All-pairs Shortest Path Problem

$W_{n\times n}$ $\quad G \rightarrow n$ no. of vertices

$$\hookrightarrow w_{ij} = \begin{cases} 0, & \text{if } i=j \\ \text{the weight of directed edge } (i,j), & \text{if } i\neq j \text{ and } (i,j)\in E \\ \infty, & \text{if } i\neq j \text{ and } (i,j)\notin E \end{cases}$$

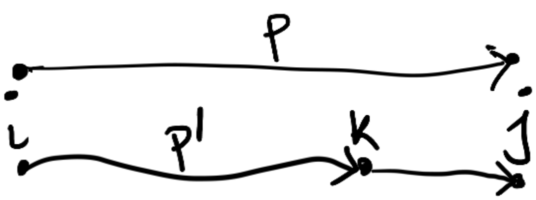
Negative-weighted edges are allowed, but the i/p graph will not contain any -ve-weighted cycle.

Structure of a shortest path for all-pairs shortest path problem:

$G(V,E)$ [All subpaths of a shortest path are also shortest paths]

a – d ⇒ shortest path
a – b, b – c, c – d
⇓
shortest paths

Now,



$p \rightarrow$ Shortest path between i & j.
$\hookrightarrow$ Let, m edges (atmost)

Assuming, there is no -ve weighted cycles, m is finite.

Now, if $i=j$, then p has weight 0 and no edges. But, if $i\neq j$, we can decompose path p into $i \overset{p'}{\rightsquigarrow} k \longrightarrow j$, so p' now contains atmost $m-1$ edges. So, p' is a shortest path from i to k,

shortest path, $\delta(i,j) = \delta(i,k) + w_{kj}$

-ve-weighted edges may be present but -ve-weighted cycles will not be there.

Floyd-Warshall Algorithm

All-pairs shortest path problem (dynamic programming)

Structure of a shortest path:

path, $p = \langle v_1, v_2, \ldots, v_l \rangle$

As this algorithm considers intermediate vertices of a shortest path, so, Any vertex of p, other than v_1 or v_l is called intermediate vertex.

$G \rightarrow V = \{1, 2, \ldots, n\}$

Considering a subset of vertices $\{1, 2, \ldots, k\}$

Considering a pair of vertices $i, j \in V$ and all paths from i to j whose intermediate vertices are from $\{1, 2, \ldots, k\}$ and let p be a minimum-weight path among them. Depending on whether or not k is an intermediate vertex of path p, two possible cases.

① If k is not an intermediate vertex of path p, then all intermediate vertices of path p are in the set $\{1, 2, \ldots, k-1\}$. Thus a shortest path from vertex i to vertex j with all intermediate vertices in the set $\{1, 2, \ldots, k-1\}$ is also a shortest path from i to j with all intermediate vertices in the set $\{1, 2, \ldots, k\}$

(2) If k is an intermediate vertex of path P, the we can decompose P like $i \overset{P_1}{\rightsquigarrow} k \overset{P_2}{\rightsquigarrow} j$.
According to the theorem, P_1 is a shortest path from i to k with all intermediate vertices in the set $\{1, 2, \cdots, k-1\}$. Also, P_2 is a shortest path from k to j with all intermediate vertices in the set $\{1, 2, \cdots, k-1\}$.