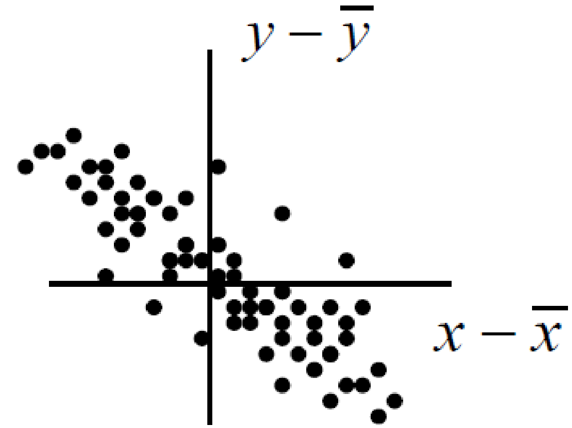
Data Analysis View: $C_{xy} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \bar{x})(y_i - \bar{y})$



Positive Correlation
"Best Friends"



Zero Correlation
i.e. uncorrelated
"Complete Strangers"



Negative Correlation
"Worst Enemies"

GPA
&
Starting Salary

Height
&
$ in Pocket

Student Loans
&
Parents' Salary

Independent vs. Uncorrelated

X & Y are Independent

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

PDFs Separate

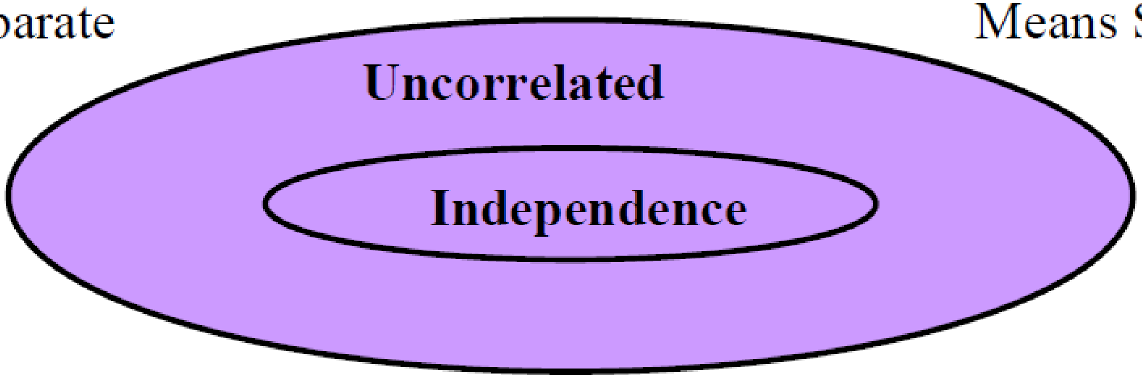
$\xRightarrow{\text{Implies}}$

$\cancel{\Leftarrow}$

X & Y are Uncorrelated

$$E\{XY\} = E\{X\}E\{Y\}$$

Means Separate



INDEPENDENCE IS A STRONGER CONDITION !!!!

Confusing Covariance and Correlation Terminology

Covariance : $\boxed{\sigma_{XY} = E\{(X - \overline{X})(Y - \overline{Y})\}}$

Correlation : $\boxed{E\{XY\}}$

Same if zero mean

Correlation Coefficient : $\boxed{\rho_{XY} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}}$

$\boxed{-1 \leq \rho_{XY} \leq 1}$

Covariance and Correlation for Random Vectors

$$\mathbf{x} = [X_1 \ X_1 \ \cdots \ X_N]^T$$

Correlation Matrix :

$$\boxed{\mathbf{R_x} = E\{\mathbf{x}\mathbf{x}^T\} = \begin{bmatrix} E\{X_1X_1\} & E\{X_1X_2\} & \cdots & E\{X_1X_N\} \\ E\{X_2X_1\} & E\{X_2X_2\} & \cdots & E\{X_2X_N\} \\ \vdots & \vdots & \ddots & \vdots \\ E\{X_NX_1\} & E\{X_NX_2\} & \cdots & E\{X_NX_N\} \end{bmatrix}}$$

Covariance Matrix :

$$\boxed{\mathbf{C_x} = E\{(\mathbf{x} - \overline{\mathbf{x}})(\mathbf{x} - \overline{\mathbf{x}})^T\}}$$

Lecture Overview

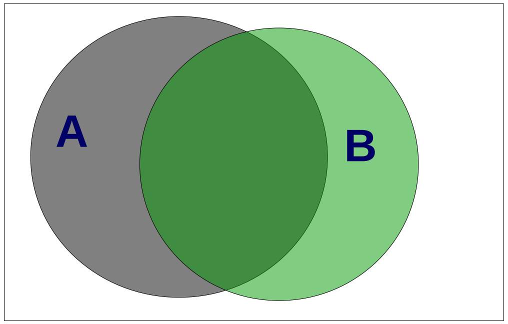
- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Basic Elements of a Statistical Decision

1. **A set of hypotheses** that characterize the possible true states of nature
2. **A test** in which data are obtained from which we wish to infer the truth,
3. **A decision rule** that operates on the data to decide in an optimal fashion which hypothesis best describes the true state of nature
4. **A criterion of optimality**

Bayes’ Theorem

- The mathematical foundations of hypothesis testing rest on Bayes' theorem



- *Conditional probability: P(A|B) = P(A,B)/P(B)*
- *Equivalently: P(B|A) = P(A,B)/P(A)*
- *Equating: P(AIB)P(B) = P(BIA)P(A) = P(A, B)*
- *Dividing: {P(AIB)P(B) = P(BIA)P(A) = P(A, B)}1/P(B)*
- Which gives us:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- Bayes' theorem allows us to infer the conditional probability, *P(A|B),* from the conditional probability *P(B|A)*

Discrete Form of Bayes' Theorem

Bayes' theorem can be expressed in discrete form, as follows:

$$P(s_i|z_j) = \frac{P(z_j|s_i)P(s_i)}{P(z_j)} \qquad \begin{array}{l} i = 1, \ldots, M \\ j = 1, \ldots \end{array}$$

where

$$P(z_j) = \sum_{i=1}^{M} P(z_j|s_i)P(s_i)$$

- s_i is the i^{th} signal class, from a set of *M* classes
- Z_j is the j^{th} sample of a received signal.
- $P(s_i)$, before the experiment, is called the *a priori probability.*
- *After* the experiment, we can compute the *a posteriori probability, $p(s_i|z_j)$,* which can be thought of as a "refinement" of our prior knowledge of nature.
- $P(z_j)$ is the probability of the received sample, Z_j, over the entire space of signal classes.
- $P(z_j)$, can be thought of as a scaling factor, since its value is the same for *each* signal class.

Use of Bayes' Theorem

- Two boxes of parts:
  - Box 1 contains 1000 parts, 10% defective
  - Box 2 contains 2000 parts,   5% defective
  - A box is randomly chosen and then a part is randomly chosen from it, tested, and found to be good, what is the probability that the part came from box 1?
- Solution:

$$P(\text{box } 1|\text{GP}) = \frac{P(\text{GP}|\text{box } 1)P(\text{box } 1)}{P(\text{GP})}$$

where GP means "good part."

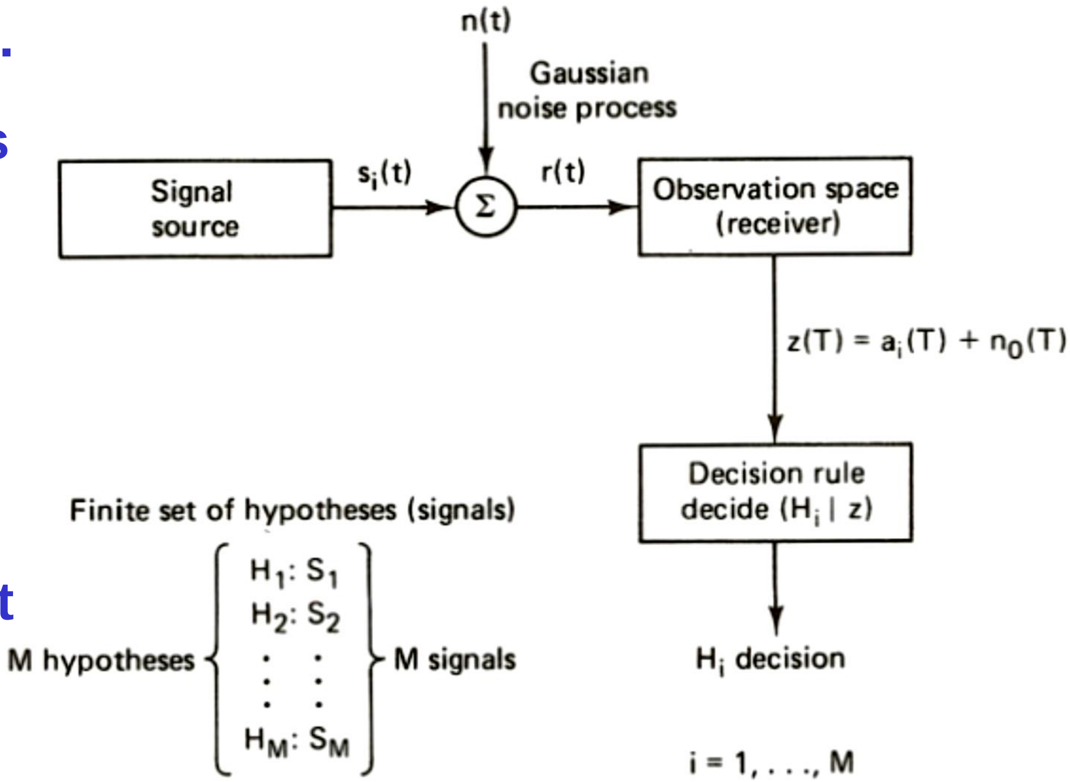
$$\begin{aligned} P(\text{GP}) &= P(\text{GP}|\text{box } 1)P(\text{box } 1) + P(\text{GP}|\text{box } 2)P(\text{box } 2) \\ &= (0.90)(0.5) + (0.95)(0.5) \\ &= 0.450 + 0.475 = 0.925 \end{aligned}$$

$$P(\text{box } 1|\text{GP}) = \frac{0.450}{0.925} = 0.486$$

Components of the Decision Theory Problem in a Communications System

- The signal source at the transmitter consists of a set *{$S_i(t)$}*, i = 1, ... , *M,* of waveforms (or hypotheses).
- A signal waveform *$r(t) = s_i(t) + n(t)$* is received, *$n(t)$* is channel AWGN.
- At the receiver, the waveform is a single number, *$z(t = T)$*, that may appear anywhere on the z-axis.
- Because noise is a Gaussian process and the receiver is linear, the output *$z(t)$* is a Gaussian process and the number, *$z(T)$*, is a ***continuous-valued random variable:***

$$z(T) = a_i(T) + n_0(T)$$



Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
  - Likelihood Ratio Test
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Target Detection Introduction

Radar functions:
- Search
- Track
- Image

Detection/Search:
- Range
- Azimuth
- Elevation
- Doppler
- Cue for other radars

Radar scan types:
- Mechanical
- Electronically Steered Array (ESA)

Concepts:
- Dwell time: Time to transmit and receive n pulses
- Coherent Processing Interval (CPI): Time interval for coherent processing

Target Detection Introduction

Antenna dwell time vs. CPI:

- Typical 10 CPIs for antenna dwell time
- Half power beamwidth θ_3
- Scanning at ω radians/sec
- T_d is the dwell time
- T_{ad} is the antenna dwell time

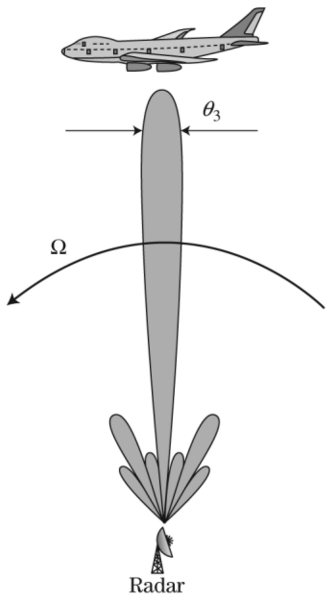
$$T_{ad} = \frac{\theta_3}{\omega}$$

$$n_{CPI} = \frac{T_{ad}}{T_d} = \frac{\theta_3}{\omega T_d}$$

Example

Example:
- Mechanical scan
- Beamwidth 50 mrad (3 degrees)
- Scan at 90 degrees per second
- => Dwell time of 33 milliseconds
- PRF = 10 kHz (0.1 millisecond)
- 32 pulses in a CPI

 => CPI=3.2 milliseconds
 => 10 CPI's in one dwell



"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Correlation Properties of Target and Noise

- Determines integrated S/N
- Noise pulses tend to be uncorrelated
- Target pulses tend to be correlated
- Result: An integrated S/N gain or coherent “pre-detection” integration gain

Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Search

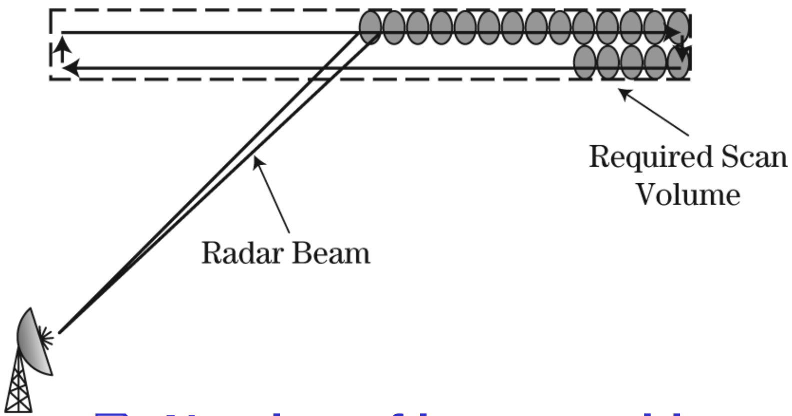


FIGURE 1-29 ■ Coverage of a search volume using a series of discrete beam positions.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

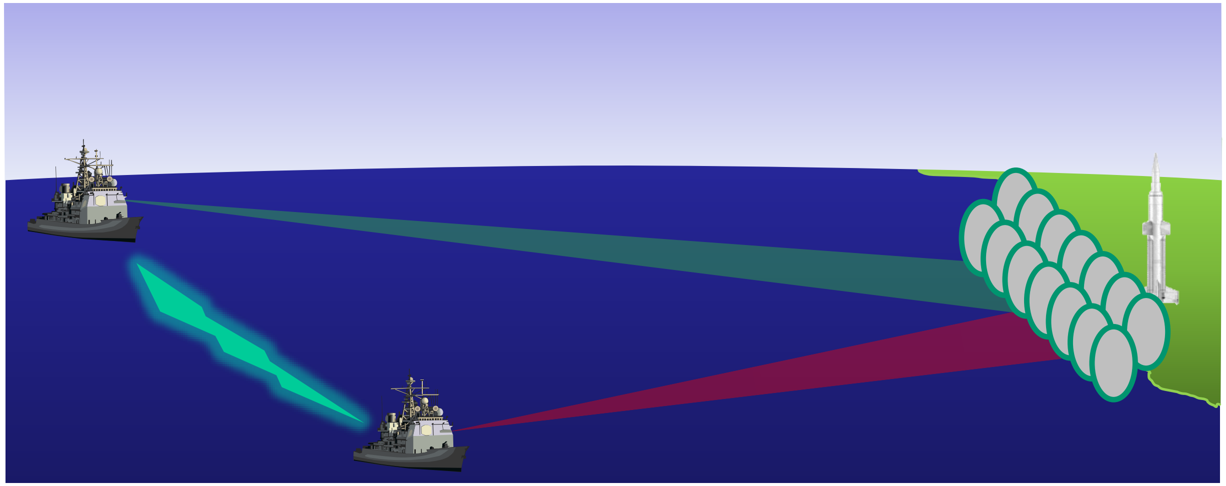
- Number of beam positions:

$$\frac{\theta_{el}\theta_{az}}{(\theta_{3dB})^2}$$

- Mechanical (air traffic control radar/weather radar):
  - Continuous and smooth scan pattern
- Electronically Scanned Array (ESA)
  - Step incrementally, with scan pattern adjustable
  - Scan and track interleaved (one radar vs. scan and track radars with mechanical)

Search Volume

- Search a given solid angle volume within a given time
  - Artillery rounds:
    - Above horizon over a 90 degree azimuth sector to 30 km
  - Limited search volume – low elevations only (cruise missile defense):
    - Search fence:



  - Volume – detect targets over large area / volume

Total Search Time

- ESA: total frame search time T_{fs} for a given volume with m pointing positions

$$T_{fs} = mT_d$$

- Number of beam positions (with solid angle to be scanned Ω and azimuth and elevation beamwidths θ_{az} and θ_{el}

$$m = \frac{\Omega}{\theta_{az}\theta_{el}}$$

Total Search Time

- Example:
  - Search volume: 90 degrees azimuth, 4 degrees elevation
  - 2 degree beamwidth (azimuth and elevation)

  =>45 positions in azimuth, 2 in elevation (90 total)

  Total frame scan time:

$$T_{fs} = \frac{\Omega T_d}{\theta_{az}\theta_{el}}$$

  If 10 msec antenna dwells per position:

  $\Rightarrow$ Total scan time 900 msec

  If PRF=20 kHz, 40 pulses per CPI => CPI =2 msec, each beam position has 5 CPIs

Total Search Time

- Antenna beams spaced at -3 dB point
  - $\Rightarrow$ 6 dB beamshape/scalloping loss
    (could reduce beam spacing, but that increases the number of beam positions)

Mechanical scan

Scan rate determined by total angular search time and total angle

Antenna dwell time < time for beam to scan past a position

Example:

2 degree beam, 45 positions at 10 msec => 90 degrees scanned in 450 msec

=> ω = 90 degrees/450 msec = 200 degrees/sec

Phased Array Antenna Issues

- Mechanical scanning => beamwidth independent of scan angle
- Electronically Scanned Array (ESA)
  - Beamwidth increases (gain decreases) with scan angle
    - Effective aperture reduced by cosine of scan angle
    - 45 degrees => 3 dB SNR decrease (1.5 dB X2)

    - Antenna element gain decrease with scan angle

    ⇒Increase dwell time with scan angle

    ⇒(note: beamshape loss decreases with scan angle)

Search Regimens

- Electronically Scanned Array (ESA):
  - Incorporate multiple interleaved functions on a single radar
    - Save space/power/etc.
    - Scanning and tracking
- Track-while-scan
    - Constant scanning protocol
    - Tracking done from each dwell
    - Example:
      - Antenna rotates at 10 rev/min (360 degrees) => 6 sec/az direction
      - Can track an arbitrary number of targets
    - Good for nonthreatening targets (commercial aircraft), not for threats (missiles)

Search Regimens

- Search-and-track
  - Resource manager
    - Search, then allocate resources for tracking
      - Some dwells for tracking only
  - Easier to implement on ESA
  - Example:
    - 90 degree azimuth, 4 degree elevation search
    - 5% to tracking a single target – 95% search

    => 10% to tracking two targets – 90% search

    ⇒10 targets use 50% of resources – double search frame

After target is detected, need to confirm or qualify target

100 potential targets, 100 msec to qualify

⇒ 10 seconds to qualify all targets (then need to track so not detected/qualified again)

Lecture Overview

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals

Overview of Detection Fundamentals

Detection: determine at azimuth/elevation if target is present

Set threshold and see if received signal (usually more than one pulse) is above threshold

If due to noise => False Alarm

If noise is known *a priori* (thermal noise) => fixed threshold

If interference => adjustable threshold

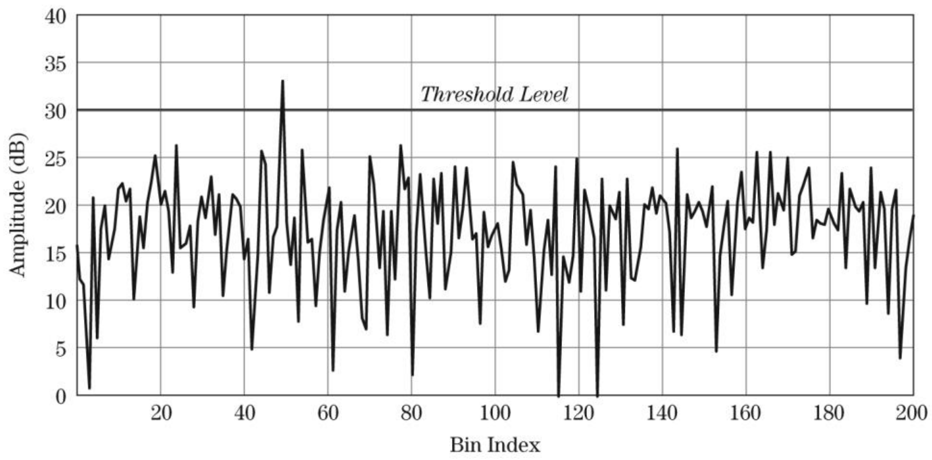
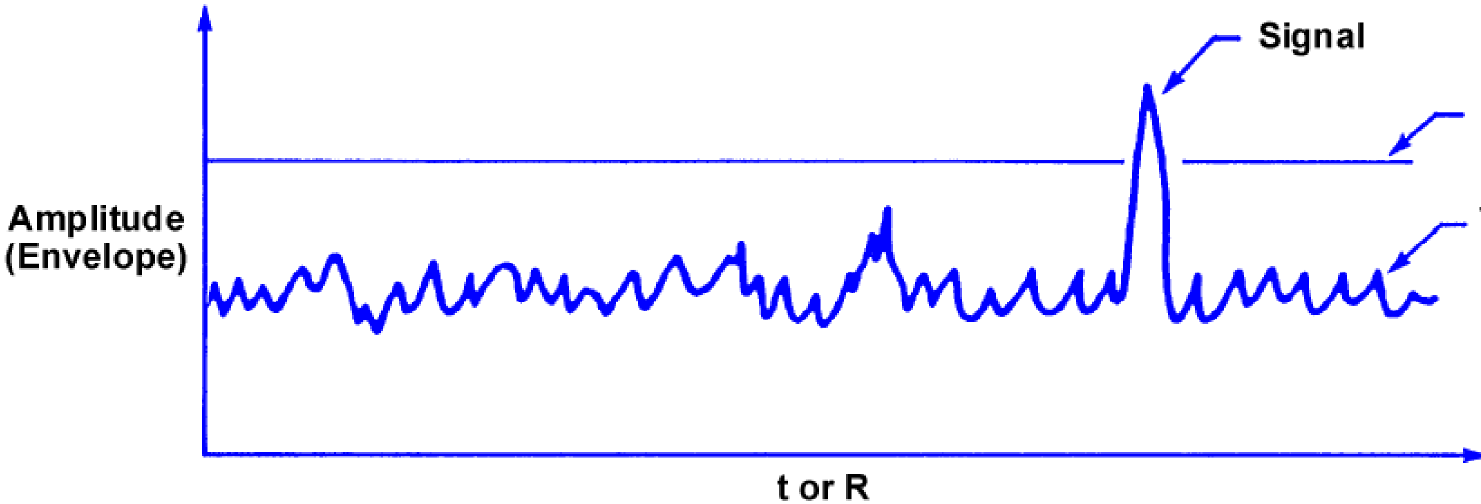


FIGURE 3-2 ■ Concept of threshold detection. In this example, a target would be declared at bin #50.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Detection Processing



- *One objective is to set the detection threshold low so that small signals can be detected*
- *Another objective, but contradictory one, is to set the detection threshold high so few false alarms can occur due to thermal noise*

If a signal is large enough, it is detected and passed on to the Radar Control Computer to Schedule Confirmation and Track Dwells

Interference

- If clutter is present, then noise can exceed signal:
  - Moving Target Indication (MTI), pulse Doppler (based on movement of target)
- Jammers
  - Angle of arrival estimation
    - Phase notching
    - Sidelobe cancellation
    - Adaptive beamforming
- Combination using Space-Time Adaptive Processing (STAP)

Probability of False Alarm and Detection

Probability of Detection, P_D
Probability of False Alarm, P_{fa}

Noise varies randomly pulse to pulse
Signal can vary randomly (due to multiple scatterers)

$$\text{Probability}\{v > V_t\} = \int_{V_t}^{\infty} p(v)\,dv$$

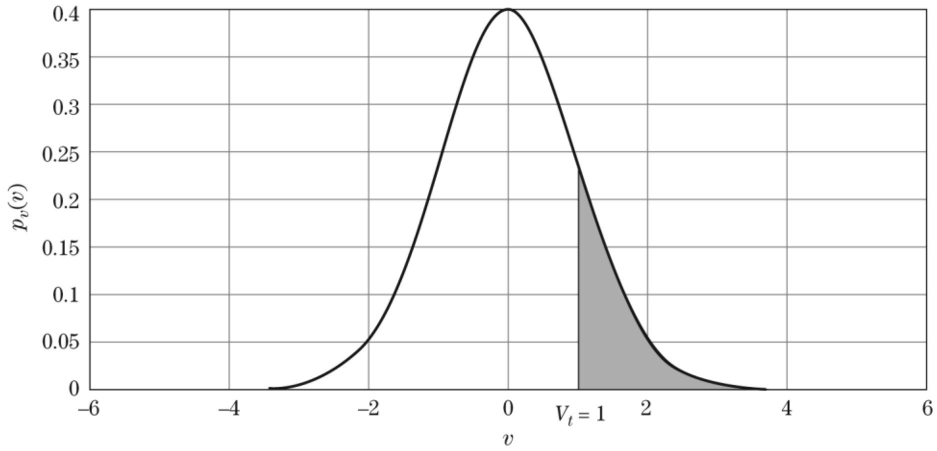


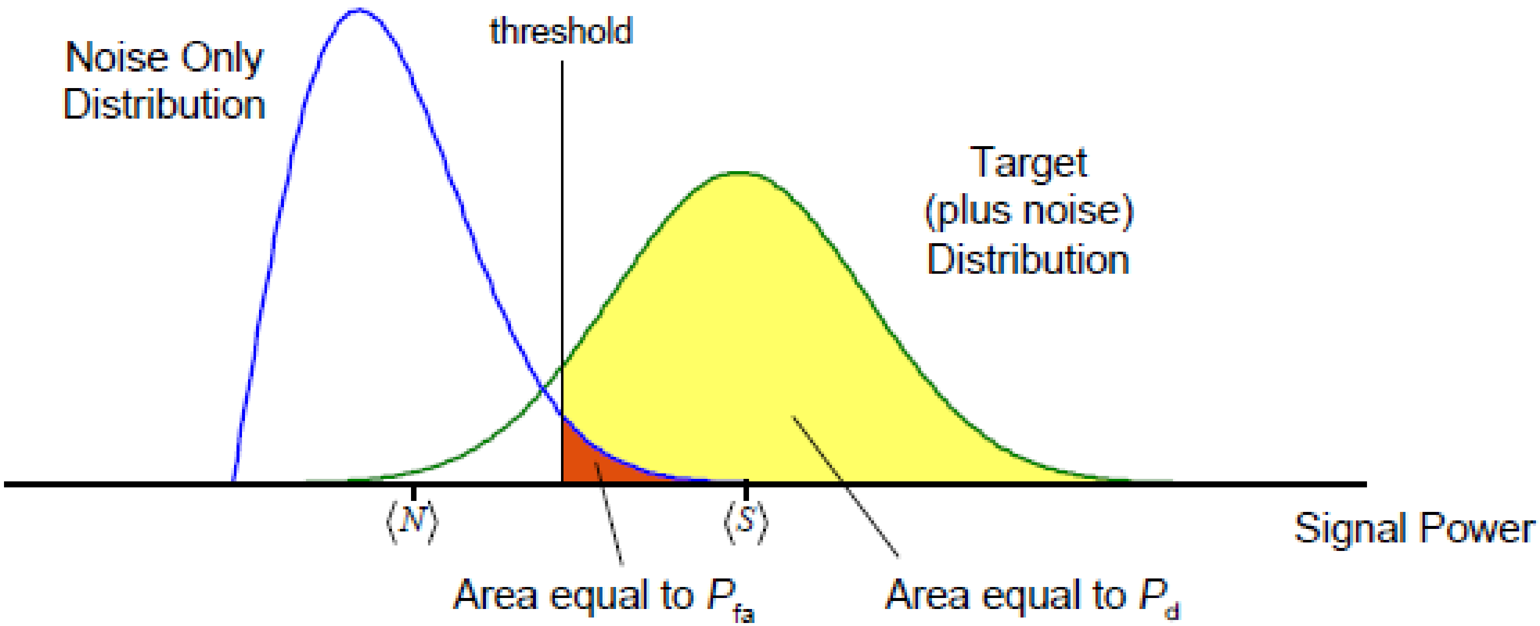
FIGURE 3-3 ■ Gaussian PDF for a voltage, v.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Probability of False Alarm

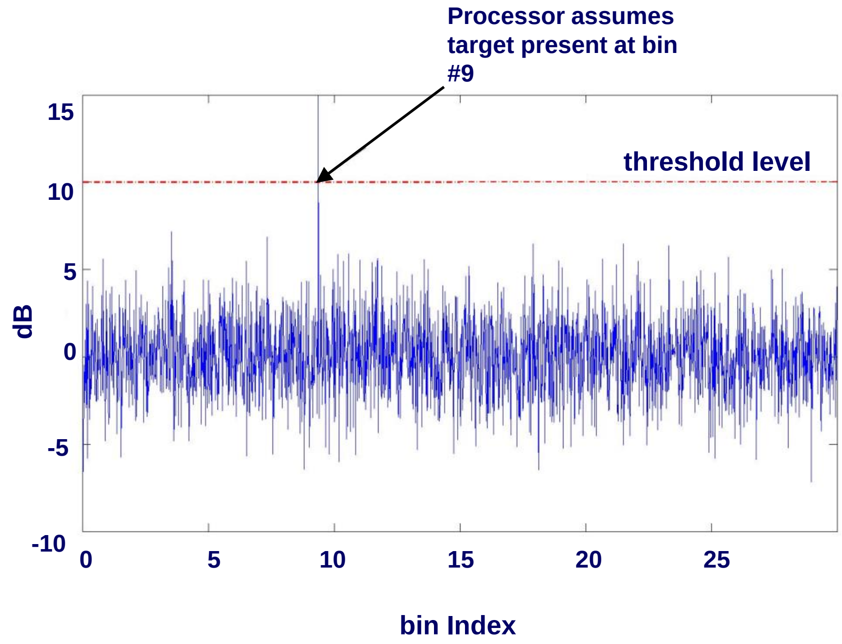
- Detection threshold attempts to minimize the area of the probability distribution function for noise beyond the threshold while maximizing the area for the target return

$$P_{FA} = \int_{Vt}^{\infty} p_i(v)\, dv$$



Threshold Detection

- Almost all radars ultimately use threshold detection to decide whether a target is present
- Threshold answers the question:
  - Is this data value “small” enough that it is probably just interference?
  - Or “big” enough that it is probably *not* just interference?
- IMPORTANT: this answer can be right or wrong!
  - False alarms, missed detections
- Data may be preprocessed to improve S/I ratio first
  - Makes the target stick up above the noise better
- Threshold detection may be applied to:
  - Raw echoes from a single pulse
  - Sum of echoes from same range over several pulses
  - FFT outputs after Doppler processing
  - etc.



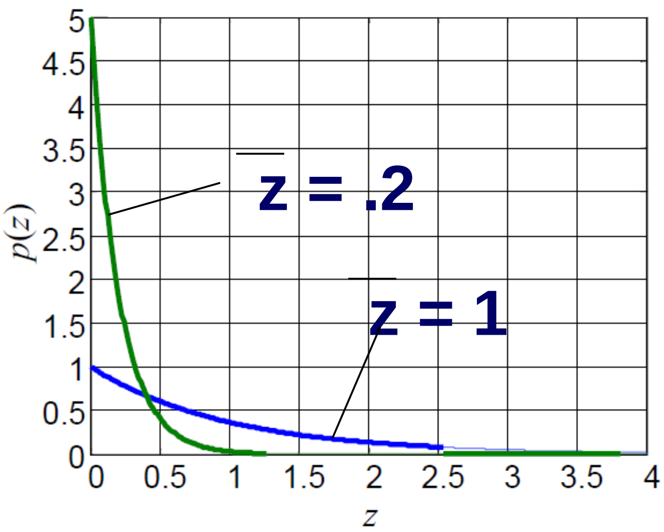
Example: Series of Range bins from a single pulse

Setting the Threshold

- Radar systems usually set the threshold using the Neyman-Pearson criterion
  - First ensure that the probability of false alarm, P_{FA}, does not exceed some specified value
  - Then maximize the probability of detection, P_D, given that value of P_{FA}
- P_{FA} depends only on the noise
- Assuming we use a square law detector $|y|^2$, the statistics of a single noise sample follow an exponential pdf:

$$p_z(z) = \frac{1}{\bar{z}} \exp(-z/\bar{z})$$

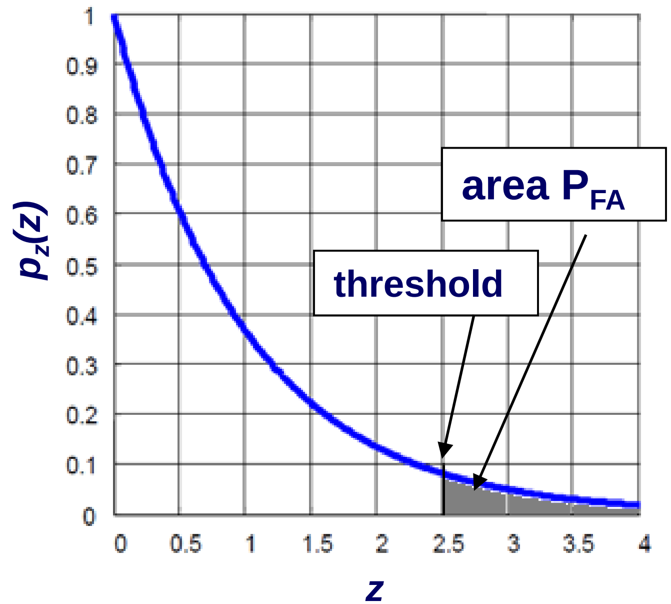
$$z = |y|^2, \quad \bar{z} = mean(z)$$



Setting the Threshold (cont'd)

- P_{FA} is the area under the noise pdf from the threshold to $+\infty$

$$p_z(z) = \exp(-z) \quad \{\bar{z} = 1\}$$



$$P_{FA} = \exp(-T/\bar{z}) \longrightarrow T = -\bar{z}\ln(P_{FA})$$

- Sometimes we integrate N pulses before the threshold test; in this case…

$$P_{FA} = \int_T^{\infty} \frac{z^{N-1}}{(N-1)!} e^{-z} dz$$

$$= 1 - I\left[\frac{T}{\sqrt{N}}, N-1\right]$$

$$(\bar{z} = 1)$$

$$I(u, M) = \int_0^{u\sqrt{M+1}} \frac{e^{-\tau}\tau^M}{M!} d\tau$$

(Incomplete Gamma Function)

How do we choose P_{FA}?

- Requirements for P_{FA} flow down from system requirements
- May depend on:
  - Number of detection opportunities
  - Consequences of a false alarm
  - Subsequent processing steps
- Result typically ranges from 10^{-2} to 10^{-8}

Probability of False Alarm and Detection

- Signal is complex:
  - In-phase I and quadrature Q components of return
- Amplitude of return:

$$r = \sqrt{I^2 + Q^2}$$

- Linear detector
  - Noise is Gaussian (zero mean) => r is Rayleigh

$$p_i(r) = \frac{r}{\sigma_n^{\ 2}} \exp\left(\frac{-r^2}{2\sigma_n^{\ 2}} \right)$$

- Square-law detector => Noise is chi-square with 2 degrees of freedom (exponential)
- Log detector
- => All three give same P_D and P_{fa}

Probability of False Alarm

Choose threshold for a given P_{fa}

For Rayleigh,

$$P_{FA} = \int_{V_T}^{\infty} \frac{A}{\sigma_n^2} e^{-A^2/2\sigma_n^2} dA = e^{-V_T^2/2\sigma_n^2}$$

or

$$V_T = \sqrt{2\sigma_n^2 \ln(1/P_{FA})}$$

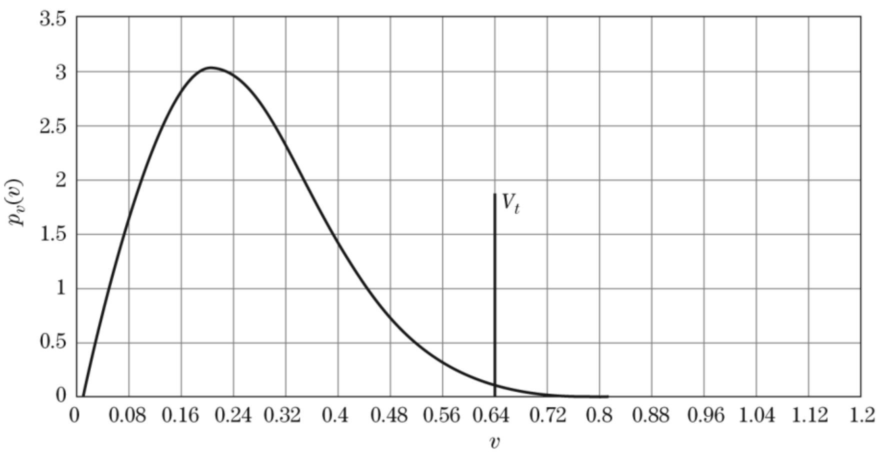


FIGURE 3-4 ■ Rayleigh distribution with an arbitrary threshold.

“Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)”

Probability of False Alarm

- Search systems:
  - If detection made, confirm on subsequent dwells
  - If confirm after n consecutive detections:

$$P_{FA}(n) = \left[P_N(1)\right]^n$$

$\Rightarrow$ If $P_{fa} = 10^{-4}$, then after two trials $P_{fa} = 10^{-8}$

Example:

- 90 beam positions, 333 range bins/position, 32 Doppler bins per range/azimuth position

$\Rightarrow$ 959,040 opportunities for false alarms

If $P_{fa} = 10^{-5}$ => 10 false alarms/scan

If verify once, $P_{fa} = 10^{-10}$ => 1 false alarms per 10,000 scans

Signal-Plus-Noise PDF: Target Detection

- First consider non-fluctuating target

FIGURE 7-1 ■ Examples of RCS calibration spheres. (Courtesy of Professor Nadav Levanon, Tel-Aviv University.)

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

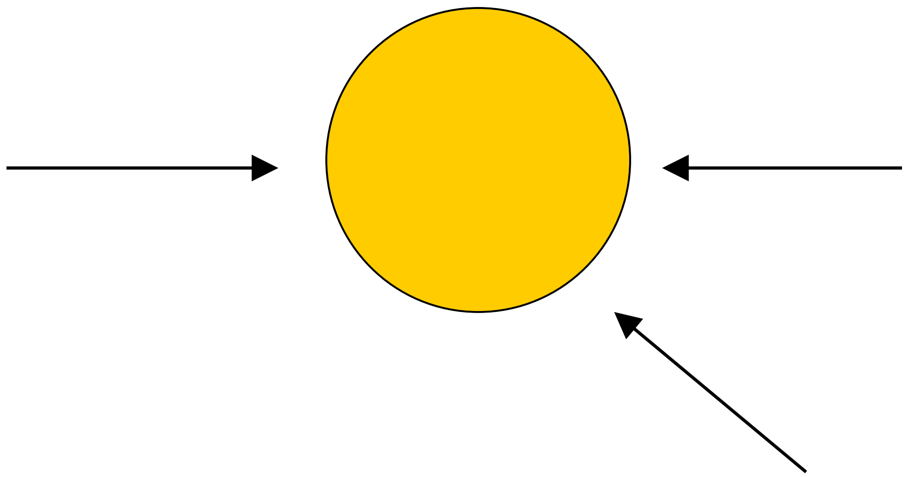
- Target plus noise is Rician (uses modified Bessel function of first kind and zero order):

$$p_{sig}(r) = \frac{r}{\sigma_n^2} \exp\left[-\left(r^2 + r_{sig}^2\right) / 2\sigma_n^2\right] I_0\left(r\, r_{sig} / \sigma_n^2\right)$$

Non-Fluctuating Target

**Nonfluctuating target
("Swerling Case 0")**

Sphere



Scattering response is independent of orientation

Signal-Plus-Noise PDF: Rician

- Non-fluctuation target:
  - P_{fa} same as before
  - But:

$$P_D = \int_{V_t}^{\infty} p_{sig}(r)dr = \int_{V_t}^{\infty} \frac{r}{\sigma_v^{\ 2}} \exp\left[-\left(r^2 + r_{sig}^{\ \ 2}\right)/2\sigma_n^{\ 2}\right] I_0(r\, r_{sig}/\sigma_n^{\ 2})dr$$

  - No closed form, but defined by Marcum's Q function (MATLAB)

Signal-Plus-Noise PDF: Rician

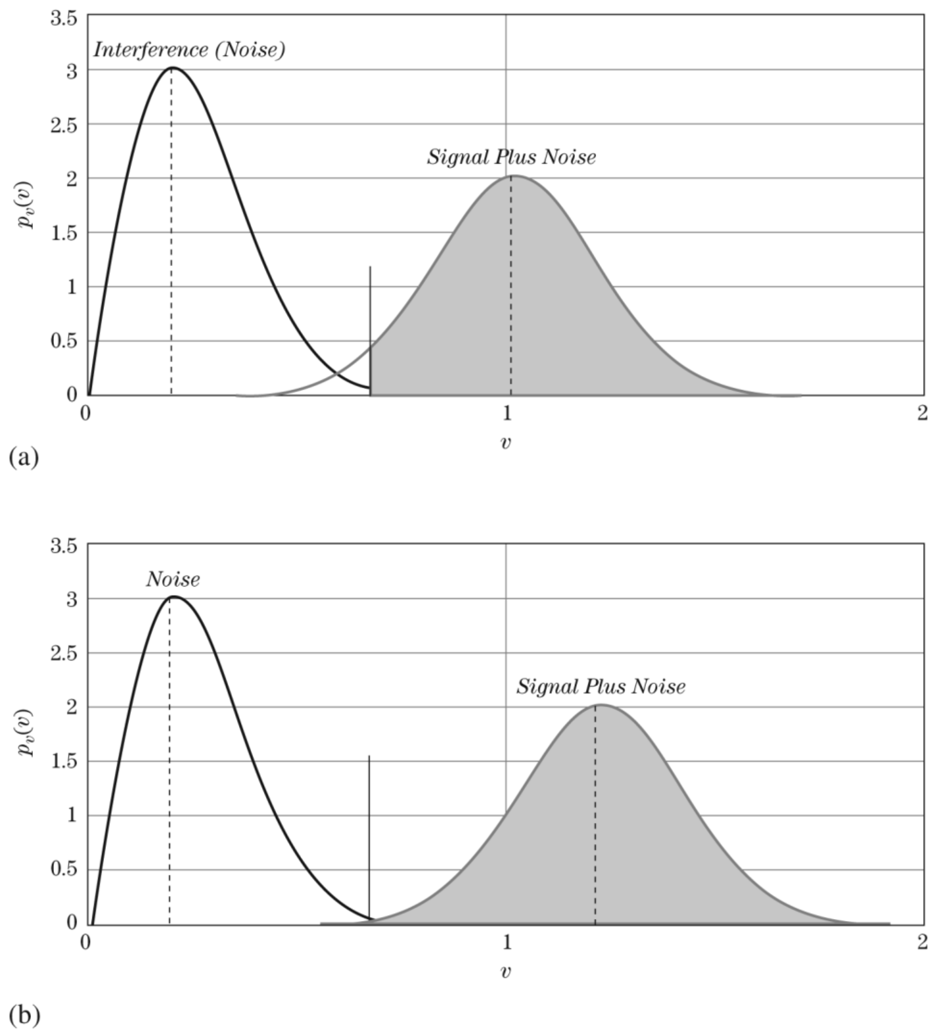


FIGURE 3-5 ■
(a) Noise-like distribution, with target-plus-noise distribution.
(b) Noise-like distribution, with target-plus-noise distribution, demonstrating the higher P_D achieved with a higher SNR.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Typical Values

- Generally,
  - $P_{fa} = 10^{-4}$ to 10^{-4}
  - P_D = 50% to 90%

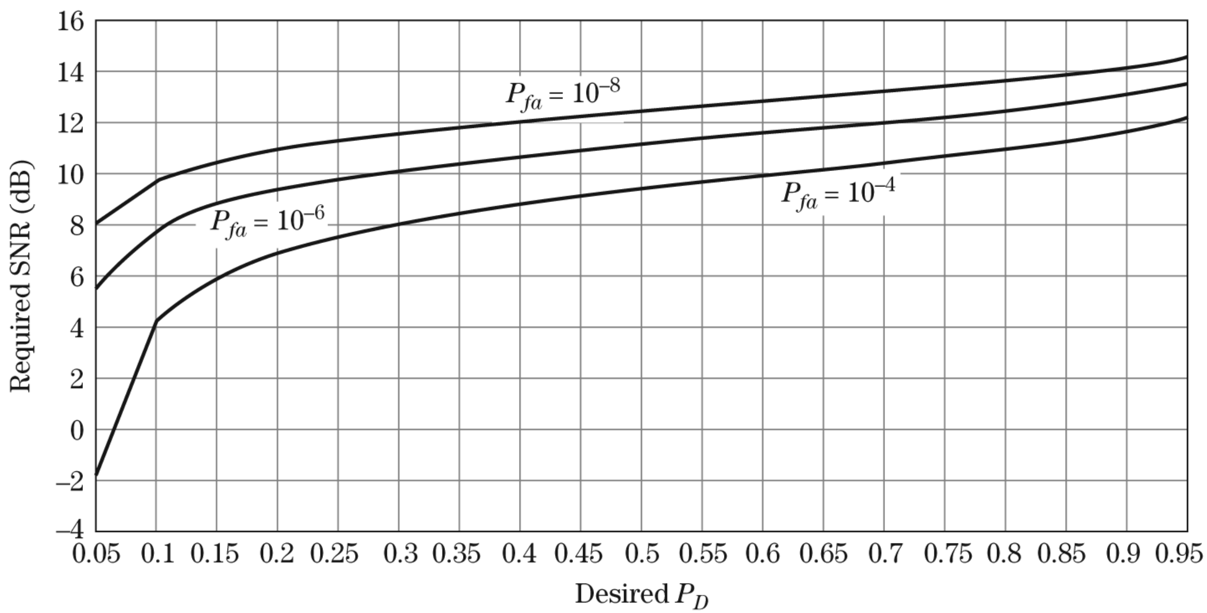
- If conditions not met, then need to adjust parameters of the system

Receiver Operating Curves

- ROC of P_D, P_{fa}, and SNR.

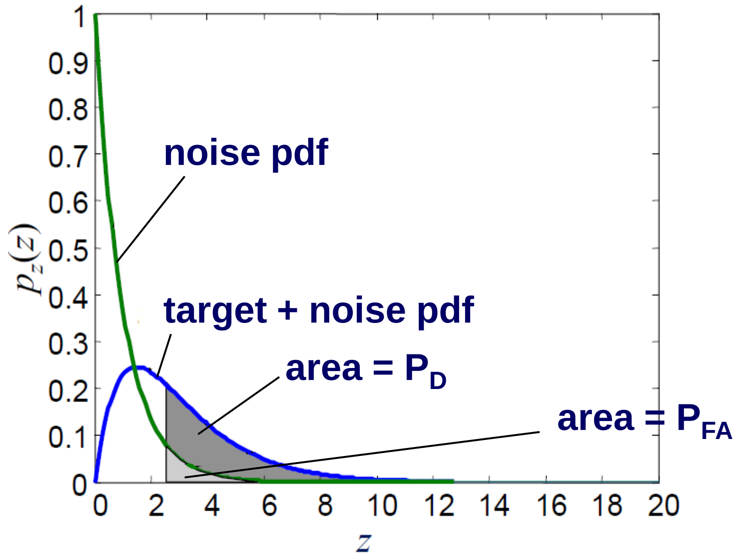
FIGURE 3-6 ■ SNR required to achieve a given P_D, for several P_{FA}'s, for a nonfluctuating (SW0) target in noise.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"



Target Fluctuation Models

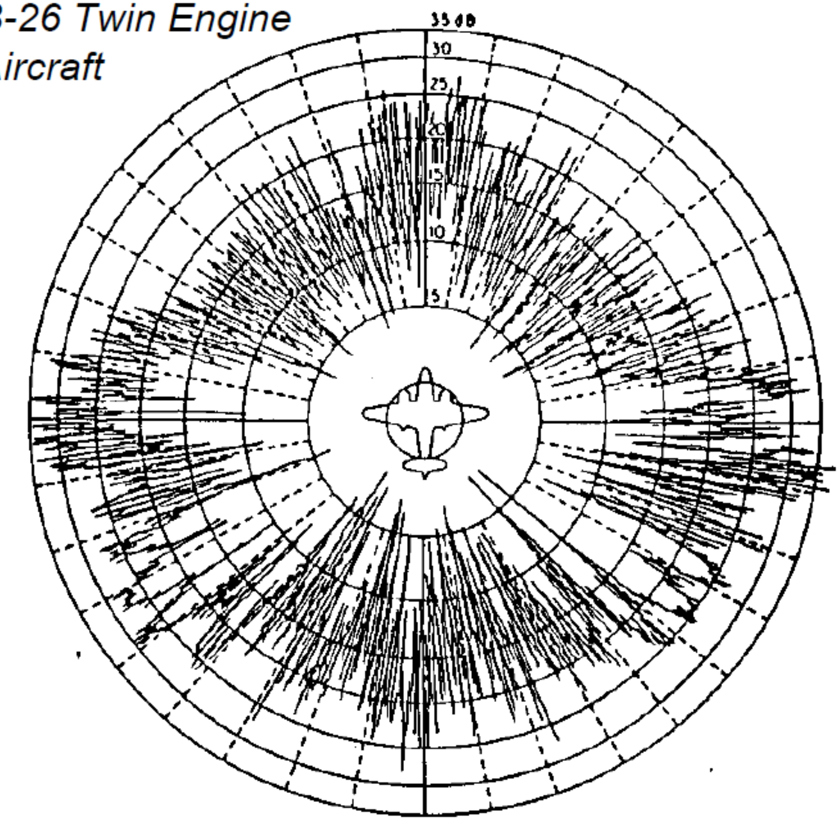
- When target is present, the received data is sum of target and noise contributions
- Probability of detection, P_D, will be integral of target + noise PDF from threshold to $+\infty$
  - We already have our threshold T from setting P_{FA}



- To get a pdf for target + noise, we need to consider target fluctuation models:
  - Radar cross section (RCS) fluctuations
  - Correlation properties

RCS Fluctuations of Complex Targets

B-26 Twin Engine Aircraft



- λ = 10 cm (3 GHz, X band)
- RCS variation can be 10-15 dB over a fraction of a degree

- Radar return from a typical "real world" target is the result of multiple scattering processes
- RCS varies with viewing angle in a very complex manner
  - As orientation changes, the nature of orientation and number of scatterers contributing to the return also changes
- Vibration also causes RCS fluctuations
- RCS of return is generally treated as <u>random</u>
  - Too complicated to model any other way
- *So we need a random process model…*

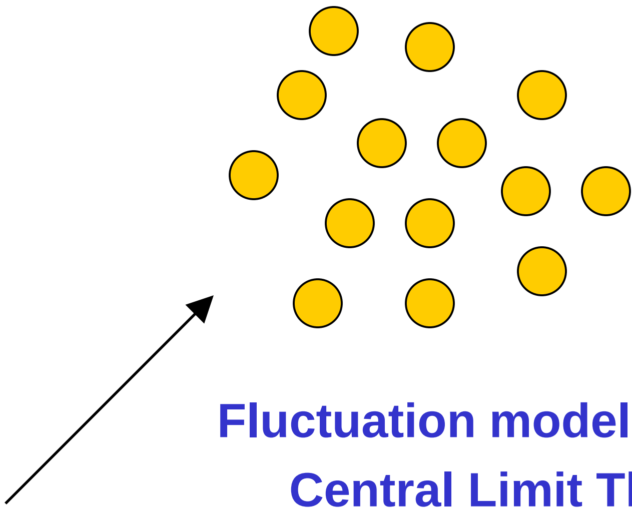
Statistical RCS Model

- Complexity of RCS fluctuations requires a statistical model of amplitude
- For a superposition with a "large number" of scatterers with…
  - Random spatial locations (therefore random phase)
  - Fixed RCS
- The RCS σ is an exponentially distributed random variable and the magnitude ς = sqrt(σ) is a Rayleigh-distributed random variable:

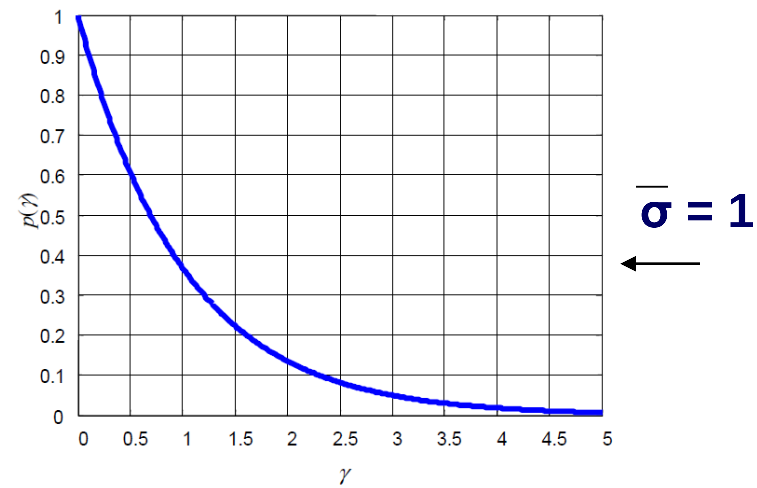
$$p(\varsigma) = \frac{2\varsigma}{\overline{\sigma}} \exp\left[-\frac{\varsigma^2}{\overline{\sigma}}\right], \quad \varsigma \geq 0 \qquad p(\sigma) = \frac{1}{\overline{\sigma}} \exp\left[\frac{-\sigma}{\overline{\sigma}}\right], \quad \sigma \geq 0$$

Fluctuating Target Model

- **Collection of Equal Size Scatterers**



Fluctuation model follows from Central Limit Theorem



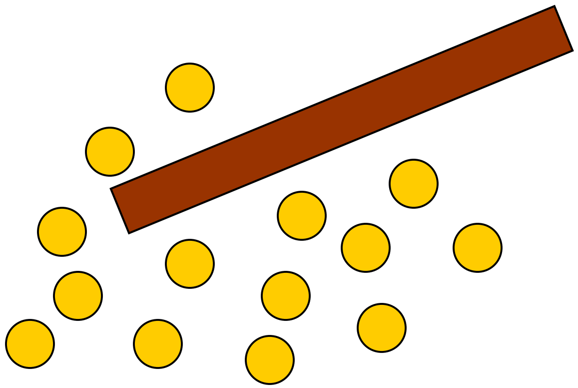
Swerling Cases 1 & 2

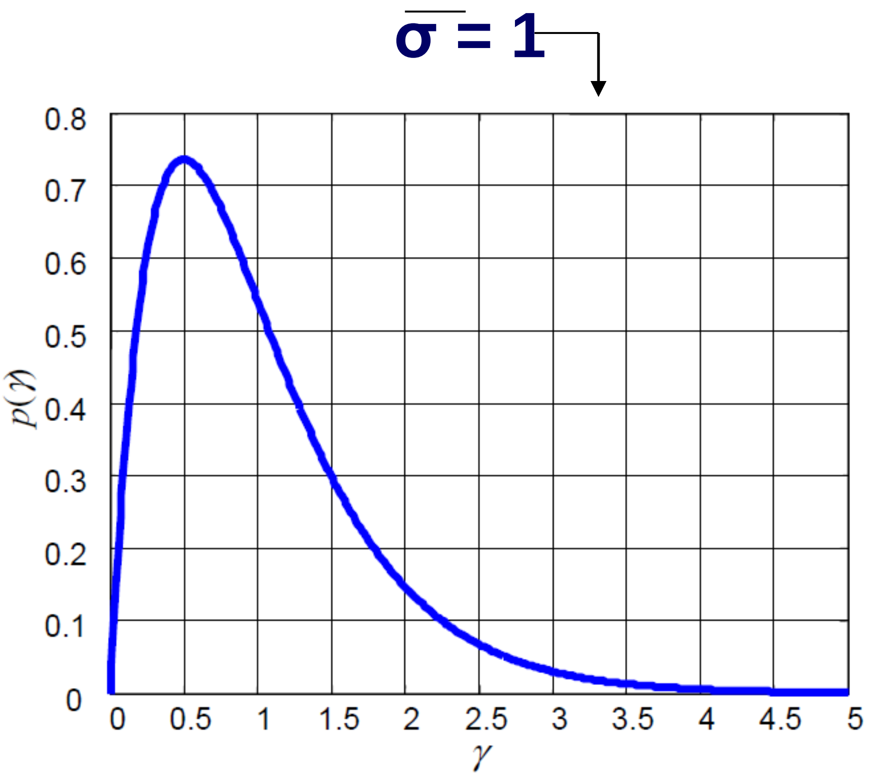
$$p(\sigma) = \frac{1}{\bar{\sigma}} e^{-\frac{\sigma}{\bar{\sigma}}}, \quad \sigma \geq 0$$

- **Exponential RCS pdf**
  - **Chi-square**
- **Sometimes called Rayleigh because corresponding voltage (not power) pdf is Rayleigh**

Another Fluctuating Target Model

- ❑ **Collection of Scatterers with One Dominant Scatterer**





Swerling Cases 3 & 4

$$p(\sigma) = \frac{4\sigma}{\bar{\sigma}^2} e^{-\frac{2\sigma}{\bar{\sigma}}}, \quad \sigma \geq 0$$

- ❑ **Chi-square with n=4 pdf for RCS**

Empirical Models for RCS Fluctuations

- Swerling models
  - Probability density function (pdf) model + correlation model
  - Primary use is detection performance prediction
- Many other statistical models
  - Log-normal
    - Empirical distribution chosen to model targets not covered by original Swerling cases
  - Weibull, K distribution, etc.
- We are going to discuss Swerling models
  - Simplest
  - Most traditional

The Swerling Models

- Four combinations of
  - Two RCS pdfs: chi-square with n = 2 (Rayleigh/exponential) or 4
  - Two correlation classes: pulse-to-pulse or dwell-to-dwell
- Nonfluctuating case sometimes called “Swerling 0” or “Swerling 5”

Probability Density Function of RCS	Decorrelation	
	scan-to-scan	pulse-to-pulse
Rayleigh/exponential	Case 1	Case 2
Chi-square, degree 4	Case 3	Case 4

Which Swerling Model Applies?

- Choice of pdf requires knowledge of target RCS characteristics
  - Choose chi-square if there is a "dominant" scatterer
  - Choose exponential if there is not a dominant scatterer
  - Particularly in high-resolution radars, neither Swerling model may be very good
- Choice of correlation model depends on geometry of the encounter over a dwell
  - Will the aspect angle change enough to decorrelate the target over the N pulses to be noncoherently integrated?
  - Is frequency agility being used to deliberately decorrelate the target?

Receiver Operating Characteristics

- ROC of P_D, P_{fa}, and SNR - fluctuating models
  - Higher SNR required for fluctuating models

Table 3-1. Required SNR for various target fluctuation models.

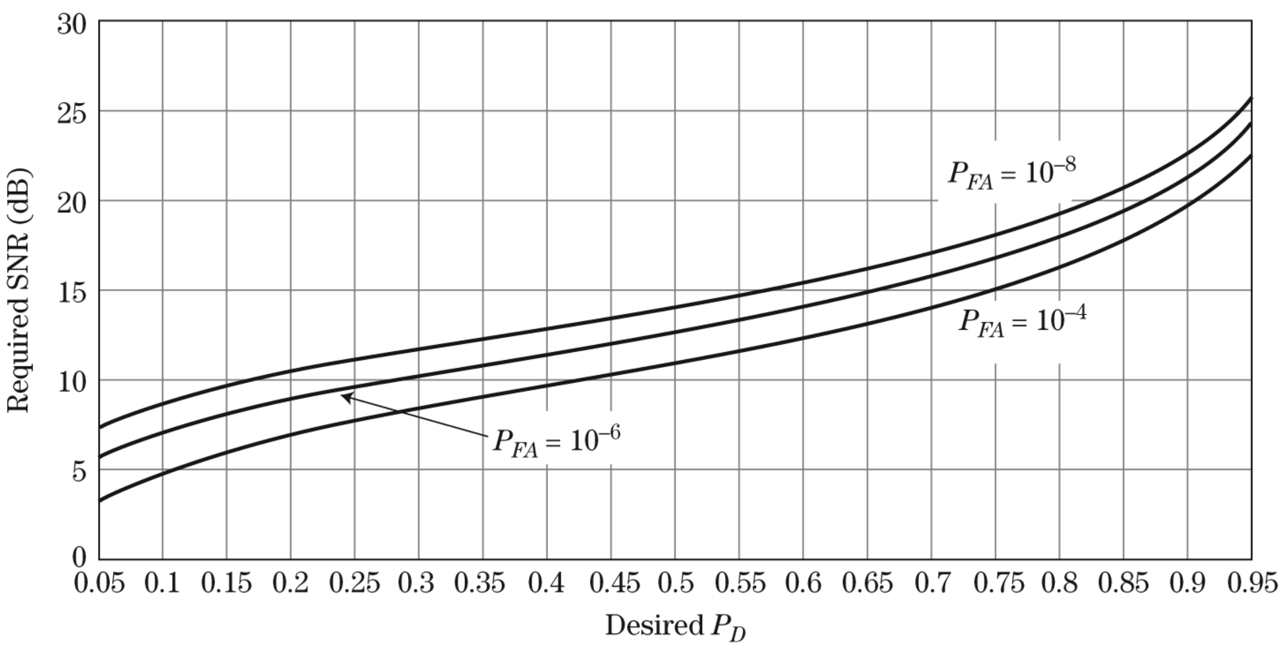
	Pd	SW0	SW1	SW2	SW3	SW4
$P_{fa}=10^{-4}$	50	9.2	10.8	10.5	11	9.8
	90	11.6	19.2	19	16.5	15.2
$P_{fa}=10^{-6}$	50	11.1	12.8	12.5	11.8	11.8
	90	13.2	21	21	17.2	17.1

Receiver Operating Curves

- ROC of P_D, P_{fa}, and SNR - fluctuating model

FIGURE 3-7 ■ SNR required to achieve a given P_D, for several values of P_{FA}, for fluctuating (SW1) target in noise.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"



Receiver Operating Curves

- ROC of P_D, P_{fa}, and SNR - fluctuating models
  - Large difference for high P_D

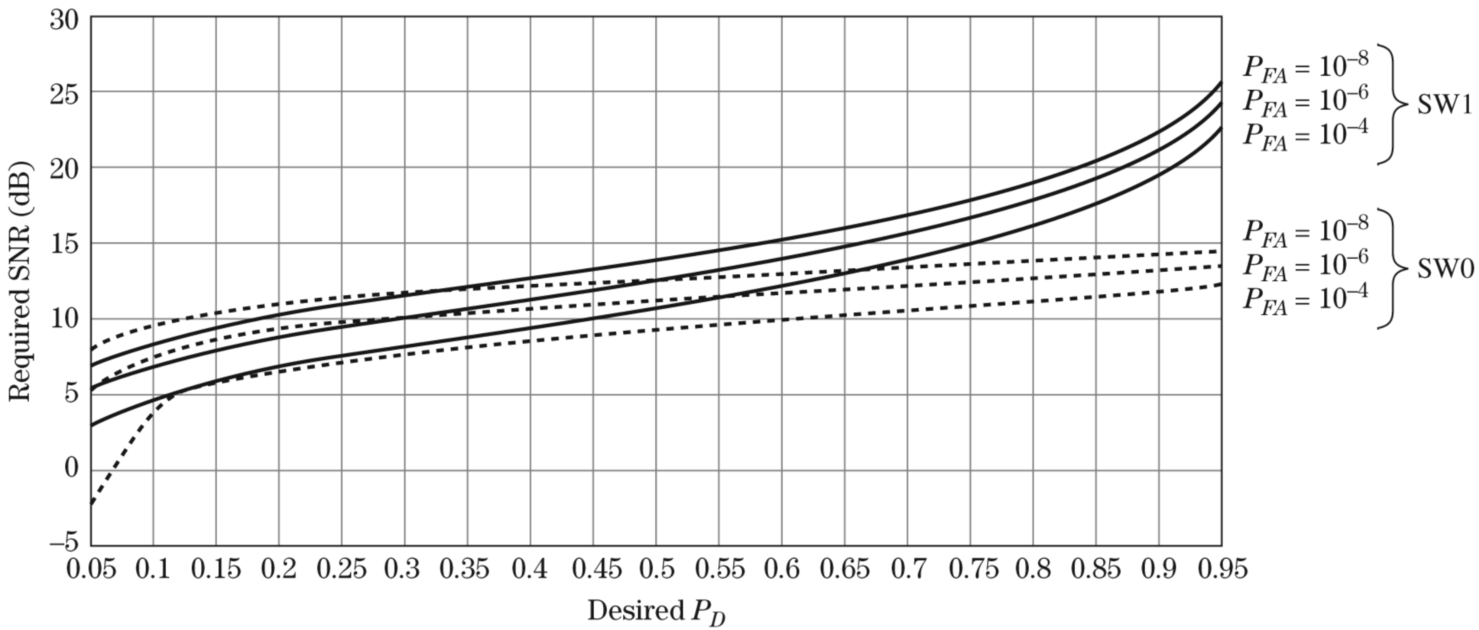
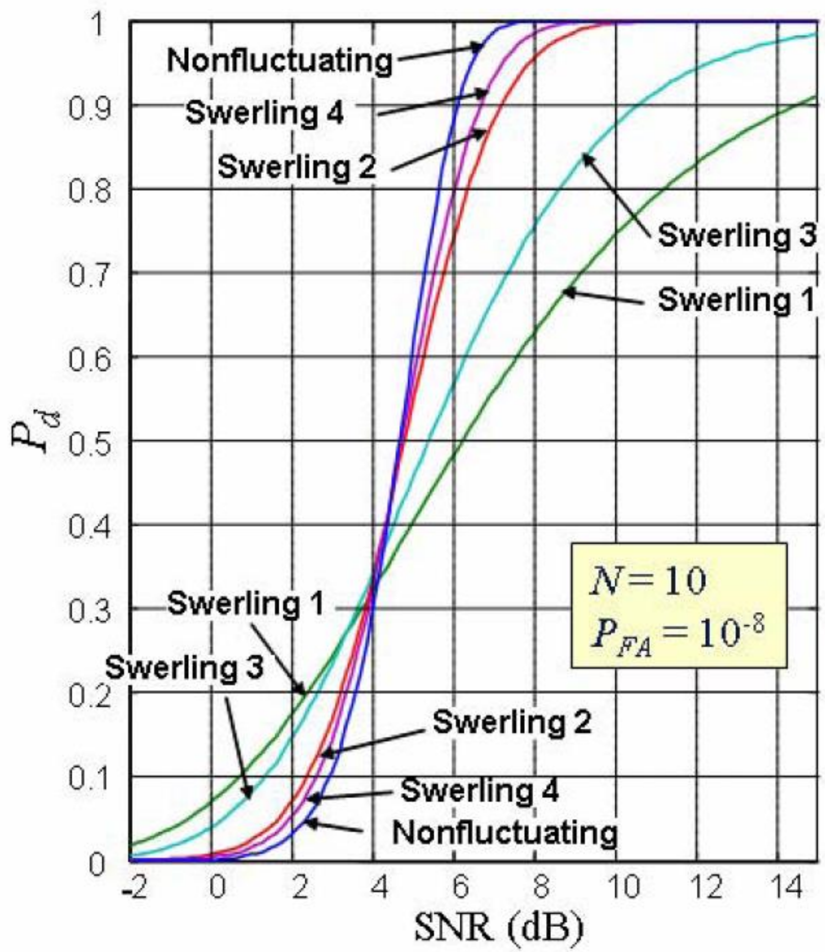


FIGURE 3-8 ■ SNR required to achieve a given P_D, several values of P_{FA}, for nonfluctuating (SW0) and fluctuating (SW1) target models.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Detection Performance



- For typical radar scenarios (e.g. $P_{FA} = 10^{-6}$) and detection probabilities > .5 (approximately)
  - Target fluctuations make detection more difficult
    - Nonfluctuating target is easier to detect than any of the Swerling targets
  - Pulse to pulse fluctuations help target detectability at high SNR
    - Swerling 2 and 4 targets are easier to detect than Swerling 1 and 3 targets
- For detection probabilities < .5, converse is true

Interference other than Noise

- Noise is Rayleigh if thermal or noise jammer
- Clutter: Weibell, log-normal, or K-distributed
  - Longer tails => increases P_{fa}
  - Need to reduce clutter (e.g., MTI)

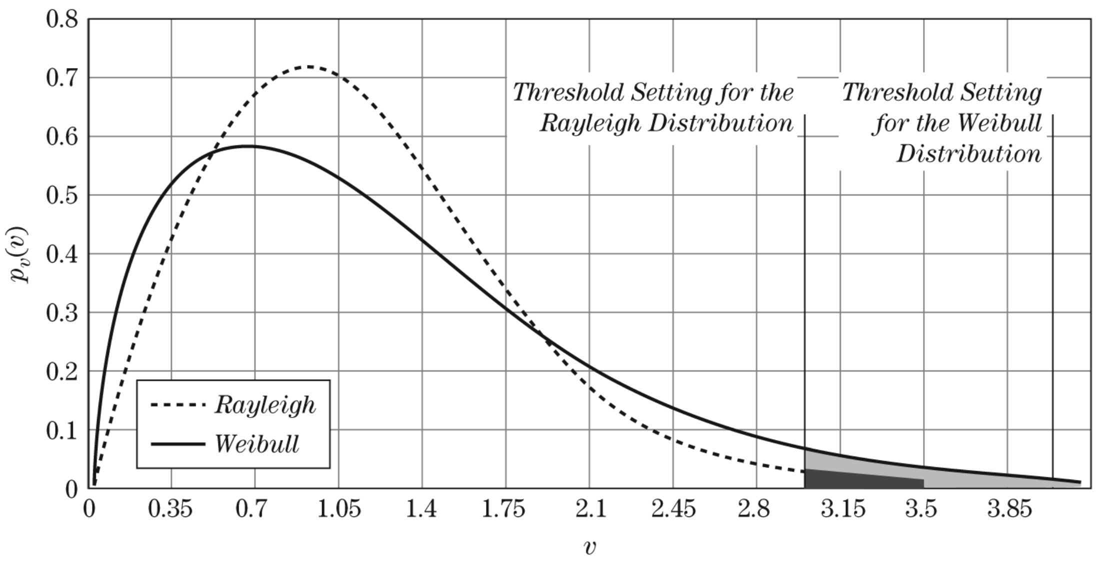


FIGURE 3-9 ■ Example clutter PDF compared with noise.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Detection in Interference

- The ability to detect weak target signals is limited by the presence of interfering signals
  - Receiver noise
  - Clutter
  - Electronic attack (EA)
    - Formerly electronic countermeasures (ECM)
    - Also known as jamming
  - Electromagnetic interference (EMI)
- Receiver noise always present
- Clutter, EA, and EMI *not* always present…
- …but if they are, they are likely much stronger than noise
- Next few lectures will address detection when clutter and EA present

Some Closed-Form Solutions

- Previous equations use Marcum's Q function and require use of MATLAB
- Useful (sometimes) to have spreadsheet or calculator-solvable solution
- For non-fluctuating targets, Albersheim's equation (empirical for N independent samples with noncoherent integration):

$$SNR = -5\log_{10} N + \left(6.2 + \frac{4.54}{\sqrt{N+0.44}}\right)\log_{10}(A = 0.12AB + 1.7B)$$

where:

$$A = \ln(0.62 / P_{FA})$$

and

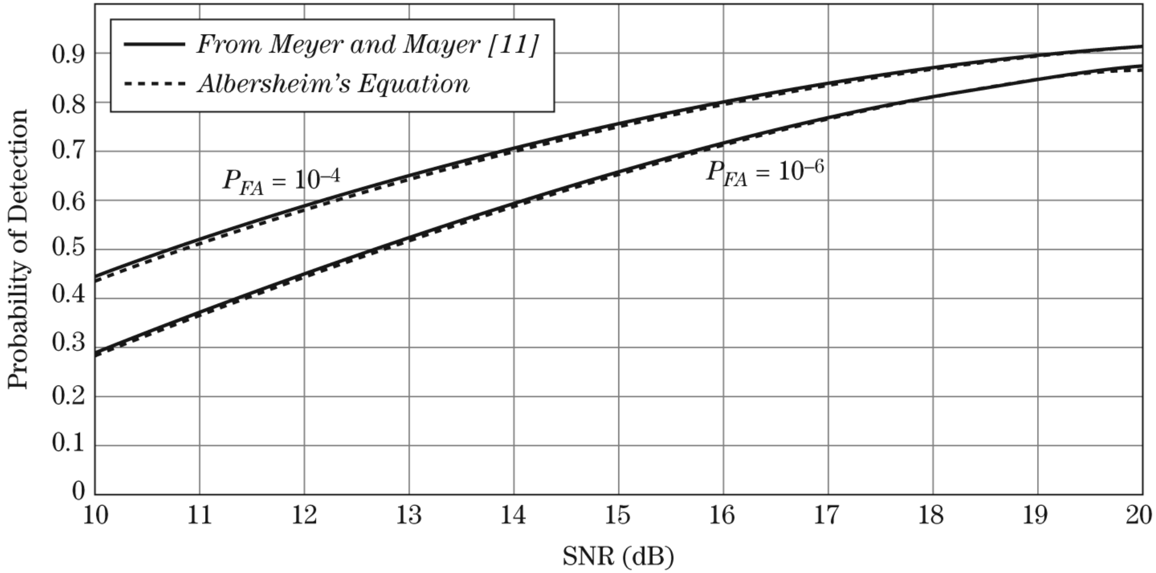
$$B = \ln\left(\frac{P_D}{1-P_D}\right)$$

Some Closed-Form Solutions

- Results with Albersheim's equation show good agreement with theory

FIGURE 3-10 ■ SNR versus P_D for a Swerling 0 target using Albersheim's equation, plotted with tabulated results from Mayer and Meyer [10].

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"



Swerling 1 Target Model

- Noise and target fluctuations are both Gaussian in I and Q, and thus PDF of target plus noise is Rayleigh:

$$p_{sig}(r) = \frac{r}{S+\sigma_n^2}\exp\left(\frac{-r^2}{2\left(S+\sigma_n^2\right)}\right),$$

and the probability of detection is:

$$P_D = \int_{V_t}^{\infty} P_{sig}(r)dr = \int_{V_t}^{\infty} \frac{r}{S+\sigma_n^2}\exp\left[-\frac{r^2}{2\left(S+\sigma_n^2\right)}\right]dr \quad \rightarrow \quad P_D = \exp\left[\frac{-V_t}{1+SNR}\right]$$

Thus, for one sample (or N coherent pulses or CPIs):

$$P_D = \left(P_{FA}\right)^{\frac{1}{1+SNR}}$$

Swerling 1 Target Model

- For L noncoherent pulses, no simple form, but:

$$P_D = 1 - F_{\chi^2_{2L}}\left(\frac{1}{SNR+1} F^{-1}_{\chi^2_{2L}}\left(1 - P_{fa}\right)\right)$$

which is easily implemented in MATLAB

Multiple-Dwell Detection Principles: Cumulative P_D

- P_D of 90% may not seem to be adequate
- But if use n dwells, then

$$P_D(n) = 1-(1-P_D(1))^n$$

- Example: With 90% for one dwell, 2 dwells is 99%, and 3 dwells is 99.9%

- Example: 99% for one dwell with Swerling 2 => SNR=24 dB versus 18 dB for 90%
- But (although increase in SNR for same P_{fa} is small),

$$P_{FA}(n) \cong n \cdot P_{FA}(1)$$

m-of-n Detection Criterion

- Detect based on at least m out of n tries (with detection probability P on a single try):

$$P(m,n) = \sum_{k=m}^{n} \frac{n!}{k!(n-k)!} P^k (1-P)^{n-k}$$

Example: 2 out of 3

$$P(2,3) = 3P^2 - 2P^3$$

Example: 2 out of 4

$$P(2,4) = 6P^2 - 8P^3 + 3P^4$$

m-of-n Detection Criterion

- Over the desired range of P_{fa} is reduced while P_D is increased

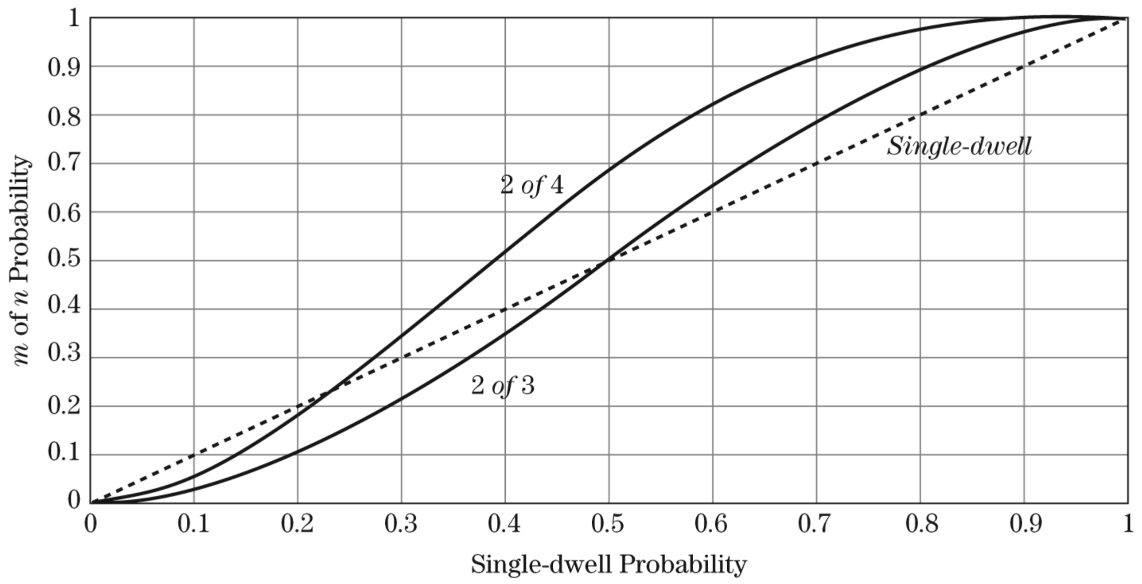


FIGURE 3-11 ■ 2-of-3 and 2-of-4 probability of threshold crossing versus single-dwell probability.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

m-of-n Detection Criterion

- Even better results with larger n.
- Example: Swerling 1, $P_{fa} = 10^{-6}$, $P_D = 95\%$, single dwell => SNR=24.3 dB but only 13.2 dB if 3-of-6 (11.1 dB gain, at $P_D = 99\%$ 20 dB gain)

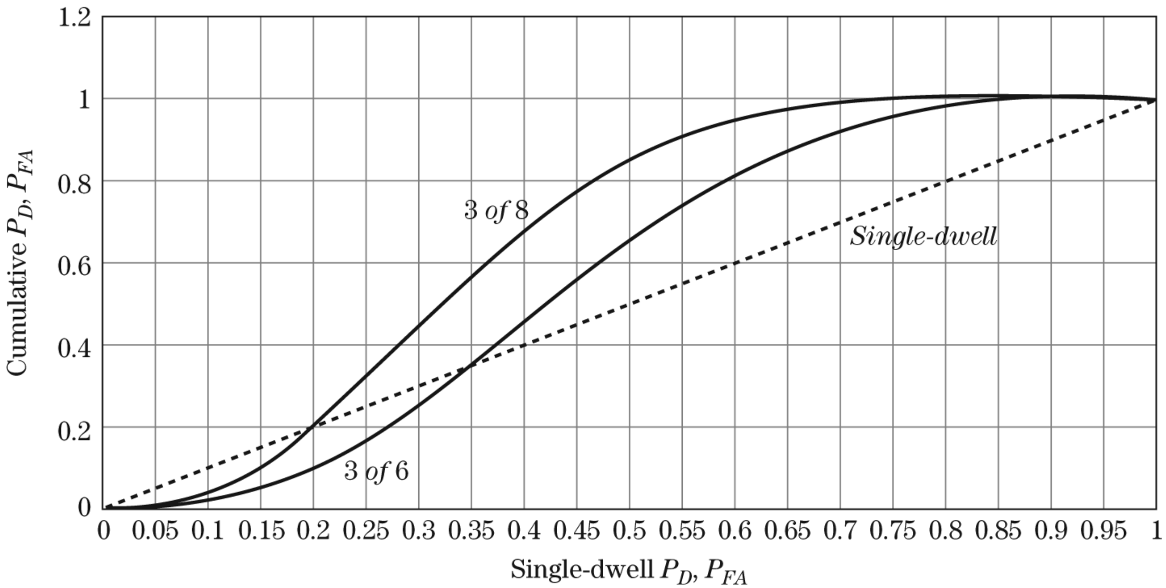


FIGURE 3-12 ■ 3-of-6 and 3-of-8 probability of threshold crossing versus single-dwell probability.

"Used with Permission from Richards et al Principles of Modern Radar, Basic Principles (www.scitechpub.com)"

Lecture Summary

- Probability
  - Random Variables
  - Expected Values
  - Functions of Random Variables
  - Correlation
  - Decision Theory
- Search and Target Detection
  - Introduction
  - Search Mode Fundamentals
  - Overview of Detection Fundamentals