Q1) Which ones are form?

a) $(f(u))^2$ b) $\int_a^6 f(u)\, du$ c) $\frac{5 * f(u)}{f(u)}$

d) $f(5)$ e) $\int f(u)\, \overbrace{\delta(x-5)}^{\text{Delta function}}\, du$

f) $\int \nabla f \cdot \vec{u}\, du$ g) $\nabla f \cdot \vec{u}$

h) $\nabla \times \vec{F}$ i) $(\nabla \times \vec{f}) \cdot (\nabla \times \vec{g})$

j) $\int (\nabla \times \nabla \times \vec{f}) \cdot \vec{v}\, du$

Q2) Identify if form is linear/nonlinear

i)

$$a(u,v) = \int \vec{u} \cdot \vec{v}\, du$$

$$b(u,v) = \int (\nabla u) \cdot (\nabla v)\, du$$

$$c(u,v) = \int v \nabla^2 u\, du$$

$$d(u,v) = \int v^2 (\nabla^2 u)\, du$$

$$e(u,v) = \int v^2 \, (\nabla^2 u)^2 \, du$$

$$X(u,v) = \int (1+v^2) \, \|\vec{u}\| \, du$$

$$Y(u,v) = \int (1+u^2) \, \nabla u \cdot \nabla v \, du$$

$$F(u,v) = \int (\nabla \times \vec{u})^2 \, du$$

$$G(u,v) = \int (\nabla \times \vec{u}) \cdot (\nabla \times \vec{v}) \, du$$

$$I(u,v) = u(s) \times v(s)$$

$$H(u,v) = \int (\nabla u \cdot \nabla v - vf) \, du$$

$$N(u,v) = \int \left[(1+u^2) \, \nabla u \cdot \nabla v - vf \right] du$$

$$P(u,v) = \int \left[(1+u^2) \, \nabla u \cdot \nabla v - e^u v \right] du$$

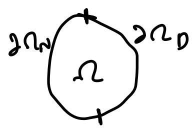
$$S(u,v) = \int \left[q(u) \, \nabla u \cdot \nabla v - f(u) \, v \right] du$$

Q3) Express PDE in Weak Form

Example

$$\nabla^2 u = f(x,y) \quad \Omega$$

$$u = u_D(x,y) \quad \partial\Omega_D \quad \text{Dirichlet}$$



$$\nabla u \cdot \hat{n} = g(x,y) \quad \partial\Omega_N \quad \text{Neumann}$$

$$\partial\Omega = \partial\Omega_D + \partial\Omega_N$$

Using Identity I for ∇^2

$$v\, \nabla^2 u = v f$$

$$\Rightarrow \int_\Omega v\, \nabla^2 u \, dx = \int v f \, dx$$

This is also a form equation:—

$$a(u,v) = \int_\Omega v\, \nabla^2 u \, dx$$

$$L(v) = \int_\Omega v f \, dx$$

$$a(u,v) = L(v)$$

But we want symmetry —

Using integration by parts:— (Identity 1)

$$\int_\Omega v\, \nabla^2 u \, dx = -\int_\Omega (\nabla v \cdot \nabla u) \, dx + \int_{\partial\Omega} v\, (\nabla u \cdot \hat{n}) \, ds$$

we use $v=0$ on $\partial\Omega_D$

$$-\int_{\Omega} (\nabla v \cdot \nabla u)\, d\Omega + \int_{\partial\Omega} v\,(\nabla u \cdot \hat{n})\, ds$$

$$= \int_{\Omega} vf \, d\Omega$$

(1)

$$a_1(u,v) = -\int_{\Omega} (\nabla v \cdot \nabla u)\, d\Omega$$

$$a_2(u,v) = \int v\,(\nabla u \cdot \hat{n})\, ds$$

$$L(v) = \int_{\Omega} vf \, d\Omega$$

$$\Rightarrow \quad a_1(u,v) + a_2(u,v) = L(v)$$

$$a(u,v) = a_1 + a_2 =$$

$$-\int_{\Omega} (\nabla v \cdot \nabla u)\, d\Omega + \int v\,(\nabla u \cdot \hat{n})\, ds$$

$$a(u,v) = L(v)$$

2)

$$\varepsilon \nabla^2 T = \beta \cdot \nabla T \qquad \Omega$$

$$T = u_D \quad \text{on } \partial\Omega_D$$

$$\frac{\partial T}{\partial n} = u_N \quad \text{on } \partial\Omega_N$$

$$\downarrow$$

(T is scalar
so v is scalar)

$$\varepsilon\, v\, \nabla^2 T = v\, \vec{\beta}\cdot\nabla T$$

Hint: Next Step is an integral

$$\underset{\text{Form 1}}{\underline{\qquad\qquad}} = \underset{\text{Form 2}}{\underline{\qquad\qquad}}$$

Hint: Identity 1 for ∇^2

$$\underset{\text{Volume form 1}}{\underline{\qquad\qquad}} + \underset{\text{surface term 1 Neumann}}{\underline{\qquad\qquad}} = \underset{\text{Form 2}}{\underline{\qquad\qquad}}$$

$$a_1(T,v) + a_2(T,v) = a_3(T,v)$$

Q>

$$\nabla^2 \vec{A} = k^2 \vec{A}$$

dot multiply by v & integrate

$$\int \vec{v}\cdot\nabla^2\vec{A} = \int \nabla\vec{v}:\nabla\vec{A}$$
+ surface term

$$\underline{\qquad\qquad} = \underline{\qquad\qquad}$$

make it symmetric

$$\underline{\qquad\qquad} = \underline{\qquad\qquad}$$

Q) $-\nabla \times \nabla \times \vec{E} = \omega^2 c^2 \vec{E}$

dot multiply by v & integrate

Valid Form

$$\int_\Omega \vec{v} \cdot \nabla \times \nabla \times \vec{E} \; d\vec{u} = \int_\Omega \omega^2 c^2 \vec{E} \cdot \vec{v} \; d\vec{u}$$

Identity 3

$\int \vec{A} \cdot \nabla \times \nabla \times \vec{B}$

$\underset{\text{Symmetric volume term } (\Omega)\; du}{\rule{4cm}{0.4pt}} + \underset{\text{surface term } (\partial\Omega)\; ds}{\rule{3cm}{0.4pt}} = \underset{\text{volume term } (\Omega)\; du}{\rule{4cm}{0.4pt}}$

Q) Elasticity Teaser

$\vec{u}$: displacement

$\bar{\varepsilon}$: strain

$\bar{\varepsilon} = g(\nabla \vec{u})$

$\bar{\sigma} = E\,\bar{\varepsilon}$

$$\nabla \cdot \bar{\sigma} = \vec{F}$$

$$\vec{v} \cdot (\nabla \cdot \bar{\sigma}) = \vec{v} \cdot \vec{F}$$

Identity 2

$$\int_\Omega \vec{A} \cdot (\vec{\nabla} \cdot \bar{B}) \; du = \int_{\partial\Omega} (\bar{B} \cdot \hat{n}) \cdot \vec{A} \; ds - \int_\Omega \bar{B} : \nabla \vec{A} \; du$$

Use this

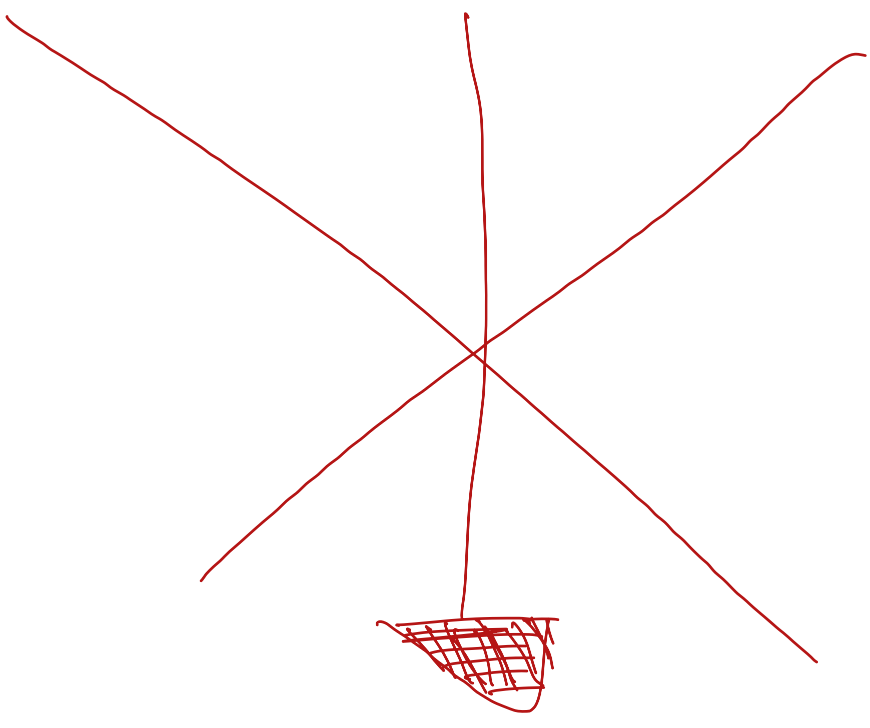
$\rule{3cm}{0.4pt} + \rule{3cm}{0.4pt} = \rule{3cm}{0.4pt}$

Ω (double dot) $\partial\Omega$ Ω (from $\nabla \cdot F$)

SOLUTIONS

- DO TRY TO SOLVE YOURSELF FIRST

- AT LAST RESORT



1) $-\varepsilon \int_{\Omega} \nabla T \cdot \nabla v \, dx + \int_{\partial\Omega} v \, \nabla T \cdot \hat{n} \, dS$

$a_1 \qquad a_2$

$= \int_{\Omega} \beta \cdot \nabla T \, v \, d\Omega$

a_3

$a_1(T,v) + a_2(T,v)$
$= a_3(T,v)$
$\Rightarrow F(T,v) = 0$
$F = a_1 - a_2 + a_3$

2) $-\int_{\Omega} \nabla \vec{v} : \nabla \vec{A} \, dx + \int_{\partial\Omega} \vec{v} \cdot (\nabla \vec{A} \cdot \hat{n}) \, dS = k \int_{\Omega} \vec{A} \cdot \vec{v} \, dx$

$a_1 \qquad a_2 \qquad a_3$

$a_1(\vec{v}, \vec{A}) + a_2(\vec{v}, \vec{A}) = a_3(\vec{v}, \vec{A}) \Rightarrow F(\vec{A}, \vec{v}) = 0$
$F = a_1 + a_2 - a_3$

3) $\int_{\Omega} (\nabla \times \vec{v}) \cdot (\nabla \times \vec{E}) \, dx - \int_{\partial\Omega} \vec{v} \cdot (\nabla \times \vec{E} \times \hat{n}) \, dS = \omega^2 c^2 \int_{\Omega} \vec{E} \cdot \vec{v} \, dx$

$a_1 \qquad a_2 \qquad a_3$

$a_1(\vec{E}, \vec{v}) - a_2(\vec{E}, \vec{v}) = a_3(\vec{E}, \vec{v})$

$F = a_1 - a_2 - a_3 = 0$

$a_1 \qquad a_2$

4) $\int_{\partial\Omega} \vec{v} \cdot (\bar{\sigma} \cdot \hat{n}) \, dS - \int_{\partial\Omega} \bar{\sigma} : \nabla \vec{v} \, dx$

L

$= \int \vec{F} \cdot \vec{v} \, dx$

$\bar{\sigma} = h(\vec{u})$

in short

$a_1(\vec{u}, \vec{v}) - a_2(\vec{u}, \vec{v})$

$= L(\vec{v})$