Well posedness of PDEs :-

- One and Unique Solution exists for the PDE.
- The solution depends on initial values and boundary conditions.

Is it relevant?

— If PDE is not well posed then there is missing or incorrect physics.

Navier – Stokes

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} - \nu \nabla^2 \vec{u} = -\nabla h$$

- No Mathematical Proof of Pointwise Solution in general.
- Weak solution exists (Not known if unique)
- Not sure if — i) we might be missing some physics

  ii) we might be using wrong laws

- If the PDE is not well posed, it might not have a solution!

Norms a.k.a. Distance

For real numbers, a_1 & a_2

$$d_{12} = |a_1 - a_2|$$

$$d_{12} = \frac{|a_1 - a_2|}{\sqrt{1 + a_1^2 a_2^2}}$$

For (x_1, y_1) & (x_2, y_2)

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$d = \sqrt[3]{(x_1 - x_2)^3 + (y_1 - y_2)^3}$$

For functions $f(\vec{x}), g(\vec{x})$

$$d_{12} = \left[\int_0^1 (f(x) - g(x))^2 \, dx\right]^{1/2}$$

$$d_{12} = \left[\int_0^1 (f(x) - g(x))^3 \, dx\right]^{1/3}$$

These 'distances' are known as norms

1) $||f - g||_{L_2}$

2) $||f - g||_{L_3}$

For FEA

Q 1) WEAK FORMULATION — WELL POSED?

A :— I) Lax - Milgram

$a \to$ continuous

for $a(u,v) = L(v)$

if $a(u,u) \geq \alpha \|u\|^2 \quad \forall u$

key Question :— Is the PDE Coercive?

$u \in V$:→ V is a Hilbert Space

II) inf-sup Condition

$u \in W \to$ Reflexive Banach space

$v \in V \to$ Banach space

for $a(u,v) = L(v)$

$\exists \alpha$

$$\alpha \|w\|_W = \sup_V \frac{a(w,u)}{\|u\|_V}$$

key Question :— Does the PDE satisfy inf-sup condition?

Also known as

BNB Theorem

→ Banach-Necas-Babuska Theorem

Q2 >

WEAK FORMULATION in FINITE-ELEMENT BASIS

- Finite Element Spaces are finite dimensional

$$V \longrightarrow V_h$$
$$W \longrightarrow W_h$$
$$a(\cdot,\cdot) \longrightarrow a_h(\cdot,\cdot)$$
$$L(\cdot) \longrightarrow L_h(\cdot)$$

We need BNB for finite dimensional case.

$\cong$ $a_h \rightarrow$ (becomes a matrix) being invertible.

Ideas

Conformity $\quad W_h \subset W$, $V_h \subset V$

Approximability $\quad \forall w \in W$

$$\lim_{h \to 0} \left(\inf_{w_h \in W_h} \| w - w_h \|_{W(h)} \right) = 0$$

Consistency $\quad a_h(u, v_h) = f_h(v_h) \quad \forall v_h \in V_h$

Asymptotic Consistency

consistency

i) a_h is uniformly bounded on $W_h \times V_h$

$\exists \, \Pi_h : W \to W_h$

(interpolation operator)

$$\| \Pi_h w - w \|_{W(h)} \le C \inf_{w_h \in W_h} \| w - w_h \|_{W(h)}$$

(There is a state of minimum distance)

$$R_h(u) = \sup_{v_h \in V_h} \left| \frac{f_h(v_h) - a_h(\Pi_h u, v_h)}{\|v_h\|_{V_h}} \right|$$

$$\lim_{h \to 0} R_h(u) = 0$$

- There is a state of minimum distance that satisfies the functional equation as $h \to 0$.

$\lim_{h \to 0} \Pi_h w$ might not lie in W.

Q3) APPROXIMATION?

Nonconsistent - • Finite quadratures for integrals

nonconformal - a_h, l_h not properly defined in W, V

ERROR ANALYSIS

Basic Idea : $\|u - u_h\| < E_h(u, u_h)$

if $h \longrightarrow 0$ & $E_h(u, u_h) \longrightarrow 0$

$\Rightarrow u = \lim_{h \to 0} u_h$

Basic Theorem

i) Finite BNB Theorem holds
Finite Element Problem is wellposed. BNB way

ii) a_h is uniformly bounded
There is a maximum for $|a_h|$

iii) asymptotically consistent

as $h \to 0$

$$\frac{|a_h(u_h, v_h) - L_h(v_h)|}{|v_h|} \to 0$$

iv) Approximability

as $h \to 0$

we can find u_h s.t.

$$\|u - u_h\| \to 0$$

$$\|u - u_h\|_{W(h)} \leq \frac{1}{\alpha_h} R_h(u) + c \inf_{w_h \in W_h} \|u - w_h\|_{W(h)}$$

as $h \to 0$

$$\|u - u_h\|_{W(h)} \to 0$$

When Things are Non-consistent, Non-conformal

then There is a Minimum Error.

Non-consistent – u is not a solution of a_h

Non-conformal – $W_h - W \neq \phi$

These are Theorems that give an error estimate: –

i) Non-consistent & Nonconformal

2nd Strang Lemma

ii) Non-consistent & conformal

eg:-

- $a_h(u_h, \dots)$ involves point values of u_h
- unable to perform a_h for some w in V.

1st Strang Lemma.