Weak forms are called weak because they hold in some average sense.

Pointwise: $f(x)$

Average: $\frac{1}{b-a}\int_a^b f(x)\,dx$

Differential Equation in Strong

$$\frac{df}{dx}=0$$

on Ω

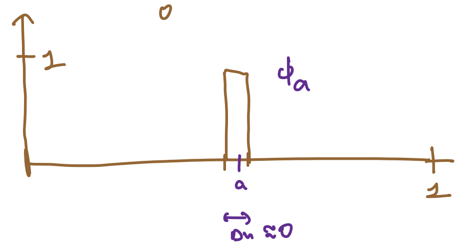
Problem: Find f such that

$$\frac{df}{dx}=0$$

at all x in Ω

Weak

$$\int_0^1 \frac{df}{dx}\,\phi_a\,dx = 0$$



Weak Problem:

Find f such that

$$\int_0^1 \frac{df}{dx}\,\phi_a\,dx = 0$$

for all a

$a_0 = 0$

$a_1 = \Delta x$

$a_2 = 2\Delta x$

$\vdots$

$a_{N-1} = (N-1)\Delta x$

$a_N = 1$

I) So, weak forms are an Average in some sense.

Average value of function is a number → Weak forms take in functions and give out numbers.

$\int_a^b f(x)\,dx$ → simplest weak form

i) takes $f(x)$

ii) gives out real number.

$$\int_\Omega \nabla f \cdot \vec{v}\, d\vec{x}$$

$d\vec{x} \equiv dx\,dy\,dz$

$$\int_a^b v \frac{df}{dx}\, dx$$

1D

$$\iint \left(\frac{df}{dx}\hat{i} + \frac{df}{dy}\hat{j}\right) \cdot \left(v_x\hat{i} + v_y\hat{j}\right) dx\,dy$$

2D

↕

$$\iint_\Omega \nabla f \cdot \vec{v}\, d\vec{x}$$

3D

$$\iiint \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(v_x\hat{i} + v_y\hat{j} + v_z\hat{k}\right)$$

$$dx\,dy\,dz$$

$$\updownarrow$$

$$\int_{\Omega} \nabla f \cdot \vec{v}\; d\vec{x}$$

Examples

$$\int_{\Omega} f\,dx \quad , \quad \int \nabla f\cdot\vec{v}\; d\vec{x} \quad , \quad \int (\nabla\times\vec{F})\cdot\vec{v}\; d\vec{x}$$

$$\int (\nabla\cdot f)\; d\vec{x}$$

$$\int f^2\,dx \quad , \quad \int \nabla f\cdot\nabla f\; d\vec{x}$$

$$\int \sin(\nabla\cdot\vec{F})\; d\vec{x}$$

$$\int_{\Omega} \log(\|\nabla\times\vec{F}\|)\; d\vec{x}$$

III) Weak forms can be written is shorthand

Let $f, v, \vec{F}, \vec{v}$ be functions!

$$a_1(f) = \int_0^1 f(u)\, du$$

$$a_2(f, \vec{v}) = \int_0^1 \nabla f \cdot \vec{v}\, d\vec{u}$$

$$a_3(f, \vec{v}) = \int_0^1 (\nabla \times \vec{F}) \cdot \vec{v}\, d\vec{u}$$

$$a_4(\vec{F}) = \int_0^1 (\nabla \cdot F)\, d\vec{u}$$

$$a(\underbrace{u, v, f, g}_{\text{Depends on functions } u, v, f, g})$$

Name ↓ a

Depends on functions u, v, f, g

IV) Some Forms are Linear some Non Linear.

Consider

$$F(u,v) = \int_0^1 u v^2 \, dx$$

Check:

$$F(c_1 u_1 + c_2 u_2, v)$$

$$= c_1 F(u_1, v) + c_2 F(u_2, v)$$

But

$$F(u, v_1 + v_2) \neq F(u, v_1) + F(u, v_2)$$

$\therefore$ The form $F(u,v)$ is

i) linear in u

ii) non-linear in v

Examples :-

Linear in u,v

$$\int uv \, dx \, , \quad \int \nabla u \cdot \nabla v \, dx$$

$$\int v \nabla \cdot (\nabla u) \, dx \qquad \int v \nabla^2 u$$

Check $\longrightarrow$ $\displaystyle\int (1 + x^2 + y^2 + z^2) \, v \, \nabla^2 u \, d\vec{x}$

Non-Linear in u,v

$$\int (u^2 + v^2) \, dx \, , \quad \int \log u \cdot v^2 \, dx$$

$$\int e^u e^v \, dx \qquad \int (\nabla u \cdot \nabla v)^2 \, dx$$

$$\int (1 + u^2) \, u \, \nabla^2 v \, dx$$

$$\int (\nabla^2 u)^2 \, v \, dx$$

$$\int (1+u^2+v^2)\,(Du \cdot Dv)\,dx$$

$$\int (1+u^2+v^2)\,(v\;D^2u)\,dx$$

Summary

I) Weak forms convert pointwise (PDE) into 'averages' in some sense (Integral)

II) These take in functions and give out numbers

III) These can be written in short hand $\rightarrow$ $a(u,v,w)$, $F(u)$

IV) These can be <u>linear</u> or <u>non-linear</u>.