* Solve: $(x+2y-3)dx = (2x+y-3)\,dy$

→ Solution:

$$\frac{dy}{dx} = \frac{x+2y-3}{2x+y-3} \qquad \text{——— (i)}$$

To reduce this ODE to the homogeneous form, put

$x = x_1 + h$ and $y = y_1 + k$, h, k are constants.

$$\Rightarrow \frac{dy}{dx} = \frac{dy_1}{dx_1}$$

∴ Equation (i) becomes $\dfrac{dy_1}{dx_1} = \dfrac{x_1+2y_1+(h+2k-3)}{2x_1+y_1+(2h+k-3)}$. ——— (ii)

choose the constants h and k such that

$$h+2k-3=0,$$
$$2h+k-3=0$$

This give $h=k=1$ and the equation (ii) reduces

to $\dfrac{dy_1}{dx_1} = \dfrac{x_1+2y_1}{2x_1+y_1}$ ——— (iii) which is a homogeneous equation.

Now put $y_1 = vx_1 \Rightarrow \dfrac{dy_1}{dx_1} = v + x_1\dfrac{dv}{dx_1}$

From (iii), $v + x_1\dfrac{dv}{dx_1} = \dfrac{1+2v}{2+v}$

$$\Rightarrow x_1\frac{dv}{dx_1} = \frac{1+2v}{2+v} - v = \frac{1+2v-2v-v^2}{2+v}$$

$$\Rightarrow \frac{2+v}{1-v^2}\,dv = \frac{dx_1}{x_1}$$

$$\Rightarrow \left\{\frac{2}{1-v^2} - \frac{1}{2}\frac{(-2v)}{1-v^2}\right\}dv = \frac{dx_1}{x_1}$$

Integrating both sides, we get

$$\ln\left(\frac{1+v}{1-v}\right) - \frac{1}{2}\ln(1-v^2) = \ln(cx_1)$$, where c is ~~an~~ arbitrary constant.

putting $v = \frac{y_1}{x_1}$, we get

$$\ln\left(\frac{x_1+y_1}{x_1-y_1}\right) - \frac{1}{2}\ln\left(\frac{x_1^2-y_1^2}{x_1^2}\right) = \ln(cx_1)$$

$$\Rightarrow \quad \frac{x_1+y_1}{x_1-y_1} \cdot \frac{x_1}{\sqrt{x_1^2-y_1^2}} = cx_1$$

Now put $x_1 = x-1$ and $y_1 = y-1$, we get

$$\frac{x+y-2}{x-y} \cdot \frac{1}{\sqrt{(x-1)^2-(y-1)^2}} = c$$

$$\Rightarrow \quad (x+y-2)^2 = c^2(x-y)^2\left[(x-1)^2-(y-1)^2\right]$$

$$\Rightarrow \quad (x+y-2) = c^2(x-y)^3$$

$\boxtimes$