Department of Mathematical Sciences

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DIFFERENTIAL EQUATIONS (MA 121)

Week 4 / April 2022 | Problem Set 4 | G.R.

■ **Notations :** ODE ≡ Ordinary Differential Equation, $y' \equiv \frac{dy}{dx}$, $y'' \equiv \frac{d^2y}{dx^2}$, $y^{(i)} \equiv \frac{d^iy}{dx^i}$ for $i = 3, 4, 5, \ldots$.

■ Solution of second order linear ODE

1. Find the general solution of the following ODEs. (Use Reduction of Order)

   (a) Given that $y = x + 1$ is a solution of $(x+1)^2\dfrac{d^2y}{dx^2} - 3(x+1)\dfrac{dy}{dx} + 3y = 0$.

   (b) Given that $y = x$ is a solution of $(x^2-1)\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$.

   (c) Given that $y = e^{2x}$ is a solution of $(2x+1)\dfrac{d^2y}{dx^2} - 4(x+1)\dfrac{dy}{dx} + 4y = 0$.

2. If $f = f_1 + if_2$ is a complex-valued solution of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), \tag{1}$$

   with p, q and r as real-valued function on a certain interval I, then f_1 and f_2 are real solutions of (1).

3. Find the general solution of the following ODEs.

   (a) $2\dfrac{d^3y}{dx^3} - 7\dfrac{d^2y}{dx^2} + 7\dfrac{dy}{dx} - 2y = 0$

   (b) $\dfrac{d^2y}{dx^2} - 10\dfrac{dy}{dx} + 25y = 0$

   (c) $4\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = 0$

   (d) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 4y = 0$

   (e) $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

   (f) $\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = 0$

   (g) $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$

4. The differential equation

$$x^2\frac{d^2y}{dx^2} + \alpha x\frac{dy}{dx} + \beta y = 0 \tag{2}$$

is known as *Euler's equation.*

(a) Show that $y = x^r$ is a solution of equation (2) if $r^2 + (\alpha - 1)r + \beta = 0$.

(b) Find the general solution of $x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} - 5y = 0, \quad x > 0$.

[**Note** : Use the transformation $x = e^t$ or $t = \ln x$. Then $\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$, $\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$. Substituting $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the equation (2), we obtain $\frac{d^2y}{dt^2} + (\alpha - 1)\frac{dy}{dt} + \beta y = 0$, which is an ODE with constants coefficients.]

5. Let a, b and c be positive numbers. Prove that every solution of the differential equation $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$ approaches zero as x approaches infinity.

6. Prove that if u is a solution of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

and v is a solution of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = r(x), \tag{3}$$

then $u + v$ is also a solution of the non-homogeneous equation (3).

7. Find the general solution of the following ODEs. (Use the method of undetermined coefficients/judicious guessing)

(a) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = 3x - 20\sin 2x$

(b) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$

(c) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} = 2x^2 + 1$

(d) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 12 + 25x^2$

(e) $\frac{d^2y}{dx^2} - 9y = x + e^{2x} - \sin 2x$

(f) $\frac{d^2y}{dx^2} + 4y = \sin 2x$

8. Find the general solution of $x^2\frac{d^2y}{dx^2} - 2y = x^2$. [Hint : Follow question number 4, and use method of variation of parameter to find a particular integral.]

9. One solution of the equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0 \tag{4}$$

is $(1+x)^2$ and the Wronskian of any two solutions of (4) is constant. Find the general solution of

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 1 + x.$$

10. Find the general solution of the following ODEs. (Use the method of variation of parameter)

(a) $\dfrac{d^2y}{dx^2} + y = \sec x, \quad -\dfrac{\pi}{2} < x < \dfrac{\pi}{2}$

(b) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = xe^{2x}$

(c) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = \dfrac{e^{-x}}{x^2}$

(d) $\dfrac{d^2y}{dx^2} - y = \dfrac{2}{1+e^x}$

(e) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^2e^x$

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