Q: Obtain the differential equation of the system of conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, in which λ is the arbitrary parameter and a, b are given fixed constants.

Ans:

Given $$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \quad \text{———} \quad (i)$$

Note that λ is the only parameter of (i), so the required differential equation will be of order 1.

Differentiating equation (i) w.r.t. x, we get

$$\frac{2x}{a^2+\lambda} + \frac{2yy'}{b^2+\lambda} = 0 \qquad \text{where } y' = \frac{dy}{dx}$$

$$\Rightarrow \frac{x}{a^2+\lambda} = -\frac{yy'}{b^2+\lambda} = \frac{1}{k}\,(\text{say}) \quad \text{———} \quad (ii)$$

Using (ii) in (i) we obtain

$$\frac{x^2}{kx} - \frac{y^2}{kyy'} = 1$$

$$\Rightarrow k = -\frac{1}{y'}(y - xy')$$

From (ii) $$a^2+\lambda = kx = \frac{x^2y' - xy}{y'} \quad \text{———} \quad (III)$$

$$b^2+\lambda = -kyy' = y(y - x') \quad \text{———} \quad (IV)$$

(iii) − (IV) $$\Rightarrow a^2 - b^2 = \frac{x^2y' - yx}{y'} + (xy' - y)y$$

$$\Rightarrow (a^2 - b^2)y' = (xy' - y)(x + yy')$$