Fields — 'functions' over entire space.

For our purposes: —

a) Space only

— $\vec{u}(\vec{r})$, $\chi(\vec{r})$, $\mathbb{T}(\vec{r})$

vector, scalar, Tensor

* We don't look at wavenumber, time $\vec{u}(\vec{k}, t)$ because that is an ODE

b) Space, Time

— $\vec{u}(\vec{r}, t)$, $\chi(\vec{r}, t)$

$\mathbb{T}(\vec{r}, t)$

c) Space, frequency.

$\vec{u}(\vec{r}, w)$, $\chi(\vec{r}, w)$

...

PDEs :—

i) Space-filling

eg:- $\nabla^2 u = f$ Equation

in Ω Region

Dirichlet $u = V$ on $\partial\Omega$ Boundary Condition

eg:- fixed Temperature

Neumann

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{n} \quad \text{on } \partial\Omega$$

$$= V$$

eg:- energy scale constant.

Robin $\alpha u + \beta \frac{\partial u}{\partial n} = V$ on $\partial\Omega$

Cauchy $u = V_1$ $\frac{\partial u}{\partial n} = V_2$ on $\partial\Omega$

2) Space – time

↳ needs Initial Values

$$\frac{\partial u}{\partial t} = \nabla^2 u$$ requires $u(t=0)$ on $\partial\Omega$

$$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$ requires
$u(t=0)$ on $\partial\Omega$
$u'(t=0)$ on $\partial\Omega$

3) Space – Frequency

- Frequency is a parameter

Sommerfeld radiation condition

- $$\lim_{x\to\infty} |x|^{\frac{n-1}{2}} \left(\frac{\partial}{\partial |x|} - ik \right) u(x) = 0$$

$n = \dim \{x\}$

FINITE DIFFERENCES

Forward $$\left(\frac{\partial u}{\partial t}\right)_{n+1} = \frac{u_{n+2} - u_{n+1}}{\Delta t}$$

Backward $$\left(\frac{\partial u}{\partial t}\right)_{n+1} = \frac{u_{n+1} - u_{n}}{\Delta t}$$

Symmetric
$$\left(\frac{\partial u}{\partial t}\right)_{n+1} = \frac{u_{n+2} - u_n}{2\,\Delta t}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{n+1} = \frac{u_{n+2} - 2u_{n+1} + u_n}{(\Delta t)^2}$$

FINITE ELEMENTS

Interlude 1 :
1) Continuous Fourier Transform.
2) Lagrange Polynomials
3) Legendre Polynomials

Basis — a) Completeness — • approximate
• exact
b) orthogonality

Interlude 2 :
1) Functionals
• functions as points
eg:- $\int_a^b (\cdot)\, dx$, $\int_a^b \frac{d}{dx}(\cdot)\, dx$
2) Linear Functionals
(Matrix Analogy)

In F.E.A.

we use
I) i) A complete Basis
(need not be orthogonal)

$$u(\vec{x}, t) = \sum \phi_j(\vec{x})\, c_j(t)$$

II) we break the space into

FINITE ELEMENTS

There is only 1 Basis function
that 'prefers' an element.

Periodic Table of the Finite Elements

1) Polyhedral cell of Mesh : Elements
2) Finite dimensional space of : Shape functions
polynomial functions
on each
element
3) Unisolvent set of functionals
on the shape functions
of each element : Degrees of Freedom

(Any Linear functional has
unique solution in terms
of these functionals)

ELEMENTS

1) Simplex 2) Cuboidal 3) Polygonal

SHAPE FUNCTIONS

1) Nodal Basis — Lagrange Polynomials
2) Heirarchial Basis — Legendre Polynomials
Bessel Polynomials.

DEGREE OF FREEDOM

Linear Functional $\{\gamma_i\}$

Shape Functions $\{\phi_j\}$

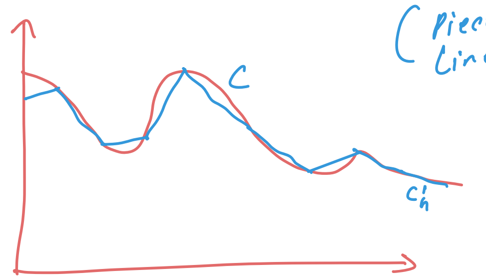
$$\gamma_i(\phi_j) = \delta_{ij}$$

eg:- Polynomial space - P^1

Approximation Space - P^1_h

h - interval length

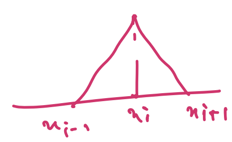
(Piecewise Linear Continuous)



1) Any curve C can be approximated by P^1_h curve C_h

2) The approximation gets better as $h \to 0$.

shape functions $\phi_i \to$ hat functions



For any curve C

$$\gamma_i(C) = C(x_i)$$

Note. $\gamma_i : C \longmapsto C(x_i)$

Space of functions

Real Numbers

Approximation Space :-

P_h^k — Piecewise Continuous polynomials of degree k

↓

Nodal Basis

Evaluation Functional Basis

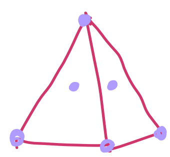
P_d^k — Discontinuous Piecewise polynomials of degree k

↓

Heirarchial Basis

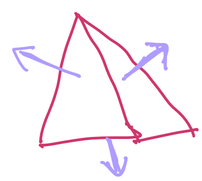
Inner Product Functional Basis.

Q- What are we computing at each point of element!



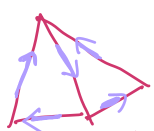
Field values

↳ $\nabla(\,)$ functional



Flux values

↳ $\nabla\cdot(\,)$ functional



edge integrals

↳ $\nabla\times(\,)$ functional

SUMMARY

1) Elements — Polyhedra
2) Approximation Space — P_d^k , P_h^k

3) Basis $\longrightarrow$ 1) Nodal Linear Functional

2) Modal

Shape Degrees of freedom

Quantity – 1) nodes (grad) 2) flux (div) 3) lines (curl)

We want Interpolation Error Theorem to hold

i.e. Difference(Solution – Projection)

$< h^{\alpha} X$

as $h \longrightarrow 0$ $X \longrightarrow$ higher derivatives

Error $\longrightarrow 0$

What we finally get?

- Piecewise Continuous Functions.
- Discontinuous at edges

(But Derivatives!)

Hence people talk of Sobolev Spaces

(where step functions are allowed at derivatives.

But not Delta functions)

NOTE

DIFFERENCE (Solution − Projection)

$< h^{\alpha} X$

has constraints on

i) Solutions (How rough? or smooth?)

ii) Approximation Space (• $P^{k}_{d/h}$ degree?

• Element type continuous discontinuous?)

iii) X (properties of higher Derivatives)