Date ______
Page ______

Exploration of Dynamics

The dynamics of the map

$$x_{n+1} = f_1(x_n) = 4x_n(1-x_n) \quad \text{If } x_n < x_{n-1}$$
$$x_{n+1} = f_2(x_n) = r x_n(1-x_n) \quad \text{If } x_n > x_{n-1}$$

without memory dependence, the logistic map $x_{n+1} = 4x_n(1-x_n)$ is chaotic with lyapunov Index of $\ln(2)$ and an Invariant density distribution which is continious in $0 \le x \le 1$

The map $x_{n+1} = r x_n(1-x_n)$, by itself shows fixed points for $r < 3$ followed by cycles of period 2^n and so on.

In the presence of memory, the above map does not begin with a fixed point but with a 2-cycle.

we have a possible 2-cycle x_1, x_2 with

$$x_2 = 4x_1(1-x_1) \quad \text{——} \quad (3.3)$$
$$x_1 = r x_2(1-x_2) \quad \text{——} \quad (3.4)$$

these changes to

$$x_2 = 4r x_2(1-x_2)(1-r x_2(1-x_2)) \quad \text{and} \quad \text{——} \quad (3.5)$$
$$x_1 = 4r x_1(1-x_1)(1-4x_1(1-x_1)) \quad \text{——} \quad (3.6)$$

x_1 & x_2 are fixed points of two Iterated function $F(x)$ & $G(x)$

$$F(x) = 4rx(1-x)(1-4x(1-x)) \quad \text{— (3.7)}$$

$$G(x) = 4rx(1-x)(1-rx(1-x)) \quad \text{— (3.8)}$$

Stability of $r < 1/4$:-

For $r < 1/4$, the fixed point $x = 0$ is stable for both $F(x)$ & $G(x)$. To verify this.

① Fixed points :- $F(x) = x$ and $G(x) = x$ & Solve for 'x'

② Derivative of fixed point: Calculate the derivative at $x = 0$

for $F(x)$

$F(0) = 0 \rightarrow$ So '0' is fixed point.

Lets Calculate $F'(x)$

$$F(x) = 4rx(1-x)(1-4x(1-x))$$

$$F'(x) = \frac{d}{dx}(4rx(1-x)(1-4x(1-x)))$$

$$\frac{d}{dx}(4rx(1-x)(1-4x+4x^2))$$

$$\frac{d}{dx}((4rx-4rx^2)(1-4x+4x^2))$$

$$\frac{d}{dx}((4rx - 16rx^2 + 16rx^3 - 4rx^2 + 16rx^3 + 16rx^4)$$

$$F'(0) = 4r$$

Since $|F'(0)| = 4r < 1$ for $r < 1/4$ $x = 0$ is stable.

Two-cycle Elements for $r > 1/4$

For $r > 1/4$, $x = 0$ becomes unstable, and a two-cycle emerges

The Two-cycle points x_1 and x_2 such that

$$F(F(x)) = x$$

$$G(G(x)) = x$$

This implies solving the second iteration of the function to find these points.

Destabalization of pitchfork Bifurcation occurs at (using $(F(x))$)

$$-1 = 4r_c\left[1 - 8x_c(1 - x_c)\right](1 - 2x_c)$$

$$1 - 2x_c = \frac{1 - 2r_c + \sqrt{4r_c^2 + 4r_c - 9}}{2(4r_c - 5)}$$

$$-1 = 4r_c\left[1 - 8\left(\frac{1 - 2r_c + \sqrt{4r_c^2 + 4r_c - 9}}{2(4r_c - 5)}\right)\left(1 - \left(\frac{1 - 2r_c + \sqrt{4r_c^2 + 4r_c - 9}}{2(4r_c - 5)}\right)\right)\right]$$

$$\left[\left(1 - \frac{1 - 2r_c + \sqrt{4r_c^2 + 4r_c - 9}}{2(4r_c - 5)}\right)\right]$$

By solving this complex expression we get.

$$r_c \approx 1.113.$$

Cycle Bifurcation

Above $r = r_c$ the system undergo from 2-cycle to a 4-cycle to an 8-cycle and Eventually into chaos. These Bifurcations are characterised by the Period-doubling route to Chaos.

Fixed points for $r \approx 1$

Near $r = 1$, another fixed point Solution Exists. For a restricted basin of attraction, we can find a stable solution to $f^2(x) = x$.

Basin of Attraction and Crisis

For $r < 3$, this solution is stable. At $r_0 = 2.7460$ the Basin of attraction for this fixed point Collides with that of the period cycle - 8 cycle, leading to Crisis. After the Crisis, only the fixed point attractor is observed.

Period Adding Bifurcations

Behaviour $r=2$:-

→ For $r=2$, the map has unique feature: The behaviour of the Initial Condition relative to $x_0 = 1/2$

→ If $x_n < 1/2$ then $x_{n+1} > x_n$ but will not exceed $x_{n+1} = 1/2$. This leads to x converging to the fixed point $x = 1/2$

→ If $x_n > 1/2$, then $x_{n+1} < x_n$, which triggers a switch to a different map, and the system does not necessarily reached a fixed point.

For $r > 2$:-

1) periodic Windows:- The system starts showing periodic behaviours with very high periods.

2) periodic decrements:- As 'r' increases slightly, the period of oscillations decreases rapidly, described by inverse period adding Bifurcations.

3) Ribbon structure:- The Bifurcation diagram shows a complex ribbon-like ~~structure~~ pattern due to branches merging and splitting.

This scenario involved Border Collision bifurcation, where the fixed points of two smooth maps intersect at a critical value, x_c,

This Dynamic model system can be modelled by a set of piecewise smooth maps. One-dimensional piecewise smooth maps can be defined in the following way:-

we take two maps $g(x)$ & $h(x; r_0)$

$$g(x) = 4x(1-x) \quad \text{for } x < x_c$$

$$h(x; r_0) = (2r_0 - 4)\, x(1-x) \quad \text{for } x \geq x_c.$$

$$\left(x_{n+1} = \left[r_0 - (4 - r_0) \tanh\left(\frac{x_n - x_c}{\epsilon}\right) \right] x_n (1 - x_n) \right)$$

Border Collision and Fixed points.

A Border Collision occurs when the fixed point of these maps cross the border at $x = x_c$. we need to analyze the fixed points x_L^* and x_R^* of $g(x)$ and $h(x; r_0)$ respectively, determine the Conditions under they Collide at $x = x_c$

Fixed points of $g(x)$

The fixed points of $g(x)$ is obtained by solving:

$$x = g(x) = 4x(1-x)$$

Solving this Equation:-

$$x = 4x(1-x)$$

$$x = 4x - 4x^2$$

$4x^2 - 3x = 0.$

$x(4x-3) = 0.$

So, the fixed points are

$x = 0$, (or) $x = 3/4$

we denote the fixed point within the relevant interval $x < x_c$ as x_L^*. For the given conditions, we assume $x_c = 1/2$.

Since $x_L^* = 3/4$

Since $x_L^* = 3/4 > 1/2$ the fixed point x_L^* cannot cross the border.

Fixed points for $h(x; r_0)$

The fixed points of $h(x; r_0)$ is obtained by solving:

$x = h(x; r_0) = (2r_0 - 4)x(1-x)$

solving the equation.

$x = (2r_0 - 4)\, x\,(1-x)$

$x = (2r_0-4)x - (2r_0-4)x^2.$

$(2r_0-4)x^2 - (2r_0-5)x = 0.$

$x\left((2r_0-4)x - (2r_0-5)\right) = 0$

So, the fixed points are $x = 0$ (or) $x = \dfrac{2r_0-5}{2r_0-4}.$

We denote the fixed point within the relevent Interval

$x \geq x_c$ as x_R^*

$$x_R^* = \frac{2r_0 - 5}{2r_0 - 4}$$

Collision Condition

For a border collision to occur, the fixed points x_R^* must Collide with the border $x_c = 1/2$.

$$x_R^* = x_c$$

$$\frac{2r_0 - 5}{2r_0 - 3} = 1/2$$

Solving for r_0:-

$$2(2r_0 - 5) = 2r_0 - 4$$

$$4r_0 - 10 = 2r_0 - 4$$

$$4r_0 - 2r_0 = 6$$

$$2r_0 = 6$$

$$\boxed{r_0 = 3}$$

→ At $r_0 = 3$ x_R^* Collides with $x_c = 1/2$, resulting in a border collision bifurcation.

→ As r_0 Increases beyond '3', the fixed points x_R^* moves past x_c, leading to stabalized fixed point & the Emergence of a Chaotic Band.

i.e ($r_0 = 3$)

As we move towards $r = 2$ from above, the distance between two consecutive nodes decreases. If we define the quantity.

$$\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n}$$

feigenbaum Constant.

where r_n is the value of 'r' at $(n-1)$ cycle goes to n-cycle,

then δ Converges to 1.414.

In ribbon-like structure, Every node is a fixed point of the return maps $f_2^{\,n-6}(f(f_2(x_m)))$ and $f_2^{\,n-6}(f_2(f(x_m)))$

where 'n' denotes the no. of cycles executed at that particular 'r' and $f(x_m)$ is the Composite function of $f_1(x_m)$ and $f_2(x_m)$

Convergence of the fixed point from left to right at nodes ensures that these nodes are also fixed points of $f_2(x_m)$.

So for Each node.

$$f_2^{\,n-6}(f_2(f(x_m^*))) = f_2(x_m^*) = x_m^*.$$

i.e $x_m^* = 1 - 1/r$

$$f_2^{n-6}\left(f_2\left(f\left(1-\tfrac{1}{r}\right)\right)\right) = 1-\frac{1}{r}$$

r-value $\rightarrow r_n$ at the n^{th} cycle.

This can be decomposed as.

$$\left[f_2^{n-7}\left(f_2\left(f\left(1-\tfrac{1}{r}\right)\right)\right) - \left(1-\tfrac{1}{r}\right)\right]$$

$$\times\left[r f_2^{n-7}\left(f_2\left(f\left(1-\tfrac{1}{r}\right)\right)\right) - 1\right] = 0.$$

we get.

$$r f_2^{n-7}\left(f_2\left(f\left(1-\tfrac{1}{r}\right)\right)\right) = 1$$

if for $n = N$ the period adding phenomenon stops, then

$$r f_2^{N-6}\left(f\left(1-\tfrac{1}{r}\right)\right) = 1$$

$$r f_2^{N-5}\left(f\left(1-\tfrac{1}{r}\right)\right) = 1$$

when period stops $r_N \rightarrow r_{N+1}$ the above gives

$$r_N f_2\left(\frac{1}{r_N}\right) = 1$$

which gives.

$$\frac{1}{r_N} = r_N\left(\frac{1}{r_N}\right)\left(1-\frac{1}{r_n}\right)$$

$$\frac{1}{r_n} = 1 - \frac{1}{r_n}$$

$$\frac{2}{r_N} = 1$$

$$\boxed{r_N = 2}$$

So, at $r=2$, the period adding phenomenon stops.

Let find δ_n

Near $r=2$, let $r_n = 2+\Delta_n$ then

$$r_n^{\,n}\,\Delta_n^2 = 1$$

$$r_{n+1}^{\,n}\,\Delta_{n+1}^2 = 1$$

$$r_n \approx r_N$$

$$\delta_n = \frac{\Delta_n - \Delta_{n-1}}{\Delta_{n+1} - \Delta_n}$$

$\delta_n = \sqrt{r_n} = \sqrt{2} \rightarrow$ for large value of 'n'.

→ Beyond $r = 2.5427$, there is a chaotic band with the Emergence of a periodic window at

$$r = 2.7244$$

For $2.7564 \leq r \leq 2.7876$, we see a cycle of period 11,

Each Element of the '11' cycle Exhibits a Sequence of period doubling bifurcation for $r < 2.7564$ and $r > 2.7876$

~~For one particular Element of the 11-cycle the two sets of~~

As we further Increase the value of 'r', there are two large periodic windows. At $r = 2.9499$, a 6-cycle is formed,

period-6 fixed points bifurcate and take period doubling route to chaos.

Like a logistic map a period 3-cycle is formed and takes an Intermittent route to chaos.

At $r = 3.8384$ period becomes Stable and goes through period doubling bifurcation