Probability:

living with uncertainty

Topics

Curwin & Slater Ch8

- The language of probability
- Quantifying & analysing uncertainty
- Using probabilities to manage uncertainty
- Decision analysis: identifying the best course in an uncertain world

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Stochastic Systems

- Governed by the laws of probability
- Involves at least one random variable

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Stochastic Systems

- A dynamical system is deterministic if the future of that system is completely predictable from knowledge of its present state.
- If there is some intrinsic randomness in the system it makes the perfect prediction of the future of that system impossible and so it is considered a stochastic system.
- There may exist strong trends or correlations in such systems but there is always some element of uncertainty.

The language of probability

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- From gambling in 18th century France to modern management
- Consider two options:
  - a conservative investment with safe but low returns (eg. enhancing an existing product)
  - a radical investment with the possibilities of a loss or big profits (eg. developing a new technology)
- Probability theory provides a structure for comparing such options

Defining probabilities

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$$P(\text{event}) = \frac{\text{number of ways the event can occur}}{\text{total number of possible outcomes}}$$

eg. P(a dice throw >4) = 2/6 = 0.333

P(7) = 0

P(>0) = 1

<u>expected</u> number of occurrences of event E =
P(E) * number of <u>trials</u>

eg. if 5 dice are thrown:
E(number of throws>4) = 0.333 * 5 = 1.667

Quantifying uncertainty

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- A complete knowledge of all possible outcomes (a priori)
- Previous experience (empirical)
- Subjective

- What does “P(project being a success)=1” mean?

Assessing risk

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A company is choosing between 4 research projects:

project	cost (£M)	value* (£M)	P(success)
A	1.5	3	0.9
B	1.5	4	0.8
C	1.5	25	0.1
D	1.5	5	0.6

*if successful, otherwise value = 0

Expected values

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A company is choosing between 4 research projects:

project	cost (£M)	value* (£M)	P(success)	expected profit (£M)	
A	1.5	3	0.9	-1.5 + 0.9*3 =	+1.2
B	1.5	4	0.8	-1.5 + 0.8*4 =	+1.7
C	1.5	25	0.1	-1.5 + 0.1*25 =	+1.0
D	1.5	5	0.6	-1.5 + 0.6*5 =	+1.5

*if successful, otherwise value = 0

The expected never happens

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Expected value of project D = **+£1.5 million**

- If successful, gain £5 million
- If fail, lose £1.5 million

Expected value = long term average

Will we be here in the long term?

Calculating probabilities

Consider throwing two dice:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

an example of a "sample space"

Calculating probabilities

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

event 1 = total score is 5

$$P(\text{event}(1)) = \frac{\text{number of outcomes of event}(1)}{\text{total number of outcomes}}$$

$$= 4/36 = 0.111$$

Calculating probabilities

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

event(2) = total score is 6

P(event(2)) = $\dfrac{\text{number of outcomes of event(2)}}{\text{total number of outcomes}}$

= 5/36 = 0.139

Combination events

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

"What is the probability that the first and the second dice will be 5's?"

The law of multiplication

What is the probability that the first and the second dice will be 5's?

event(3) = the first is a 5
event(4) = the second is a 5

P(event(3) and event(4)) = P(event(3)) x P(event(4))
= 1/6 x 1/6 = 1/36

But only if the two events are independent

Example: investing in two research projects

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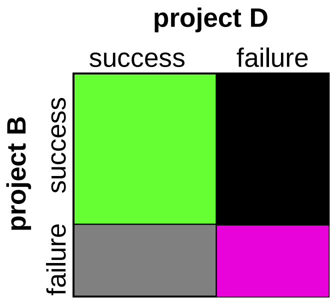
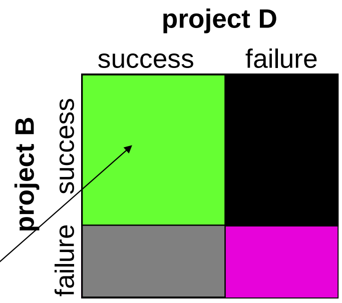


Company decides to invest in both projects B and D

B = event "B is successful", P(B) = 0.8
D = event "D is successful", P(D) = 0.6

What is the probability of both projects being successful?



if independent

P(B and D) = P(B) * P(D)

P(B and D) = 0.8 * 0.6 = 0.48

The addition law

Probability that the total score is 5 or 6:

$$P(\text{event}(1) \text{ or } \text{event}(2)) = \left[\frac{4}{36} + \frac{5}{36}\right] = \frac{9}{36} = 0.25$$

In general:

P(event(A) or event(B)) = P(event(A)) + P(event(B))

But only true if the events are "mutually exclusive", ie. they cannot both occur

The general addition law

"What is the probability that if you throw two dice, one of them is a 6?"

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

P(at least one 6) = 11/36 (not 1/6 + 1/6 = 12/36)

In general:
P(event(A) or event(B)) = P(event(A)) + P(event(B)) - P(event(A) and event(B))

What is the probability of project B or project D being successful?

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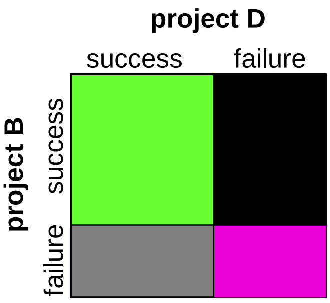

B, D are not mutually exclusive, ie. both could occur

P(B or D) = P(B) + P(D) - P(B and D)

= 0.8 + 0.6 - 0.8 * 0.6

= 0.92

P(0 successful projects)

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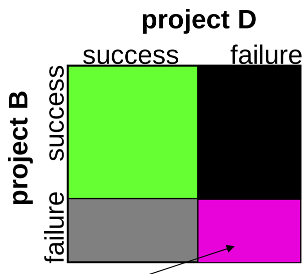

if independent

P(not B and not D) = P(not B) * P(not D)

= (1-P(B)) * (1-P(D))

= 0.2 * 0.4

= 0.08

= 1 - P(B or D)

Prior knowledge

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	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

“I have thrown two dice, <u>the first is revealed to be a 6</u>, what is the probability that the total score is 10 or more?”

Conditional probability

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	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

E: event "the total score is 10 or more"
F: event "the first dice is a 6"

$$P(\text{E given F}) = \frac{P(\text{E and F})}{P(\text{F})} = \frac{3/36}{6/36} = 0.5$$

General law of multiplication

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Rewriting the expression for the conditional probability:

P(X and Y) = P(X given Y) * P(Y)

if X and Y are independent, P(X given Y) = P(X)

P(X given Y) = P(X|Y)

P(B and D): not independent

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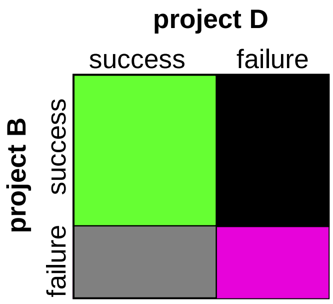

Perhaps the two projects are dependent on the same technology, i.e. if D is successful, B is very likely to succeed as well: P(B given D) = 0.9

P(B and D) = P(B given D) * P(D)

P(B and D) = 0.9 * 0.6 = 0.54

Reliability of a power unit: Risk of Failure?

generator → electric motor

P(generator working on any single day) = P(G) = 0.94
P(electric motor working) = P(M) = 0.98

P(power unit working) = P(generator <u>and</u> electric motor working)

P(G <u>and</u> M) = P(G given M) * P(M)

If independent, P(power unit working) = 0.94*0.98 = 0.92

But should investigate further, perhaps failure is not independent, i.e. P(G given M) > P(G)?

Probability is a function of available information

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- As more information is obtained so the outcome is more certain (for good or bad)
- Given more relevant information, the probability changes
- But information/ research costs money
- Cannot continue research until 100% certain (P=1.0)

Probability of Aero-Engine Failure

Early Jet engines:

- One failure in 2000hrs operation
- For a 10hr flight 1/200 probability of failure (0.005)

Intercontinental Aircraft Required 4 engines:

P(Fa+Fb+Fc+Fd) = P(Fa)*P(Fb)*P(Fc)*P(Fd)
P(aircraft failure) = 0.005 x 0.005 x 0.005 x 0.005
= 0.000000000625

- Or an aircraft failure once every 1.6 billion hours