Fourier Transformation

Welcome To The World of Signals

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What is Fourier Transformation?(Recap)

The generalized form of the complex Fourier series is referred to as the Fourier transform. It helps to expand the non-periodic functions and convert them into easy sinusoid functions.

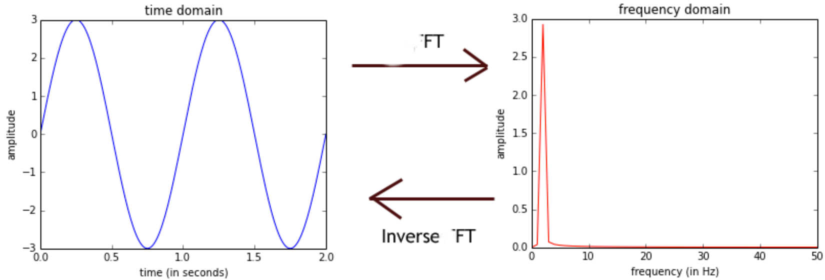

Figure: Fourier Transformation

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Fourier Transform Type

There are two types of Fourier transform i.e., forward Fourier transform and inverse Fourier transform.
The forward and inverse Fourier transform is used to decompose a function or a signal into its constituent frequencies and times respectively.

Forward Fourier Transform

The forward Fourier transform is a mathematical technique used to transform a time-domain signal into its frequency-domain representation. The forward Fourier transform of a continuous-time signal $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

where $X(\omega)$ is the Fourier transform of $x(t)$, ω is the angular frequency, j is the imaginary unit($\sqrt{-1}$), and t is the time.

Inverse Fourier Transform

The inverse Fourier transform is the process of converting a frequency-domain representation of a signal back into its time-domain form. The inverse Fourier transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

where $x(t)$ is the time-domain signal and $X(\omega)$ is the Fourier transform of $x(t)$, ω is the angular frequency, j is the imaginary unit($\sqrt{-1}$), and t is the time.

Fourier Transform Notation

For convenience, we will write the Fourier transform of a signal $x(t)$ as

$$X(f) = \mathcal{F}\{x(t)\}$$

Fourier Transform Notation

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and the inverse Fourier transform of $X(f)$ as

$$x(t) = \mathcal{F}^{-1}\{X(f)\}.$$

Fourier Transform Notation

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$$X(f) = \mathcal{F}\{x(t)\}$$

and the inverse Fourier transform of $X(f)$ as

$$x(t) = \mathcal{F}^{-1}\{X(f)\}.$$

Note that

$$\mathcal{F}^{-1}\{\mathcal{F}\{x(t)\}\} = x(t)$$

at points of continuity of $x(t)$.

Example of Fourier Transform

Let $x(t) = rect(t)$ where rect(t) is the rectangular pulse function defined as

$$rect(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

Example of Fourier Transform

Let $x(t) = rect(t)$ where rect(t) is the rectangular pulse function defined as

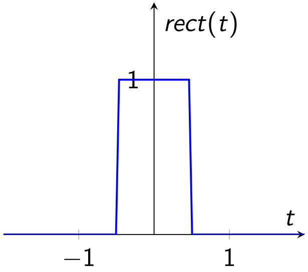
$$rect(t) = \begin{cases} 1 & \text{if } |t| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$$

and $sinc(\omega)$ is the sinc function defined as

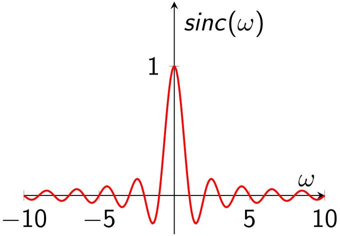
$$sinc(\omega) = \begin{cases} \frac{sin(\omega/2)}{\omega/2} & \text{if } \omega \neq 0 \\ 1 & \text{if } \omega = 0 \end{cases}$$

Example of Fourier Transform

The forward Fourier transform of $rect(t)$ is $sinc(\omega)$ and the inverse Fourier transform of $sinc(\omega)$ is $rect(t)$.



(a) Inverse fourier transform of $sinc(\omega)$



(b) forward fourier transform of $rect(t)$

Properties of Fourier Transform

- ⋆ Linearity
- ⋆ Time Shifting
- ⋆ Time Scaling
- ⋆ Time Reversal
- ⋆ Differentiation
- ⋆ Integration
- ⋆ Convolution

Properties of Fourier Transform

- ⋆ Linearity
- ⋆ Time Shifting
- ⋆ Time Scaling
- ⋆ Time Reversal
- ⋆ Differentiation
- ⋆ Integration
- ⋆ Convolution

Properties of Fourier Transform

Linearity

If $x_1(t)$ and $x_2(t)$ are two signals with fourier transform $X_1(\omega)$ and $X_2(\omega)$ respectively, then the Fourier transform of a linear combination of the signals is linear:

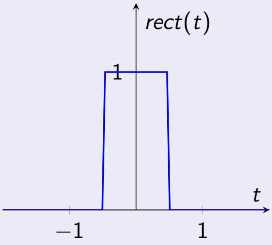
$$\mathcal{F}\{ax_1(t) + bx_2(t)\} = aX_1(\omega) + bX_2(\omega)$$

where a and b are constants.

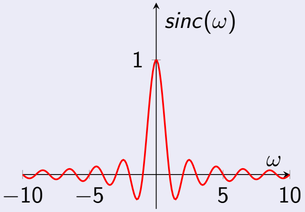
Properties of Fourier Transform

Linearity

Let $x_1(t) = rect(t)$ and $x_2(t) = rect(t)$ be two signals and let $X_1(\omega) = \mathcal{F}\{x_1(t)\} = sinc(\omega)$ and $X_2(\omega) = \mathcal{F}\{x_2(t)\} = sinc(\omega)$ be their fourier transforms respectively.



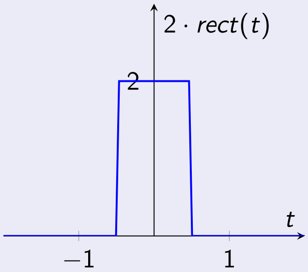
(a) $x_1(t)$ and $x_2(t)$



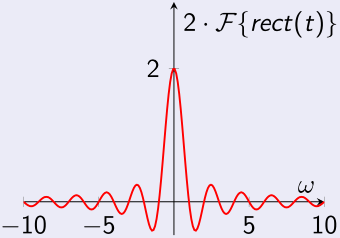
(b) $\mathcal{F}\{x_1(t)\}$ and $\mathcal{F}\{x_2(t)\}$

Properties of Fourier Transform

Linearity



(a) $2 \cdot x_1(t)$



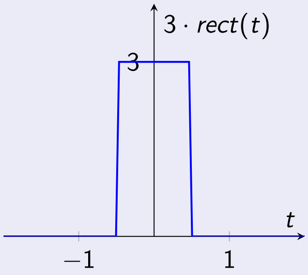
(b) $\mathcal{F}\{2 \cdot x_1(t)\}$

Here,

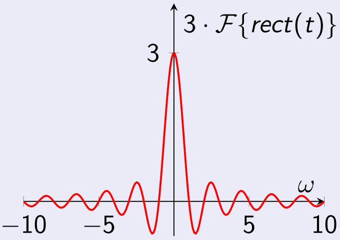
$$\mathcal{F}\{2 \cdot x_1(t)\} = 2 \cdot \mathcal{F}\{x_1(t)\}$$

Properties of Fourier Transform

Linearity



(a) $3 \cdot x_2(t)$



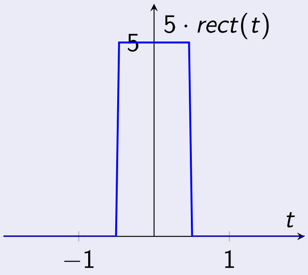
(b) $\mathcal{F}\{3 \cdot x_2(t)\}$

Here,

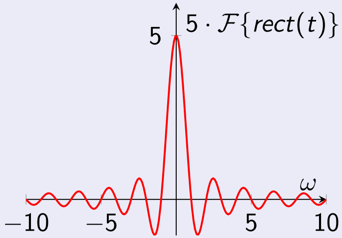
$$\mathcal{F}\{3 \cdot x_2(t)\} = 3 \cdot \mathcal{F}\{x_2(t)\}$$

Properties of Fourier Transform

Linearity



(a) $2 \cdot x_1(t) + 3 \cdot x_2(t)$



(b) $\mathcal{F}\{2 \cdot x_1(t)\} + \mathcal{F}\{3 \cdot x_2(t)\}$

Here,

$$\mathcal{F}\{2 \cdot x_1(t) + 3 \cdot x_2(t)\} = 2 \cdot \mathcal{F}\{x_1(t)\} + 3 \cdot \mathcal{F}\{x_2(t)\}$$

Properties of Fourier Transform

- ⋆ Linearity
- ⋆ Time Shifting
- ⋆ Time Scaling
- ⋆ Time Reversal
- ⋆ Differentiation
- ⋆ Integration
- ⋆ Convolution

Properties of Fourier Transform

Time Shifting

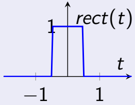
If $x(t)$ is a signal with Fourier transform $X(\omega)$, then shifting the signal in time corresponds to a phase shift in its Fourier transform:

$$x(t - t_0) \xrightarrow{\text{FT}} X(\omega)e^{-j\omega t_0}$$

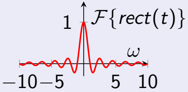
Properties of Fourier Transform

Time Shifting

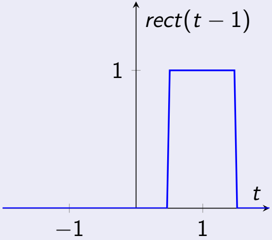
Let $x(t) = rect(t)$ be a signal with Fourier transform $X(\omega) = \mathcal{F}\{rect(t)\}$.



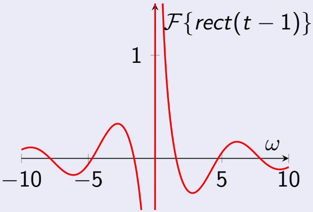
(a) $rect(t)$



(b) $\mathcal{F}\{rect(t)\}$



(a) $rect(t-1)$



(b) $\mathcal{F}\{rect(t-1)\}$

Properties of Fourier Transform

Time Shifting

Here,

$$rect(t-1) \xrightarrow{\text{FT}} X(\omega)e^{-j\omega 1}$$

Properties of Fourier Transform

- Linearity
- Time Shifting
- Time Scaling
- Time Reversal
- Differentiation
- Integration
- Convolution

Properties of Fourier Transform

Time Scaling

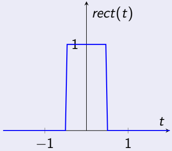
If $x(t)$ is a signal with Fourier transform $X(\omega)$, then stretching or compressing a signal in time inversely scales its frequency spectrum:

$$x(at) \xrightarrow{\text{FT}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

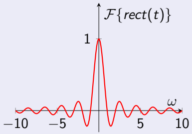
Properties of Fourier Transform

Time Scaling

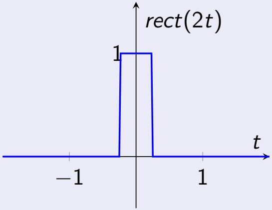
Let $x(t) = rect(t)$ be a signal with Fourier transform $X(\omega) = \mathcal{F}\{rect(t)\}$.



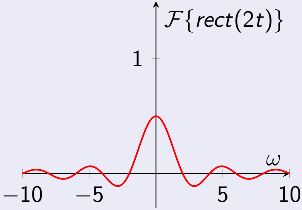
(a) $x(t)$



(b) $\mathcal{F}\{x(t)\}$



(a) $x(2t)$



(b) $\mathcal{F}\{x(2t)\}$

Properties of Fourier Transform

Time Scaling

Here,

$$rect(2t) \xrightarrow{\text{FT}} \frac{1}{2} sinc(\frac{\omega}{2})$$

Properties of Fourier Transform

- ⋆ Linearity
- ⋆ Time Shifting
- ⋆ Time Scaling
- ⋆ Time Reversal
- ⋆ Differentiation
- ⋆ Integration
- ⋆ Convolution

Properties of Fourier Transform

Time Reversal

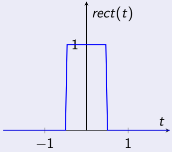
If $x(t)$ is a signal with Fourier transform $X(\omega)$, then time reversal of the signal in the time domain corresponds to frequency reversal in the frequency domain:

$$x(-t) \xrightarrow{\text{FT}} X(-\omega)$$

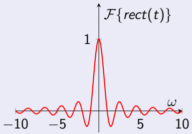
Properties of Fourier Transform

Time Reversal

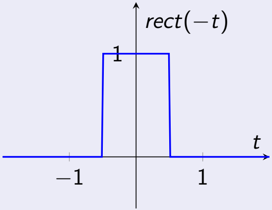
Let $x(t) = rect(t)$ be a signal with Fourier transform $X(\omega) = \mathcal{F}\{rect(t)\}$.



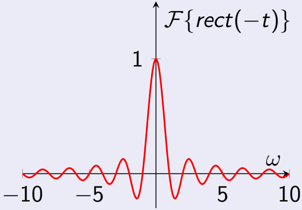
(a) $x(t)$



(b) $\mathcal{F}\{x(t)\}$



(a) $x(-t)$



(b) $\mathcal{F}\{x(-t)\}$

Properties of Fourier Transform

Time Reversal

Both functions are even hence they remain same. Here,

$$rect(-t) \xrightarrow{\text{FT}} \mathcal{F}\{x(-t)\}$$

Properties of Fourier Transform

- ⋆ Linearity
- ⋆ Time Shifting
- ⋆ Time Scaling
- ⋆ Time Reversal
- ⋆ **Differentiation**
- ⋆ Integration
- ⋆ Convolution

Properties of Fourier Transform

Differentiation

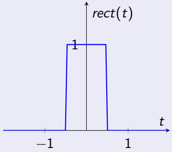
If $x(t)$ is a signal with Fourier transform $X(\omega)$, then differentiation in the time domain corresponds to multiplication by $j\omega$ in the frequency domain:

$$\frac{d}{dt}x(t) \xrightarrow{\text{FT}} j\omega X(\omega)$$

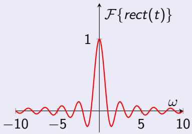
Properties of Fourier Transform

Differentiation

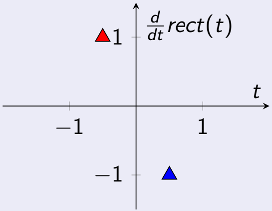
Let $x(t) = rect(t)$ be a signal with Fourier transform $X(\omega) = \mathcal{F}\{rect(t)\}$.



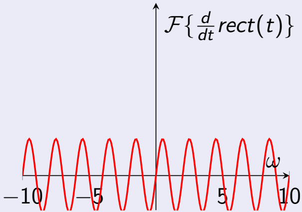
(a) $x(t)$



(b) $\mathcal{F}\{x(t)\}$



(a) $\frac{d}{dt}x(t)$



(b) $\mathcal{F}\{\frac{d}{dt}x(t)\}$

Properties of Fourier Transform

- ⋆ Linearity
- ⋆ Time Shifting
- ⋆ Time Scaling
- ⋆ Time Reversal
- ⋆ Differentiation
- ⋆ Integration
- ⋆ Convolution

Properties of Fourier Transform

Integration

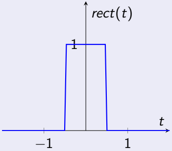
If $x(t)$ is a signal with fourier transform $X(\omega)$, then integration in the time domain corresponds to multiplication by $\frac{1}{j\omega}$ in the frequency domain:

$$\int x(t)dt \xrightarrow{\text{FT}} \frac{1}{j\omega}X(\omega)$$

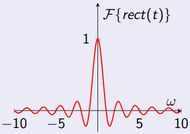
Properties of Fourier Transform

Integration

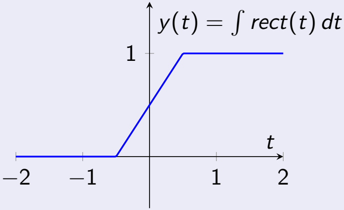
Let $x(t) = rect(t)$ be a signal with Fourier transform $X(\omega) = \mathcal{F}\{rect(t)\}$.



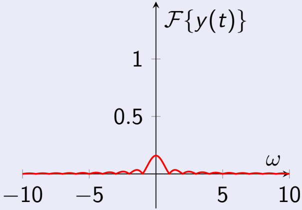
(a) Signal



(b) $\mathcal{F}\{rect(t)\}$



(a) $y(t) = \int x(t)dt$



(b) $\mathcal{F}\{y(t)\}$

Properties of Fourier Transform

- Linearity
- Time Shifting
- Time Scaling
- Time Reversal
- Differentiation
- Integration
- Convolution

Properties of Fourier Transform

Convolution

If $x(t)$ and $y(t)$ are two signals with fourier transform $X(\omega)$ and $Y(\omega)$, then convolution of the two signals in the time domain corresponds to multiplication in the frequency domain:

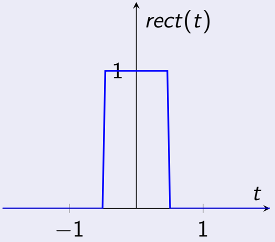
$$x(t) * y(t) \xrightarrow{\text{FT}} X(\omega) \cdot Y(\omega)$$

Properties of Fourier Transform

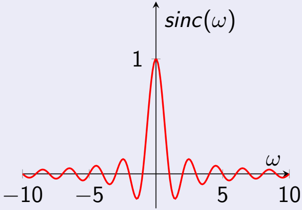
Convolution

Let $x(t) = rect(t)$ and $y(t) = rect(t)$ be signals with Fourier transforms $X(\omega) = \mathcal{F}\{rect(t)\}$ and $Y(\omega) = \mathcal{F}\{rect(t)\}$. The convolution in the time domain corresponds to the product in the frequency domain:

$$z(t) = x(t) * y(t) \quad \leftrightarrow \quad Z(\omega) = X(\omega) \cdot Y(\omega).$$



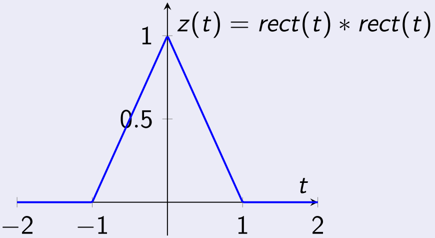
(a) $rect(t)$



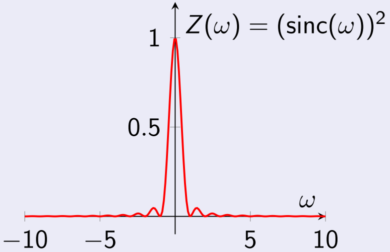
(b) $\mathcal{F}\{rect(t)\}$

Properties of Fourier Transform

Convolution



(a) $z(t) = rect(t) * rect(t)$



(b) $Z(\omega) = \mathcal{F}\{z(t)\}$

Here,

$$rect(t) * rect(t) = X(\omega) \cdot Y(\omega)$$

Fourier Transform Properties (Table 1)

Property	Time Domain	Fourier Transform
Linearity	$x(t) = Ax_1(t) + Bx_2(t)$	$X(j\omega) = AX_1(j\omega) + BX_2(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Differentiation in Time	$\frac{d^n x(t)}{dt^n}$	$(j\omega)^n X(j\omega)$
Differentiation in Frequency	$-jtx(t)$	$\frac{dX(j\omega)}{d\omega}$
Time Integration	$\int_{-\infty}^{t} x(\tau) d\tau$	$\frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega)$

Fourier Transform Properties (Table 2)

Property	Time Domain	Fourier Transform
Time Scaling	$x(at)$	$\frac{1}{\lvert a \rvert} X\left(\frac{j\omega}{a}\right)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Frequency Shifting	$x(t)e^{j\omega_0 t}$	$X(j(\omega - \omega_0))$
Duality	$X(t)$	$2\pi x(-j\omega)$
Time Convolution	$x(t) * h(t)$	$X(j\omega)H(j\omega)$
Parseval's Theorem	$\int_{-\infty}^{\infty} \lvert x(t) \rvert^2 dt$	$\frac{1}{2\pi} \int_{-\infty}^{\infty} \lvert X(j\omega) \rvert^2 d\omega$
Modulation	$z(t) = x(t)y(t)$	$Z(\omega) = \frac{1}{2\pi} X(j\omega) * Y(j\omega)$

Fourier Transform Table

Signal in Time Domain	Fourier Transform
$\delta(t)$	1
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$
$\delta(t - t_0)$	$e^{-j\omega t_0}$
$te^{-at}u(t)$	$\frac{1}{(a+j\omega)^2}$
$u(-t)$	$\pi\delta(\omega) - \frac{1}{j\omega}$
$e^{at}u(-t)$	$\frac{1}{a-j\omega}$

Fourier Transform Table

Signal in Time Domain	Fourier Transform
$e^{-a\|t\|}$	$\frac{2a}{a^2+\omega^2}$
$\cos(\omega_0 t)$	$\pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\sin(\omega_0 t)$	$-j\pi[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$
$\frac{1}{a^2+t^2}$	$e^{-a\|\omega\|}$
$\mathrm{Sgn}(t)$	$\frac{2}{j\omega}$
1 (for all t)	$2\pi\delta(\omega)$