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DIFFERENTIAL EQUATIONS (MA 121)

Week 2 / April 2022 Problem Set 2 G.R.

■ **Notations :** ODE $\equiv$ Ordinary Differential Equation, $y' \equiv \frac{dy}{dx}$, $y'' \equiv \frac{d^2y}{dx^2}$, $y^{(i)} \equiv \frac{d^iy}{dx^i}$ for $i = 3, 4, 5, \ldots$.

■ Introduction of ODE, Solutions of First order and First degree ODE

1. Solve the initial value problems

   (a) $\dfrac{dy}{dx} + y = f(x)$ where $f(x) = \begin{cases} 2, & 0 \le x < 1, \\ 0, & x \ge 1, \end{cases}$ $\qquad y(0) = 0$

   $\left[\text{Ans: } y(x) = \begin{cases} 2\left(1 - e^{-x}\right), & 0 \le x < 1, \\ 2\left(e - 1\right)e^{-x}, & x \ge 1. \end{cases}\right]$

   (b) $\dfrac{dy}{dx} + y = f(x)$ where $f(x) = \begin{cases} e^{-x}, & 0 \le x < 2, \\ e^{-2}, & x \ge 2, \end{cases}$ $\qquad y(0) = 1.$

   $\left[\text{Ans: } y(x) = \begin{cases} (1 + x)\,e^{-x}, & 0 \le x < 2, \\ 2e^{-x} + e^{-2}, & x \ge 2. \end{cases}\right]$

2. Consider the ODE $a\dfrac{dy}{dx} + by = k\,e^{-\lambda x}$, where a, b and k are positive constants and λ is a non-negative constant.

   (a) Solve this equation.

   (b) Show that if $\lambda = 0$ every solution approaches $\frac{k}{b}$ as $x \to \infty$, but if $\lambda > 0$ every solution approaches 0 as $x \to \infty$.

3. The equation

$$\frac{dy}{dx} = A(x)y^2 + B(x)y + C(x) \tag{1}$$

   is called *Riccati's equation*. Note that (or check yourself) if $A(x) = 0$ for all x, then equation (1) is a linear equation, whereas if $C(x) = 0$ for all x, then equation (1) is a Bernoulli equation.

   (a) Show that if f is any solution of (1), then the transformation

$$y = f + \frac{1}{v}$$

   reduces equation (1) to a linear equation in v (dependent variable) and x (independent variable).

(b) Using the above, solve the following ODE
$$\frac{dy}{dx} = -y^2 + xy + 1,$$
given solution $f(x) = x$.

4. Solve the following ODEs by finding integrating factor.

(a) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)\,dy = 0$ $\left[\text{Ans: } x^3y^2 + \frac{x^2}{y} = c\right]$

(b) $(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0$ $\left[\text{Ans: } x = cye^{\frac{1}{xy}}\right]$

(c) $(xy^2 - x^2)\,dx + (3x^2y^2 + x^2y - 2x^3 + y^2)\,dy = 0$
$\left[\text{Ans: } \left(\frac{1}{2}x^2y^2 - \frac{1}{3}x^3 + \frac{1}{6}y^2 - \frac{1}{18}y + \frac{1}{108}\right)e^{6y} = c\right]$

(d) $\left(xy^2 - e^{\frac{1}{x^3}}\right)dx - x^2y\,dy = 0$ $\left[\text{Ans: } 3y^2 - 2x^2e^{\frac{1}{x^3}} = cx^2\right]$

5. Show that $\dfrac{1}{(x+y+1)^4}$ is an integrating factor of the ODE
$$\left(2xy - y^2 - y\right)dx + \left(2xy - x^2 - x\right)dy = 0$$
and hence solve it. $\left[\text{Ans: } xy = c\,(x+y+1)^3\right]$

6. Find the orthogonal trajectories of the family of curves $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, where a is a parameter. $\left[\text{Ans: } x^{\frac{4}{3}} - y^{\frac{4}{3}} = k\right]$

7. Show that the family of conics $\dfrac{x^2}{a^2+\lambda} + \dfrac{y^2}{b^2+\lambda} = 1$ is self-orthogonal, where λ is a parameter. $\left[\text{Try to show : } x^2 - y^2 + xy\left(y' - \dfrac{1}{y'}\right) = a^2 - b^2\right]$

8. Find the orthogonal trajectories of the family of cardioides $r = a(1-\cos\theta)$, a being a parameter. [Ans: $r = c(1+\cos\theta)$]

■ **NOTE** [Trajectories in polar co-ordinate] : In cartesian co-ordinate, slope of the tangent of a curve at the point (x, y) is $\frac{dy}{dx}$; whereas in polar co-ordinate, slope of the tangent of a curve at the point (r, θ) is $r\frac{d\theta}{dr}$

1. *Orthogonal Trajectories.*

Step 1. From the given family of curves $f(r, \theta, c) = 0$, eliminating c, we get the ODE of the given family as
$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \tag{2}$$

Step 2. In the ODE (2), replace $\mathbf{\frac{dr}{d\theta}}$ by $\mathbf{-r^2\frac{d\theta}{dr}}$. This gives the ODE
$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0 \tag{3}$$
of the orthogonal trajectories.

Step 3. Solve the ODE (3) and obtain a one-parameter family $g(r, \theta, k) = 0$, which is the desired family of orthogonal trajectories of the given family of curves.

2. *Oblique Trajectories (angle α).*

Step 1. From the given family of curves $f(r, \theta, c) = 0$, eliminating c, we get the ODE of the given family as

$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \tag{4}$$

Step 2. In the ODE (4), replace $\mathbf{\frac{dr}{d\theta}}$ by $\dfrac{\mathbf{r \mp r^2 \frac{d\theta}{dr}} \tan\alpha}{\mathbf{r\frac{d\theta}{dr}} \pm \tan\alpha}$. This gives the ODE

$$F\left(r, \theta, \frac{r - r^2 \frac{d\theta}{dr} \tan\alpha}{r\frac{d\theta}{dr} + \tan\alpha}\right) = 0 \tag{5}$$

of the oblique trajectories.

Step 3. Solve the ODE (5) and obtain a one-parameter family $g(r, \theta, k) = 0$, which is the desired family of oblique trajectories of the given family of curves.

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