Weak Forms

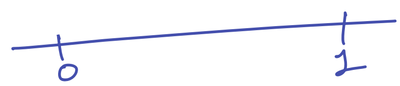
$x^{1/2}=1 \longleftrightarrow x^2=1$

After manipulations, we get another equivalent equation.

Some common solutions.

Case Problem

$$\frac{df}{dx}=0$$

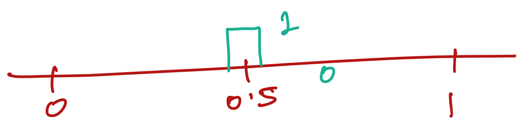
on 

$h=0.0000001$

if f is smooth over 

$$\Leftrightarrow$$

$$\int_0^h \frac{df}{dx}\,dx=0,\ \int_h^{2h}\frac{df}{dx}\,dx=0,\ \int_{2h}^{3h}\frac{df}{dx}\,dx=0,\ \dots\ \int_{10^8 h}^{1}\frac{df}{dx}\,dx=0$$

$\phi_{0.5}$: 

$$\int_{0.5}^{0.5+h}\frac{df}{dx}\,dx=0 \Leftrightarrow \int_0^1 \frac{df}{dx}\phi_{0.5}\,dx=0$$

$$\phi_{0.5} \longrightarrow \phi_a$$

$$\frac{df}{dx}=0 \Leftrightarrow \int_0^1 \frac{df}{dx}\phi_a\,dx=0$$

a in 

Problem : $\frac{df}{dx} = 0$ on $x \in [0, 1]$

Equivalent Problem: $\int_0^1 \frac{df}{dx} \phi_a \, dx = 0$

for all ϕ_a

The equivalent problem is called Weak Formulation.

For every differential equation, There is a Weak Form.

eg:- $\nabla^2 u = f$

we use finite elements basis ϕ_a space V

with trial function v

$$v \nabla^2 u = vf$$

$$\int v \nabla^2 u \, dx = \int vf \, dx$$

$$\Rightarrow \underbrace{\int \nabla u \cdot \nabla v \, dx}_{\hat{a}(u, v)} = \underbrace{\int vf \, dx}_{\hat{L}(v)}$$

$\hat{a}(u, v) = \hat{L}(v)$ for all v in Approximation Space

1) $L(v)$ is linear in v

$$L(c_1 v_1 + c_2 v_2) = c_1 L(v_1) + c_2 L(v_2)$$

2) $a(u,v)$ is linear in u & v

u: $$a(c_1 u_1 + c_2 u_2, v) = c_1 a(u_1, v) + c_2 a(u_2, v)$$

v: $$a(u, c_1 v_1 + c_2 v_2) = c_1 a(u, v_1) + c_2 a(u, v_2)$$

FEA: $a(u,v) = L(v)$

becomes a linear equation.

$u = \sum c_i \phi_i$ we want to find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_N \end{bmatrix}$

$$a\left(\sum c_i \phi_i, v\right) = L(v)$$

for all v

$\Rightarrow$ for all ϕ_i

let $v = \phi_j$

$$\Rightarrow \quad a\left(\sum c_i \phi_i, \phi_j\right) = L(\phi_j)$$

$$\Rightarrow \sum_i c_i \, a(\phi_i, \phi_j) = L(\phi_j) \quad \forall j$$

$$\Rightarrow \begin{bmatrix} a(\phi_i, \phi_j) \end{bmatrix}_{i,j} \begin{bmatrix} c_i \end{bmatrix}_i = \begin{bmatrix} L(\phi_j) \end{bmatrix}_j$$

So PDE $\longrightarrow$ Weak form
$\downarrow$
Algebraic Equation

$$u(x_1, y_1, z_1) = \sum c_i \, \phi_i(x_1, y_1, z_1)$$

Why Not $\int (\nabla^2 u) v \longrightarrow \int \nabla u \cdot \nabla v$ why

$$a_2(u,v) = \int v(\nabla^2 u)\, dx \qquad a_1(u,v) = \int \nabla u \cdot \nabla v \, dx$$

$a_1(u,v)$ is symmetric in (u,v)

$a_2(u,v)$ is not.

Symmetric expressions are better studied.