Introduction to Uncertainty Quantification
ME 597/MA 598, Spring 2018
Homework 1
Due: 01/23/2018

Ilias Bilionis (instructor, ibilion@purdue.edu)
Rohit Tripathy (TA, rtripath@purdue.edu)

References

[1] Lecture 1 slides

[2] Lecture 2 slides

[3] Lecture 3 slides

[4] Lecture 4 slides

[5] E. T. Jaynes, *Probability Theory: The Logic of Science*, Cambridge, 2002.

[6] Karl Popper wikipedia entry.

Instructions

1. Print this document.
2. Use a pen/pencil to fill in the answer in the space provided marked as either "Proof" or "Answer." Please try to keep it neat. Use another piece of paper for your early attempts.
3. Scan your document in black and white. You can use Purdue's library facilities to do this. Alternatively, you can use your smart phone. In the latter case you have (at least) two options. You can use the dropbox app (hit the "+" sign on the first screen and select "Scan Document"). The other option is an app called Scannable. Use whatever you want as soon as the resulting files are black and white and of a moderate file size. **Do not upload unprocessed pictures you took with your phone. We will reject those.**
4. Please write your name on every page (starting from this one).
5. The total homework points are 100. Please note that the problems are not weighed equally.

First Name: Gourav
Last Name: Jhanwar
Email: gjhanwar@purdue.edu

First Name: Gourav Last Name: Jharway

Problem 1. (32 points) This exercise demonstrates that probability theory is actually an extension of logic. Assume that you know that "A implies B". That is, your prior information is:

$$I = \{A \implies B\}.$$

Please answer the following questions in the space provided:

A. (4 points) $p(AB|I) = p(A|I)$.
Proof:

Since $A \Rightarrow B$, A is a subset of B.
∴ Intersection of a set, which is a subset of other set, is the set itself. $AB = A$

∴ $p(AB|I) = p(A|I)$

B. (4 points) If $p(A|I) = 1$, then $p(B|I) = 1$.
Proof:

Since A is a subset of B,
& $p(A|I) = 1$
It implies both A & B are same to
∴ $p(B|I) = 1$

C. (4 points) If $p(B|I) = 0$, then $p(A|I) = 0$.
Proof:

$A \Rightarrow B$
A is a subset of B.
If $p(B|I) = 0$, it means event B havent occured.
∴ If B havent occured implies, event A has also not taken place.
∴ $p(A|I) = 0$

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D. (4 points) B and C show that probability theory is consistent with Aristotelian logic. Now, you will discover how it extends it. Show that if B is true, then A becomes more plausible, i.e.

$$p(A|BI) \geq p(A|I).$$

Proof: From B & C, we can see that, if event B occurs, probability of event A occuring increases.
∴ $p(A|BI)$ = probability of A given B.
Since A is subset of B, & if B becomes true, A becomes more plausible.
∴ $p(A|BI) \geq p(A|I)$

E. (4 points) Give at least two examples of D that apply to various scientific fields. To get you started, here are two examples: $p(A|B)$

a. A: It is raining. B: There are clouds in the sky. Clearly, $A \implies B$. D tells us that if there are clouds in the sky, raining becomes more plausible.

b. A: General relativity. B: Light is deflected in the presence of massive bodies. Here $A \implies B$. Observing that B is true makes A more plausible.

Answer:

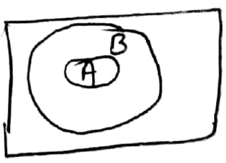
a. A: The sky is over cast
B: The sun is not visible

b. A: Gravity
B: Apple falling down the tree
Here $A \Rightarrow B$, observing apple falling down tree, gravitation becomes more plausible.

F. (4 points) Show that if A is false, then B becomes less plausible, i.e.:

$$p(B|\neg AI) \leq p(B|I).$$

Proof: A is subset of B, $A \subseteq B$.
∴ if A is false, event that B will occur will also reduce. (using the venn diag.)



Mathematically it can be represented as,
$p(B|\neg AI) \leq p(B|I)$

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G. (4 points) Can you think of an example of scientific reasoning that involves F? For example: *A*: It is raining. *B*: There are clouds in the sky. F tells us that if it is not raining, then it is less plausible that there are clouds in the sky.

Answer:

A: Gravity

B: Apple is falling down the tree

F: If there is no gravity, then it is less pausible that apple will be falling down the tree.

H. (4 points) Do D and F contradict Karl Popper's [6] *principle of falsification*, "A theory in the empirical sciences can never be proven, but it can be falsified, meaning that it can and should be scrutinized by decisive experiments."

Asnwer:

No — D&F do not contradict Karl Popper's principle of falsification

In example D&F, it is shown that if B is true, A becomes more plausible. This thing can be extended to principle of falsification. ie a theory in the empirical science can be falsified, if there is any experiment, which contradicts the theory, thus making the theory in empirical science less pausible (falsified)

e.g. Gravity theory in the empirical sciences can never be proven, but it can falsified by decisive experiments like apple going up instead of falling down the tree.

xample: A:
it is less

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Problem 2. (20 points) Consider the medical diagnosis example from Lecture 3.

A. (10 points) Compute the probability that a patient that tested negative has tuberculosis. Does the test change our prior state of knowledge about about the patient?

Answer:

A: Test is positive

B: patient has disease

From question, $p(B|I) = 0.004$, $p(A|B,I) = 0.8$, $p(A|\neg B I) = 0.1$ .. $(p(\neg A|\neg B I) = 0.9)$

Now, using the sum rule,

probability that the test is +ve $= p(A) = p(A|BI) \cdot p(B) + p(A|\neg BI) \cdot p(\neg B)$

$(\because p(\neg B|I) = 0.996)$

$\therefore p(A|I) = 0.8 \times 0.004 + 0.1 \times 0.996$

$= \underline{0.1028}$

$\therefore p(\neg A|I) = 1 - p(A|I) = 1 - 0.1028 = \underline{0.8972}$

probability that test is negative & the patient has TB $= 0.2 \times 0.004 = 0.0008 = p(\neg A|B) p(B)$

probability that tested negative has TB $= p(B|\neg A I) = \dfrac{p(B \neg A|I)}{p(\neg A)}$

$\therefore$ probability that tested $= \dfrac{0.0008}{0.8972} = \underline{\underline{0.00089}}$

B. (10 points) What would a good test look like? Find values for $p(A|B, I) = p(\text{test is positive}|\text{has tuberculosis}, I)$, and $p(A|\neg B, I) = p(\text{test is positive}|\text{does not have tuberculosis}, I)$, so that $p(B|A, I) = p(\text{has tuberculosis}|\text{test is positive}, I) = 0.99$. There are more than one solutions. How would you pick a good one? Thinking in this way can help you set goals if you work in R&D. If you have time, try to figure out whether or not there exists such an accurate test for tuberculosis.

Answer:

We know that,

$p(B|I) = \underline{0.004}$ $\therefore p(\neg B|I) = 1 - 0.004 = \underline{\underline{0.996}}$

$\therefore p(\text{has tuberclosis} | \text{test is +ve}, I) = 0.99$

$p(B|A) = 0.99$

$\dfrac{p(BA|I)}{p(A)} = 0.99$ .. (using product rule) — ①

But $p(A|I) = \underbrace{p(A|BI)}_{\text{let } x} p(B|I) + \underbrace{p(A|\neg B, I)}_{y} \cdot p(\neg B|I)$ .. Using sum rule

$= x \times 0.004 + 0.996y$

$\therefore$ From ①

$\dfrac{p(A|BI) \cdot p(B|I)}{p(A)} = 0.99$

$\dfrac{0.004x}{0.004x + 0.996y} = 0.99$

$\therefore$ After solving we get,

$0.00004x = 0.98604y$

$\boxed{x = 24,651y}$

Let $x = 1$ $\therefore y = \dfrac{1}{24651} = \underline{4.05 \times 10^{-5}}$

Probabilities with above values will be good test because probability that test is +ve given patient has turberclosis is 1 (too high) making the test more effective

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Problem 3. (20 points) Let A and B be independent conditional on I. Prove that:

$$A \perp B|I \iff p(AB|I) = p(A|I)p(B|I).$$

Hint: Use the fact that $A \perp B|I$ means that $p(A|B, I) = p(A|I)$ and $p(B|A, I) = p(B|I)$.

Answer:

Since A & B are independent, i.e. $A \perp B | I$

$P(A|B,I) = p(A|I)$ - ① since probability of A does not depend upon B

Now, using multiplication rule,

$p(AB|I) = p(A|B,I) \; p(B)$

But from ①, $p(A|B) = p(A|I)$

$\therefore \boxed{p(AB|I) = p(A|I) \cdot p(B|I)}$

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Problem 4. (20 points) Let X be a continuous random variable and $F(x) = p(X \leq x)$ be it's cumulative distribution function. Using only the basic rules of probability, prove that:

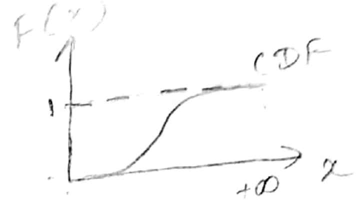
A. (4 points) The CDF starts at 0 and goes up to 1:

$$F(-\infty) = 0 \text{ and } F(\infty) = 1.$$

Proof:

$F(x) = P(X \leq x)$

$= \int_{-\infty}^{x} f(x)\,dx$



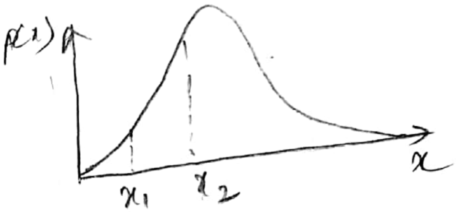
$\therefore F(-\infty) = \int_{-\infty}^{-\infty} f(x)\,dx = 0$

$\& F(+\infty) = \int_{-\infty}^{\infty} f(x)\,dx = 1$ (since, $\int_{-\infty}^{\infty} f(x)\,dx = 1$)

B. (8 points) $F(x)$ is a monotonically increasing function of x, i.e.,

$$x_1 \leq x_2 \implies F(x_1) \leq F(x_2).$$

Proof:



$x_1 \leq x_2$

$F(x_1)$ represents the area behind line x_1

$\& F(x_2)$ represents the area behind line x_2. $x_2 > x_1$

But, $\therefore$ area$(x_2) >$ area(x_1)

$\therefore F(x_2) > F(x_1)$

C. (8 points) The probability of X being in the interval $[x_1, x_2]$ is:

$$p(x_1 \leq X \leq x_2|I) = F(x_2) - F(x_1).$$

The CDF is defined on semi-infinite intervals of the form $(-\infty, x_1]$ & $(-\infty, x_2]$.

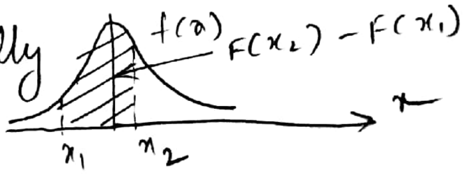
Here the events $A = \{x \in (-\infty, x_1]\}$, $B = \{x \in (-\infty, x_2]\}$ $\& C \in \{X \in (x_1, x_2]\}$ are in relationship

$B = A + C$

Since A & C are disjoint & $B = A \cup C$ $\therefore P(B) = P(A) + P(C)$.

$\therefore P(C) = P(B) - P(A)$

$p(x_1 \leq X \leq x_2) = F(x_2) - F(x_1)$ → Graphically



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Problem 5. (8 points) Let X be a random variable. Prove that:

$$\mathbb{V}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 .$$

Proof: $\mathbb{E}(x) = \int x p(x) dx$

We know that, $V(x) = \mathbb{E}[(X - E(X))]^2$

$= \mathbb{E}(X^2 - 2X\mathbb{E}(X) + (\mathbb{E}(X))^2)$

But we know that $\mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y)$ & $\mathbb{E}(cX) = c\mathbb{E}(X)$

$\therefore V(x) = \mathbb{E}(X^2) - 2\mathbb{E}(X)\mathbb{E}(X) + \mathbb{E}((\mathbb{E}(X))^2)$

constant

$\therefore (E[X+c] = E(X) + c)$

$\therefore V(x) = \mathbb{E}(X^2) - 2(\mathbb{E}(X))^2 + (\mathbb{E}(X))^2$

$$\boxed{V(x) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2}$$

Feedback

How much time (approximately) did it take you to finish this assignment?
Answer: