Theorem :

Let M and N have continuous partial derivatives at all points (x,y) in a domain D. Then the ODE $M(x,y)\,dx + N(x,y)\,dy = 0$ is exact in D iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \forall (x,y) \in D$$

Proof: If the equation $M(x,y)\,dx + N(x,y)\,dy = 0$ ——— (i) be exact, then there must be a function $u(x,y)$, such that

$$M\,dx + N\,dy = du. \qquad \text{——— (ii)}$$

We have

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy, \qquad \text{——— (iii)}$$

x and y being independent variable.

Comparing (ii) and (iii), we obtain

$$\frac{\partial u}{\partial x} = M \quad \text{and} \quad \frac{\partial u}{\partial y} = N.$$

$$\therefore \quad \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y\,\partial x}$$

and

$$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial^2 u}{\partial x\,\partial y}.$$

As M and N have continuous partial derivatives,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Conversely, let $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

We have to find u such that

$$M\,dx + N\,dy = du.$$

Let $P = \int M\,dx \Rightarrow \frac{\partial P}{\partial x} = M$

Then $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} = \frac{\partial^2 P}{\partial y\,\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}\right)$.

$\therefore N = \frac{\partial P}{\partial y} + f(y)$, where $f(y)$ is a function of y.

Now, $M\,dx + N\,dy$

$$= \frac{\partial P}{\partial x}dx + \frac{\partial P}{\partial y}dy + f(y)\,dy$$

$$= d\left(P + F(y)\right),$$ where $dF(y) = f(y)\,dy$ or $F(y) = \int f(y)\,dy$.

$\therefore$ We have $u(x,y) = P(x,y) + F(y)$ and we can write

$$M\,dx + N\,dy = du.$$

$\square$

Solve: $4x^3y\,dx + (x^4 + y^4)\,dy = 0$ [Homogeneous + Exact]

Solution: $Mx + Ny =$ constant.