Hello World

Poisson's Equation

$$-\nabla^2 u(x) = f(x) \quad x \text{ in } \Omega$$
$$u(x) = u_D(x) \quad x \text{ on } \partial\Omega$$

$$x = (x(0), x(1), x(2) \ldots)$$

Pipeline

i) Identify the Ω
PDE
$\partial\Omega$
f

ii) Reformulate the PDE as a Weak Form.

iii) Define all these in code using FEniCS objects.

iv) Call FEniCS to solve the problem and postprocess the results.

Step iii) & iv) are fairly short in FEniCS

Program

Create Mesh and define Element Space

Define Boundary Condition

Define Variational Problem

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx$$

Compute Solution

plot solution and mesh.

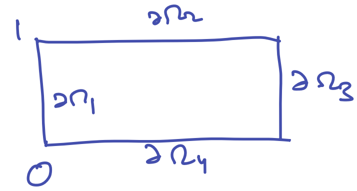
Compute error in L^2 norm.

Compute errors at vertices.

<u>Advection - Diffusion</u>

$$\varepsilon \nabla^2 T = \beta \cdot \nabla T$$

P_1 .



$$\varepsilon \nabla^2 T v = (\beta \cdot \nabla T) v$$

$$\Rightarrow \int_\Omega \varepsilon (\nabla T \cdot \nabla v) \, dx = \int_\Omega \beta \cdot \nabla T v \, dx$$

$$T = 0 \quad : \quad \partial\Omega_1$$
$$T = 1 \quad : \quad \partial\Omega_2, \partial\Omega_4$$
$$\frac{\partial T}{\partial y} = 0 \quad : \quad \partial\Omega_3$$

$\beta = (4x_2(1-x_2), 0) \longrightarrow$ flow velocity.

<u>Spectral Problem</u>

- Wave equation

$\varphi \in H_0^1(\Omega)$ $\quad \varphi \neq 0$

$\lambda \in \mathbb{R}$

$$\int \nabla \varphi \cdot \nabla v = \lambda \int \varphi v$$

<u>Euler Bernoulli Equation</u>

$\varphi \in H_0^2(\Omega)$

$u \rightarrow$ displacement

$$\int_0^L u''v''\,dx = \int_0^L f v\,dx$$

$u(0)=0 \qquad u(L)=0$

$u'(0)=0 \qquad u'(L)=0$

Stress equation

$$\nabla \cdot \bar{\sigma} = f$$

$\sigma \rightarrow$ stress tensor

$f \rightarrow$ force density.

$$\sigma = \lambda \, tr(\varepsilon) I + 2\mu \varepsilon$$

$$\varepsilon = \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$$

$$a(u,v) = L(v) \qquad \forall v \in \hat{V}$$

$$a(u,v) = \int_\Omega \sigma(u) : \nabla v \,dx$$

$$\sigma(u) = \lambda(\nabla \cdot u) I + \mu\left(\nabla u + (\nabla u)^T\right)$$

$$L(v) = \int_\Omega f \cdot v \,dx + \int_{\partial\Omega_T} T \cdot v \,ds$$

Nonlinear Poisson Equation

$$-\nabla \cdot (q(u) \nabla u) = f \qquad \Omega$$

$$u = u_D \quad \text{on} \quad \partial\Omega$$

$$F(u;v) = 0 \qquad \forall\, v$$

$$F(u;v) = \int_\Omega \left(q(u)\nabla u \cdot \nabla v - fv\right) dx$$

Time-Dependent

1) Diffusion Equation

i) $\frac{\partial u}{\partial t} = \nabla^2 u + f$ in $\Omega \times (0,T]$

$u = u_D$ on $\partial\Omega \times (0,T)$

$u = u_0$ at $t = 0$

Implicit Euler

$$a(u^{n+1}, v) = L_{n+1}(v)$$

$$a(u^{n+1}, v) = \int (u^{n+1} v + \Delta t \nabla u^{n+1} \cdot \nabla v) dx$$

$$L_{n+1}(v) = \int_\Omega (u^n + \Delta t f^{n+1}) v \, dx$$

Initial condition

$$a_0(u,v) = L_0(v)$$

$$a_0(u,v) = \int_\Omega u v \, dx$$

$$L_0(v) = \int_\Omega u_0 v \, dx$$

2) Wave propagation from a source point

$$U_{tt} = c^2 \left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right)$$

$$\frac{U^1 - 2U^0 + U^{-1}}{\Delta t^2} = c^2 \Delta U$$

$$\Rightarrow \quad U^1 - c^2 \Delta t^2 \Delta U^1 = 2U^0 - U^{-1}$$

$$\Rightarrow \quad a(u_1, v) = L(v)$$

$$a(u_1, v) = \int v U' + c^2 \Delta t^2 \nabla U \cdot \nabla v)\, dv$$

$$L(v) = \int (2U^0 - U^{-1}) v \, du$$

Initial conditions: 0

B.C. cond: point source at left Boundary.

Time-Harmonic Maxwell Equations

$$\nabla \times \mu^{-1} \nabla \times \vec{E} - \omega^2 \varepsilon \vec{E} = 0$$

$$\nabla \cdot \vec{E} = 0$$

$$(\vec{E}, \phi) \in H(\text{curl}) \times H^1/\mathbb{R}$$

① $$\int_\Omega (\nabla \times \vec{E}) \cdot (\nabla \times v)\, du + \int_\Omega \nabla \phi \cdot v \, du$$

$\forall v$ $$= \omega^2 \varepsilon \int_\Omega \vec{E} \cdot \vec{v}\, du$$

② $$\int_\Omega \vec{E} \cdot \nabla \varphi \, du = 0$$

$$\phi = 0 \quad \partial\Omega$$

- Nedelec Elements
- Projection to P_1 to study solution

Incompressible Navier Stokes

$$\rho\left(\frac{\partial u}{\partial t} + u \cdot \nabla u\right) = \nabla \cdot \sigma(u,p) + f$$

$$\nabla \cdot u = 0$$

$$\sigma(u,p) = 2\mu \varepsilon(u) - pI$$

$$\varepsilon(u) = \frac{1}{2}(\nabla u + \nabla u^T)$$

Splitting, Chorin's Method
$\Downarrow$
IPCS

Refer FEniCS manual

Multiphysics

Advection – Diffusion – Reaction

Physics 1 >

$$\rho\left(\frac{\partial w}{\partial t} + w\cdot\nabla w\right) = \nabla\cdot\sigma(w,p) + f$$

$$\nabla\cdot w = 0$$

Physics 2 >

- $$\frac{\partial u_1}{\partial t} + w\cdot\nabla u_1 - \nabla\cdot(\varepsilon\nabla u_1) = f_1 - k u_1 u_2$$

- $$\frac{\partial u_2}{\partial t} + w\cdot\nabla u_2 - \nabla\cdot(\varepsilon\nabla u_2) = f_2 - k u_1 u_2$$

- $$\frac{\partial u_3}{\partial t} + w\cdot\nabla u_3 - \nabla\cdot(\varepsilon\nabla u_3) = f_3 + k u_1 u_2 - k u_3$$

Gray Scott Reaction Diffusion

$$U + 2V \longrightarrow U + 3V$$

$$V \longrightarrow Q$$

$$u_t = D_1 \Delta u - uv^2 + \gamma(1-u)$$

$$v_t = D_2 \Delta v + uv^2 - (\gamma + k)v$$

$$\frac{\partial u}{\partial n} = 0, \quad \frac{\partial v}{\partial n} = 0$$

$$u(x,y,0) = 1 - 2v(x,y,0)$$

$$v(x,y,0) = \begin{cases} \frac{1}{4}\sin^2(4\pi x)\sin^2 4\pi y & 1 \le x, y \le 1.5 \\ 0 & \text{otherwise} \end{cases}$$

$$\Omega = (0,2) \times (0,2)$$

$$Dt = 0.5$$

$$D_1 = 8 \times 10^{-0.05}$$

$$D_2 = 4 \times 10^{-0.05}$$

$$r = 0.024$$

$$k = 0.06$$