Department of Mathematical Sciences

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DIFFERENTIAL EQUATIONS (MA 121)

Week 3 / April 2022 | Problem Set 3 | G.R.

■ **Notations :** ODE ≡ Ordinary Differential Equation, $y' \equiv \frac{dy}{dx}$, $y'' \equiv \frac{d^2y}{dx^2}$, $y^{(i)} \equiv \frac{d^iy}{dx^i}$ for $i = 3, 4, 5, \ldots$.

■ Existence and uniqueness of solutions of First order and First degree ODE, Basic theory of linear second order ODE

1. Show that each of the following functions satisfies Lipschitz condition in the rectangle D defined by $|x| \leq a$, $|y| \leq b$.

   (a) $f(x, y) = x^2 + y^2$

   (b) $f(x, y) = x \sin y + y \cos x$

   (c) $f(x, y) = x^2 e^{x+y}$

   (d) $f(x, y) = A(x)y^2 + B(x)y + C(x)$, where the functions A, B and C are continuous on $|x| \leq a$

   (e) $f(x, y) = x\,|y|$.

2. Show that each of the following functions does not satisfy Lipschitz condition in any domain which includes the line $y = 0$.

   (a) $f(x, y) = y^{\frac{2}{3}}$

   (b) $f(x, y) = \sqrt{|y|}$

3. Discuss the existence and uniqueness of a solution of the following initial-value problems.

   (a) $\frac{dy}{dx} = x^2 + y^2, \quad y(x_0) = y_0$

   (b) $\frac{dy}{dx} = P(x)y^2 + Q(x)y, \quad y(x_0) = y_0$, where $P(x)$ and $Q(x)$ both are polynomials of degree 3.

   (c) $\frac{dy}{dx} = y^{\frac{4}{3}}, \quad y(x_0) = y_0$

   (d) $\frac{dy}{dx} = y^{\frac{2}{3}}, \quad y(x_0) = y_0$.

4. Suppose that f and g are linearly independent on an interval I. Prove that $y_1 = f + g$ and $y_2 = f - g$ are are also linearly independent on I.

5. Consider the differential equation

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0. \tag{1}$$

   (a) Show that e^{2x} and e^{3x} are linearly independent solutions of the equation (1) on the interval $-\infty < x < \infty$.

   (b) Write the general solution of the equation (1).

   (c) Find the solution that satisfies the conditions $y(0) = 2$, $y'(0) = 3$. Explain why this solution is unique. Over what interval is it defined?

6. Consider the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0. \tag{2}$$

   (a) Show that e^x and xe^x are linearly independent solutions of the equation (2) on the interval $-\infty < x < \infty$.

   (b) Write the general solution of the equation (2).

   (c) Find the solution that satisfies the conditions $y(0) = 1$, $y'(0) = 4$. Explain why this solution is unique. Over what interval is it defined?

7. Consider the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0. \tag{3}$$

   (a) Show that x and x^2 are linearly independent solutions of the equation (3) on the interval $0 < x < \infty$.

   (b) Write the general solution of the equation (3).

   (c) Find the solution that satisfies the conditions $y(1) = 3$, $y'(1) = 2$. Explain why this solution is unique. Over what interval is this solution defined?

8. Consider the differential equation

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0. \tag{4}$$

   Show that $y(x) = x^2$ can never be a solution of (4) if the functions $p(x)$ and $q(x)$ are continuous at $x = 0$.

9. (a) Let $y_1(x)$ and $y_2(x)$ be solutions of (4) on the interval I with $y_1(x_0) = 1$, $y_1'(x_0) = 0$, $y_2(x_0) = 0$, and $y_2'(x_0) = 1$ for some $x_0 \in I$. Show that $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions of (4) on the interval I.

   (b) Show that $y(x) = a_0\, y_1(x) + b_0\, y_2(x)$ is the solution of (4) satisfying $y(x_0) = a_0$ and $y'(x_0) = b_0$ for some $x_0 \in I$.

10. Suppose that *Wronskian* of any two solutions of (4) is constant in some interval I. Prove that ODE (4) becomes $\frac{d^2y}{dx^2} + q(x)y = 0$ on I.

11. Let y_1 and y_2 be solutions of Bessel's equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$$

on the interval $(0, \infty)$, with $y_1(1) = 1$, $y_1'(1) = 0$, $y_2(1) = 0$, and $y_2'(1) = 1$. Compute $W[y_1, y_2](x)$.

12. Assume that p and q are continuous, and that the functions y_1 and y_2 are solutions of the ODE

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

on the interval I.

(a) Prove that if y_1 and y_2 vanish at the same point in the interval I, then they cannot form a fundamental set of solutions on this interval.

(b) Prove that if y_1 and y_2 achieve a maximum or minimum at the same point in the interval I, then they cannot form a fundamental set of solutions on this interval.

(c) Suppose that y_1 and y_2 are a fundamental set of solutions on the interval $(-\infty, \infty)$. Show that there is one and only one zero of y_1 between consecutive zeros of y_2.

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