Can we trust the Solution?

- A computer will give wrong answer with 19 digit precision.

Errors

i) Rounding Error — finite Precision Arithmetic

$$\begin{bmatrix} 10^{-12} & 1 \\ 1 & 0 \end{bmatrix}^{-1} \neq \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

ii) Data uncertainty — uncertainty in values

$$x \pm \Delta x$$

iii) Approximation / Truncation error

— $f(x+h) \simeq f(x) + h f'(x)$

Error Propagation

Stability: — Does the error blow up over time

Explicit:

$Y(t+\Delta t) = F(Y(t))$

$$\frac{dy}{dt} = Ay$$

$$y_{n+1} - y_n = A y_n \Delta t$$

$$\Rightarrow \quad y_{n+1} = y_n(1 + A\Delta t)$$

$$e_{n+1} = y_{n+1} - \hat{y}_{n+1}$$

$$e_n = y_n - \hat{y}_n$$

$$\Rightarrow \quad e_{n+1} = e_n(1 + A\Delta t)$$

$$\Rightarrow \quad e(k) = e(0)\, e^{At}$$ The error blows up

Implicit:

$G(\, Y(t), Y(t+\Delta t)) = 0$

$$\frac{y_{n+1} - y_n}{\Delta t} = A\, y_{n+1}$$

$$\Rightarrow \quad y_{n+1} - A\Delta t\, y_{n+1} = y_n$$

$$\Rightarrow \quad y_{n+1}(1 - A\Delta t) = y_n$$

$$\Rightarrow \quad y_{n+1} = \frac{y_n}{1 - A\Delta t}$$

Propagated Error $\longrightarrow 0$

$$\Rightarrow \quad e_{n+1} = \frac{e_n}{1 - A\Delta t}$$

Truncated Error is the determining factor.

$$\Rightarrow \quad e_{n+2} = \frac{e_n}{(1 - A\Delta t)^2}$$

$$\Rightarrow \quad e_{n+k} = \frac{e_n}{(1 - A\Delta t)^k}$$

$$\Rightarrow \quad e_{n+k} = \frac{e_n}{c^k}$$

$$\Rightarrow \quad e(k) = \frac{e_0}{c^k}$$

Implicit - Explicit

$$Y(t+\Delta t) = F(\, Y(t+\Delta t)) + G(Y(t))$$

- Implicit is 'Implicit' if the non-linear solver for $Y(t+\Delta t)$ calculates it to below threshold error iteratively.

- Any explicit calculation of $Y(t+\Delta t)$ makes the implicit problem explicit.

Conditions

eg:- Courant-Friedrichs-Lewy

For wave phenomenon,

(Time step for amplitude calculation) < (Time for wave to travel from 1 grid point to another)

General Error Scaling

$$\left| u(t) - u_{h,k}(t) \right|_{L_2} < C_1 \left(h^m + k^n \right) e^{C_2 t}$$

Higher Derivatives

spatial term

time term

Generic instability

CASE STUDY :

FINITE TIME BLOW UP

ISSUE: All Numerical Methods & Mathematical Problems assume the solutions are 'well behaved' for them to exist.

($x(t_0)=\infty$) A Blowup Singularity violates All of them.

- We can't trust the Numerical Methods

So How do we proceed?

MULTISCALE MODEL. SIMUL.
Vol. 12, No. 4, pp. 1722–1776

TOWARD THE FINITE-TIME BLOWUP OF THE 3D AXISYMMETRIC EULER EQUATIONS: A NUMERICAL INVESTIGATION*

GUO LUO† AND THOMAS Y. HOU‡

Abstract. Whether the three-dimensional incompressible Euler equations can develop a singularity in finite time from smooth initial data is one of the most challenging problems in mathematical fluid dynamics. This work attempts to provide an affirmative answer to this long-standing open question from a numerical point of view by presenting a class of potentially singular solutions to the Euler equations computed in axisymmetric geometries. The solutions satisfy a periodic boundary condition along the axial direction and a no-flow boundary condition on the solid wall. The equations are discretized in space using a hybrid 6th-order Galerkin and 6th-order finite difference method on specially designed adaptive (moving) meshes that are dynamically adjusted to the evolving solutions. With a maximum effective resolution of over $(3 \times 10^{12})^2$ near the point of the singularity, we are able to advance the solution up to $\tau_2 = 0.003505$ and predict a singularity time of $t_s \approx 0.0035056$, while achieving a *pointwise* relative error of $O(10^{-4})$ in the vorticity vector ω and observing a (3×10^8)-fold increase in the maximum vorticity $\|\omega\|_\infty$. The numerical data are checked against all major blowup/non-blowup criteria, including Beale–Kato–Majda, Constantin–Fefferman–Majda, and Deng–Hou–Yu, to confirm the validity of the singularity. A local analysis near the point of the singularity also suggests the existence of a self-similar blowup in the meridian plane.

Euler equation

$$\dot{\vec{u}} + \vec{u}\cdot\nabla\vec{u} = -\nabla p$$

$$\nabla\cdot\vec{u} = 0$$

u – velocity
p – pressure

$\vec{u}(t_0) = \infty$, True or False

Beale, Kato & Majda

- smooth $\vec{u}$ blows up at $t=T$

$\vec{\omega} = \nabla \times \vec{u}$

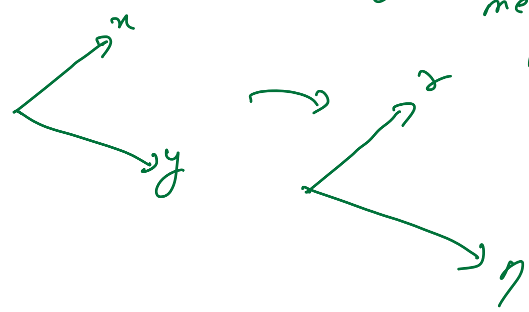
if $\displaystyle\int_0^T \left(|\omega_x| + |\omega_y| + |\omega_z|\right)dt = \infty$

Problem:–
To determine ω divergent.

Custom Moving Mesh

Setup a custom mesh so that:—

i) Any variables that depend on mesh are exactly known.



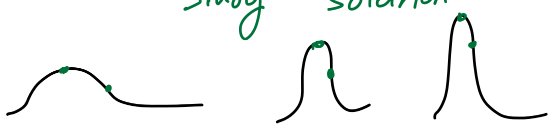
$\gamma(x)$

$\eta(y)$

How?

Study solution on an uniform mesh

cylindrical symmetry



FINITE-TIME SINGULARITY OF 3D EULER 1733

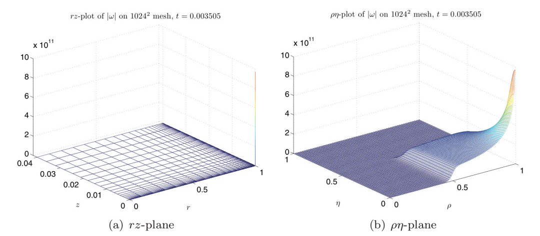


FIG. 1. *The vorticity function $|\omega|$ on the 1024×1024 mesh at $t = 0.003505$, in (a) rz-coordinates and (b) ρη-coordinates, where for clarity only one-tenth of the mesh lines are displayed along each dimension.*

- Update Mesh at each step so that—

  50% function change has same number of points.

- Solution in 1 mesh is projected onto other mesh.

$U = \sum c_i \phi_i$ $\quad c_i$ known.

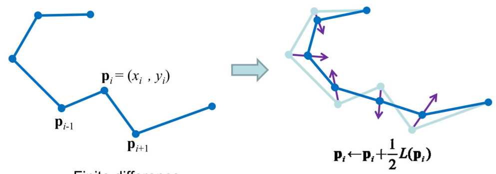
$= \sum b_i \psi_i$ $\quad \psi_i$ known

Custom Mesh avoids Mesh Smoothing

- Mesh Smoothing removes high frequency noise but also high frequency signal.

Laplacian Smoothing

An easier problem: How to smooth a curve?



$$\mathbf{p}_i \leftarrow \mathbf{p}_i + \frac{1}{2} L(\mathbf{p}_i)$$

Finite difference discretization of second derivative = Laplace operator in one dimension $\longrightarrow$ $L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i)$

16

To keep high resolution information, we avoid mesh smoothing in custom mesh.

Solver

- Same as F.E.A

Basis:– each element has B-splines basis of 6th order

i.e. piecewise polynomials of 5th order

Time Stepping

Advection :– $\mu = \frac{u \Delta t}{\Delta h}$ & ~~[illegible]~~

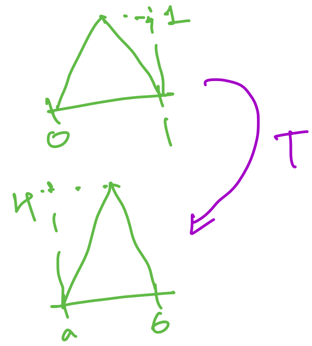
CFL is kept low

$$\left| \frac{u_{t+1}}{u_t} \right| \leq 1.05$$

Explicit RK-45

Some Derived Quantities

derivatives: —



$$\frac{\partial u}{\partial r}, \frac{\partial u}{\partial \eta}, = \frac{Q\ (\text{6th order polynomial fit})}{\left(\frac{\partial T}{\partial r}\right)_i}$$

$\left(\frac{\partial T}{\partial r}\right)_i$ known because T comes from custom mesh.

Default is Bad

when $\frac{\partial T}{\partial r} = C \approx 0$

Also called catastrophic cancellation → • bad cancellation happens i.e cancellation with a lot of error.

large relative errors

$\approx (1e18 - 1.0000000001e18)$

• Conventional meshing uses mesh smoothing. — lose resolution.

• Conventional meshing uses approximation of T. — more errors

What Happens

as we approach $w(T)=\infty$

- time steps become small
- mesh scale becomes ~~sm~~ finer.
- The error in simulation increases as assumptions break down.

So simulation goes on till we reach termination thresholds, t_e, h_e

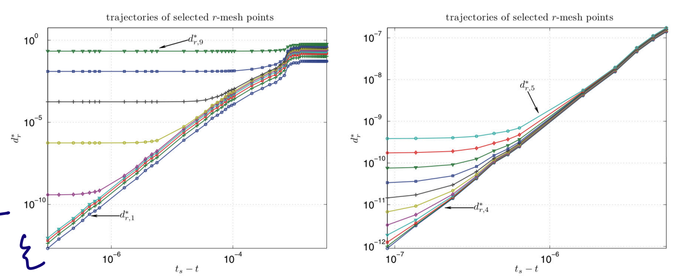
precheck 1: Is the adaptive mesh effective.

- mesh compression ratio :- $10^9 : 1$

  resolution :- $10^{12} : 1$

Far from ∞

close to ∞



Transition

(a) full view (b) detailed view

FIG. 2. *Trajectories of selected r-mesh points on the 2048 × 2048 mesh in log-log scale (see text for explanation). The last time instant shown in the figure is t_e, the stopping time.*

Conclusion: Decent sampling is done in entire domain.

$\rightarrow$ Solution is valid.

Precheck 2: Numerical Error Propagation

solving $Ax=b$ $\rightarrow$ $\hat{x}$ computational solution

$$w = \frac{|A\hat{x} - b|}{||A||x| + |b||}$$

k = condition number

$$\frac{\|\delta x\|_\infty}{\|x\|_\infty} \leq wk \simeq 10^{-9} \rightarrow 10^{-12}$$

(on mesh refinement)

Precheck 3:

$\log(\log|w|_{max})$ with t

does it converge with refinement.

Precheck 4: Is it accurate?

mathematical quantities like

Global
i) energy
ii) $\int |w|^2 \, dx$,
iii) Fourier T. $(\bar{u}^2(\bar{k}))$

Pointwise — Since none is available. we look at error with mesh refinement $(h \rightarrow 0)$

$$e = \left| u_{h_1}(x_1) - u_{h_2}(x_2) \right|$$

TABLE 10

Maximum (relative) change of kinetic energy E, minimum circulation Γ_1, and maximum circulation Γ_2 over the time interval $[0, 0.003505]$. The initial value of each quantity, measured on the finest 2048×2048 mesh, is indicated in the last row "Init. value" of the table.

Mesh size	$t = 0.003505$		
	$\|\delta E\|_{\infty,t}$	$\|\delta \Gamma_1\|_{\infty,t}$	$\|\delta \Gamma_2\|_{\infty,t}$
1024×1024	1.5259×10^{-11}	4.3525×10^{-17}	1.2485×10^{-14}
1280×1280	4.1730×10^{-12}	3.3033×10^{-17}	7.7803×10^{-15}
1536×1536	2.0787×10^{-12}	3.1308×10^{-17}	9.9516×10^{-15}
1792×1792	6.4739×10^{-13}	2.7693×10^{-17}	2.1351×10^{-14}
2048×2048	6.6594×10^{-13}	2.5308×10^{-17}	3.4921×10^{-14}
Init. value	55.9309	0.0000	6.2832×10^{2}

Recheck 5 : Trial Problems.

i) take $\vec{u}(x, t)$ as known solution (with singularity)

and simulate.

i.e. Check for fake similar solution.

Recheck 6 : Order of Meshing - Asymptotic

since mesh is of order 6

check $|u - u_h| < h^6$

$\log|u - u_h| \approx 6 \log h$

$$\frac{\log|u - u_h|}{\log h} = \beta$$

$\beta \simeq 6$ away from blowup

$\beta \simeq 4$ near blowup due to error accumulation.

Precheck : Order of timestepping

$$\frac{\log (u-u_f)}{\log k} = \alpha$$

$$\alpha \simeq 3.5$$

$\therefore$ we ~~can~~ have 3 significant digits.

Study for Singularity

$$\int_0^T \|\vec{\omega}\| \, dt = \infty$$

generic ansatz,

$$\|\omega\|_\infty \sim c(T-t)^{-\nu}$$

! (not exact) ! $\quad \nu > 1$

Basic idea: To fit $\|\omega\|$ with $a(T-t)^{-\nu}$

Although finite-time singularities were frequently reported in numerical simulations of the Euler equations, most such singularities turned out to be either numerical artifacts or false predictions, as a result of either insufficient resolution or inadvertent data analysis procedure (more to follow on this topic in section 4.4). Indeed, by

How to do it properly?

Find $[t_1, t_2]$ before T

i) wrong t_1, t_2 can give false singularities.

2) we don't know if $(t_s - t)$ ' is valid.

we are guessing.

Solution:— t_2 :— error in FEA > Threshold.

$$x \in [\tau_0, t_2]$$

1) fit (x, t_2)
2) choose $t_1 = x$ for which R^2 is maximum.

fit is successful if:—

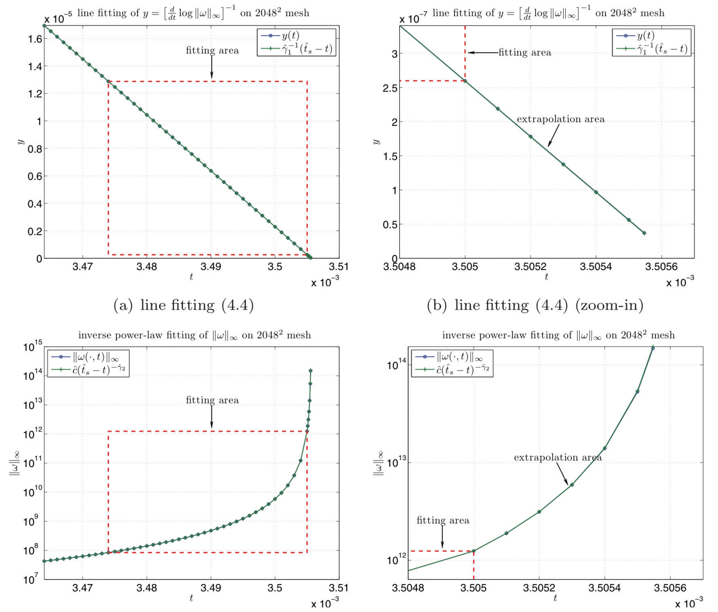
t_2 & $T_{estimated}$ converge to $T_{simulation}$

The fit equation:—

$$\underbrace{\left[\frac{d}{dt}\log\|w(x_0,t)\|\right]^{-1}}_{y} = \underbrace{\frac{-1}{\gamma}}_{m}\,(\underbrace{t}_{x}\,\underbrace{-T}_{c})$$

$\therefore$

Post-Check 1



(a) line fitting (4.4) (b) line fitting (4.4) (zoom-in)

(c) line fitting (4.6) (d) line fitting (4.6) (zoom-in)

FIG. 9. *Blowup of the maximum vorticity:* (a) *inverse log t-derivative of* $\|\omega\|_\infty$ *and its line fit* $\hat{\gamma}_1^{-1}(\hat{t}_s - t)$*;* (b) *a zoom-in view of* (a) *in the extrapolation interval;* (c) *maximum vorticity* $\|\omega\|_\infty$ *and its inverse power-law fit* $\hat{c}(\hat{t}_s - t)^{-\hat{\gamma}_2}$*;* (d) *a zoom-in view of* (c) *in the extrapolation interval. All results shown in this figure are computed on the* 2048×2048 *mesh.*

Paper 2

That was all suggestive. Not a Proof!

For Proof! –

STABLE NEARLY SELF-SIMILAR BLOWUP OF THE 2D BOUSSINESQ AND 3D EULER EQUATIONS WITH SMOOTH DATA I: ANALYSIS

JIAJIE CHEN AND THOMAS Y. HOU

ABSTRACT. Inspired by the numerical evidence of a potential 3D Euler singularity [65, 66], we prove finite time blowup of the 2D Boussinesq and 3D axisymmetric Euler equations with smooth initial data of finite energy and boundary. There are several essential difficulties in proving finite time blowup of the 3D Euler equations with smooth initial data. One of the essential difficulties is to control a number of nonlocal terms that do not seem to offer any damping effect. Another essential difficulty is that the strong advection normal to the boundary introduces a large growth factor for the perturbation if we use weighted L^2 or H^k estimates. We overcome this difficulty by using a combination of a weighted L^∞ norm and a weighted $C^{1/2}$ norm, and develop sharp functional inequalities using the symmetry properties of the kernels and some techniques from optimal transport. Moreover we decompose the linearized operator into a leading order operator plus a finite rank operator. The leading order operator is designed in such a way that we can obtain sharp stability estimates. The contribution from the finite rank operator to linear stability can be estimated by constructing approximate solutions in space-time. This enables us to establish nonlinear stability of the approximate self-similar profile and prove stable nearly self-similar blowup of the 2D Boussinesq and 3D Euler equations with smooth initial data and boundary.

1. INTRODUCTION

The question whether the 3D incompressible Euler equations can develop a finite time singularity from smooth initial data of finite energy is one of the most outstanding open questions in

For proof also, they used finite elements.

To solve

$$\frac{\partial f_1(t)}{\partial t} = A f_1 \qquad f_1(0) = f_0$$

$$\frac{\partial f_2(t)}{\partial t} = B f_2 + a(u) \int_\Omega f_1 g \, dx$$

Apriori error:- $|u - u_{h,l}| < C_1(h^m + k^n) e^{C_2 t}$

Not a good bound.

Post computation method

Solve $g(t,u) = e^{Bt} a(u)$

numerically,

gives us $\hat{g}(t,u)$

since $\hat{g}(t,u)$ is numeric

$\rightarrow$ there is error

let $e(t,u) = \left(\frac{\partial}{\partial t} - B\right) \hat{g}$

approx $\rightarrow \hat{f}_2(t,u) = \int_0^t P(f_1(s)) \hat{g}(t-s) \, ds$

We know $\hat{f}_2$, it is not f_2

$$f = f_1 + f_2 = f_1' + \hat{f}_2 \quad (1)$$

then $\partial_t \hat{f}_2 = \mathcal{L} \hat{f}_2 + a(n) P(f_1) + R(f_1, t)$

By (1) $\Rightarrow$ $\partial_t f_1 = A f_1 - R(f_1, t)$

. if $e(t, u)$ is small } Can be checked

. $R(f_1, t)$ is small

How small?

$\downarrow$

Given by Theorem.

Then

Perturbation Theorems for Ordinary Differential Equations

Aaron Strauss and James A. Yorke*

Department of Mathematics,
University of Maryland, College Park, Maryland

Received July 28, 1965

Using Theorems

$f_1 \to 0$

Consider the following systems of ordinary differential equations:

$$x' = f(t, x), \qquad (N)$$

$$x' = f(t, x) + g(t, x), \qquad (P)$$

$$x' = f(t, x) + g(t, x) + h(t), \qquad (P_1)$$

where $f(t, x)$ is continuous, satisfies a Lipschitz condition on some semi-cylinder, $f(t, 0) = 0$, and $x = 0$ is uniform asymptotically stable for (N). Let $g(t, x)$ and $h(t)$ be sufficiently smooth for local existence and uniqueness. Consider the conditions

(H_1): There exists $r > 0$ such that if $|x| \leqslant r$, then $|g(t, x)| \leqslant \gamma(t)$ for all $t \geqslant 0$, where

$$G(t) = \int_t^{t+1} \gamma(s)\, ds \to 0 \quad \text{as} \quad t \to \infty. \qquad (1.1)$$

(H_3): There exists a continuous, nonincreasing function $H(t)$ satisfying

$$\lim_{t \to \infty} H(t) = 0$$

such that $|\int_{t_0}^{t} h(s)\, ds| \leqslant H(t_0)$ for every $0 \leqslant t_0 \leqslant t \leqslant t_0 + 1$.
Then we prove: If $g(t, x)$ satisfies (H_1) and $h(t)$ satisfies (H_3), then there exists $T_0 \geqslant 0$ and $\delta_0 > 0$ such that if $t_0 \geqslant T_0$ and $|x_0| < \delta$, the solution $F(t, t_0, x_0)$ of (P_1) approaches zero as $t \to \infty$. In particular, if $x = 0$ is a solution of (P_1), then it is uniform asymptotically stable. Furthermore, if $g(t, x) \equiv 0$ and $h(t)$ does not satisfy (H_3), then *no* solution of (P_1) can approach zero as $t \to \infty$.

In the case $f(t, x) = Ax$, where A is a constant matrix, the above results

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* The second author holds an NSF Co-op. Fellowship.