An Introduction to Stirling's Number of the Second Kind

ChatGPT & S. H.

October 20, 2024

1 Introduction

In combinatorics, *Stirling's number of the second kind* $S(n, k)$ is the number of ways to partition a set of n elements into k non-empty subsets [1]. These numbers arise in various areas of mathematics and have applications in **set theory, number theory, and even computer science**[1].

Stirling's numbers of the second kind can be defined recursively and have many interesting properties, which we will explore in this document.

2 Properties of Stirling Numbers

2.1 Definition

The Stirling number of the second kind, denoted by $S(n, k)$, is defined as the number of ways to divide a set of n elements into k non-empty subsets. It can be written recursively as $S(n, k) = k \cdot S(n-1, k) + S(n-1, k-1)$, for $n > 0$, with the boundary conditions $S(0, 0) = 1, \quad S(n, 0) = 0$ for $n > 0$, and $S(n, k) = 0$ for $k > n$.

2.2 Combinatorial Interpretation

Stirling numbers of the second kind have a natural combinatorial interpretation. They count the ways to partition a set of n elements into k non-empty subsets. For example, consider the set $\{1, 2, 3\}$. The number of ways to partition this set into two subsets is given by $S(3, 2) = 3$. These partitions are:

- $\{1\}, \{2, 3\}$
- $\{2\}, \{1, 3\}$
- $\{3\}, \{1, 2\}$

[1] `https://en.wikipedia.org/wiki/Stirling_numbers_of_the_second_kind`

2.3 Closed Form

Stirling numbers of the second kind can be described by the following equation:

$$S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n$$

2.4 First Few Examples

Table 1 shows the values of the Stirling numbers of the second kind, $S(n,k)$, for small values of n and k:

$n\backslash k$	1	2	3	4	5
1	1				
2	1	1			
3	1	3	1		
4	1	7	6	1	
5	1	15	25	10	1

Table 1: Stirling Numbers of the Second Kind for $n \leq 5$.

2.5 Conclusion

~~“Don’t forget to practice more problems involving Stirling numbers to fully understand their applications!”~~

References

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science*, 2nd. USA: Addison-Wesley Longman Publishing Co., Inc., 1994.