NLD Report

Period adding bifurcation in a logistic map with memory

Introduction

Period-Adding Bifurcation

Period-adding bifurcation is a phenomenon in dynamical systems where the system exhibits an increasing sequence of periodic orbits as a control parameter is varied. This results in the addition of a period to the existing orbit, forming a new orbit with a period that is the previous period plus an integer. This bifurcation is common in systems with piecewise continuous maps or those exhibiting border-collision bifurcations.

Key Characteristics :

- **Sequence of Increasing Periods**: The system transitions through a series of periodic orbits (e.g., period-2, period-3, period-4) as the control parameter changes.
- **Control Parameter Influence**: Changes in a system parameter, such as feedback strength or a system constant, drive the bifurcation.

The Logistic Map

The logistic map is a mathematical function used to model population dynamics under resource limitations. It is defined by the iterative equation:

$$x_{n+1} = r * x_n * (1 - x_n)$$

where x_n represents the normalized population at generation n and r is a parameter representing the growth rate.

the logistic map's behavior varies with change in growth rate r:

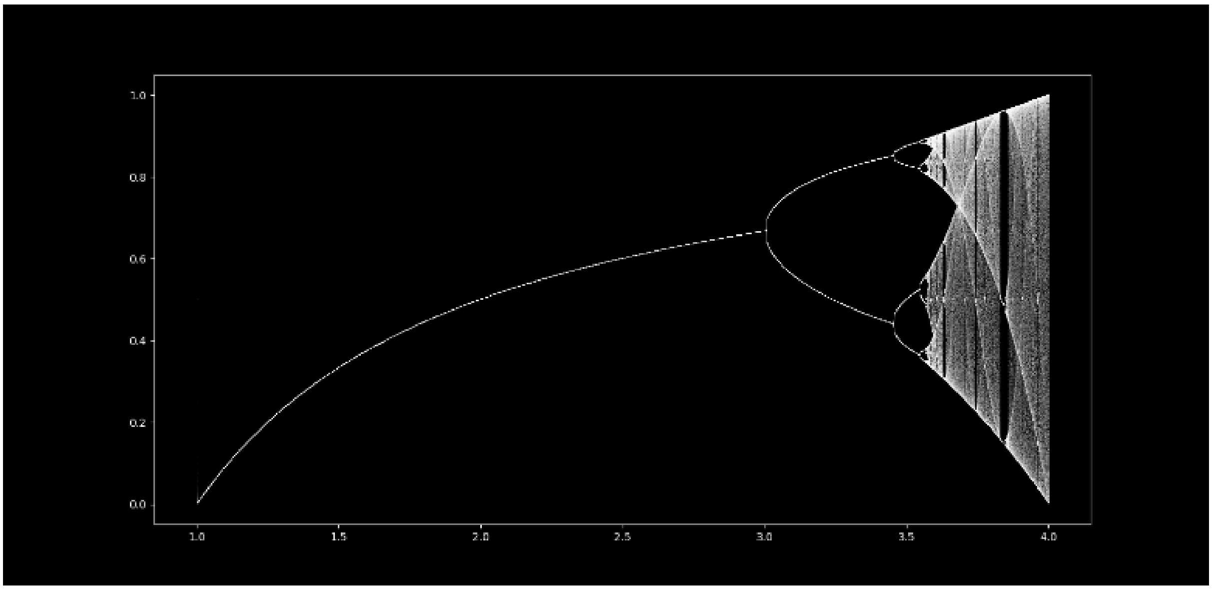
1. **for** $0 < r \leq 1$

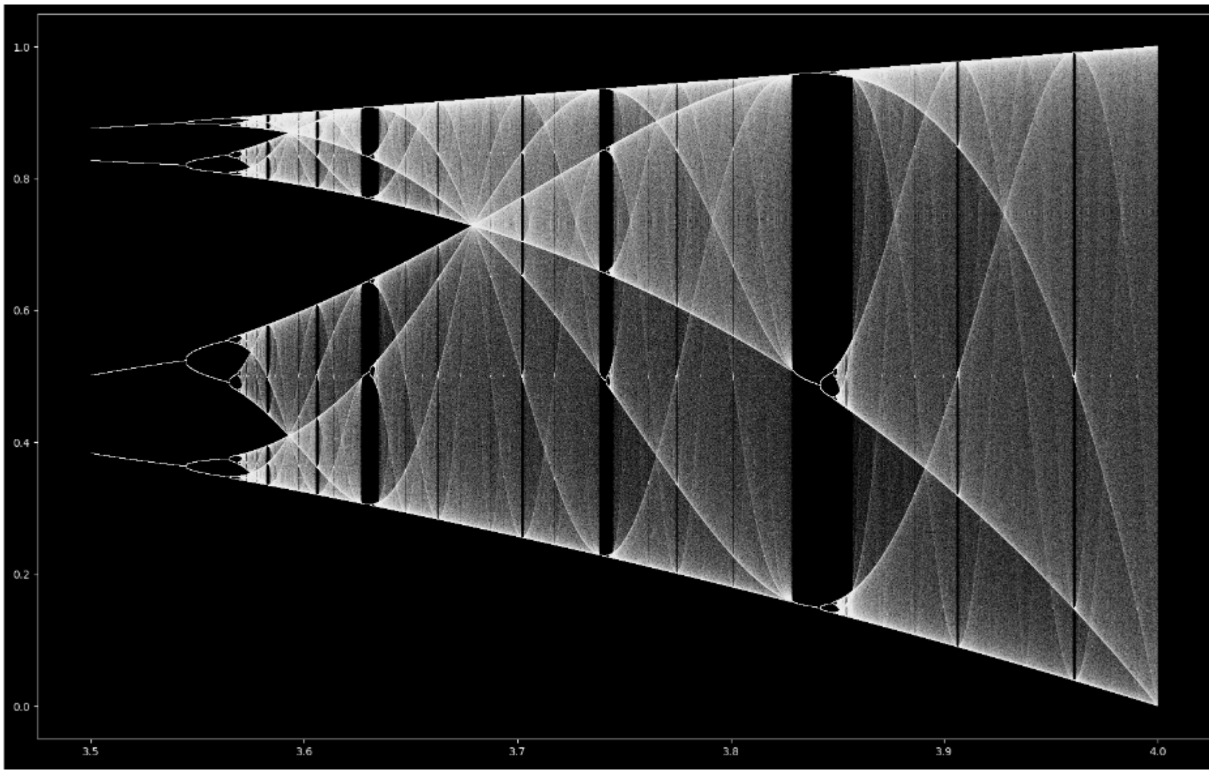
- The population goes extinct, independent of initial conditions.

2. **for** $1 < r \leq 2$
   - The population quickly approaches $\frac{r-1}{r}$, regardless of initial conditions.
3. **for** $2 < r \leq 3$
   - The population approaches $\frac{r-1}{r}$ after initial fluctuations. Convergence is linear, except at r = 3, where it slows significantly.
4. **for** $3 < r \leq 3.44949$
   - The population oscillates permanently between two values dependent on r.
5. **for** $3.44949 < r \leq 3.54409$
   - The population oscillates among four values.
6. **for** $r \geq 3.54409$
   - The population undergoes a period-doubling cascade, oscillating among 8, 16, 32, etc., values.
7. **for** $r \approx 3.56995$
   - Chaos emerges, with slight variations in initial conditions leading to dramatically different outcomes.
8. **for** $r > 3.56995$
   - The system exhibits chaotic behavior, with isolated ranges of non-chaotic behavior known as islands of stability.
9. **for** $r = 3.8284$
   - A stable period-3 cycle appears.
10. **for** $r > 4$

- Almost all initial values diverge from the interval \([0,1]\), forming a Cantor set, indicating chaotic dynamics.

Plots of logistic maps





The Model

The authors introduced a modified logistic map in their study, incorporating memory-dependent feedback. This modification aimed to provide a deeper understanding of the system's behavior under different conditions. The modified equations, featuring a history-dependent cutoff value x_c and a non-linearity parameter r_0, offer valuable insights into the dynamics of the system. Through two distinct conditions for x_c - predefined and dependent on the previous step's state - the authors explored how past states influence the system's evolution. The choice of the logarithmic function $tanh(x_n - x_c/\epsilon)$ in the equations allows for the modulation of r_0 based on historical information, resulting in diverse patterns in the system's behavior. This modified logistic map provides a useful framework for understanding the impact of memory dependence on dynamical systems.

$$x_{n+1} = \left[r_0 - (4 - r_0)\tanh\left(\frac{x_n - x_c}{\epsilon}\right)\right] x_n(1 - x_n)$$

Here, x_{n+1} represents the population at the next time step, r_0 denotes the non-linearity parameter, and x_c signifies the history-dependent cutoff value.

We explored two distinct conditions for x_c:

1. **Predefined Cutoff**:
   - In this scenario, x_c remains constant throughout the iterations. It serves as a predetermined threshold influencing the system's dynamics. For instance, setting x_c = 0.5 provides a clear cutoff point, affecting how the system evolves over time.
2. **Dependent on Previous Step**:
   - Alternatively, we considered a condition where x_c is determined by the previous step's value x_{n-1}. This introduces a memory-dependent feedback mechanism, where the system's state at each iteration influences the cutoff value for the next iteration. Consequently, the dynamics become inherently history-dependent, with past states shaping the evolution of the system.

The choice of the logarithmic function, $tanh(x_n - x_c/\epsilon)$, in the equations plays a crucial role in modulating the non-linearity parameter r_0 based on historical information. As x_c varies, the logarithmic function adjusts the value of r_0 accordingly, leading to distinct patterns in the system's behavior.

In summary, our model equations incorporate memory dependence through a history-dependent cutoff value x_c, influencing the system's dynamics under different conditions. By exploring predefined and dependent cutoff scenarios, we gain insights into how past states affect the evolution of the system, offering valuable perspectives on complex dynamical phenomena.

Exploration of Dynamics

We begin the exploring the dynamics of the map

$$x_{n+1} = f_1(x_n) = 4x_n(1 - x_n) \quad \text{if } x_n < x_{n-1} \tag{3.1}$$
$$x_{n+1} = f_2(x_n) = rx_n(1 - x_n) \quad \text{if } x_n > x_{n-1} \tag{3.2}$$

With 0<r<4 and r = 2*r_0− 4

Without memory dependence, the logistic map $x_{n+1} = 4 * x_n * (1 - x_n)$ is fully chaotic with lyapunov index of $ln2$ and a invariant density didtribution which is continious in $0 \leq x \leq 1$

The map $x_{n+1} = r * x_n * (1 - x_n)$ by itself shows fixed point for r < 3 followed by cycles of period 2^n and so on

In the presence of memeory the above map doesnt begin with a fixed point but witha 2-cycle,we can see the existense of it by noticing that for r < 1 i.e r_0 < 2.5

x_{n+1} will be smaller than if we use

Eq. (3.2) and then Eq. (3.1) will have to be used at the next step and

x_{n+2} > x_{n+1} which forces the use of Eq. (3.2) at the next step. Thus,

we have a possible 2-cycle x_1 , x_2 with

$$x_2 = 4x_1(1 - x_1) \tag{3.3}$$
$$x_1 = rx_2(1 - x_2). \tag{3.4}$$

by solving these two equations further we get:

$$x_2 = 4rx_2(1 - x_2)(1 - rx_2(1 - x_2)) \quad \text{and} \tag{3.5}$$
$$x_1 = 4rx_1(1 - x_1)(1 - 4x_1(1 - x_1)) \tag{3.6}$$

Thus x_1 and x_2 are the fixed points of two iterated functions F(x) and G(x) given by

$$F(x) = 4rx(1 - x)(1 - 4x(1 - x)) \tag{3.7}$$
$$G(x) = 4rx(1 - x)(1 - rx(1 - x)). \tag{3.8}$$

These functions have zero as the stable fixed point for r <1/4. The two cycle elements will be non zero for r > ¼ and can be found from Eqs. (3.7) and (3.8) and the cycle will be stable so

long as the slopes of F(x) and G(x) at the fixed point are greater than − 1. Destabilization through a

pitchfork bifurcation occurs at (using F(x))

$$-1 = 4r_c[1 - 8x_c(1 - x_c)][1 - 2x_c]$$

using Eq. (3.7), we substitute for x_c, from above and find $r_c = \frac{\sqrt{(160)} - 4}{8} \approx 1.113$

i.e $r_0 = 2.565$

At r_c = 1.113 i.e r_0 = 2.565 the 2-cycle bifurcates to 4-cycle and then to 8-cycle

Near r = 1 there is another possible fixed point

solution. For a restricted basin of attraction we get a stable solution

of $f_2(x) = x$. This solution is stable for r < 3 and at r_0 = 2 . 7460

the basin of attraction for this fixed point collides with that of the

period 8-cycle and crisis occus. After that we see only the fixed point attractor.

Period adding bifurcation

Interesting Dynamics at r=2

At r=2, the map exhibits a bifurcation behavior where different initial conditions lead to different outcomes:

- **Initial Conditions $x_n < 1/2$:**
  For
  $x_n < 1/2$, iterates x_{n+1} are greater than x_n but do not exceed 1/2. This leads to convergence towards the fixed point x=1/2.
- **Initial Conditions $x_n > 1/2$:**
  For
  $x_n > 1/2$**:**, iterates x_{n+1} are less than x_n , and subsequent iterations switch between maps (3.1) and (3.2). This alternation can prevent convergence to a fixed point.

For r=2, certain initial conditions result in a fixed point while others do not, indicating a bifurcation point where the behavior of the system changes.

Behavior for $r = 2 \pm \epsilon$

For $r = 2 - \epsilon$ (with ϵ being a small positive number), the map predominantly converges to the fixed point as a sole outcome.

For $r = 2 + \epsilon$**,** the map enters a periodic window of very large period, which rapidly decreases as r increases slightly. This phenomenon is characterized by the inverse of period adding bifurcation, forming a ribbon-like structure in the bifurcation diagram.

3. Piecewise Smooth Maps and Border Collision Bifurcation

The map can be analyzed using piecewise smooth maps, which exhibit border collision bifurcations:

- **Piecewise Smooth Maps:**
  The system can be described by:

$$
\begin{aligned}
x_{n+1} &= f(x_n;\mu) \\
&= \begin{cases} g(x_n) = 4x_n(1-x_n) & \text{if } x_n < x_c \\ h(x_n;r_0) = (2r_0-4)x_n(1-x_n) & \text{if } x_n \geq x_c. \end{cases}
\end{aligned} \tag{4.2}
$$

- **Border Collision:**
  Border collision occurs when the fixed points of these maps cross the border
  $x_n = x_c$. For the first map, the fixed point is $x_L{}^* = 3/4$, which does not cross the border. For the second map, the fixed point $x_R{}^* = \frac{2*r_0-5}{2*r_0-4}$ can cross the border as r_0 varies. When $x_R{}^*$ collides with x_c, bifurcation occurs, leading to stabilization of the fixed point and the emergence of a chaotic band.

For $x_c = 1/2$, the collision occurs at $r_0 = 3$

4. Two-Dimensional Piecewise Smooth Maps

These maps can be extended to two dimensions:

$$
\begin{aligned}
&y_{n+1} = x_n \\
&x_{n+1} = 4x_n(1-x_n) \quad \text{if } x_n < y_n \qquad (4.4) \\
&\text{and} \\
&y_{n+1} = x_n \\
&x_{n+1} = (2r_0-4)x_n(1-x_n) \quad \text{if } x_n \geq y_n \qquad (4.5)
\end{aligned}
$$

Here, the fixed points of both maps lie on the border, so border collision does not occur.

5. Period Adding Phenomenon

As r approaches 2 from above, the distance between consecutive nodes in the bifurcation diagram decreases. The quantity δ is defined as:

$$
\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} \tag{4.6}
$$

This converges to approximately 1.414

1. **Ribbon-like Structure and Fixed Points**:
   - The ribbon-like structure in the bifurcation diagram is characterized by fixed points at each node. These nodes are points where the map transitions through different cycles.
   - For a given value of *r*, the fixed point $x_m{}^*$ satisfies $f_2(x_m{}^*) = x_m{}^*$
2. **Composite Functions and Fixed Points**:
   - The fixed points of the return maps are described by composite functions of $f_1 and f_2$:

     $f_2^{n-6}(f(f_2(x_m)))$

     $f_2^{n-6}(f_2(f(x_m)))$
   - Here, $f(x_m)$ represents the combination of $f_1 and f_2$:

3. **Convergence of Fixed Points**:
   - The convergence of fixed points across the nodes ensures that these nodes are also fixed points of f_2:

$$f_2^{n-6}(f_2(f(x_m^*))) = f_2(x_m^*) = x_m^* \tag{4.7}$$

i.e. $x_m^* = 1 - \frac{1}{r}$, and hence

$$f_2^{n-6}\left(f_2\left(f\left(1 - \frac{1}{r}\right)\right)\right) = 1 - \frac{1}{r} \tag{4.8}$$

4. **Calculation of *r* values**:
   - The value of *r* at the nth node, r_n, can be derived using:

$$\left[f_2^{n-7}\left(f_2\left(f\left(1-\frac{1}{r}\right)\right)\right)-\left(1-\frac{1}{r}\right)\right]$$
$$\times\left[rf_2^{n-7}\left(f_2\left(f\left(1-\frac{1}{r}\right)\right)\right)-1\right]=0 \tag{4.9}$$

where the first factor gives the solution for the n − 1 cycle and the second produces the new (nth) node. So Eq.(4.8) gives all the nodes up to the nth order.
We get the value of r for the nth node (
r_n) from

$$rf_2^{n-7}\left(f_2\left(f\left(1-\frac{1}{r}\right)\right)\right)=1. \tag{4.10}$$

If for n = N the period adding phenomenon stops, then

$$rf_2^{N-6}\left(f\left(1-\frac{1}{r}\right)\right)=1 \tag{4.11}$$

5. **Stopping of Period Addition**:
   - When period adding stops, the *r*value reaches a specific point where further bifurcation doesn't occur:

     r

     rN=2*rN*=2
   - This implies that at *r*=2, the period-adding phenomenon stops, which helps to clarify the bifurcation diagram.

     r=2

Calculation of $\delta\delta$

The ratio $\delta\delta$ for the nth cycle can be calculated using the bifurcation points rnrn:

1. **Ratio Definition**:

$$\delta = \lim_{n\to\infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

2. **Finding Roots r_n**:

    - By finding the r values at each node, r_n, we can determine δ:

      r

      r_n

      δ

      $$\delta_n = \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

    - This requires calculating the r values for successive bifurcations and using them to find the ratio.

      r