Module 5 :

Introduction to graph theory.

→ Definitions and Examples.

→ Subgraphs.

→ complement of a graph.

→ Graph Isomorphism

→ vertex degree.

→ Euler trails and circuits.

Graphs :- A Graph is a pair (V,E), where V is a non-empty set and E is a set of unordered pairs of elements taken from the set V.

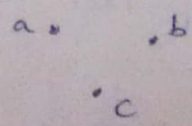
The elements of V are called Vertices and the set V is called Vertex set. The elements of E are called undirected edges (or) just edges and the set E is called Edge set.

Note :- 1) The Vertex set of a graph/digraph has to be non-empty but the edge set can be empty.

2) (A,B) denote an ordered pair of A & B (or) a directed edge.
$\{A,B\}$ denote an unordered pair of A & B (or) an undirected edge.

Null Graph :- A graph/digraph containing no edges is called a Null Graph.

Ex:-



$V = \{a,b,c\}$, $E = \phi$ (or) $\{\ \}$

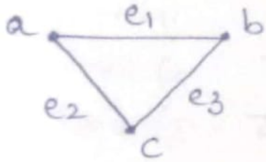
Trivial Graph :- A null graph with only one Vertex is Called a trivial Graph.

Ex :- $\cdot$ a $\quad V = \{a\}$ & $E = \{\ \}$

Finite Graph / digraph :- A graph (or) digraph with only a finite no of Verticies and edges is Called a finite graph/digraph.

End Verticies :- If V_i and V_j denote two Verticies of a graph and if e_k denotes the edge joining V_i and V_j, then V_i & V_j are Called the End Verticies of e_k.

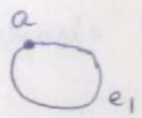
Ex :-



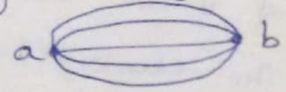
a & b are end Verticies of e_1
a & c " ——————— " e_2
b & c " ——————— " e_3

Loop :- An edge whose end Verticies are Same is Called a Loop

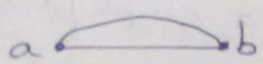
Ex :-



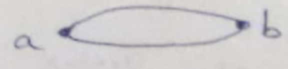
Multiple edges :- Two (or) more edges having same end Verticies are Called Multiple edges.



Parallel edges :- Two edges having same end verticies are called parallel edges.



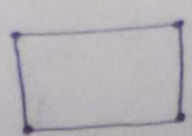
(or)



Simple Graph :- A graph which does not contain loops and multiple edges is Called Simple Graph.

Loop-free :- A graph which does not contain loop, is called Loop-free.

Ex :-



Simple Graph / Loop-free

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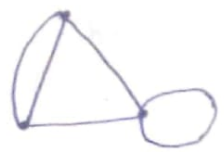
Multigraph :- A graph which contains multiple edges but no loops is called Multi-graph.

General Graph :- A graph which contains multiple edges (or) loops (or both is called a General graph.

Multigraph



General Graph

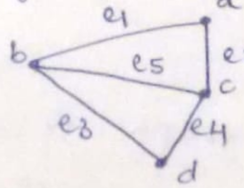
Incidence :- If 'V' is an end vertex of an edge 'e' in a graph G then the edge 'e' is incident on (to) the Vertex 'V'.

Note :- 1) Every edge is incident on two verticies one at each end.
2) The two end verticies are coincident if the edge is a loop.

Adjacent Verticies :- Two Verticies are said to be adjacent verticies if there is an edge joining them.

Adjacent edges :- Two non-parallel edges are said to be adjacent edges if they are incident on a common vertex.

Ex :-

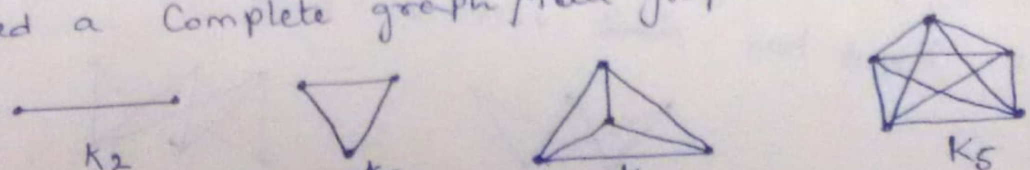


Adj-verticies :- (a, b) (a, c) (b, c) (b, d) (c, d)

Adj-edges :- (e_1, e_5) (e_1, e_2) (e_2, e_5) (e_2, e_4) (e_4, e_5) (e_3, e_1) (e_3, e_4) (e_5, e_3)

Complete Graph :- A simple graph of order ≥ 2 (ie no of verticies ≥ 2) in which there is an edge b/w every pair of verticies is called a Complete graph / full graph. It is denoted by K_n.

Ex :-



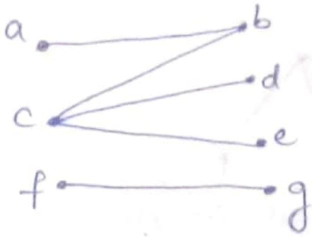
K_2 K_3 K_4 K_5

K_5 is called the Kuratowski's first graph.

Bipartite Graph :- A simple Graph G in which its vertex set V is the Union of two of its mutually disjoint non-empty SubSets V_1 and V_2 and each edge in G joins a vertex in V_1 and a vertex in V_2, then G is said to be a bipartite graph.

Ex :-

G :



In the above Graph G,

$$V = \{a, b, c, d, e, f, g\}$$

$$E = \{ab, cb, cd, ce, fg\}$$

V is the union of two of its subsets $V_1 = \{a, c, f\}$ $V_2 = \{b, d, e, g\}$

which are such that

(i) V_1 & V_2 are disjoint $[\Rightarrow V_1 \cap V_2 = \phi]$

(ii) every edge in G joins a vertex in V_1 & a vertex in V_2.

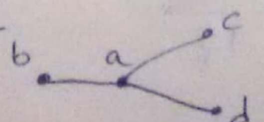
& (iii) G contains no edge that joins two vertices both of which are in V_1 (or) V_2.

Thus G is a bipartite Graph with V_1 & V_2 as bipartites (or partitions).

Complete Bipartite Graph:- A bipartite graph $G = (V_1, V_2; E)$ is said to be a complete bipartite graph if there is an edge b/w every vertex in V_1 & V_2. It is denoted by $K_{r,s}$ where r is the no of verticies in V_1 & s is the no of Verticies in V_2, with $r \leq s$.

Thus, $K_{r,s}$ has $r+s$ verticies and rs edges.

Ex :-



$K_{1,3}$

$V_1 = \{a\}$, $V_2 = \{b, c, d\}$

$K_{2,3}$

$K_{3,3}$

$K_{3,3}$ is called as Kuratowski's 2^{nd} Graph.

Problems :-

1) If $G = G(V,E)$ is a simple Graph, prove that $2|E| \leq |V|^2 - |V|$

Soln :- In a simple Graph, there are no multiple edges.

Each edge of a graph is determined by a pair of verticies.

The no of edges cannot exceed the no of Pairs of verticies.

The no of pair of verticies that can be choosen from n verticies is

$$^nC_2 = \frac{n!}{(n-2)!\,2!} \qquad \left[\because {}^nC_r = \frac{n!}{(n-r)!\,r!}\right]$$

$$= \frac{n(n-1)(n-2)!}{2\,(n-2)!} = \frac{n(n-1)}{2}$$

Thus for a simple Graph with $n (\geq 2)$ verticies, the no of edges cannot exceed $\frac{1}{2}n(n-1)$.

Thus if a graph $G = G(V,E)$ has n-verticies & m edges, then

$$m \leq \frac{1}{2}n(n-1)$$

$$2m \leq n^2 - n$$

ie $2|E| \leq |V|^2 - |V|$.

Note :- In a simple Graph, for a pair of verticies, we can have only one edge
ie 2 verticies $\longleftrightarrow$ 1 edge
Hence no of edges cannot exceed no of pair of verticies.

2) Show that a Complete graph with n verticies, namely K_n has $\frac{1}{2}n(n-1)$ edges.

Soln :- In a complete graph, there exists exactly one edge b/w every pair of verticies.

$\therefore$ the no of edges in a complete graph = the no of pair of verticies.

If the no of verticies is n, then no of pair of verticies is

$$^nC_2 = \frac{n!}{(n-2)!\,2!} = \frac{n(n-1)(n-2)!}{(n-2)!\,2!} = \frac{n(n-1)}{2}$$

Thus in a complete Graph with n verticies, no of edges, $m = \frac{1}{2}n(n-1)$

3) Show that a simple Graph of order $n=4$ & size $m=7$ & a complete graph of order $n=4$ & size $m=5$ do not exist.

Soln :- For a simple Graph,

$$2m \leq n^2 - n.$$
$$2(7) \leq 16-4$$
$$14 \leq 12 \quad \# \text{ not true.}$$

For a complete graph,

$$m = \frac{1}{2}n(n-1)$$
$$5 = \frac{1}{\cancel{2}} \cdot \cancel{4}^{2} \times 3$$
$$5 = 6 \quad \# \text{ not true.}$$

4) (a) How many verticies and how many edges are there in the complete bipartite graphs $K_{4,7}$ & $K_{7,11}$?

(b) If the graph $K_{r,12}$ has 72 edges, what is r?

Soln :- A complete Bipartite Graph $K_{r,s}$ has $r+s$ verticies and rs edges.

(a) The Graph $K_{4,7}$ has $4+7=11$ verticies and $4 \times 7 = 28$ edges

& the Graph $K_{7,11}$ has $7+11 = 18$ verticies & $7 \times 11 = 77$ edges

(b) If the graph $K_{r,12}$ has 72 edges, then

$$12r = 72 \Rightarrow \boxed{r=6}$$

5) Let $G=(V,E)$ be a simple graph of order $|V|=n$ & size $|E|=m$. If G is a bipartite graph, P.T $4m \leq n^2$.

Soln :- Let V_1 and V_2 be bipartites of G, with $|V_1| = r$ & $|V_2| = s$.

Since for the Given graph, $|V|=n$, we should have $r+s=n$ so that ~~$r=n-s$ and~~ $s=n-r$.

The graph G has the maximum no of edges when each of the r-verticies in V_1 is joined by an edge to each of the s-verticies in V_2 & this maximum is equal to rs.

$\Rightarrow \quad |E| = m \leq rs$. (max

ie $\quad m \leq rs = r(n-r)$

$= rn - r^2$

$= -(r^2 - nr)$

by completing the square, we have

$\cancel{m \leq}\ rs = -\left[\left(r - \frac{1}{2}n\right)^2 - \left(\frac{1}{2}\cdot n\right)^2\right]$

$\cancel{m \leq}\ rs = -\left[\left(r - \frac{n}{2}\right)^2 - \left(\frac{n}{2}\right)^2\right]$

(or) $\quad rs = \left(\frac{n}{2}\right)^2 - \left(r - \frac{n}{2}\right)^2$, this is maximum when $r = \frac{n}{2}$.

Thus when $r = \frac{n}{2}$, $rs = \frac{n^2}{4}$.

Hence $\quad m \leq rs = \frac{n^2}{4}$

$\Rightarrow \quad m \leq \frac{n^2}{4} \Rightarrow 4m \leq n^2$

6) Show that a simple graph of order $n=4$ & size $m=5$ cannot be a bipartite graph.

Soln :- for a bipartite graph, $4m \leq n^2$. $\qquad 4(5) \leq 4^2$

$20 \leq 16$ #

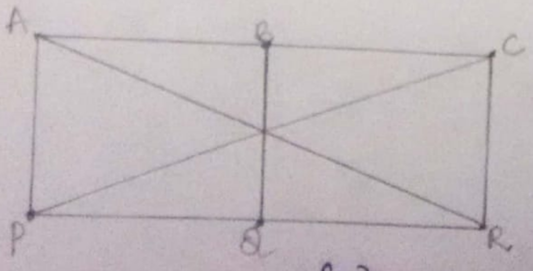
Here $4m = 4(5) = 20$ & $n^2 = 4^2 = 16$.

$4m > 16$ #

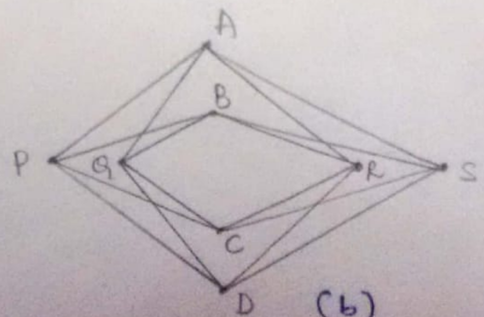
Hence not a bipartite Graph.

Hw

1) Verify that the foll are bipartite Graphs? what are their bipartites?



(a)



(b)

2) Company X has offices in cities B, D & K; Company Y in cities B & M; Company Z in cities C & M. Represent this situation by a bipartite Graph. is this a complete bipartite Graph?

3) State whether the foll Graphs can exist or cannot exist:

✓ (a) Simple Graph of order 3 & size 2

× (b) " ——— " ——— 5 & " 12

✓ (c) complete " ——— " 5 & size 10

✓ (d) Bipartite " ——— " 4 & " 3

× (e) " ——— " ——— 3 & " 4

✓ (f) complete Bipartite Graph of order 4 & size 4.

✓ → exist | × → Not

Note :- 1) for a simple Graph, $2|E| \leq |V|^2 - |V|$

2) for a complete Graph, $m = \frac{1}{2} n(n-1)$.

3) for a complete Bipartite Graph, $K_{r,s}$, $r+s$ vertices & rs edges exists.

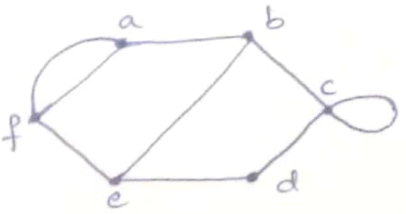
4) for a Bipartite Graph, $4m \leq n^2$.

Vertex Degree :-
Let G be a graph & v be a vertex of G. Then the no of edges of G that are incident on v with the loops counted twice is called the degree of the vertex v and is denoted by deg (v) (or) d(v).

Degree Sequence :- The degrees of the verticies of a graph arranged in non-decreasing order is called degree sequence.

Degree of the graph :- The minimum of the degrees of verticies of a graph is called the degree of the graph.

Ex :-



d(a) = 3
d(b) = 3
d(c) = 4
d(d) = 2
d(e) = 3
d(f) = 3

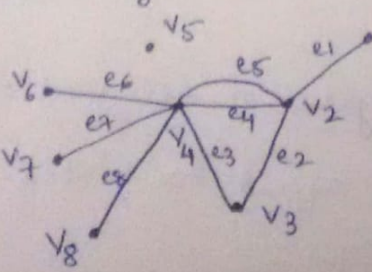
∴ The degree sequence is 2, 3, 3, 3, 3, 4 and the degree of the graph is '2'.

Isolated Vertex :- A vertex in a graph which is not an end vertex of any edge of the graph is called an isolated vertex.
ie a vertex is an isolated vertex if & only if its degree is zero.

Pendant Vertex :- A vertex of degree 'one' is called a pendant vertex.

Pendant edge :- An edge incident on a pendant vertex is called a pendant edge.

Ex :-



$V_5 \to$ isolated vertex.
V_1, V_6, V_7, V_8 are pendant verticies.
e_1, e_6, e_7, e_8 are pendant edges.

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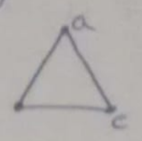
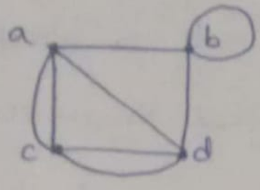
Regular Graph :- A graph in which all the vertices are of same degree 'K' is called a regular graph of degree 'K' (or) a K-regular Graph.

Ex :-



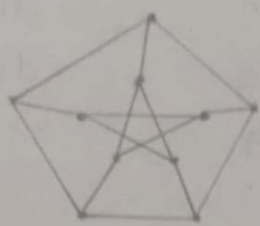
$d(a) = d(b) = d(c) = 2$.

$\Rightarrow$ 2-regular Graph.

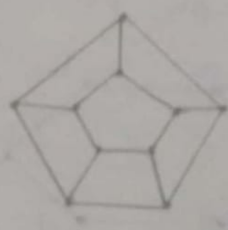


$d(a) = 4 = d(b) = d(c) = d(d)$.

Petersen Graph :- A 3-regular Graph which contains 10 vertices and 15 edges is called petersen Graph.



(or)

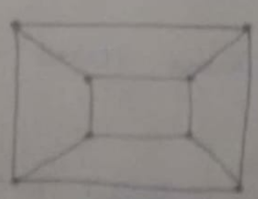


Three-dimensional hypercube :-

A cubic graph with $8 = 2^3$ vertices is called 3-dimensional hypercube & is denoted by Q_3

In General, for any +ve integer k, a loop-free k-regular graph with 2^k vertices is called the k-dimensional hypercube & is denoted by Q_k.

Ex :-



3-dimensional hypercube

Hand shaking Property :- The sum of the degrees of all vertices in a graph is an even no, and this no is equal to twice the no of edges in the Graph.

ie for a graph $G = (V, E)$, $\sum_{v \in V} deg(v) = 2|E|$

Proof :- The above property is obvious from the fact that while counting the degrees of verticies, each edge is counted twice (once at each end). (8)

Just say this
Note :- The name "Hand shaking property" because "if several people shake hands, then the total no of hands shaken must be even, since two hands are involved in each handshake".

Theorem :- In every graph, the no of verticies of odd degree is Even.

Proof :- Consider a graph with 'n' verticies. Suppose 'k' of these verticies are of odd degree then the remaining n-k verticies are of even degree.

Let $v_1, v_2 \ldots v_k$ be the verticies of odd degree and $v_{k+1}, v_{k+2} \ldots v_n$ be " ——— even degree.

then $$\sum_{i=1}^{n} \deg(v_i) = \sum_{i=1}^{k} \deg(v_i) + \sum_{i=k+1}^{n} \deg(v_i) \rightarrow (1)$$

In view of hand shaking property, the sum on the LHS of the above expression is equal to twice the no of edges in the graph. This sum is even.

The second sum in the RHS, is the sum of the degree of verticies with even degree. This sum is also even.

$\therefore$ the 1^{st} sum in the RHS must also be even.

ie $d(v_1) + d(v_2) + \ldots + d(v_k) = \text{Even} \rightarrow (2)$.

but each of $d(v_1), d(v_2) \ldots d(v_k)$ is odd

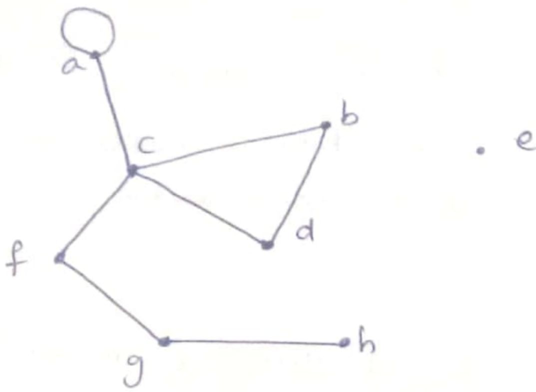
$\therefore$ the no of terms in the LHS of (2) must be even [$\because$ odd no's added even no of times, the result is Even]

ie k is Even.

Hence the proof.

Problems:-

1) For the given graph, indicate the degree of each vertex & verify the handshaking property.



Soln:- deg (a) = 3, deg (b) = 2, deg (c) = 4, deg (d) = 2
d(e) = 0, deg (f) = 2, deg (g) = 2, deg (h) = 1

The Sum of the degrees of verticies = 16.

The graph has 8 edges.

Thus the Sum of the degrees of verticies = twice the no of edges.

16 = 2 (8) True.

Hand shaking property is verified.

2) Can there be a graph with 12 verticies such that two of the verticies have degree 3 each & the remaining 10 verticies have degree 4 each? If so, find |E| ? ie its size

Soln:- Sum of the deg of verticies = (2×3) + (10×4)
= 6 + 40
= 46, which is Even.

By Hand Shaking property, sum of the deg of verticies = twice the no of edges.

ie 46 = 2|E| ⇒ |E| = 23 ~~property holds~~.

Hence there can be a graph of the desired type (whose size is 23)

3) In a graph $G = (V, E)$, what is the largest possible value for $|V|$ if $|E| = 19$ and $\deg(v) \geq 4$ for all $v \in V$?

Soln :- Given :- all verticies are of degree greater than (or) equal to 4.

$\therefore$ the sum of the degrees of verticies $\geq 4n$

where $n = |V|$ & this sum = twice the no of edges.

$$\therefore \quad 2|E| \geq 4n$$
$$2 \times 19 \geq 4n \quad \Rightarrow \quad n \leq \frac{38}{4} = 9.5 < 10$$

Thus the largest possible value of $|V| = 9$

ie given graph can have atmost 9 verticies.

4) Prove that the hypercube Q_n has $n2^{n-1}$ edges.

Determine the no of edges in Q_8.

Soln :- In the hypercube Q_n, the no of verticies is 2^n and each vertex is of degree 'n'.

$\therefore$ the sum of the degrees of verticies of Q_n is $n \times 2^n$.

By Hand shaking property, we have

Sum of the deg of verticies = 2 × no of edges

ie $$n \times 2^n = 2|E|$$
$$\Rightarrow \quad |E| = \frac{1}{2} \times n \times 2^n = n2^{n-1}$$

$\therefore$ Q_n has $n2^{n-1}$ no of edges.

Also, No of edges in $Q_8 = 8 \times 2^7 = 1024$.

5) (a) what is the dimension of the hypercube with 524 288 edges?

(b) How many verticies are there in a hypercube with 4980736 edges?

P.T.O.

Soln:- For the k-dimensional hypercube Q_k, the no of verticies is 2^k & no of edges is $k \cdot 2^{k-1}$.

(a) To find k, given $k \cdot 2^{k-1} = \cancel{54}\,524288$

ie $k \cdot 2^{k-1} = 2^{19}$

$= 2^4 \times 2^{15}$

$k \cdot 2^{k-1} = 16 \times 2^{15}$ of the form $k \cdot 2^{k-1}$

$\Rightarrow \boxed{k = 16}$

For (b), keep dividing by 2, until you get a no not divisible by 2. whatever asked, express in terms of '2' power

Thus the dimension of the hypercube with 524288 edges is

$k = 16$

(b) we have $4980736 = 19 \times 2^{18}$ of the form $k \cdot 2^{k-1}$, which indicates that Q_k has 4980736 edges when $k = 19$.

$\therefore$ In this hypercube, no of verticies is $2^k = 2^{19} = 524288$.

6) (a) If k is odd, show that the no of verticies in a k-regular graph is even

(b) Show that it is not possible to have a set of nine people at a party such that each one knows exactly five of the others in the party.

Soln:-(a) In a k-regular graph, the degree of each vertex is k.

$\therefore$ if such a graph has n verticies, then the sum of degrees is nk, and this has to be an Even no (by handshaking property)

If k is odd, n must be Even to satisfy the above.

Hence the proof.

(b) Let G be a graph with 9 verticies, each vertex representing a person in the given set, and each edge representing an acquiantance (given situation).

P.T.O

If each person in the set (party) knows exactly five other persons (10)
in the set (party), then there will be exactly five edges
incident on each vertex & the Graph G will be 5-regular.

This is not possible b'coz G has an odd no of verticies.
(Since by (a), if k is odd, n must be Even).

Hence the graph G of the desired type does not exist.

7) (a) If a graph with n verticies & m edges is k-regular,
Show that $m = kn/2$.

(b) Does there exist a cubic graph with 15 verticies?

(c) " ————— " 4-regular graph with 15 edges?

Soln :- (a) Given :- The Graph G is k-regular.

$\Rightarrow$ the degree of every vertex is k.

$\therefore$ if G has n verticies, then the sum of deg of verticies is nK.
By Hand Shaking property, this must be equal to 2m (if G has m edges)

ie $nK = 2m \Rightarrow m = \frac{nK}{2}$.

(b) If there is a cubic graph (3-regular graph) with 15 verticies,
the no of edges it should have is $m = \frac{kn}{2}$

$m = \frac{3 \times 15}{2} = \frac{45}{2}$ (not an integer)

Thus the graph of desired type does not exist.

(c) If there is a 4-regular graph with 15 edges (ie $K=4, m=15$),
the no of ~~edges~~ verticies it should have is $n = \frac{2m}{k}$.

$\Rightarrow n = \frac{2 \times 15}{4} = \frac{30}{4}$ (not an integer)

Thus the graph of desired type does not exist.

8) (a) Show that in a complete Graph of n verticies (namely K_n), the degree of every vertex is $(n-1)$ & that the total no of edges is $\frac{n(n-1)}{2}$.

(b) If K_n has 'm' edges, S.T $n(n+1) = 2(n+m)$

Soln :- A complete graph is a simple graph in which every ~~edge~~ vertex is joined with every other vertex through exactly one edge.

∴ If there are 'n' verticies, each vertex is joined to $(n-1)$ verticies through exactly one edge.

Hence there occur $(n-1)$ edges at every vertex

$\Rightarrow$ degree of every vertex is $(n-1)$.

∴ sum of degree of verticies is $n(n-1)$, this sum must be equal to $2m$ (by H.S.P)

i.e $n(n-1) = 2m \Rightarrow m = \frac{1}{2}n(n-1)$.

Thus K_n has $\frac{1}{2}n(n-1)$ edges.

(b) If K_n has m edges, then

$$m = \frac{1}{2}n(n-1)$$

add n on b.s

$$n+m = n + \frac{1}{2}n(n-1)$$

$$2(n+m) = 2n + n^2 - n = n^2 + n$$

$$2(n+m) = n(n+1)$$

9) Show that there is no graph with 12 verticies & 28 edges in the foll cases :-

(a) The degree of a vertex is either 3 (or) 4

(b) The degree of a " ——— " 3 (or) 6.

Soln :- Suppose there is a graph with 28 edges & 12 verticies, of which k verticies are of deg 3 (each), then :

Scanned by CamScanner

(a) If all the remaining $(12-k)$ verticies have degree 4,

then $3k + 4(12-k) = 2 \times 28$

$3k + 48 - 4k = 56$

$-k = 8 \Rightarrow k = -8$ #

(b) If all the remaining $(12-k)$ verticies have degree 6,

then $3k + 6(12-k) = 56$

$3k + 72 - 6k = 56.$

$-3k = -16 \Rightarrow k = \frac{16}{3}$ #

Hence in both cases, the graph of the desired type cannot exist.

10) Determine the order $|V|$ of the graph $G = (V, E)$ in the foll

Cases :- (a) G is a cubic graph with 9 edges

(b) G is regular with 15 edges

(c) G has 10 edges with 2 verticies of deg 4 & all others of deg 3.

Soln :- (a) Suppose the order of G is n,

Since G is a cubic graph, all verticies of G have deg 3.

$\therefore$ Sum of degrees of verticies is $3n$.

Since G has 9 edges, by HSP,

$3n = 2 \times 9 = 18$

$\Rightarrow n = 6 \Rightarrow |V| = 6.$

(b) Given: G is a regular graph

$\Rightarrow$ all verticies of G must be of same degree, say k.

Let G be of order 'n', then the sum of the degrees of verticies is nk.

Since G has 15 edges, by HSP

$nk = 2 \times 15 \Rightarrow n = \frac{30}{k}$

since k is a +ve integer, it follows that 'n' must be a divisor of 30.

ie n must be 1, 2, 3, 5, 6, 10, 15 and 30 (possible orders of G)

(c) suppose the order of G is 'n'.

Since 2 verticies of G are of deg 4 & all others are of deg 3, sum of the degrees of verticies is

$$(2\times 4)+(n-2)3 = 8+3(n-2).$$

By HSP,

$$8+3(n-2) = 2\times 10$$

$$3n-6 = 20-8 = 12$$

$$3n = 18 \Rightarrow \boxed{n=6}$$

ie $|V| = 6$.

Hw

1) Consider a graph having n verticies & m edges. If p no of verticies are of deg k & the remaining verticies are of deg k+1, P.T $p = (k+1)n - 2m$.

2) P.T there is no simple graph with 7 verticies, one of which has deg 2, 2 have deg 3, 3 have deg 4 & the remaining vertex has deg 5.

11) for a graph with 'n' vertices and 'm' edges, if δ is the minimum, Δ is the maximum of degree of vertices, show that $\delta \le \frac{2m}{n} \le \Delta$.

Soln:- Let $d_1, d_2 \ldots d_n$ be the degrees of $1^{st}, 2^{nd} \ldots n^{th}$ vertex resp, then

$\Sigma \deg(v) = d_1+d_2+\ldots+d_n$

by HSP, $\Sigma \deg(v) = 2|E| = 2m$

$\therefore d_1+d_2+\ldots+d_n = 2m \rightarrow$ ①.

Given that δ is min of $d_1, d_2 \ldots d_n$

ie $\delta \le d_1, \delta \le d_2, \ldots \delta \le d_n$.

$\therefore \delta+\delta+\ldots+\delta_{(n\,times)} \le d_1+d_2+\ldots+d_n$

$\Rightarrow n\delta \le 2m$ (by ①)

$\Rightarrow \delta \le \frac{2m}{n} \rightarrow$ ②

Also given that Δ is max of $d_1, d_2 \ldots d_n$

ie $\Delta \ge d_1, \Delta \ge d_2 \ldots \Delta \ge d_n$

$\Rightarrow \Delta+\Delta+\ldots+\Delta_{(n\,times)} \ge d_1+d_2+\ldots+d_n$

$\Rightarrow n\Delta \ge 2m$ (by ①)

$\therefore \Delta \ge \frac{2m}{n}$ (or) $\frac{2m}{n} \le \Delta \rightarrow$ ③

from ② & ③,

$$\delta \le \frac{2m}{n} \le \Delta.$$

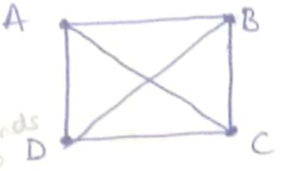
Isomorphism :-

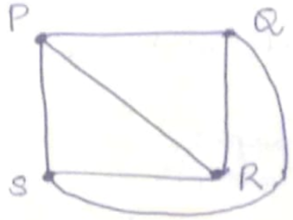
Two graphs G and G' are said to be isomorphic if there is a one-one correspondance b/w their vertices & b/w their edges such that the adjacency of vertices is preserved.

Such graphs will have the same structure, differing only in the way their verticies & edges are labelled (or) only in the way they are represented geometrically.

If G & G' are isomorphic, we write $G \cong G'$.

Ex :-





$A \leftrightarrow P$, $B \leftrightarrow Q$, $C \leftrightarrow R$, $D \leftrightarrow S$ – 1-1 correspondance b/w verticies.

$\{A, B\} \leftrightarrow \{P, Q\}$
$\{B, C\} \leftrightarrow \{Q, R\}$
$\{C, D\} \leftrightarrow \{R, S\}$
$\{A, D\} \leftrightarrow \{P, S\}$
$\{A, C\} \leftrightarrow \{P, R\}$
$\{B, D\} \leftrightarrow \{Q, S\}$

1-1 Correspondence b/w edges

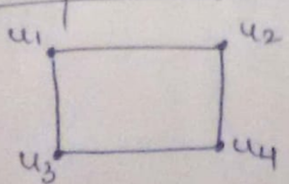
and the adjacency of verticies is preserved.

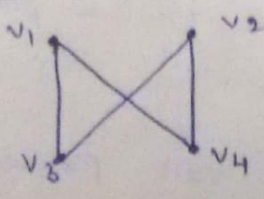
Two digraphs D_1 & D_2 are said to be Isomorphic if there is a one-one correspondence b/w their vertices & b/w their edges such that adjacency of verticies along with the direction is preserved.

Problems :-

Verify the two graphs given below are isomorphic :-

1)



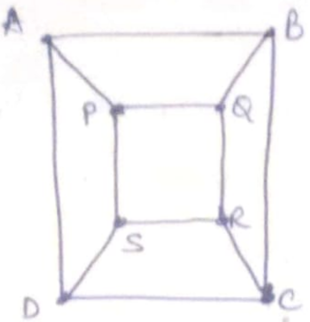


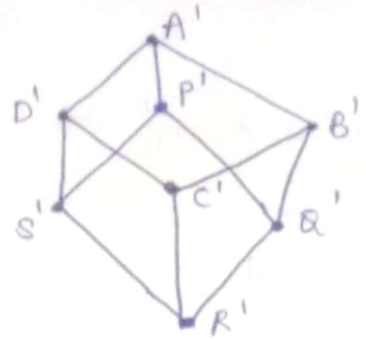
$u_1 \leftrightarrow v_1$
$u_2 \leftrightarrow v_4$
$u_3 \leftrightarrow v_3$
$u_4 \leftrightarrow v_2$

$\{u_1, u_2\} \leftrightarrow \{v_1, v_4\}$
$\{u_2, u_4\} \leftrightarrow \{v_4, v_2\}$
$\{u_3, u_4\} \leftrightarrow \{v_3, v_2\}$
$\{u_1, u_3\} \leftrightarrow \{v_1, v_3\}$

1-1 correspondence b/w the edges & vertices of 2 graphs exist. adjacent vertices in the 1^{st} graph correspond to adjacent verticies in the 2^{nd} graph & vice-versa. Hence it is Isomorphic

2)





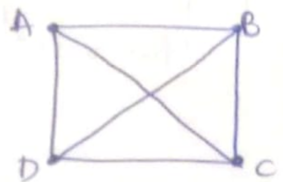
$A \longleftrightarrow A'$, $B \leftrightarrow B'$, $C \longleftrightarrow C'$, $D \longleftrightarrow D'$,
$P \longleftrightarrow P'$, $Q \leftrightarrow Q'$, $R \longleftrightarrow R'$, $S \leftrightarrow S'$.
$\{A, B\} \longleftrightarrow \{A', B'\}$, $\{B, C\} \longleftrightarrow \{B', C'\}$ $\{C, D\} \longleftrightarrow \{C', D'\}$

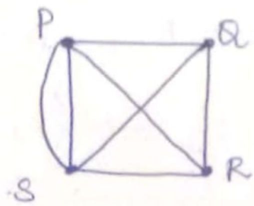
etc ie 1-1 correspondence b/w the verticces & b/w the edges exist.

Also adjacency of vertices is preserved.

$\Rightarrow$ Isomorphic graphs.

3)



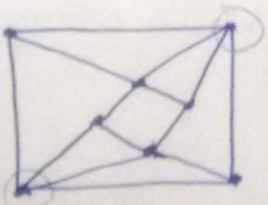


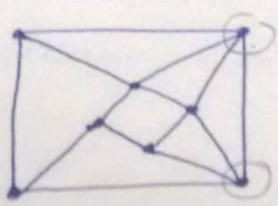
first Graph has 4 verticies & 6 edges;
Second " ——— " ——— " but 7 edges.

Thus the 1-1 correspondence b/w the edges is not possible.

$\Rightarrow$ Not Isomorphic

4)



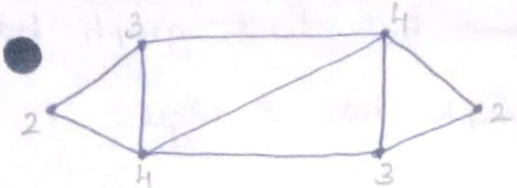


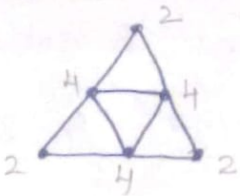
First Graph has a pair of verticies of degree 4, which are not adjacent whereas second graph has a pair of verticies of degree 4, which are adjacent.

$\therefore$ Adjacency of verticies is not preserved.

$\Rightarrow$ Not Isomorphic.

5>



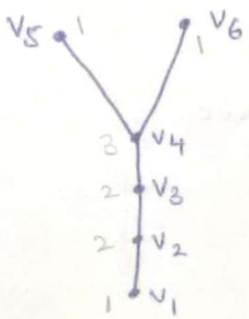


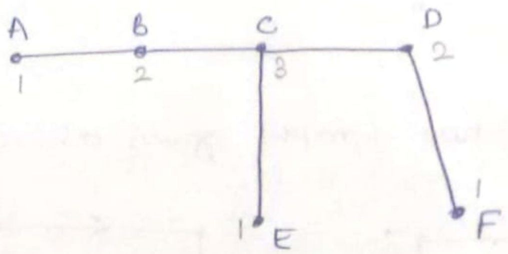
Both the Graphs has 6 verticies and 9 edges. But first Graph has 2 verticies of deg 4 whereas 2nd Graph has 3 verticies of deg 4.

∴ there cannot be one-one correspondence b/w the verticies & b/w the edges of the 2 graphs, which preserves the adjacency of verticies.

⇒ Not Isomorphic.

6>





Both the Graphs has 6 verticies and 5 edges.

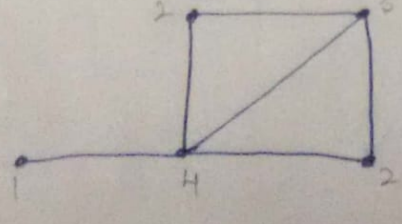
$V_1 \leftrightarrow A$, $V_2 \leftrightarrow B$, $V_3 \leftrightarrow D$, $V_4 \leftrightarrow C$, $V_5 \leftrightarrow E$, $V_6 \leftrightarrow F$

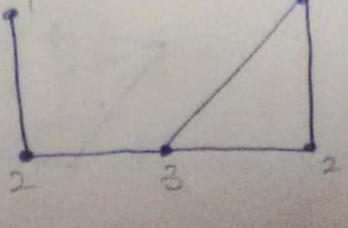
In the first Graph, vertex V_4 ~~is~~ (which is of deg 3) is adjacent to verticies V_5, V_6, V_3 which are of deg 1, 1, 2 resp whereas in the second Graph, vertex C (which is of deg 3) is adjacent to verticies B, D, E which are of deg 2, 2, 1 resp.

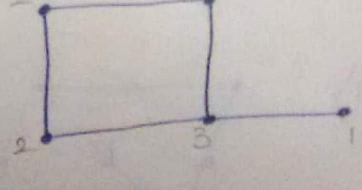
∴ Adjacency of verticies is not preserved.

⇒ Not Isomorphic.

7> Show that no two of the foll 3 graphs are isomorphic:







All the 3 graphs has 5 verticies, ~~and~~ but first graph has 6 edges ~~and~~ whereas 2^{nd} & 3^{rd} graphs has 5 edges.

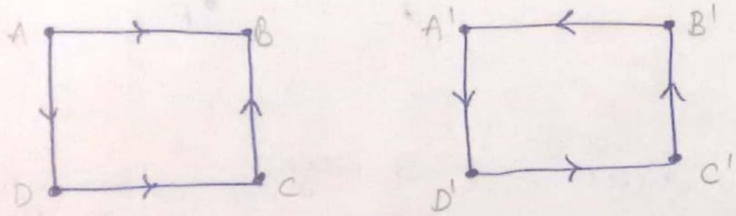
$\therefore$ 1–1 correspondence b/w the edges does not exist for 1^{st} & 2^{nd} graph, as wel as, 1^{st} and 3^{rd} graphs.

In the 2^{nd} Graph, a vertex of deg 3 is adjacent to 3 verticies of deg 2, whereas in the 3^{rd} Graph, vertex of deg 3 is adjacent to 2 verticies of deg 2 and 1 vertex of deg 1.

$\therefore$ Adjacency of verticies is not preserved b/w 2^{nd} & 3^{rd} graphs.

Hence No two Graphs given above are isomorphic.

8)



The two digraphs have same no of verticies ie 4 and same no of directed edges ie 4.

~~Here A ⟺ A'~~

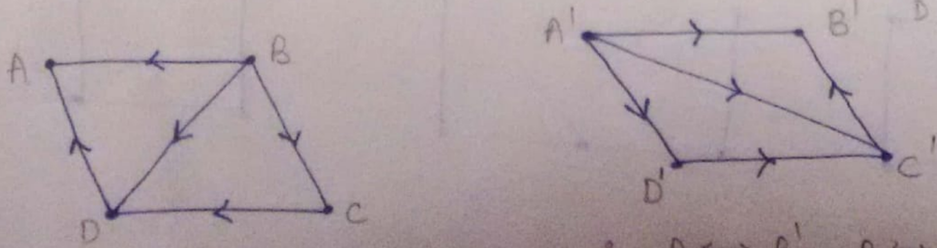
we observe that, vertex A of the 1^{st} graph has 2 out degrees and no indegree, whereas there is no such ~~grap~~ vertex in the 2^{nd} graph.

$\therefore$ there is no 1–1 correspondence b/w the verticies of the 2 digraphs which preserves the direction of edges.

$\Rightarrow$ Not Isomorphic.

A → 2 in, 0 out
B → 0 in, 3 out
C → 1 in, 1 out
D → 2 in, 1 out

9)



A' → 0 in, 3 out
B' → 2 in, 0 out
C' → 2 in, 1 out
D' → 1 in, 1 out

$\therefore A \leftrightarrow B', B \leftrightarrow A', C \leftrightarrow D', D \leftrightarrow C'$

• $A \leftrightarrow B'$, $B \leftrightarrow A'$, $C \leftrightarrow D'$, $D \leftrightarrow C'$

we observe that there exists 1-1 correspondence b/w the vertices of the given digraphs.

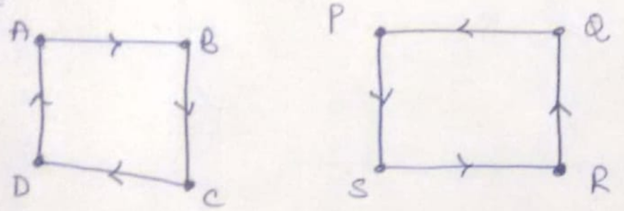
Also $(B, A) \leftrightarrow (A', B')$, $(B, D) \leftrightarrow (A', C')$, $(D, A) \leftrightarrow (C', B')$

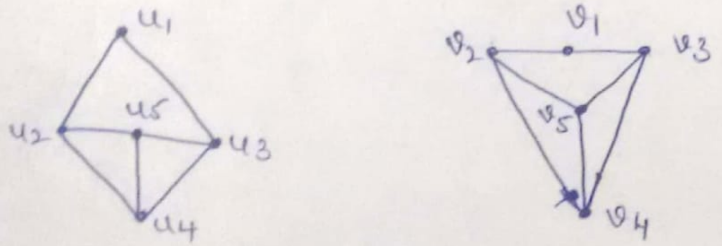
$(B, C) \leftrightarrow (A', D')$, $(C, D) \leftrightarrow (D', C')$.

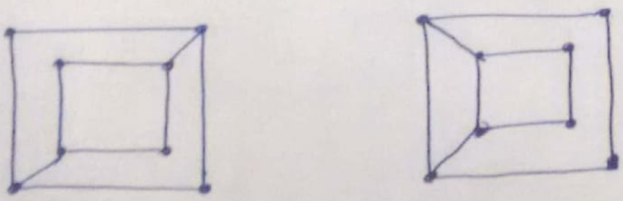
ie 1-1 correspondence b/w the edges of the given digraph exist, preserving the adjacency of verticies including directions of the edges.

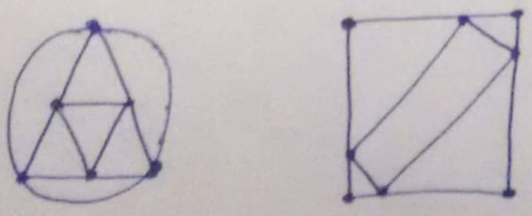
$\Rightarrow$ Isomorphic.

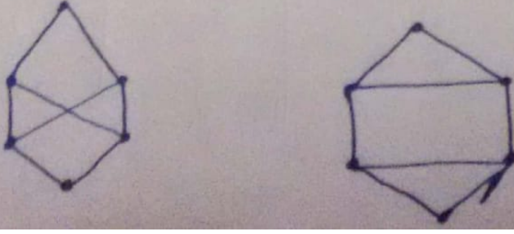
Hw

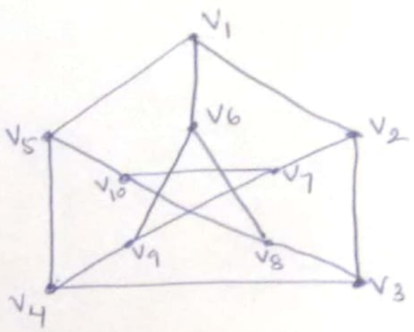
1) 

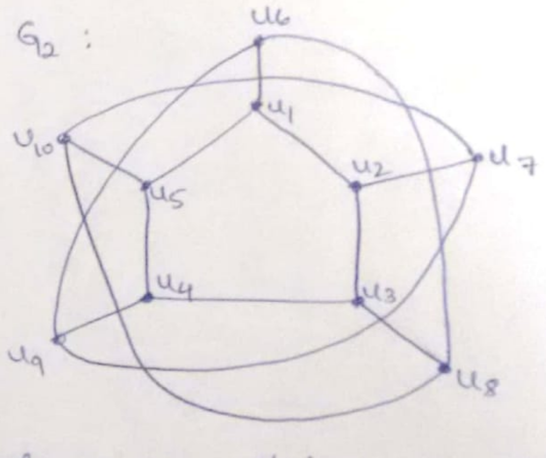
2) 

3) 

4) 

5) 

10) G_1: 

G_2: 

$v_i \leftrightarrow u_i \quad \forall\, i = 1, 2, 3 \ldots 10$

Both G_1 & G_2 have 10 vertices & 15 edges and degree of each vertex in both G_1 and $G_2 = 3$ with this correspondence, $\exists$ 1-1 correspondence b/w ~~vertices &~~ the edges of G_1 & G_2 which preserves adjacency of vertices.

$\therefore G_1 \cong G_2$.

6) 

Subgraphs :-

Given two graphs G and G_1, we say G_1 is a subgraph of G if the following conditions hold :

(i) All the verticies and all the edges of G_1 are in G.

(ii) Each edge of G_1 has the same end verticies in G as in G_1.

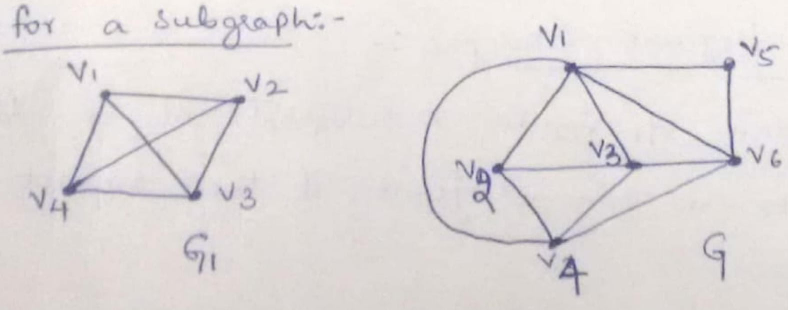
→ Give Ex Here.

Note :- Any graph isomorphic to a subgraph of a graph G is also a subgraph of G.

** Consequences of the defn of a subgraph :-

1) Every graph is a subgraph itself.
2) Every simple graph of n-verticies is a subgraph of the complete graph K_n.
3) If G_1 is a subgraph of a graph G_2 and G_2 is a subgraph of a graph G, then G_1 is a subgraph of G.
4) A single vertex in a graph G is a subgraph of G.
5) A single edge in a graph G, together with its end verticies is a subgraph of G.

Ex :- for a subgraph :-

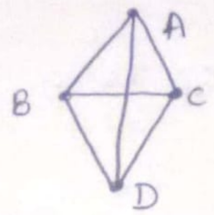


Spanning Subgraph :-

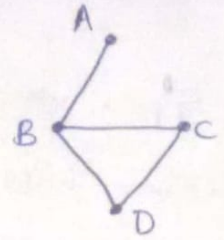
Given a graph $G = (V, E)$, if there is a subgraph $G_1 = (V_1, E_1)$ of G such that $V_1 = V$, then G_1 is called a spanning subgraph of G. [ie all the verticies of G should exist in G_1]

Note :- Every graph is its own spanning subgraph.

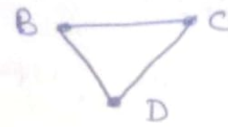
Ex :-



Graph G



Spanning Subgraph



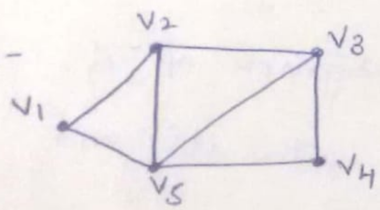
A subgraph, but not spanning Subgraph

Induced Subgraph :-

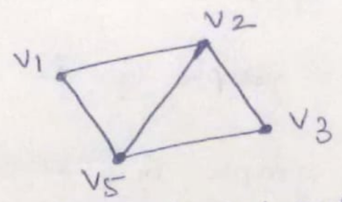
Given a graph $G=(V,E)$, suppose there is a subgraph $G_1=(V_1,E_1)$ of G such that every edge $\{A,B\}$ of G, where $A,B \in V_1$ is an edge of G_1 also. Then G_1 is called an induced subgraph of G and is denoted by $\langle V_1 \rangle$

ie a subgraph G_1 of the graph $G=(V,E)$ is called an induced subgraph if $\exists$ $V_1 \subseteq V$ and contains all the edges from G, denoted by $\langle V_1 \rangle$

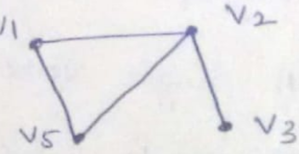
Ex :-



G



induced Subgraph



Not an induced Subgraph

Edge-disjoint and Vertex-disjoint Subgraphs :-

Let G be a graph and G_1, G_2 be 2 subgraphs of G, then

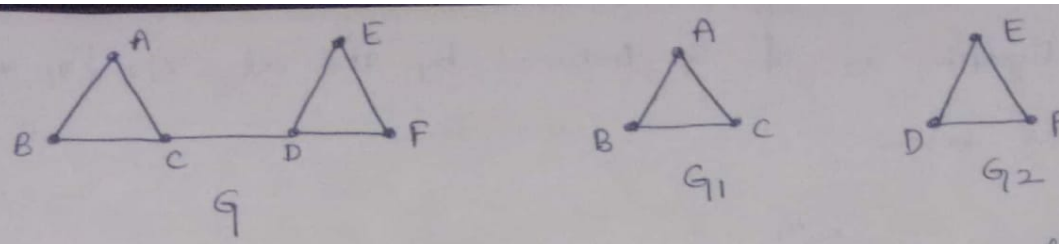
(i) G_1 & G_2 are said to be edge-disjoint if they do not have any common edge

(ii) G_1 & G_2 are said to be vertex-disjoint if they do not have any common edge & any common vertex.

Note :- Subgraphs that have no vertex in common cannot have edges in common.

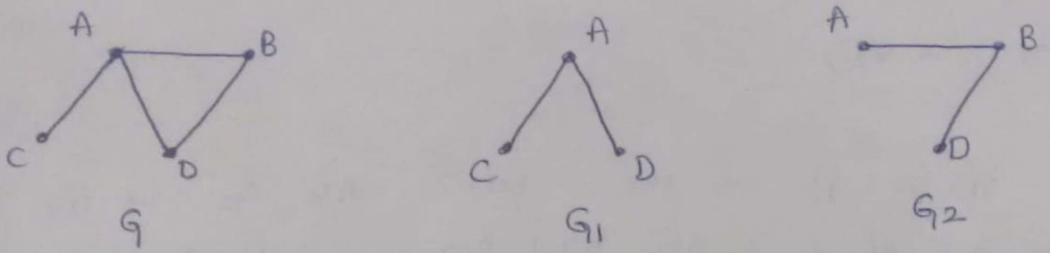
ie two vertex-disjoint subgraphs must be edge-disjoint but the converse is not true.

Ex : 1)



G_1 & G_2 are vertex-disjoint subgraphs, also edge-disjoint subgraphs.

2)



G_1 and G_2 are edge-disjoint, but not vertex-disjoint.

Problems :-

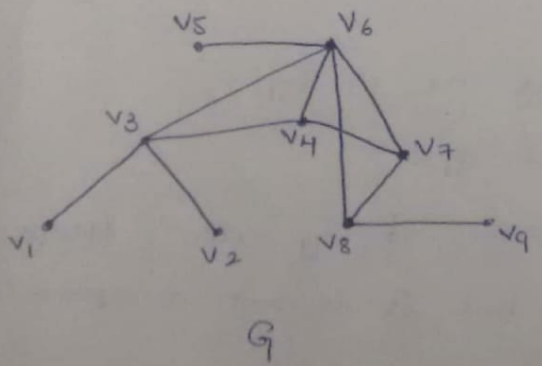
1) Consider the Graph G shown below :

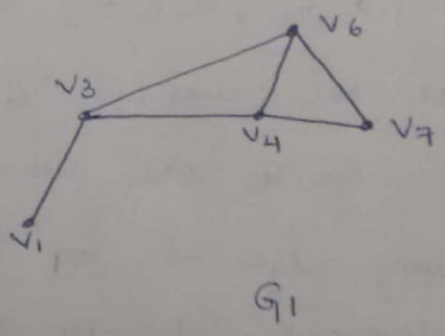
(a) Verify that the graph G_1 is an induced subgraph of G.

Is this a spanning subgraph of G ?

(b) Draw the subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v_6, v_8, v_9\}$.

Soln :-





Soln :- (a) The vertex set of the graph G_1, namely $V_1 = \{v_1, v_3, v_4, v_6, v_7\}$ is a subset of the vertex set $V = \{v_1, v_2 \ldots v_9\}$ of G.

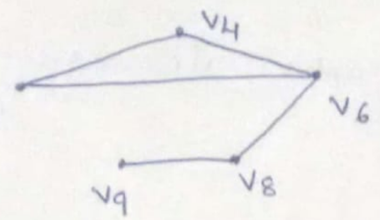
Also, all the edges of G_1 are in G. Each edge in G_1 has the same end vertices in G as in G_1.

$\therefore$ G_1 is a subgraph of G.

Every edge $\{v_i, v_j\}$ of G where $v_i, v_j \in V_1$, is an edge of G_1.

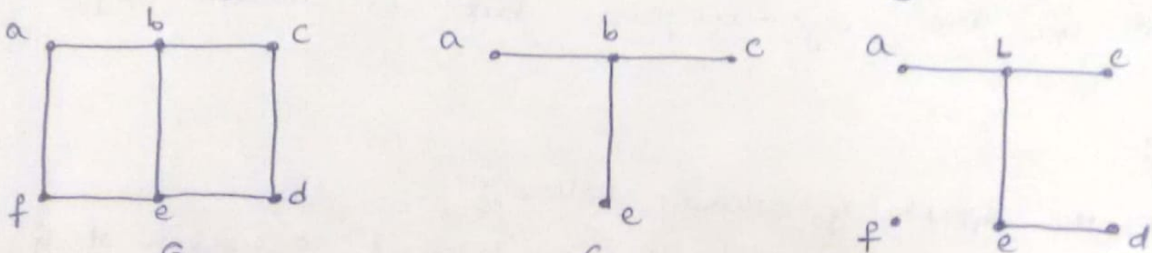
$\therefore$ G_1 is an induced subgraph of G, but not a spanning subgraph of G since $V_1 \neq V$.

(b) Subgraph G_2 of G induced by the set $V_2 = \{v_3, v_4, v$ $v_8, v_9\}$ is as below:



$G_2 = \langle V_2 \rangle$

2) Three graphs G_1, G_2, G_3 are shown below: Are G_2 and G_3 induced subgraphs of G_1? Are they spanning subgraphs?



G_1 G_2 G_3

Soln :- The vertex set of G_2 & G_3, namely $V_2 = \{a, b, c, e\}$ and $V_3 = \{a, b, c, d, e, f\}$ are subsets of the vertex set $V_1 = \{a, b, c, d, e, f\}$ of G_1.

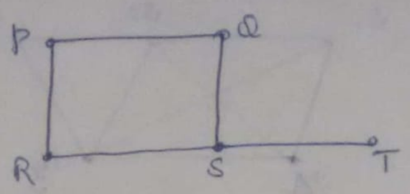
Also all the edges of G_2 & G_3 are in G_1.

$\therefore$ G_2, G_3 are subgraphs of G_1.

Since every edge of G_2 is an edge of G_1 also. Hence G_2 is an induced subgraph of G_1, but it is not a spanning subgraph of G_1 since $V_2 \neq V_1$.

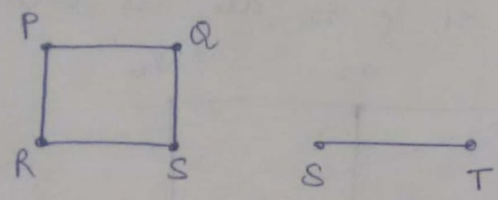
The graph G_3 does not contain all the edges of G_1. Hence G_3 is not an induced subgraph of G_1, but it is a spanning subgraph of G_1 since $V_3 = V_1$.

3) For the graph shown below, find two edge-disjoint subgraphs and two vertex-disjoint subgraphs:

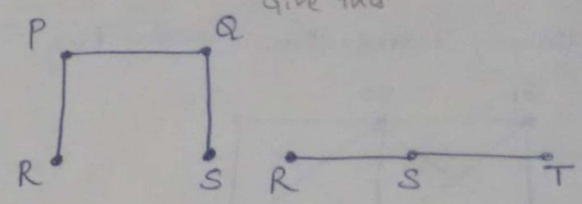


Soln :- Edge-disjoint subgraphs are as below :-

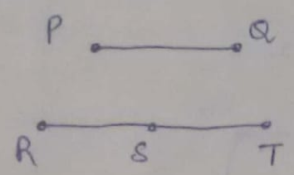
Give this



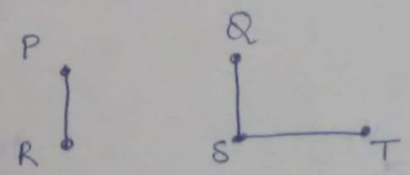
(or)



Vertex-disjoint subgraphs are as below :-



(or)



Operations on Graphs :—

Consider 2 graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, then

(i) The Graph whose vertex set is $V_1 \cup V_2$ & the edge set is $E_1 \cup E_2$ is called the union of G_1 & G_2 and it is denoted by $G_1 \cup G_2$.

Thus $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$

(ii) If $V_1 \cap V_2 \neq \phi$, the graph whose vertex set is $V_1 \cap V_2$ & the edge set is $E_1 \cap E_2$ is called the intersection of G_1 and G_2 & is denoted by $G_1 \cap G_2$.

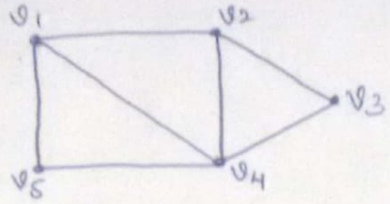
Thus $G_1 \cap G_2 = (V_1 \cap V_2, E_1 \cap E_2)$ if $V_1 \cap V_2 \neq \phi$

(iii) The graph whose vertex set is $V_1 \cup V_2$ & the edge set is $E_1 \Delta E_2$, where $E_1 \Delta E_2$ is the symmetric difference of E_1 & E_2. This graph is called the ring sum of G_1 & G_2 and is denoted by $G_1 \Delta G_2$.

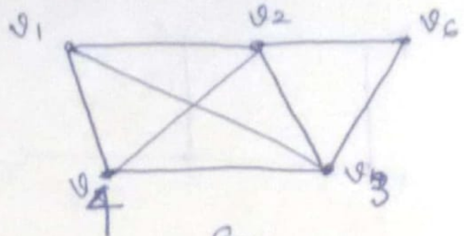
Thus $G_1 \Delta G_2 = (V_1 \cup V_2, E_1 \Delta E_2)$

Note :- $E_1 \Delta E_2$ denotes set of all edges, which are in E_1 (or) E_2, but not in both. ie $E_1 \Delta E_2 = (E_1 \cup E_2) - (E_1 \cap E_2)$

Ex :-

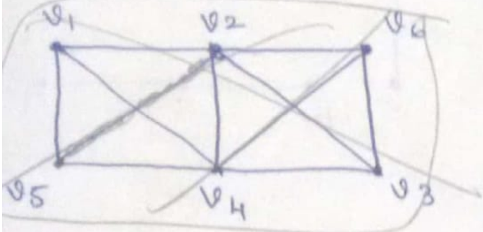


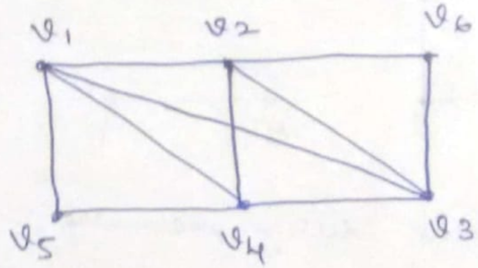
G_1



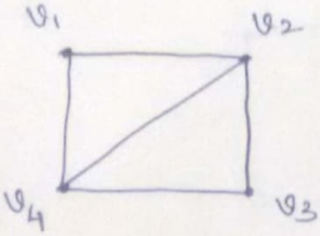
G_2

Union, Intersection and Ring sum of G_1 & G_2 are as below :-

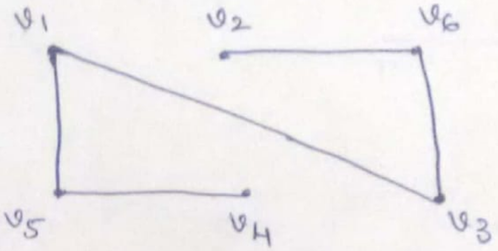




$G_1 \cup G_2$



$G_1 \cap G_2$



$G_1 \Delta G_2$

Decomposition :- The graph G is decomposed into 2 subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2 = \phi$

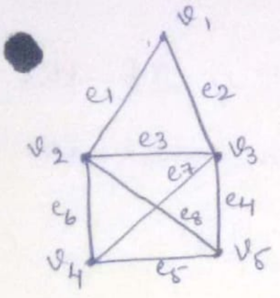
Deletion :- If v is a vertex in a graph G, then G−v denotes the subgraph of G obtained by deleting v and all edges incident on v from G. This subgraph G−v is called vertex-deleted subgraph of G.

Clearly G−v is the subgraph of G induced by $V_1 = V - \{v\}$

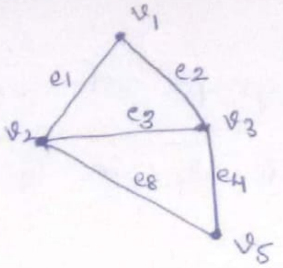
If 'e' is an edge in the graph G, then G−e denotes the subgraph of G obtained by deleting the edge 'e' from G. This graph G−e is referred to as edge-deleted subgraph of G

The deletion of an edge does not alter the no of vertices. Thus an edge deleted subgraph of a graph G is a spanning subgraph.

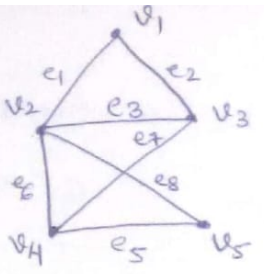
Ex •



G



$G - \{v_4\}$



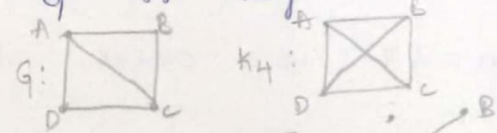
$G - e_4$

Complement of a Simple Graph :-

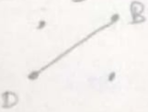
Complement of a simple graph G, denoted by $\overline{G}$, is the graph obtained by deleting those edges which are in G and adding the edges which are not in G. (Give Ex Here)

Note :- 1> G and $\overline{G}$ have the same vertex set.

2> Two verticies are adjacent in G iff they are not adjacent in $\overline{G}$.

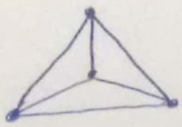


3> $\overline{\overline{G}} = G$ 4> $\overline{G} = K_n - G$. then $\overline{G} =$

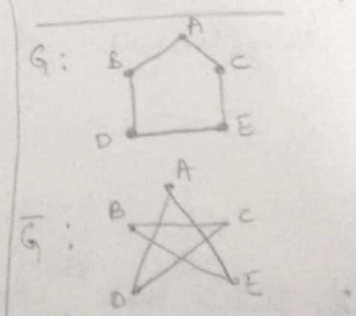


5) complement of K_n is a null graph.

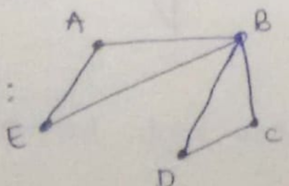
$K_n = G = K_4$ (say)



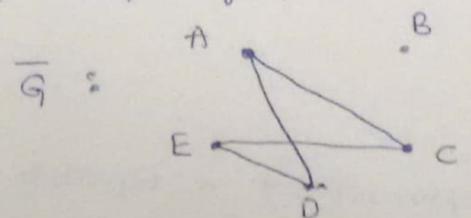
, then $\overline{G} = \{\ \}$ ie



for self-complementary

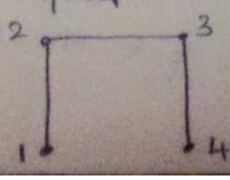


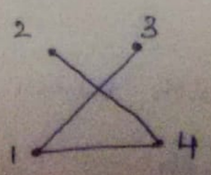
Ex :- for complement of a simple graph

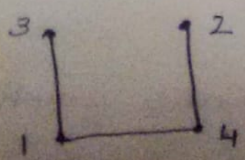


Self-Complementary Graphs :- A simple graph G which is isomorphic to its complement $\overline{G}$ is called a self-complementary Graph.

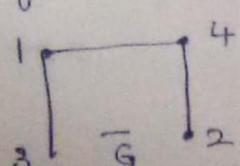
Ex :-







(or)



$G \cong \overline{G}$ $\therefore$ self-complementary.

Examples :-

1) Let G be a simple graph of order 'n'. If the no. of edges in G is 56 & in $\bar{G}$ is 80. What is 'n'?

Soln :- $\bar{G} = K_n - G$

$\therefore$ No. of edges in $\bar{G}$ = no. of edges in K_n − no. of edges in G

i.e. $56 = \frac{1}{2} n(n-1) - 80$

$(56+80)2 = n^2 - n$

i.e. $n^2 - n - 272 = 0$. $\quad = \frac{1 \pm \sqrt{1-4(1)(-272)}}{2}$

$n^2 - 17n + 16n - 272 = 0$ $\quad = \frac{1 \pm \sqrt{1+1088}}{2} = \frac{1 \pm 33}{2}$

$n(n-17) + 16(n-17) = 0$

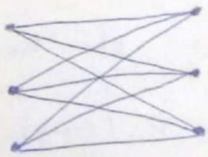
$(n-17)(n+16) = 0$ $\quad = 17, -16$

$n = 17$ (or) $n = -16$ (not possible)

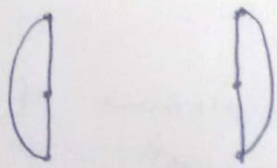
$\therefore$ $n = 17$ i.e. order of $G = 17$.

2) Find the complement of the complete Bipartite graph $K_{3,3}$.

Soln :- we have $K_{3,3}$ given by :-

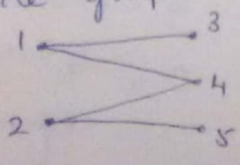


Complement of $K_{3,3}$ is as shown :

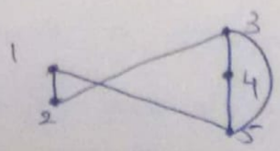


3) Show that the complement of a bipartite graph need not be a bipartite graph.

Soln :-



G



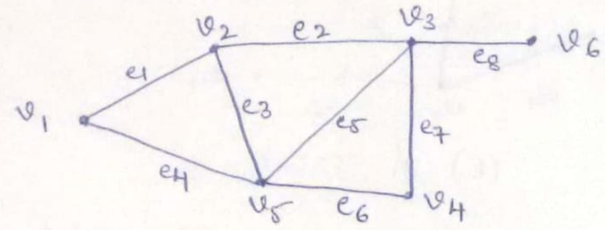
$\bar{G}$

Here G is a bipartite graph with order 5, but its complement $\bar{G}$ is not a bipartite graph.

Walk and their classification :-

Walk :- A finite alternating sequence of verticies and edges is called a walk.

Ex:-



$v_1\, e_1\, v_2\, e_2\, v_3\, e_5\, v_5\, e_6\, v_4\, e_7\, v_3\, e_8\, v_6$ is a walk.

Note :-

1> A walk begins and ends with verticies.

2> An edge in the sequence is incident on the verticies preceeding and following it in the sequence.

3> In a walk, a vertex (or) an edge (or both) can appear more than once.

4> The no of edges present in a walk is called its Length.

5> The vertex with which a walk begins is called the initial vertex (or the origin) of the walk and the vertex with which a walk ends is called the final vertex (or the terminus) of a walk. 6> The initial & final vertex of a walk are together called its terminal verticies. The terminal verticies of a walk need not be distinct (same).

7> Non-terminal verticies of a walk are called its Internal verticies.

8> A walk having 'u' as the initial vertex and 'v' as the final vertex is called a walk from u to v (or) a u-v walk.

9> A walk that begins and ends at the same vertex is called a closed walk.

10> A walk which is not closed is called an open walk.

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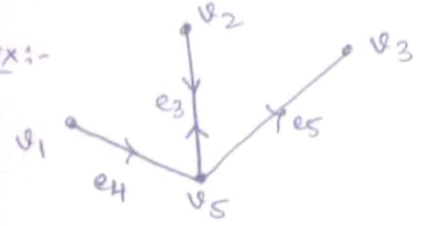
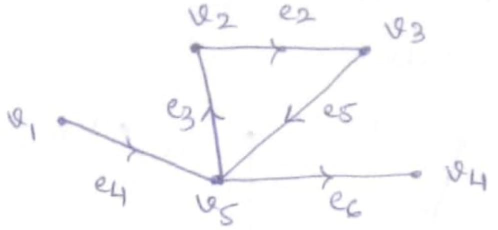
Trail :- An open walk in which no edge appears more than once is called a Trail.

Ex:-



(a) Not a trial



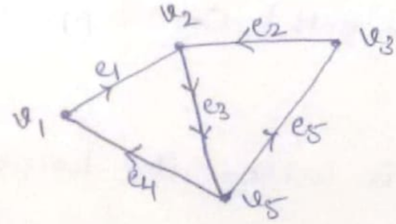
(b) A Trial

(a) $v_1\, e_4\, v_5\, e_3\, v_2\, e_3\, v_5\, e_5\, v_3$ is not a trial since the edge e_3 is repeated twice.

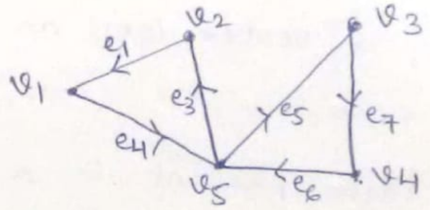
(b) $v_1\, e_4\, v_5\, e_3\, v_2\, e_2\, v_3\, e_5\, v_5\, e_6\, v_4$ is a trial

Circuit :- A closed walk in which no edge appears more than once is called a circuit.

Ex:-



(a) Not a circuit



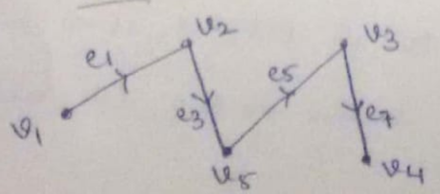
(b) A circuit

(a) $v_1\, e_1\, v_2\, e_3\, v_5\, e_5\, v_3\, e_2\, v_2\, e_3\, v_5\, e_4\, v_1$ is not a circuit since the edge e_3 is repeated twice.

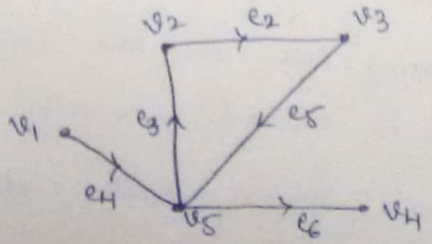
(b) $v_5\, e_3\, v_2\, e_1\, v_1\, e_4\, v_5\, e_5\, v_3\, e_7\, v_4\, e_6\, v_5$ is a circuit.

Path :- A trial in which no vertex appears more than once is called a Path.

Ex:-



(a) path



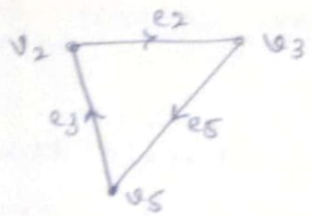
(b) Not a path.

(a) $v_1\, e_1\, v_2\, e_3\, v_5\, e_5\, v_3\, e_7\, v_4$ is a path.

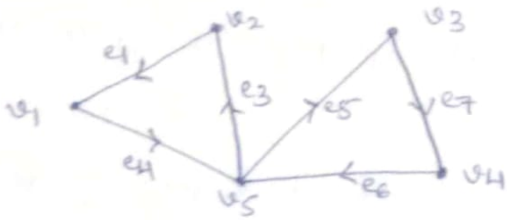
(b) $v_1\, e_4\, v_5\, e_3\, v_2\, e_2\, v_3\, e_5\, v_5\, e_6\, v_4$ is not a path since the vertex v_5 is repeated twice.

Cycle :- A circuit in which the terminal vertex does not appear as an ~~inter vertex~~ end internal vertex and no internal vertex is repeated is called a cycle.

Ex :-



(a) cycle



(b) Not a cycle

(a) $v_2 e_2 v_3 e_5 v_5 e_3 v_2$ is a cycle.

(b) $v_2 e_1 v_1 e_4 v_5 e_5 v_3 e_7 v_4 e_6 v_5 e_3 v_2$ is not a cycle since the internal vertex v_5 is repeated twice.

(OR)

$\underline{v_5} e_3 v_2 e_1 v_1 e_4 \underline{v_5} e_5 v_3 e_7 v_4 e_6 \underline{v_5}$ is not a cycle since the terminal vertex (v_5) appears as an internal vertex

Note / observations :-

1) A walk can be open (or) closed. In a walk, a vertex and/or an edge can appear more than once.

2) A trial is an open walk in which a vertex can appear more than once but an edge cannot appear more than once.

3) A circuit is a closed walk in which a vertex can appear more than once but an edge cannot appear more than once.

4) A path is an open walk in which neither a vertex nor an edge can appear more than once.

5) Every path is a ~~trial~~ trail; but a ~~trial~~ trail need not be a path.

6) A cycle is a closed walk in which neither a vertex nor an edge can appear more than once.

7) Every cycle is a circuit; but a circuit need not be a cycle.

8) If a cycle contains only one edge, it has to be a loop.

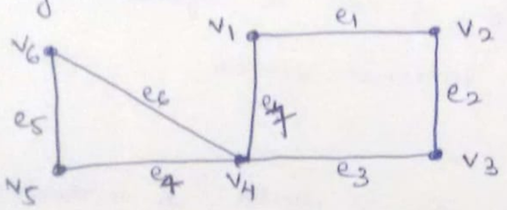
9) Two parallel edges (when they occur) form a cycle.

10) In a Simple graph, a cycle must have atleast 3 edges.

Note :- In case of digraphs, the walks, ~~trails~~ trails, circuits, paths and cycles become directed walks, directed trails, directed circuits, directed paths and directed cycles.

Problems :-

1) For the graph given below, indicate the nature of the following walks :



(i) $v_1\, e_1\, v_2\, e_2\, v_3\, e_2\, v_2$ → Open walk which is not a trial since edge e_2 is repeated.

(ii) $\underline{v_4}\, e_7\, v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, \underline{v_4}\, e_4\, v_5$ → Trial which is not a path since vertex v_4 is repeated.

(iii) $v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_4\, v_5$ → Trial which is a path.

(iv) $v_1\, e_1\, v_2\, e_2\, v_3\, e_3\, v_4\, e_7\, v_1$ → closed walk which is a cycle.

(v) $v_6\, e_5\, v_5\, e_4\, \underline{v_4}\, e_3\, v_3\, e_2\, v_2\, e_1\, v_1\, e_7\, \underline{v_4}\, e_6\, v_6$ → Closed walk which is a circuit, but not a cycle, since vertex v_4 is repeated.

2) In the graph shown below, how many paths are there from a to h ? How many of these paths have a length 5 ?

P.T.O

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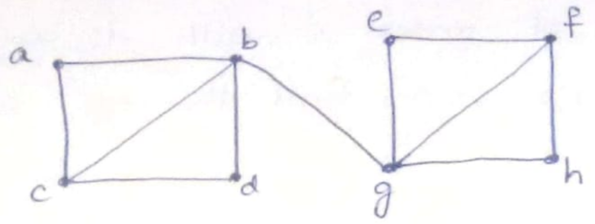


The following are the paths from a to h :—

abgh , acbgh , acdbgh ,

abgefh , abgfh , acbgefh , acbgfh,

acdbgefh , acdbgfh , whose lengths are respectively

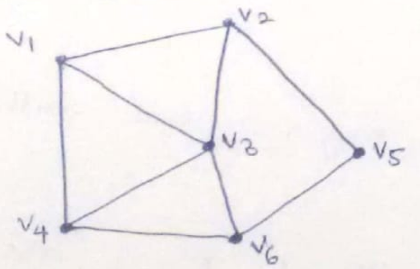
3 , 4 , 5 , 5 , 4 , 6 , 5 , 7 , 6 .

Thus the total no of paths from a to h are Nine

and the paths of length 5 are three in number.

3) Determine the no of different paths of length 2 in the graph shown below:

Q8



Soln :- The no of paths of length 2 that pass through the vertex v_1 is the no of pairs of edges incident on v_1 . $= {}^3C_2$ ($\because$ 3 edges are incident on v_1)

Since 3 edges are incident on v_1, this number $= {}^3C_2 = 3$.

$|||^{ly}$ the no of paths of length 2 that pass through the verticies v_2, v_3, v_4, v_5, v_6 resp are :

${}^3C_2 = 3$, ${}^4C_2 = 6$, ${}^3C_2 = 3$, ${}^2C_2 = 1$, ${}^3C_2 = 3$

Total no of paths of length 2 in the above graph is

$3 + 3 + 6 + 3 + 1 + 3 = 19$.

P.T.O.

4) If G is a simple graph of order n with d_i as the degree of a vertex v_i for $i = 1, 2, \ldots n$, find the no of paths of length 2 in G.

Soln :- we have $\deg(v_i) = d_i$, the no of edges incident on v_i is exactly d_i.

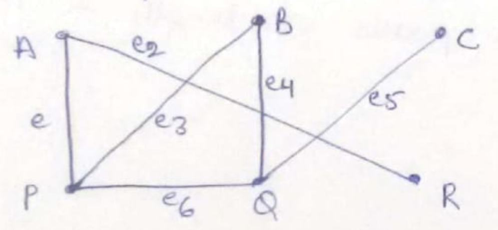
In this, every two edges give a path of length 2 which contains v_i.

combination

$\therefore \exists\ C(d_i, 2)$ paths containing v_i $\forall\ i = 1, 2 \ldots n$

$\therefore$ total no of paths of length 2 in G is $\sum_{i=1}^{n} C(d_i, 2)$.

5) Find all the cycles in the graph shown below:



Soln :- It is clear that, no cycle begin & end with the verticies A, C and R.

The cycles beginning & ending with B, P, Q are:

$B e_3 P e_6 Q e_4 B$, $P e_6 Q e_4 B e_3 P$, $Q e_4 B e_3 P e_6 Q$.

But all of these represent ~~one & only the~~ same cycle for the given graph.

Thus there is only one cycle

6) If G is a bipartite graph, show that G has no cycle of odd length.

Soln :- Since G is Bipartite, the vertex set V is partitioned into two disjoint sets V_1 & V_2.

Let $v_0 v_1 v_2 \ldots v_m v_0$ be a cycle in G

Assume that v_0 is in V_1, v_1 is in V_2, v_2 is in V_1, v_3 is in V_2 and so on.

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Thus the verticies in the cycle belong to V_1, V_2 altern- 22
-ately.

Since the terminal vertex of the cycle is v_0 & it is in V_1, the no of edges that belong to the cycle cannot be 3 or 5 or 7 or any odd number.

Thus G has no cycle of odd length.

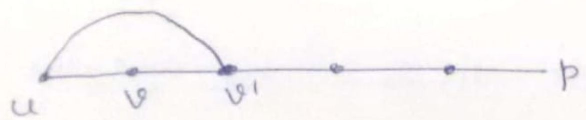
Ex:- 1)

2)

only $v_0 v_1 v_2 v_3 v_0$ is a cycle, of even length not the whole graph.

7) If G is a simple graph with no cycles, prove that G has atleast one pendant vertex. ~~QP~~

Soln :- Consider a path p in G which has a maximum no of verticies. Let 'u' be an end vertex of p, then every neighbour of u belongs to p.



If u has atleast 2 neighbours say v & v', then v & v' both belong to p and then the edges (u, v) (v, v') (v', u) form a cycle. This is not possible, because G has no cycles.

Hence u can have only one neighbour.

$\Rightarrow$ u is a pendant vertex.

Thus G has atleast one pendant vertex.

8) Prove the following :-

(i) A path with n verticies is of length (n−1)

(ii) If a cycle has n-verticies, then it has n edges.

(iii) The degree of every vertex in a cycle is two.

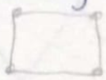
Soln :-

(i) In a path, every vertex except the last vertex followed by precisely one edge.

$\therefore$ If a path has n-verticees, it must have $n-1$ edges.

Hence its length is $n-1$.

(ii) In a cycle, every vertex is followed by precisely one edge.



$\therefore$ If a cycle has n versticies, it must have n edges.

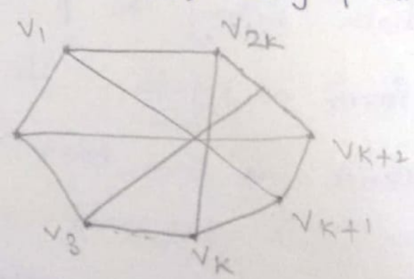
(iii) In a cycle, exactly two edges are incident on every vertex [one edge through which we enter the vertex & one edge through which we leave the vertex]

$\therefore$ degree of every vertex in a cycle is two.

9) Show that, for any +ve integer $k \geq 2$, $\exists$ a simple cubic graph of order $2k$.

Soln:- Consider a set of points $v_1, v_2 \ldots v_{2k}$ and the cycle made up of the foll $2k$ edges:

$\{v_1, v_2\} \; \{v_2, v_3\} \ldots \{v_{k-1}, v_k\} \; \{v_k, v_{k+1}\} \ldots \{v_{2k-1}, v_{2k}\} \; \{v_{2k}, v_1\}$



To this cycle, add k-edges

$\{v_1, v_{k+1}\} \; \{v_2, v_{k+2}\} \ldots \{v_k, v_{2k}\}$

Thus the ~~Generating~~ resulting graph is simple and contains $2k + k = 3k$ edges.

In this graph, exactly 3 edges are incident on every vertex v_i, namely the edges $\{v_{i-1}, v_i\}$ & $\{v_i, v_{i+1}\}$ which belong to the original cycle and the edge $\{v_i, v_{k+i}\}$ which has been added to the original cycle. Thus the simple graph constructed is of order $2k$ in which the deg of every vertex is 3. Thus the desired graph exists.

10) Prove that in a graph, there is a u-v trial iff there is a u-v path. (23)

Soln:- <u>Necessary</u>:

Given :- There is a u-v trial

To prove : ~~There is a u-v path~~. u-v trial is a path.

Let u-v be trial in G.

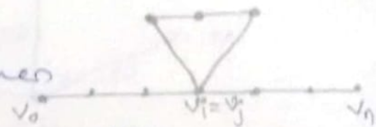
Among these trials, choose a trial of minimum length and let this be $v_0 v_1 v_2 \dots v_n \longrightarrow (1)$

where $v_0 = u$ & $v_n = v$ & the edges ~~of~~ joined between.

~~case (i) If there is only one u-v trial, then it will be the only trial of minimum length.~~

case (ii) If in ①, no vertex repeats, then it is a path from u to v & hence the proof.

<u>Case (iii)</u> If a vertex repeats say $v_i = v_j$ then,



① $\Rightarrow$ $v_0 v_1 v_2 \dots v_{i-1}\ v_i\ v_{i+1} \dots v_{j-1}\ v_j\ v_{j+1} \dots v_n$.

$v_0 v_1 v_2 \dots v_{i-1}\ v_i\ v_{j+1} \dots v_n \rightarrow (2)$ is the trial got by skipping the verticies $v_{i+1}, v_{i+2} \dots v_{j-1}\ v_j$ together with all the edges preceeding them.

Thus ② is the trial which is shorter than that of ①.

This is a contradiction since we have assumed ① to be of shorter length.

Thus a u-v trial with minimum length is a path.

u-v trial is a u-v path

<u>Sufficiency</u>:

Given : there is a u-v path

To prove : ~~there is a~~ u-v path is a trial.

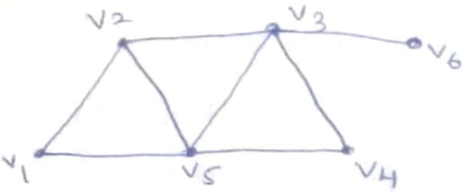
WKT Every path is a trial.

Hence if there is a u-v path, then it is a u-v trial.

Hw

1) for the Graph shown below, find the nature of the foll walks:

(a) $v_1 v_2 v_5 v_3 v_4 v_5 v_1$ (b) $v_1 v_2 v_3 v_5 v_1$



2) for the graph shown below, determine:

(i) a walk from v_2 to v_4 which is not a trail.

(ii) a $v_2 - v_4$ trail which is not a path.

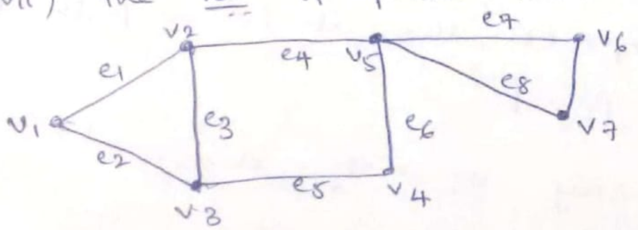
(iii) a path from v_2 to v_4.

(iv) a closed walk from v_2 to v_2 which is not a circuit.

(v) a circuit from v_2 to v_2 which is not a cycle.

(vi) a cycle from v_2 to v_2

(vii) the no of paths from v_2 to v_6.



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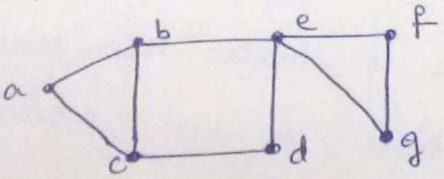
11) In the graph shown below, determine i) a walk from b to d that is not a trail; ii) b-d trail that is not a path iii) a path from b to d. iv) a closed walk from b to b that is not a circuit v) a circuit from b to b that is not a cycle vi) a cycle from b to b.

QP



Soln:. i) b-c-a-b-c-d is a walk, but not a trail since the edge {b, c} is repeated.

ii) b-a-c-b-e-d is an open walk which is a trail, but not a path since vertex b is repeated.

iii) b-c-d is a path from b to d.

iv) b-e-f-g-e-b is a closed walk from b to b, which is not a circuit since the edge {b, e} is repeated.

v) b-c-d-e-g-f-e-b is a circuit from b to b, which is not a cycle since the internal vertex e is repeated.

vi) b-a-c-b is a cycle from b to b.

Connected and Disconnected Graphs :-

A Graph G is said to be

(i) Connected if there is atleast one path b/w every two distinct verticies in G. (OR)**

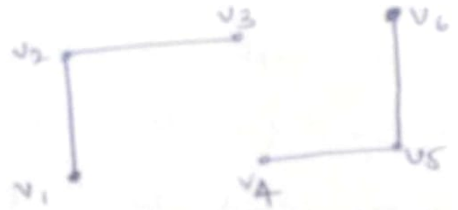
(ii) disconnected if G has atleast one pair of distinct verticies b/w which there is no path.

**(OR)

Two verticies in a graph G are said to be connected if there is atleast one path from one vertex to the other

Ex :-

Connected



disconnected

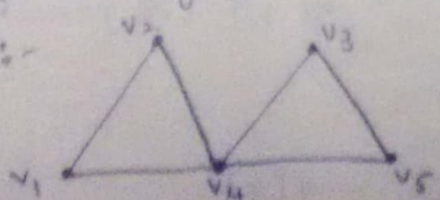
Component :- Every non-trivial graph G consists of one or more connected graphs. Each such connected graph is a subgraph of G and is called a component of G.

Note : 1) A connected graph has only one component, where as a disconnected graph has two (or) more components.

2) The No of components of a graph G is denoted by K(G).
In above Ex, K(G) = 1 & K(G) = 2 resp.

3) If u & v are 2 verticies in a connected graph, then the length of the shortest path [ie the path containing least no of edges] is called the distance b/w u & v.

Ex :-



Distance from v_1 to v_3 is 2

" ——— " v_2 to v_5 is 2

Theorem 1 :- If a graph has exactly two verticies of odd degree, then there must be a path connecting these verticies.

Proof :- Let v_1 & v_2 be two verticies of a graph G with odd degrees.

Suppose there is no path b/w v_1 & v_2, then the graph G is disconnected and v_1 & v_2 belong to two different components say H_1 & H_2 of G.

$\Rightarrow$ Each of H_1 & H_2 contains only one vertex of odd degree.

This is a contradiction because H_1 & H_2 are graphs and WKT in a graph, the no of verticies of odd degrees is always Even [by Thm in Hand shaking property].

Hence there must be a path connecting v_1 & v_2.

Hence the proof.

Theorem 2 :- A simple graph with 'n' verticies & 'k' components can have atmost $(n-k)(n-k+1)/2$ edges.

Proof :- Given G is a simple graph.

$\Rightarrow$ each of the components of G is a simple graph.

Let n_1 be the no of verticies in the 1st component

n_2 " ——— " ——— 2nd "

$\vdots$

n_k " ——— " ——— kth "

$\therefore$ max no of edges in the i^{th} component is $\frac{1}{2} n_i(n_i-1)$

Let N be the max no of edges.

$$\therefore \quad N \le \frac{1}{2} \sum_{i=1}^{k} n_i(n_i-1) \le \frac{1}{2}\left[\sum_{i=1}^{k} n_i^2 - \sum_{i=1}^{k} n_i\right] \rightarrow (1)$$

(since G has k components)

Also $n_1+n_2+\dots+n_k = n$

$$\Rightarrow \sum_{i=1}^{k} n_i = n \longrightarrow (2)$$

Now, $(n_1-1)+(n_2-1)+\dots+(n_k-1) = (n_1+n_2+\dots+n_k)-(1+1+\dots \text{ k times})$

$= n-k$.

Squaring on b.s

$$(n_1-1)^2+(n_2-1)^2+\dots+(n_k-1)^2+S = (n-k)^2 \longrightarrow (3)$$

where S is the sum of products of the form $2(n_i-1)(n_j-1)$,

$i=1,2\dots k$, $j=1,2\dots k$, $i \neq j$. we have $S \geq 0$

Since each of $n_1, n_2, \dots n_k \geq 1$, $(n-k)^2$

$\therefore$ (3) $\Rightarrow (n_1-1)^2+(n_2-1)^2+\dots+(n_k-1)^2 \le (n-k)^2$

$\Rightarrow n_1^2+n_2^2+\dots+n_k^2+(1+1+\dots \text{ k times}) - 2(n_1+n_2+\dots+n_k) \le (n-k)^2$

$\Rightarrow n_1^2+n_2^2+\dots+n_k^2+k-2n \le (n-k)^2$

$\Rightarrow n_1^2+n_2^2+\dots+n_k^2 \le (n-k)^2-k+2n = n^2+k^2-2nk-k+2n$

$\Rightarrow n_1^2+n_2^2+\dots+n_k^2 \le n^2+\{k(k-2n)-1(k-2n)\}$

$\le n^2+(k-1)(k-2n)$

(or) $\displaystyle\sum_{i=1}^{k} n_i^2 \le n^2-(k-1)(2n-k) \longrightarrow (4)$

Sub (2) & (4) in (1),

$N \le \frac{1}{2}\left[n^2-(k-1)(2n-k)-n\right] = \frac{1}{2}\left[n^2-2nk+k^2+2n-k-n\right]$

$N \le \frac{1}{2}\left[(n-k)^2+(n-k)\right] = \frac{1}{2}(n-k)(n-k+1)$.

$N \le \frac{1}{2}(n-k)(n-k+1)$

ie $N \le \frac{1}{2}(n-k)(n-k+1)$.

Thus the no of edges in G cannot exceed $\frac{1}{2}(n-k)(n-k+1)$.

Proof.

Theorem 3 :- A connected graph with 'n' verticies has atleast $(n-1)$ edges

Proof:: Since the graph is connected, $n \geq 2$.

If 'm' denote the no of edges, we have to prove $m \geq n-1$.

The proof is by mathematical Induction.

when $n = 2$

$\Rightarrow$ there are two verticies in the graph and since the graph is connected, $\exists$ atleast one edge joining these verticies

ie $m \geq 1 = (2-1) = (n-1)$

$\Rightarrow m \geq (n-1)$.

This verifies the result for $n=2$.

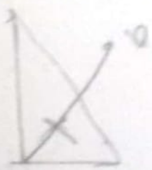
Assume that the result is true for $n=k$, with $k \geq 2$.

ie $m_k \geq k-1$ where m_k is the no of edges in G_k

To prove: The result is true for $n = k+1$

Let G_{k+1} be the graph with $(k+1)$ no of verticies.

Let G_k be the graph obtained by deleting an edge from G_{k+1} for which v is an end vertex.



Thus G_k is a connected graph with k- verticies.

Let m_k be the no of edges in G_k, then

from the assumption made above,

$$m_k \geq k-1$$

$$\Rightarrow m_k + 1 \geq (k-1)+1 \quad \text{ie} \quad m_k + 1 \geq (k+1) - 1$$

But $m_k + 1$ is the no of edges in G_{k+1} & $k+1$ is the no of verticies in G_{k+1}.

Thus the result $m \geq (n-1)$ holds for $n = k+1$.

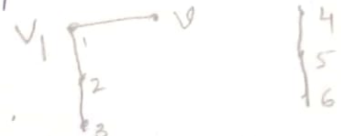
Hence the proof.

Theorem 4 :– A Graph G is disconnected iff its vertex set V can be partitioned into 2 non-empty disjoint subsets V_1 & V_2 such that $\exists$ no edge in G whose one end vertex is in V_1 and the other is in V_2.

Proof :- Necessary :

Given : G is disconnected.

Consider a vertex v in G. Let V_1 be the set of all vertices of G that are connected to v. Since G is disconnected, V_1 does not include all vertices of G



$\Rightarrow$ V_1 is a proper subset of V.

Let $V_2 = V - V_1$ then $V_1 \cap V_2 = \phi$, $V = V_1 \cup V_2$ and no vertex in V_1 is connected to any vertex in V_2.

Hence V_1 & V_2 form a partition of V of desired type.

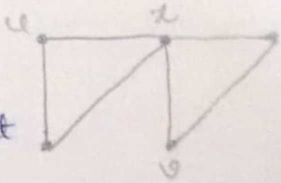
Sufficiency :

To prove : G is disconnected.

Consider 2 arbitrary vertices v & u in G such that $v \in V_1$ & $u \in V_2$ $\Rightarrow$ $\exists$ no path b/w v & u (given)

Hence G is not connected.

EX 1 :- If G is a simple graph with 'n' vertices in which the deg of every vertex is atleast $\frac{n-1}{2}$, P.T G is connected.



Soln :- Take any 2 vertices u & v of G. then they are either adjacent (or) not adjacent

If they are adjacent then G is connected.

otherwise, each vertex has atleast $\frac{n-1}{2}$ neighbours since the degree at each vertex is $\frac{n-1}{2}$.

$\therefore$ u & v taken together have atleast $(n-1)$ neighbours.

(Since G has total of n-vertices, the total no of neighbours which u & v together can have is only $n-2$)

~~or atleast one vertex say~~ Let x be a neighbour of both u & v.

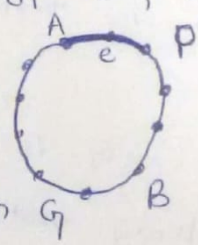
$\Rightarrow$ there is an edge b/w u & x & b/w x & v.

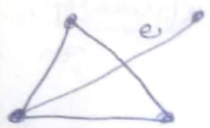
$\Rightarrow$ there is a path b/w u & v.

$\Rightarrow$ G is connected.

Ex 2 :- Prove that a connected graph G remains connected after removing an edge 'e' from G iff e is part of some cycle of G.

Soln :- Suppose 'e' is a part of some cycle 'C' in G then the removal of 'e' from G will not affect the connectivity of G since after the removal of 'e', the end verticies of 'e' remain connected.





Suppose 'e' is not a part of any cycle, then the removal of 'e' from G disconnects these end points.

Thus $G-e$ is a disconnected graph.

Hence the proof.

E[illegible]er Circuits and Euler Trails :-

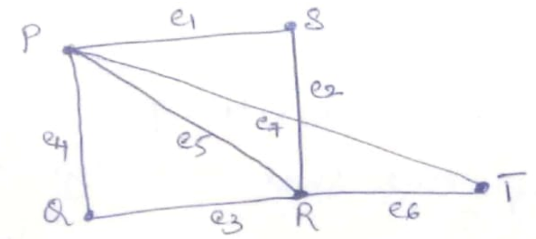
Euler Circuit :- A circuit in a connected graph G which contains all the edges of G is called an Euler circuit

Euler Trail :- A Trail in a connected graph G which contains all the edges of G is called an Euler Trail.

A Graph which contains Euler circuit is called Euler graph.

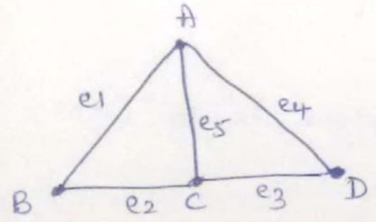
A Graph which contains Euler trail is called Semi-Euler graph

EX :- 1)



The closed walk $P e_1 S e_2 R e_3 Q e_4 P e_5 R e_6 T e_7 P$ is an Euler circuit. ∴ This graph is an Euler graph.

2)



The trail $A e_1 B e_2 C e_3 D e_4 A e_5 C$ is an Euler trail.

∴ the graph is a Semi-Euler graph.

This graph is not Euler graph since it ~~is not a~~ does not contain any Euler circuit.

Just say [This graph is not Euler graph. B'coz every sequence of edges which includes all edges starts & ends with the same vertex & which will contain atleast one repeated edge]

Theorem 1 :- A connected graph G has an Euler circuit if & only if all verticies of G are of Even degree.

QP

Proof :- Necessary :

Given :- A connected graph G has an Euler circuit

T.P.T : All verticies of G are of Even degree.

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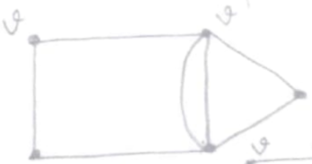
while tracing this circuit, every time the circuit meets a vertex 'v' it goes through two edges incident on 'v'

This is true for all verticies that belong to the circuit.

Since the circuit is an Euler circuit, it contains all the edges of G & meets all the verticies atleast once.

Thus the degree of every vertex is a multiple of two.

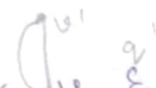
⇒ Every vertex is of Even degree.



Sufficiency :-

Given :- All verticies of G are of Even degree.

T.P.T :- G has an Euler circuit.



Construct a circuit starting at an arbitraly vertex v & going through the edges of G such that no edge is repeated

Since every vertex is of Even degree, we can depart from every vertex we enter and the tracing cannot stop at any vertex other than v. Thus a circuit 'q' is obtained having 'v' as the initial & final vertex.

If this circuit contains all the edges in G, then the circuit 'q' is an Euler circuit.

Otherwise consider the subgraph H obtained by removing all the edges that belong to q from G. The degrees of verticies in this subgraph are also Even.

Since G is connected, the circuit 'q' & the subgraph 'H' must have atleast one vertex in common (say v').

Starting from v', construct a circuit q' in H.

The two circuits q & q' together constitutes a circuit which starts & ends at 'v', and has more edges than 'q'.

If this circuit contains all the edges in G, then the circuit is an Euler circuit, otherwise repeat the process until we get a circuit that starts from v & ends at v containing all the edges in G. In this way, we obtain an Euler circuit in G.

Hence the proof.

Theorem 2 :- A connected graph G has an Euler circuit iff G can be decomposed into edge disjoint cycles.

Proof :-

Necessary : Given :- G has an Euler circuit

T.P.T : G can be decomposed into edge disjoint cycles.

G has an Euler circuit $\Rightarrow$ every vertex in G is of Even degree [by Thm 1]

Consider a vertex v_1 in G. Since v_1 is of even degree, there is atleast 2 edges incident on v_1. choose one of these edges. Let v_2 be the other end vertex of this (chosen) edge.

Since v_2 is also of even degree, $\Rightarrow$ there must be atleast one other edge incident on v_2. choose one of such edges and let v_3 be the other end vertex.

Proceeding like this, we get a vertex which has been traversed, thus forming a cycle C_1.

Remove C_1 from G. All the verticies in the resulting graph must also be of even degree, and in this graph, construct a cycle C_2 as previously done. Remove the cycle C_2 and proceed as above. The process ends when no edges are left.

Thus we get a sequence of cycles whose union is G and intersection is a null graph. Hence G has been decomposed into edge-disjoint cycles.

Sufficiency :

Given :- G can be decomposed (partitioned) into edge-disjoint cycles.

T.P.T :- G has an Euler circuit.

~~Since~~ the degree of every vertex in a cycle is two,

$\Rightarrow$ Every vertex in G is of even degree.

$\Rightarrow$ G has Euler circuit [by Thm 1]

Problems :-

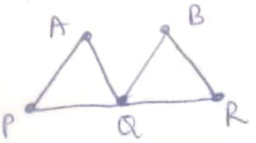
1) for what value of 'n', does K_n has an Euler trail, but not an Euler circuit? find all positive integers $n (\geq 2)$ for which the complete graph K_n contains an Euler circuit.

Soln:- For $n=2$, the graph K_n contains exactly one edge. This edge together with its end verticies constitutes an Euler trail. In this case, K_n cannot have an Euler circuit.

for $n \geq 3$, K_n contains an Euler circuit iff $n-1$ [ie deg of every vertex in K_n] is Even ; ie iff n is odd.

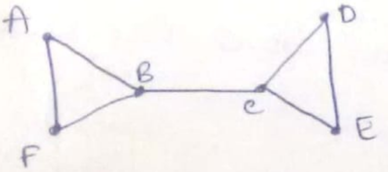
2) find the Euler Circuit in the following :-

(i)



Euler circuit → A P Q R B Q A.

(ii)

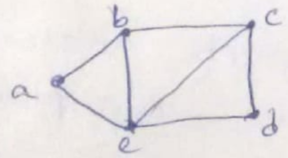


NO Euler circuit

An Euler trail exists

ie B A F B C E D C

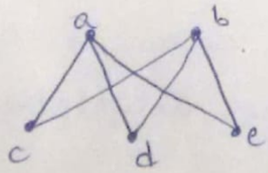
(iii)



No Euler Circuit

Euler trail → b a e c b e d c.

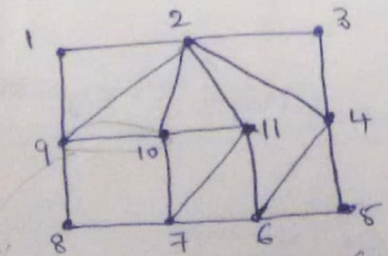
(iv)



No Euler Circuit

Euler trail → a c b e a d b

(v) QP



Euler circuit ↓

1 2 3 4 5 6 7 8 9 10 7 11 6 4 2 11 10 2 9 1

Remove the edge {9,10} & find an Euler trail.

Ans:- 10 7 11 6 4 2 11 10 2 9 1 2 3 4 5 6 7 8 9

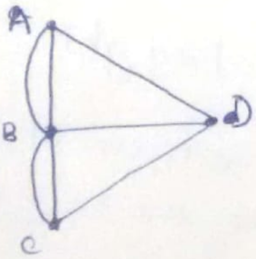
Konigsberg Bridge Problem :- (a city in Europe)

QP The city of Konigsberg was divided into 4 parts by Pregel river and seven bridges connected these regions.

The citizens of the city tried to find a way to walk along the city so as to cross each bridge exactly once & then return to the starting point. This problem is now known as Konigsberg Bridge problem.

Euler analyzed the problem with the help of graph and gave the solution.

Euler represented the 4 regions by 4 verticies & the 7 bridges by edges as shown below:



WKT the existence of Euler circuit depends on the degrees of verticies of the graph.

from the fig,

$\deg(A) = 3 = \deg(C) = \deg(D)$; $\deg(B) = 5$

All the 4 verticies are of odd degree.

Also a graph G has Euler circuit iff every vertex in G has even degree.

Thus for the above graph, Euler circuit does not exist.

$\Rightarrow$ it is not possible to walk over each of the seven bridges exactly once & return to the starting point.

Thus the Soln to the Konigsberg Bridge problem is negative.

Trees

→ Definitions, properties and Examples.

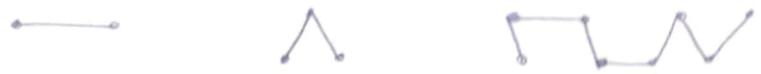
→ Roted trees.
→ Preorder and Postorder Traversal.
→ Trees Sorting – Merge Sort.
→ Spanning trees.
→ weighted trees.

→ Prefix codes – optimal Prefix code.

TREES

<u>Defn</u> :- A graph G is said to be a Tree if it is connected and has no cycles [$\Rightarrow$ Simple graph $\Rightarrow$ No loops & $\|^{el}$ edges]

<u>Ex</u> :-



* A pendant vertex of a tree is called a Leaf.

* Group of trees form a forest. ie each component of the disconnected graph is a tree. Such a graph is called a forest.

<u>Theorem 1</u> :- A graph G is a tree iff there is one & only one path b/w every pair of vertices.

<u>Proof</u> :- Necessary:

Let T be a tree

$\Rightarrow$ T is a connected simple graph

T is connected $\Rightarrow$ there must be atleast one path b/w every pair of vertices, of T.

If suppose there are 2 paths b/w a pair of vertices of T, then the union of the paths will form a cycle & T cannot be a tree. Thus b/w every pair of vertices in a tree, there must be exactly one path.

<u>Sufficiency</u>:

Since there is a path b/w every pair of vertices in G, it is obvious that G is connected.

Since there is only one path b/w every pair of vertices, G cannot have a cycle; B'coz if there is a cycle, then $\exists$ 2 paths b/w 2 vertices on the cycle.

Thus G is a connected graph containing no cycles.

$\Rightarrow$ G is a tree.

Theorem 2 :- A tree with n vertices has $n-1$ edges.

[ie for a tree of n vertices & m edges, $m = n-1$.
(or) for a tree $T=(V,E)$, $|E| = |V|-1 \Rightarrow |V| = |E|+1$]

Proof :- proof is by induction on n.

The theorem holds good for $n=1$, $n=2$, $n=3$.

Assume that the theorem holds for all trees with n vertices where $n \leq k$.

Consider a tree T with $k+1$ vertices.

In T, let 'e' be an edge with end vertices u & v.

Since T is a tree, it has no cycles, and hence there is no other edge (or) path b/w u & v.

$\therefore$ Deletion of 'e' from T will disconnect the graph., and $T-e$ consists of exactly 2 components T_1 & T_2.

Since T does not contain any cycle, both T_1 & T_2 does not contain any cycles. $\Rightarrow T_1$ & T_2 are trees.

Both these trees have less than $k+1$ vertices each.

$\therefore$ By assumption made, theorem holds for T_1 & T_2 each.

ie each of T_1 & T_2 contains one less edge than the no of verticies in it.

Since the total no of vertices in T_1 & T_2 (taken together) is $k+1$, the total no of edges in T_1 & T_2 (taken together) is $(k+1)-2 = k-1$.

But T_1 & T_2 taken together is $T-e$.

Thus $T-e$ has $k-1$ edges

$\Rightarrow$ T has exactly k edges.

Hence a tree T with $k+1$ verticies has k edges.

Hence the proof.

P.T.O.

Theorem 3 :- Any connected graph with n verticies & $n-1$ edges is a tree. [Ex cannot be given for this thm]

Proof :- Let G be a connected graph with n vertices & $n-1$ edges.

Assume that G is not a tree, then G contains a cycle say C. Let 'e' be an edge in C. Deleting the edge 'e' from G will not disconnect the graph. Thus $G-e$ is a connected graph. But $G-e$ has n verticies & $n-2$ edges, and hence it cannot be connected [by Thm, WKT a connected graph with n vertices has atleast $n-1$ edges]. This is a contradic-tion.

Hence G must not have a cycle.

$\Rightarrow$ G must be a tree.

Theorem 4 :- A connected graph G is a tree iff adding an edge b/w any 2 vertices in G creates exactly one cycle in G.

Proof :- Suppose G is a connected graph and is a tree then G has no cycles & there is exactly one path b/w any 2 vertices u & v.

If we add an edge b/w u & v, then an additional path is created b/w u & v & the 2 paths constitute a cycle. Since G has no cycles earlier, this is the only cycle which G now possesses.

Conversely,

Suppose G is connected and adding an edge b/w any 2 vertices u & v in G creates exactly one cycle in G
$\Rightarrow$ before adding this edge, exactly one path was there b/w u & v $\Rightarrow$ G is a tree.

Hence the proof.

P.T.O.

Minimally connected graphs :-

A connected graph is said to be minimally connected if the removal of any one edge from it disconnects the graph.

Theorem 5 :- A connected graph is a tree if and only if it is minimally connected.

Proof :- Suppose G is a connected graph which is not a tree. then G contains a cycle C. The removal of any one edge 'e' from this cycle will not make the graph disconnected.

$\therefore$ G is not minimally connected.

This is equivalent to saying that if a connected graph is minimally connected, then it is a tree [contrapositive].

Conversely, suppose G is a connected graph which is not minimally connected. Then $\exists$ an edge 'e' in G such that $G-e$ is still connected.

$\therefore$ 'e' must be in some cycle in G. $\Rightarrow$ G is not a tree.

This is equivalent to saying that if a connected graph is a tree, then it is minimally connected (contrapositive).

Hence the proof:

Note :- contrapositive :-

$p \to q \Rightarrow \sim q \to \sim p$.

(or) $q \to p \Rightarrow \sim p \to \sim q$.

Problems :-

1) Show that (i) the complete graph K_n is not a tree when $n>2$

(ii) the complete Bipartite graph $K_{r,s}$ is " ———— " $r \geq 2$

Soln :- (i) Let v_1, v_2, v_3 are any 3 vertices of K_n, where $n>2$.

then the closed walk $v_1 v_2 v_3 v_1$ is a cycle in K_n.

Since K_n has a cycle, it cannot be a tree.

(ii) Let V_1 & V_2 be the bipartites. (3)

Let v_1, v_2 be any 2 vertices in the first bipartite

& v_1', v_2' be " " second " of $K_{r,s}$,

where $s \geq r > 1$.

Then the closed walk $v_1 v_1' v_2 v_2' v_1$ is a cycle in $K_{r,s}$.

Since $K_{r,s}$ has a cycle, it cannot be a tree.

2) Prove that a tree with 2 or more vertices contains atleast 2 leaves (pendant vertices).

Soln:- Consider a tree T with 'n' vertices, where $n \geq 2$, then it has $n-1$ edges.

$\therefore$ By HSP, sum of degrees of n vertices $= 2(n-1)$.

Thus if $d_1, d_2, \dots, d_n$ are the degrees of vertices of T,

then $d_1 + d_2 + \dots + d_n = 2n-2$.

If each of $d_1, d_2 \dots d_n$ is ≥ 2, then $d_1 + d_2 + \dots + d_n \geq 2n$.

Since this is not true, atleast one of the d's is less than 2.

ie there is a 'd' which is equal to 1 [Since T is connected, no 'd' can be zero]. Say d_1 is 1, then

$$d_2 + d_3 + \dots + d_n = (2n-2) - 1 = 2n-3. \ (*)$$

~~is 1 one of $d_2, d_3 \dots d_n$ is 1~~

~~This is possible only if atleast~~ one more 'd' which is $= 1$.

$\therefore$ there is atleast one more 'd' with degree 1 ie there are atleast 2 vertices with degree 1

Thus in T, $\exists$ atleast 2 pendant vertices (leaves).

(*) If each of $d_2, d_3 \dots d_n \geq 2$, then $d_2 + d_3 + \dots + d_n \geq 2(n-1)$

ie $2n-3 \geq 2n-2$ (not possible)

3) Prove that a graph with n vertices, $n-1$ edges & no cycles is connected.

Soln:- Let G be a graph with n vertices, $n-1$ edges & no cycles.

Suppose G is not connected. Let the components of G be H_i, $i = 1, 2 \dots k$. If H_i has n_i vertices, we have

$$n_1 + n_2 + \dots + n_k = n.$$

Since G has no cycles, H_i's also do not have cycles, and they are all connected graphs.

$\Rightarrow$ they are trees.
$\hookrightarrow$ H_i's

$\Rightarrow$ each H_i must have $n_i - 1$ edges.

$\therefore$ Total no of edges in these H_i's is

$$(n_1-1)+(n_2-1)+\dots+(n_k-1) = (n_1+n_2+\dots+n_k) - \underbrace{(1+1+\dots+1)}_{k\text{ times}}$$

$$= n-k.$$

This must be equal to total no of edges in G ie $n-1$.

ie $n-k = n-1$.

$\Rightarrow$ $k=1$. (k is the no of components)

This is not possible since $k>1$

$\therefore$ G must be connected.

4) Let F be a forest with k components (trees). If 'n' is the no of vertices & m is the no of edges in F,

P.T $n = m+k$.

Soln :- Let $H_1, H_2 \dots H_k$ be the components of F.

Since each of these is a tree, if n_i is the no of vertices in H_i & m_i is the no of edges in H_i, then

$m_i = n_i - 1$ for $i = 1, 2 \dots k$.

$\Rightarrow$ $m_1+m_2+\dots+m_k = (n_1-1)+(n_2-1)+\dots+(n_k-1)$

$m = n-k$.

$\Rightarrow$ $m = n-k$ (or) $n = m+k$.

5) Let $T_1 = (V_1, E_1)$ & $T_2 = (V_2, E_2)$ be 2 trees. If $|E_1| = 15$ & $|V_2| = 3|V_1|$, determine $|V_1|, |V_2|$ & $|E_2|$

Soln:- we have $|V_1| = |E_1| + 1 = 15+1 = 16$.

Since $|V_2| = 3|V_1| = 3\times 16 = 48$

$\therefore$ $|E_2| = |V_2| - 1 = 48-1 = 47$.

6) If a tree has 400 vertices, find the sum of the degree of the vertices.

— solⁿ in page (4) —

8) In a tree, if the degree of every non-pendant vertex is 3, Show that the no of vertices in the tree is an Even no.

Soln :- Let 'n' be the no of vertices in a tree T.

let 'K' be the no of Pendant vertices (deg = 1) ⇒ there are $(n-K)$ non-pendant vertices.

If each non-pendant vertex is of degree 3, then the sum of deg of vertices $= (K\times 1) + 3\times(n-K) = 2(n-1)$ (HSP)

$$\Rightarrow \quad K+3n-3K = 2n-2$$

$$\Rightarrow \quad n = 2K-2$$

(or) $n = 2(K-1)$, which is an Even no.

9) Suppose that a tree T has N_1 vertices of deg 1, N_2 vertices of deg 2, N_3 vertices of deg 3, ... N_K vertices of deg K.

Prove that $N_1 = 2+N_3+2N_4+\dots+(K-2)N_K$.

Soln :- Total no of vertices $= N_1+N_2+\dots+N_K$.

& sum of deg of vertices $= (N_1\times 1)+(N_2\times 2)+(N_3\times 3)+\dots(N_K\times K)$

$= N_1+2N_2+3N_3+\dots+KN_K$.

& Total no of edges $= N_1+N_2+\dots+N_K-1$.

By HSP, $N_1+2N_2+\dots+KN_K = 2(N_1+N_2+\dots+N_K-1)$

$(3N_3-2N_3)+(4N_4-2N_4)+\dots+(KN_K-2N_K) = 2N_1-N_1-2$

$(K-2)N_K = N_1-2$.

$$\Rightarrow \quad N_3+2N_4+\dots+(K-2)N_K$$

(or) $N_1 = 2+N_3+2N_4+\dots+(K-2)N_K$.

10) If a tree has 4 vertices of deg 2, one vertex of deg 3, 2 vertices of deg 4 & one vertex of deg 5. find the no of leaves in T.

Ans $N=10$

11) Suppose that a tree T has 2 vertices of deg 2, 4 vertices of deg 3 and 3 vertices of deg 4. find the no of pendant vertices in T.

ANS $N=12$

Given $|V| = 400$

$\therefore |E| = |V| - 1 = 399$.

$\Rightarrow$ By HSP, Sum of the degrees of the vertices $= 2|E|$

$= 2 \times 399$

$= 798$

DO (8) & (9) Here.

7) If a tree has 4 vertices of deg 3, 2 vertices of deg 4 & one vertex of deg 5. Find the no of pendant vertices in T.

Soln:- Let N be the no of pendant vertices in T.
↳ degree 1

Total no of vertices $= N + 4 + 2 + 1 = N+7$.

Sum of the degrees of the vertices $= (N \times 1) + (4 \times 3) + (2 \times 4) + (1 \times 5)$

$= N + 12 + 8 + 5$

$= N + 25$

Since T has N+7 vertices, it has $(N+7) - 1 = N+6$ edges.

$\therefore$ By HSP, $N + 25 = 2(N+6)$

$N + 25 = 2N + 12$

$N = 13$

$\Rightarrow$ given tree has 13 pendant vertices.

8) Prove that every tree is a planar graph.

Soln:- Since a tree has no cycle in it, no subgraph of a tree has any cycles.

Hence no subgraph of a tree can be isomorphic to K_5 (or) $K_{3,3}$, which are non-planar.

$\therefore$ a tree cannot be non-planar

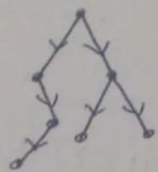
ie Every tree is a planar graph.

Rooted Trees

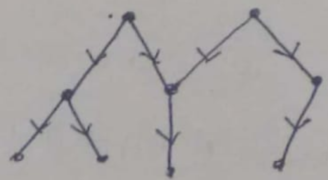
* **Directed Tree** :- A directed graph which is a tree is called a directed tree.

* **Rooted Tree** :- A directed Tree T is called a rooted tree if
  (i) T contains a unique vertex called the root whose in-degree is equal to zero.
  (ii) in-degrees of all other vertices of T are equal to one.

Ex :-



rooted tree



not a rooted tree.

* **Level number** :- A vertex 'v' of a rooted tree is said to be at the k^{th} level (or) has level number 'k' if the path from r to v is of length k.

Note :-

1) If v_1 & v_2 are 2 vertices such that v_1 has a lower level no than v_2 & there is a path from v_1 to v_2, then v_1 is an ancestor of v_2 (or) v_2 is a descendant of v_1.

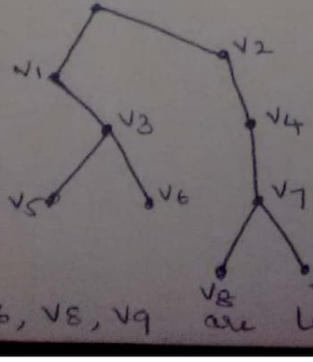
2) If v_1 & v_2 are such that v_1 has a lower level no than v_2 and there is an edge from v_1 to v_2, then v_1 is called the parent of v_2 (or) v_2 is called the child of v_1.

3) Two vertices with a common parent are called Siblings.

4) A vertex whose out-degree is zero is called a Leaf.

5) A vertex which is not a leaf is called Internal vertex.

Ex :-



v_1 is the ancestor of v_3, v_5, v_6.
(or) v_3, v_5, v_6 are the descendant of v_1.
v_1 is the parent of v_3
(or) v_3 is the child of v_1
v_5 & v_6 are siblings.
v_5, v_6, v_8, v_9 are leaves ; all other vertices are internal vertices.

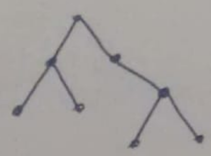
* <u>m-ary Tree</u> :- A rooted tree T is called an m-ary tree if every internal vertex of T is of out-degree $\leq m$; ie if " ——— " ——— " has atmost m children.

* <u>Complete m-ary Tree</u> :- A rooted tree T is called a complete m-ary tree if every internal vertex of T is of out-degree m ; ie if every internal vertex of T has exactly m children.

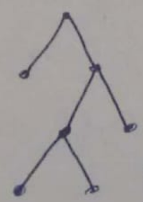
* <u>Binary Tree</u> :- An m-ary tree for which $m = 2$ is called a binary tree ie A rooted tree T is called a Binary tree if every vertex of T is of out-degree ≤ 2 $\Rightarrow$ if every vertex has atmost two children.

* <u>Complete Binary Tree</u> :- A rooted tree T is called a complete Binary tree if every internal vertex of T is of outdegree 2; ie if every internal vertex has exactly 2 children.

<u>Ex</u> :-



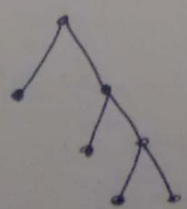
Binary tree



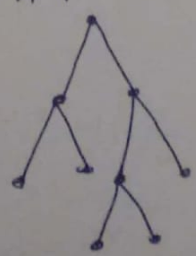
complete Binary tree.

* <u>Balanced Tree</u> :- If T is a rooted tree & h is the largest level no achieved by a leaf of T, then T is said to have height 'h'. A rooted tree of height 'h' is said to be <u>balanced</u> if the level no of every leaf is h (or) h-1.

<u>Ex</u> :-



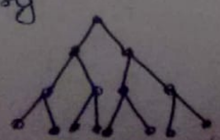
Not balanced



balanced.

* <u>Full Binary Tree</u> :- Let T be a complete Binary tree of height h. Then T is called a full Binary tree if all the leaves in T are at level 'h'.

<u>Ex</u> :-



Problems :-

1) Let T be a complete m-ary tree of order n with p leaves & q internal vertices. Prove that

(i) $n = mq+1 = \dfrac{mp-1}{m-1}$.

(ii) $p = (m-1)q+1 = \dfrac{(m-1)n+1}{m}$

(iii) $q = \dfrac{n-1}{m} = \dfrac{p-1}{m-1}$.

Soln :- Given that T is a complete m-ary tree.

$\Rightarrow$ every internal vertex is of out-degree m.

Since every vertex in a rooted tree is either a leaf or an internal vertex, the total no of vertices in T is the sum of no of leaves in T & no of internal vertices in T.

ie $n = p+q \longrightarrow (1)$.

WKT out-degree of every leaf is zero and out-degree of every internal vertex is m

$\Rightarrow$ Sum of out-degrees of all vertices in T $\Big\} = (p\times 0)+(q\times m) = qm \longrightarrow (2)$

WKT in a rooted tree of order n, the in-degree of the root is zero & the indegree of the remaining $(n-1)$ vertices are 1 each.

$\therefore$ Sum of indegrees of all vertices of T $\Big\} = (1\times 0)+(n-1)1 = n-1 \longrightarrow (3)$

By 1st thm of digraph theory,

Sum of indegrees = Sum of out-degrees.

ie $n-1 = qm$

$\Rightarrow \boxed{n = qm+1} \rightarrow (4). \Rightarrow \boxed{q = \dfrac{n-1}{m}}$

sub (4) in (1), $qm+1 = p+q$.

$\boxed{p = (m-1)q+1}$

$\Rightarrow \boxed{q = \dfrac{p-1}{m-1}}$

(4) $\Rightarrow$ $n = m\dfrac{(p-1)}{(m-1)} + 1 = \dfrac{m(p-1)+(m-1)}{m-1}$

$$\boxed{n = \frac{mp-1}{m-1}}$$

we have $q = \dfrac{p-1}{m-1}$

$\therefore$ (1) $\Rightarrow$ $n = p + \dfrac{p-1}{m-1} = \dfrac{pm-p+p-1}{m-1}$

$$n = \frac{pm-1}{m-1}$$

$$\therefore \quad \boxed{p = \frac{(m-1)n+1}{m}}$$

Note :- for a complete binary tree, the results are :- $\rightarrow$ ie $m=2$

(i) $n = 2q+1 = 2p-1$

(ii) $p = q+1 = \frac{1}{2}(n+1)$

(iii) $q = \frac{1}{2}(n-1) = p-1$.

2) Find the no. of leaves in a complete Binary tree if it has 29 vertices.

Soln :- Complete Binary tree $\Rightarrow$ $m=2$.

Given $n = 29$

$p = ?$

we have $p = \frac{1}{2}(n+1) = \frac{1}{2}(29+1) = \frac{30}{2} = 15$

$\therefore$ No. of leaves $= p = 15$.

3) Find the no. of internal vertices in a complete 5-ary tree with 817 leaves. Hence find its order.

Soln :- Given :- $m=5$; $p = 817$

$q = ?$

we have $q = \dfrac{p-1}{m-1} = \dfrac{816}{4} \Rightarrow \boxed{q = 204}$

ie given tree has 204 internal vertices.

order, $n = mq+1$

$n = (5)(204)+1$

$n = 1020+1$

$\boxed{n = 1021}$

4) A classroom contains 25 microcomputers that must be Connected to a wall socket that has 4 outlets. Connections are made by using extension cords that have 4 outlets each. Find the least no of cords needed to get this Computer set up for the class.

Soln :- Let us treat the wall Socket as the root of a complete 4-ary tree with the computers as its leaves & the internal vertices other than the root as extension cords.

i.e. $m = 4$, $p = 25$

$\Rightarrow$ No of internal vertices is $q = \frac{p-1}{m-1} = \frac{24}{3} = 8$.

$\therefore$ no of extension cords needed [ie the no of internal vertices minus the root] is $q-1 = 8-1 = 7$.

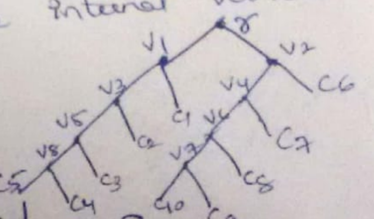
5) A complete Binary tree has 20 leaves. Find the no of vertices.

Soln :- $m = 2$; $p = 20$.

$$n = \frac{mp-1}{m-1} = \frac{2 \cdot 20 - 1}{2-1} = \underline{\underline{39}}.$$

6) The Computer laboratory of a school has 10 computers that are to be connected to a wall socket that has 2 outlets. connections are made by using extension chords that have 2 outlets each. Find the least no of cords needed to get these computers set up for use.

Soln :- Consider the complete Binary tree ($m=2$) having the wall socket as the root, the computers as the leaves & the internal vertices other than the root as extension cords.



$\therefore$ $m=2$, $p = 10$

$\Rightarrow$ No of internal vertices $= q = \frac{p-1}{m-1} = \frac{10-1}{2-1} = 9$.

$\therefore$ No of extension cords needed = no of internal vertices − root
$= 9 - 1 = 8$

Pre-order and Post-order Traversals :-

Consider a rooted tree T. Let r be its root & let $T_1, T_2 \dots T_K$ be subtrees of T as we move from left to right.

Suppose we list the vertices of T such that r is the first vertex & the subsequent vertices are selected in the foll order : vertices of T_1 read from left to right, vertices of T_2 read from left to right & so on. The list of vertices so prepared is called the pre-order Traversal of T.

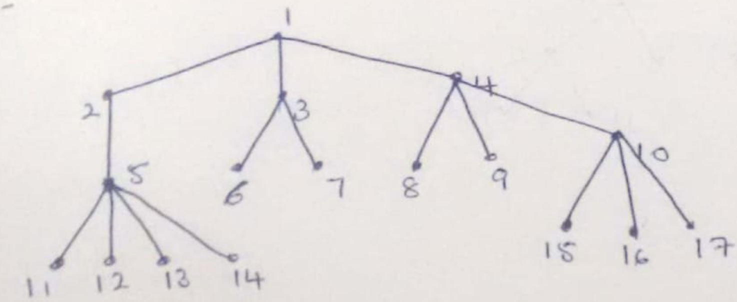
Suppose we list the vertices of T selected in the foll order : vertices of T_1 read from leftmost descendant, vertices of T_2 read from leftmost descendant & so on & finally the root r. The list of vertices so prepared is called the post-order Traversal of T.

Pre order :- node left right
NLR

Post order :- Left right node
LRN

EX :-

1)



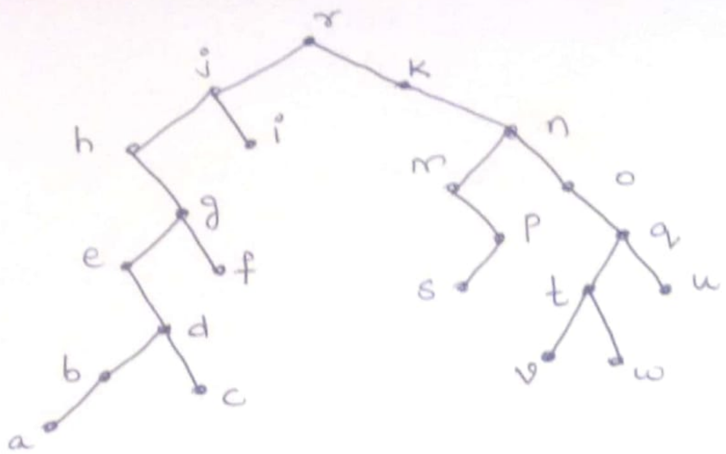
Preorder traversal :-

1 2 5 11 12 13 14 3 6 7 4 8 9 10 15 16 17

Postorder traversal :-

11 12 13 14 5 2 6 7 3 8 9 15 16 17 10 4 1

2)



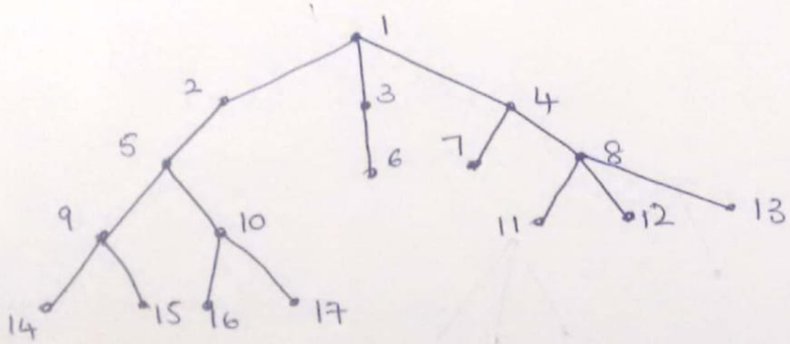
Pre order traversal :-

r j h g e d b a c f i k n m p s o q t v w u.

Postorder traversal :-

a b c d e f g h i j s p m v w t u q o n k r

3)



Pre order traversal :-

1 2 5 9 14 15 10 16 17 3 6 4 7 8 11 12 13.

Post order traversal :-

14 15 9 16 17 10 5 2 6 3 7 11 12 13 8 4 1

Sorting

The method of splitting and merging, done by the use of balanced complete trees is known as merge sort.

The process of sorting consists of two parts:

In the first part, we recursively split the given list in half until each sublist contains & all subsequent list a single element.

In the second part, we merge the sublists in non-decreasing order until the original 'n' integers have been sorted.

Ex 1:- Apply merge sort to the lists

Q) −1, 7, 4, 11, 5, −8, 15, −3, −2, 6, 10, 3.

−1, 7, 4, 11, 5, −8, 15, −3, −2, 6, 10, 3 }

Soln:-

{−1, 7, 4, 11, 5, −8, 15, −3, −2, 6, 10, 3}

{−1, 7, 4, 11, 5, −8}   {15, −3, −2, 6, 10, 3}

{−1, 7, 4}   {11, 5, −8}   {15, −3, −2}   {6, 10, 3}

{−1, 7}   {4}   {11, 5}   {−8}   {15, −3}   {−2}   {6, 10}   {3}

{−1}   {7}   {4}   {11}   {5}   {−8}   {15}   {−3}   −2   {6}   {10}   3

To merge,

−1   7   11   5   15   −3   6   10

{−1, 7}   4   {5, 11}   −8   {−3, 15}   −2   {6, 10}   3

{−1, 4, 7}   {−8, 5, 11}   {−3, −2, 15}   {3, 6, 10}

{−8, −1, 4, 5, 7, 11}   {−3, −2, 3, 6, 10, 15}

{−8, −3, −2, −1, 3, 4, 5, 6, 7, 10, 11, 15}.

∴ sorted out version of the given list is

−8, −3, −2, −1, 3, 4, 5, 6, 7, 10, 11, 15.

Ex 2 :- −9, 6, 5, −3, 4, 2, −7, 6, −5, 10, −11, 0, 1.

Soln :- {−9, 6, 5, −3, 4, 2, −7, 6, −5, 10, −11, 0, 1}

{−9, 6, 5, −3, 4, 2, −7} {6, −5, 10, −11, 0, 1}

{−9, 6, 5, −3} {4, 2, −7} {6, −5, 10} {−11, 0, 1}

{−9, 6} {5, −3} {4, 2} {−7} {6, −5} {10} {−11, 0} {1}

{−9} {6} {5} {−3} {4} {2} −7 {6} {−5} 10 {−11} {0} 1

To merge,

−9 6 5 −3 4 2 −7 6 −5 10 −11 0 1

{−9, 6} {−3, 5} {2, 4} −7 {−5, 6} 10 {−11, 0} 1

{−9, −3, 5, 6} {−7, 2, 4} {−5, 6, 10} {−11, 0, 1}

{−9, −7, −3, 2, 4, 5, 6} {−11, −5, 0, 1, 6, 10}

{−11, −9, −7, −5, −3, 0, 1, 2, 4, 5, 6, 10}.

Ex 3 :- {7, 3, 8, 4, 5, 10, 6, 2, 9}.

Ex 4 :- −1, 0, 2, −2, 3, 6, −3, 5, 1, 4.

Spanning Trees:-

Defn:- Let G be a connected graph. A subgraph T of G is called a spanning tree of G if (i) T is a tree and (ii) T contains all vertices of G.

1) Spanning tree is also called a maximal tree since a spanning tree T of a graph G is a subgraph of G that contains all vertices of G.

2) The edges of a spanning tree are called its branches.

3) If G has n-vertices, a spanning tree of G must have n vertices and hence $n-1$ edges.

4) If T is a spanning tree of a graph G, then the edges of G which are not in T are called the chords of G wrt T. The set of all chords of G is the complement of T in G. This set is called the chord set or cotree of T in G and is denoted by $\bar{T}$.
clearly $G = T \cup \bar{T}$. Ex:- ⊛ (P.T.O)

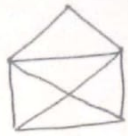
5) A spanning tree is defined only for a connected graph.

6). In a disconnected graph of n vertices each component have a spanning tree. These spanning trees together form a spanning forest.

7) Every tree is a spanning tree of itself.

8). A connected graph can have more than one spanning tree.

(*) Ex



G

T

$\bar{T}$

$\therefore G = T \cup \bar{T}$

Theorem 1 :- A graph is connected if and only if it has a spanning tree.

Proof :- Let G be a connected graph.

If G has no cycles then G is a tree & G itself is a spanning tree.

If G has cycles, delete an edge from each cycle.

The resulting graph G' is cycle-free, connected & contains all vertices. $\Rightarrow$ G' is a spanning tree of G.

Thus G has a spanning tree.

Conversely,

Suppose a graph G has a spanning tree T.

$\Rightarrow$ T is a tree. $\Rightarrow$ $\exists$ a path between every pair of vertices of T.

Since T is a spanning tree $\Rightarrow$ T contains all the vertices

$\therefore$ there is a path between every pair of vertices of in G.

$\Rightarrow$ G is connected.

Theorem 2 :- with respect to any of its spanning trees, a connected graph of n-vertices & m edges has $n-1$ branches and $m-n+1$ chords.

Proof :- Let G be a connected graph of n vertices & m edges, T be a spanning tree of G then T contains all vertices of G. $\Rightarrow$ T has n vertices.

$\Rightarrow$ T has $n-1$ edges.

$\Rightarrow$ T has $n-1$ branches.

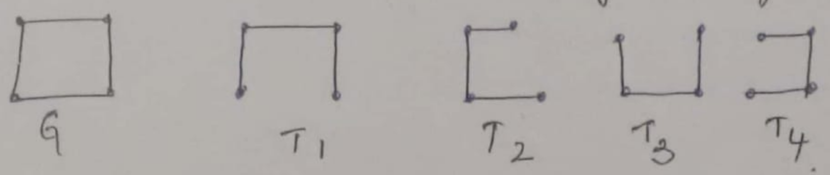
The no. of edges in G that are not in T is $m-(n-1)$

$\Rightarrow$ G has $m-n+1$ chords.

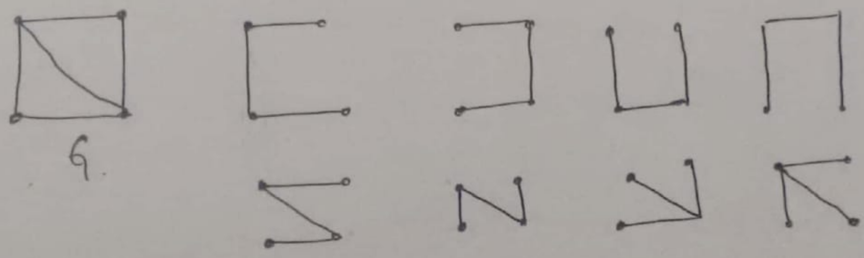
Hence the proof

Ex 1 Find all spanning trees of graph given.

(i) QP.



(ii) QP



Ex 2 prove that a pendant edge in a connected graph G is contained in every spanning tree of G.

Soln :- A pendant edge is an edge whose one end vertex is a pendant vertex.

let e be a pendant edge of a connected graph G & let v be the corresponding pendant vertex then e is the only edge that is incident on v.

Suppose there is a spanning tree T of G for which e is not a branch. Then T cannot contain the vertex v. This is not possible because T must contain every vertex of G (Hence there is no spanning tree of G for which e is not a branch.)

Hence a pendant edge in a connected graph is contained in every spanning tree of G.

Ex 3 Show that a Hamilton path is a spanning tree.

Soln:- Hamilton path P in a connected graph G. if such a path exists, then it contains every vertex of G. and that if G has n vertices then P has n-1 edges. Thus P is a connected subgraph of G with n vertices & n-1 edges.

$\Rightarrow$ P is a tree & contains all vertices of G

$\Rightarrow$ P is a spanning tree of G.

Ex 4 How many edges are to be removed from the K_5 & $K_{4.4}$ in order to obtain their spanning trees.

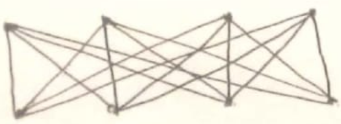
Soln:-

K_5

T

K_5 has 10 edges out of which 4 edges form a S. Tree

$\therefore$ 6 edges are removed.



$K_{4,4}$

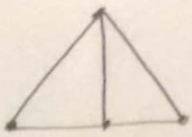
T.

$K_{4.4}$ has 16 edges.

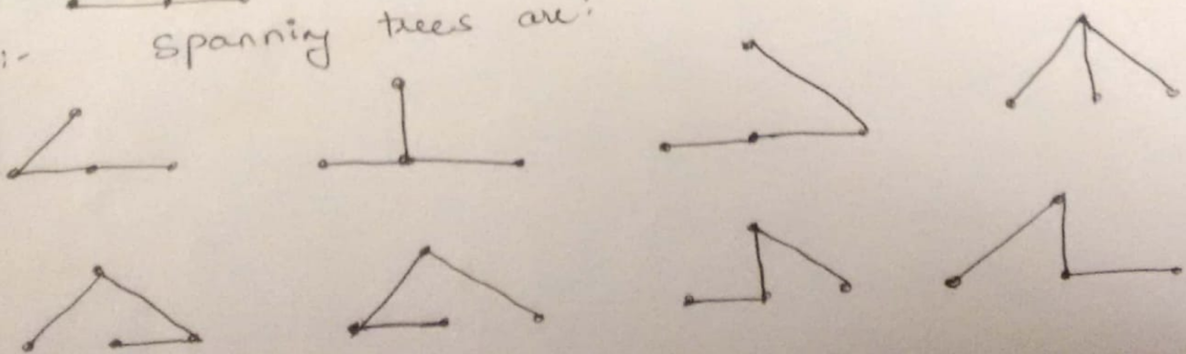
7 are there in S.T

$\therefore$ 9 edges are removed.

Ex 5 Find all the spanning trees of the graph given below:



Soln:- Spanning trees are:



Algorithms for constructing Spanning trees:-

A simple graph G has a spanning tree iff G is connected. Instead of constructing a spanning tree by removing edges. Spanning tree can be built up by successively adding edges.

Two algorithms based on this principle for finding a spanning tree are 1) Depth first Search (DFS) and
2) Breadth first search (BFS).

1) Depth first Search (DFS) :-

Step 1 :- Arbitrarily choose a vertex from the vertices of the graph and designate it as the root.

2 :- Form a path starting at this vertex by successively adding edges as long as possible where each new edge is incident with the last vertex in the path without producing any cycle.

3 : If the path goes through all vertices of the graph the tree consisting this path is a spanning tree.

otherwise, move back to the next to last vertex in the path and if possible form a new path starting at this vertex passing through vertices that were not already visited.

4 : If this cannot be done move back another vertex in the path that is two vertices back in the path & repeat.

5 : Repeat this procedure, begining at the last vertex visited, moving back up the path one vertex at a time forming new paths that are as long as possible until no more edges can be added.

6) This process ends since the graph has a finite number of edges and is connected.

Thus a spanning tree is produced.

BFS - Algorithm :-

1) Arbitrarily choose a vertex and designate it as the root. then add all edges incident to this vertex such that the addition of edges does not produce any cycle.

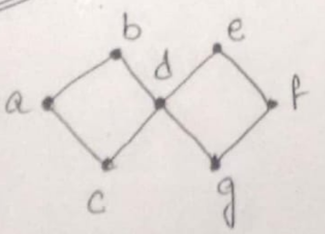
2) The new vertices added at this stage become the vertices at level 1 in the spanning tree, arbitrarily order them.

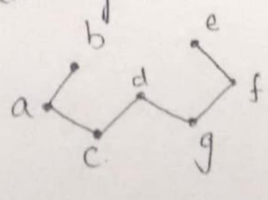
3) For each vertex at level 1 visited in order add each edge incident to this vertex to the tree as long as it does not produce any cycle

4) Arbitrarily order the children of each vertex at level 1. This produces the vertices at level 2 in the tree.

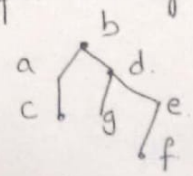
5) Continue the same produce until all the vertices in the tree have been added. A spanning tree is produced since we have produced a tree without cycle containing every vertex of the graph.

Ex. Find the spanning tree of the graph using DFS, BFS.

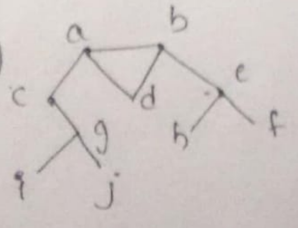
1) 

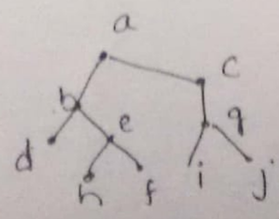


DFS

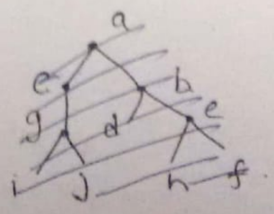


BFS

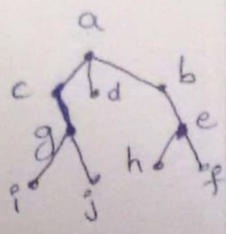
2) 



DFS



~~BFS~~.



BFS

Prefix codes and weighted trees :-

Sequence :- A sequence is a set whose elements are listed in order. The number of elements contained in a sequence is called its length.
A sequence consisting of only 0 and 1 is called a binary sequence or a binary string. Ex:- *

[NOTE :- Suppose a message consisting of the letters a, e, n, r, t is to be transmitted. If a = 1, e = 0, n = 10, r = 01, t = 101 be the coding scheme for these letters. Suppose a message 'eat' is to be transmitted then the coded form of the message is given by 01101.

This binary sequence may not be decoded properly since the sequence 01101 can be decoded as rt or rar or eaar. Hence this coding scheme cannot be used in transmissions.

Suppose a = 10, e = 0, n = 1101, r = 111, t = 1100.
The mes eat is transmitted as the binary sequence 0101100. clearly the decoding of this sequence yeilds eat & no other message. eat $\xrightarrow{\text{coding}}$ 0101100 $\xrightarrow{\text{decoding}}$ eat

∴ In the first of the coding schemes the code 1 is assigned to a, & the code 10 is to n & 101 to t
Thus the code assigned to t contains the codes assigned to a and n as prefixes.

In the second coding scheme, the code of any letter is not a prefix of the code of any other letter.]

* Ex :- 01, 001, 101, 1001, 11111 are binary strings of length 2, 3, 3, 4, 5 resp.

Binary sequences are used as codes for messages sent through transmitting channel.

Prefix codes :-

Let P be a set of binary sequences that represent a set of symbols, then P is called a prefix code if no sequence in P is the prefix of any other sequence in P.

Ex

$P_1 = \{10, 0, 1101, 111, 1100\}$ is a prefix code

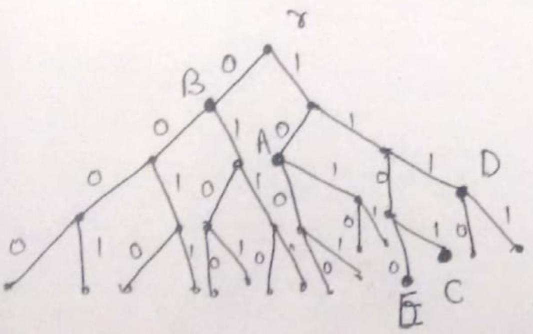
$P_2 = \{000, 001, 01, 10, 11\}$ — " —

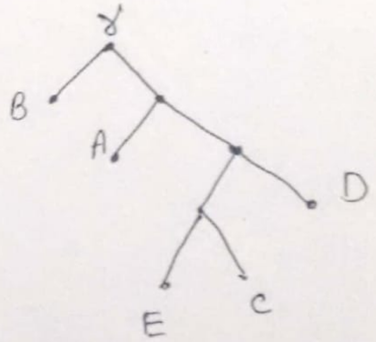
$A_1 = \{01, 0, 101, 10, 1\}$ is not a prefix code

$A_2 = \{1, 00, 01, 000, 0001\}$ is not a prefix code.

Prefix codes can be represented by binary trees as shown below:

Consider P_1, the longest sequence has length 4.
Now construct a full binary tree of height 4.
assign the symbol 0 to every left edge from its parent vertex & 1 to every right edge from its parent vertex





The subtree extracted from the ful binary tree ~~that~~ contains the root r and the vertices A . B . C . D . E
This subtree represents the prefix code given by P_1.

a = 10 e = 0 n = 1101 r = 111 t = 1100

a = 1, e = 0, n = 10, r = 01 t = 101

BEAD $\xrightarrow{\text{coding}}$ 0110010111 $\xrightarrow{\text{decoding}}$ BEAD

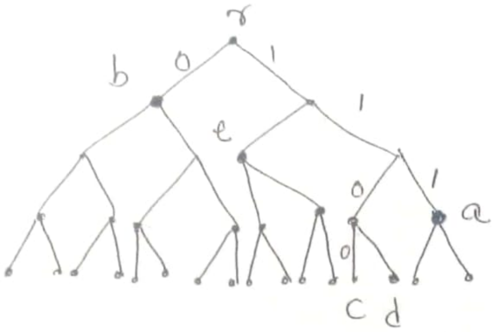
Ex. Consider the prefix code

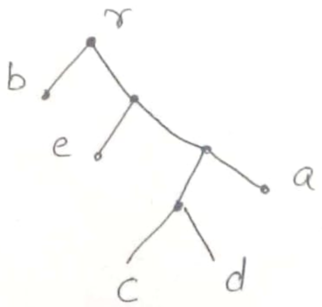
a = 111 b = 0. c = 1100 d = 1101 e = 10.

Using this code decode the following sequences and draw the binary tree that represents prefix code?

a) 100 11 1.1 1 01 (b) 10 111 1.00 1.1 00 0 1101. (c) 1101.11 /111 0010

e b a d e a e b c b d d a d e





Ex 2 given a = 00, b = 01. c = 101. d = x10. e = yz1.

where determine x y & z so that the given code is a prefix code. Hence draw the binary tree.

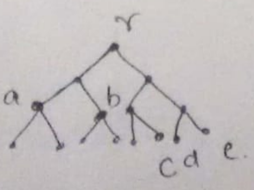
Soln:- If x = 0. then d = 010 contains the code 01 for b.

∴ x = 1 then d = 110

If y = 0	z = 0	e = 001	not	'a'
y = 0	z = 1	011	not	'b'
y = 1	z = 0	101	"	c
y = 1	z = 1.	111	is the code.	

x = 1, y = 1. z = 1.

a = 00, b = 01 c = 101. d = 110 e = 111



Q3 Find the prefix code represented by

$\{00, 011, 0100, 0101, 10, 110, 111\}$

Weighted trees :-

Consider a set of n-positive integers $w_1, w_2 \cdots w_n$ where $w_1 \le w_2 \le \cdots \le w_n$. Suppose we assign these integers to n-leaves of a complete binary tree $T = (V, E)$ in any 1-1 manner. The resulting tree is called a complete weighted binary tree with $w_1, w_2 \cdots w_n$ as weights.

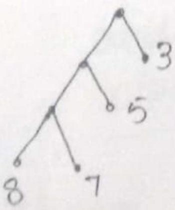
If $l(w_i)$ is the level number of the leaf of T to which the weight w_i is assigned, then $w(T)$ defined by

$$w(T) = \sum_{i=1}^{n} w_i \cdot l(w_i).$$ is called the weight of the tree T

For a given set of weights, the value of $w(T)$ depends upon the tree T chosen.

Ex

1)



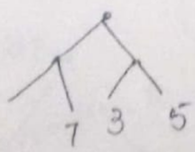
T_1

$w(T_1) = 1 \times 3 + 2 \times 5 + 7 \times 3 + 8 \times 3 = 58$

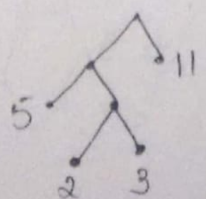
$w(T_2) = 8 \times 2 + 7 \times 2 + 3 \times 2 + 5 \times 2 = 46$

2)

T_2



3)



$11 \times 1 + 5 \times 2 + 2 \times 3 + 3 \times 3$

$11 + 10 + 6 + 6 = 3\cancel{5}$

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Optimal Tree : Given a set of weights, if we consider the set of all complete binary trees to whose leaves these weights are assigned, then tree in this set which carries the minimum weight is called an optimal tree for the weights. For a given set of weights, there can be more than one optimal tree.

Huffman's Procedure :-

An optimal tree for a given set of weights can be constructed as follows :-

1) Arrange the weights in non-decreasing order and assign them to isolated vertices.

2) Select 2 vertices with minimum weights. Add these 2 vertices to get a new vertex. Draw a tree with new vertex as root and the selected vertices as children. Leave the remaining vertices undisturbed.

3) Repeat the procedure in step (2) until all the vertices and subtrees are connected to get a complete weighted binary tree.

4) The tree obtained is the optimal tree for the given set of weights and is also known as Huffman's tree. This tree is not unique.

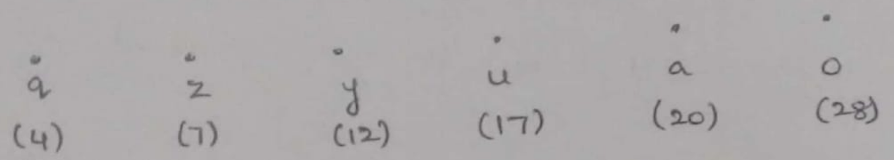
Optimal Prefix code :- optimal tree can be used to obtain a prefix code for the symbols representing its leaves. For this purpose, we first label the symbols 0, 1 to its edges by labelling procedure indicated earlier. Then all the vertices & the leaves of tree can be identified by binary sequences. The binary sequences thru' which the leaves are identified, yield a prefix code for the symbols representing the leaves. This prefix code is known as optimal prefix code.

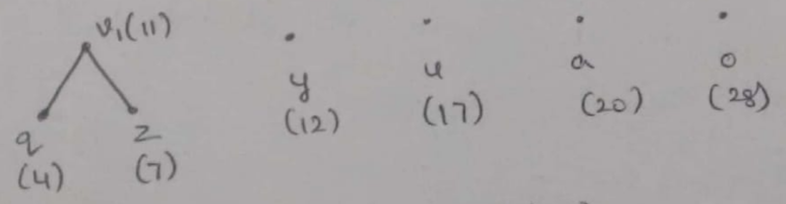
Note :- optimal prefix code is not unique since optimal tree is not unique.

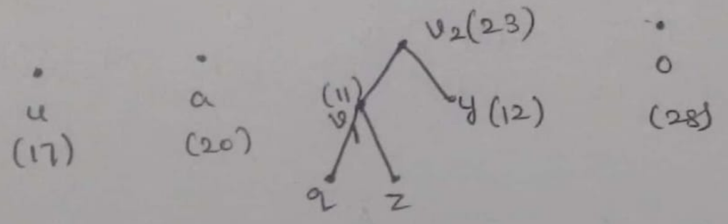
Problems :-

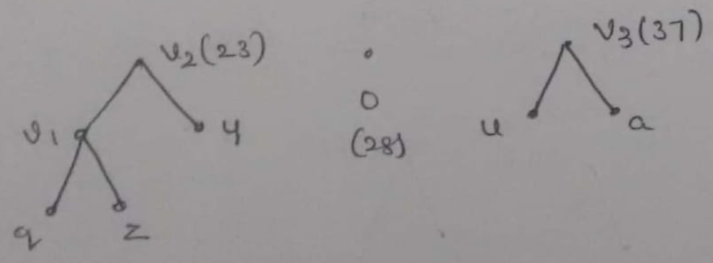
1) Construct an optimal prefix code for the symbols a, o, q, u, y, z that occur with frequencies 20, 28, 4, 17, 12, 7 respectively.

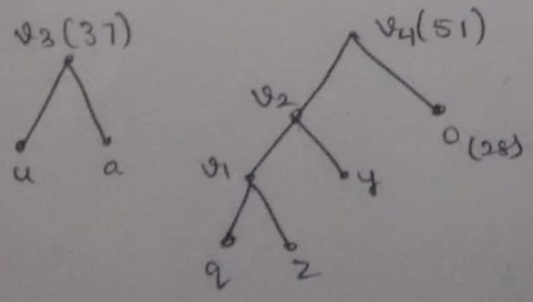
Soln:- Arranging these symbols along with their frequencies in non-decreasing order, treating frequencies as the weights and the corresponding symbols as the isolated vertices:

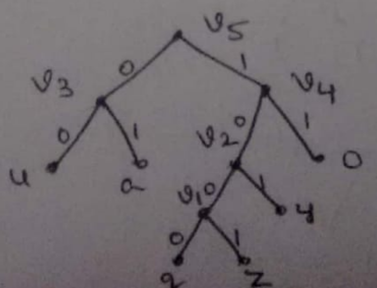












is the optimal tree (Huffman tree)

∴ optimal prefix code for the given symbols is :

a : 01   u : 00   o : 11

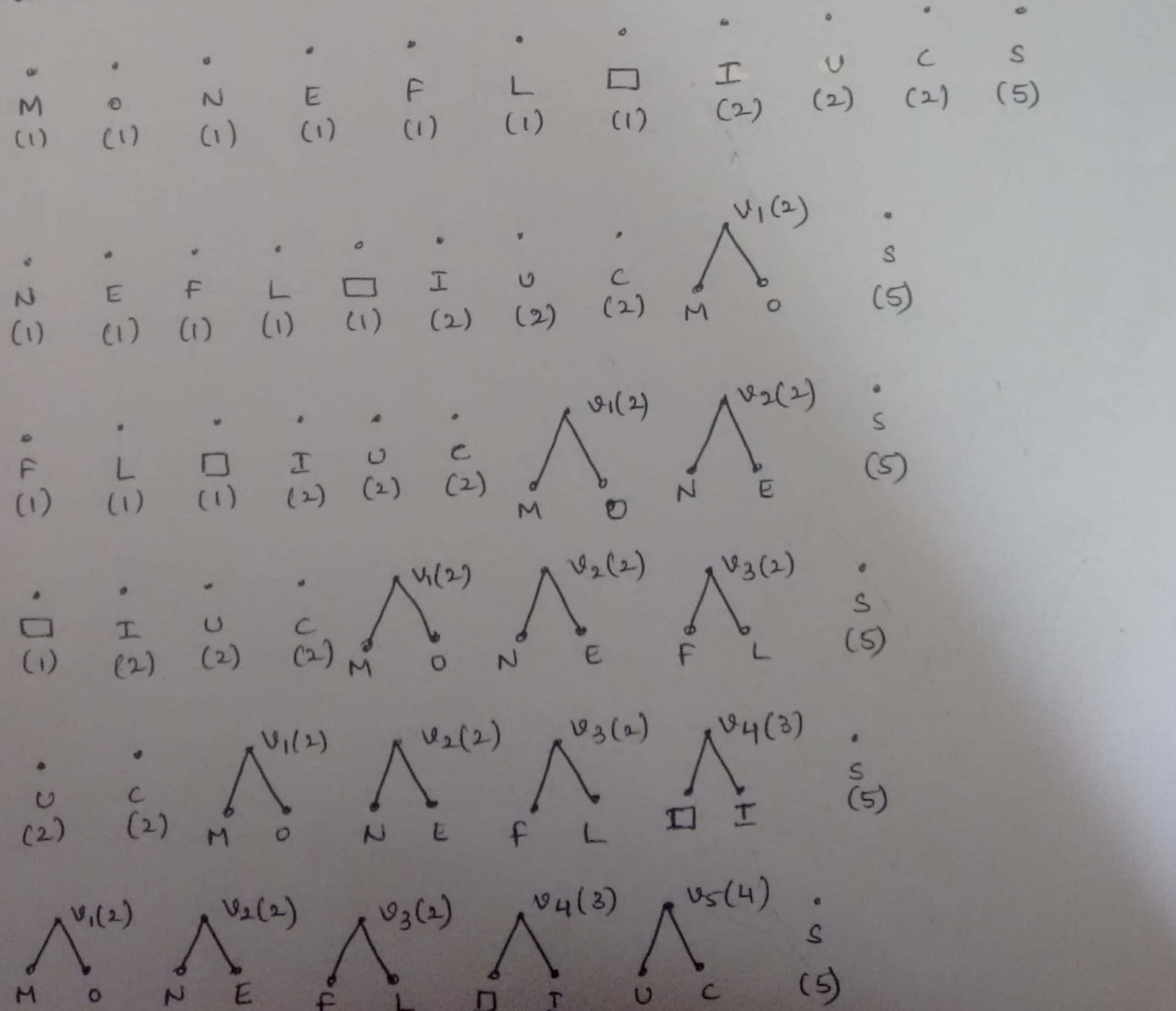
y : 101   q : 1000   z : 1001 .

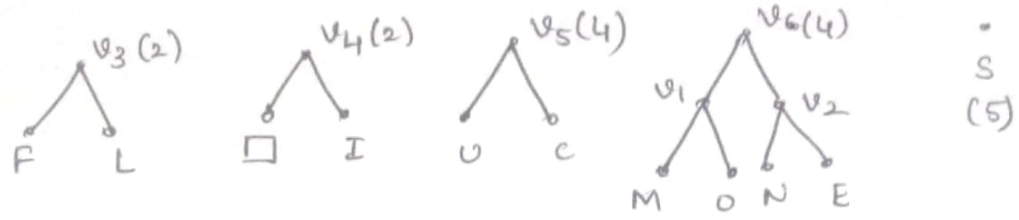
2) Obtain an optimal Prefix code for the message MISSION SUCCESSFUL . INDICATE THE CODE -

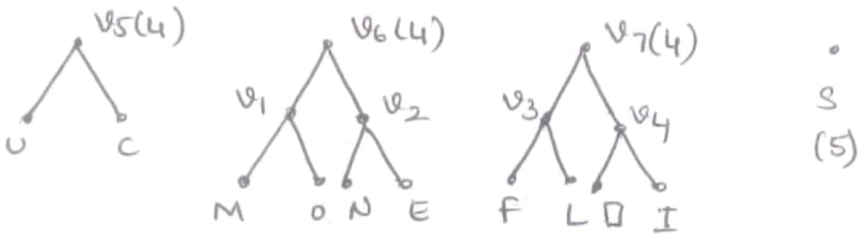
Soln :- The given message consists of the letters M, I, S, O, N, U, C, E, F, L with frequencies 1, 2, 5, 1, 1, 2, 2, 1, 1, 1 resp.

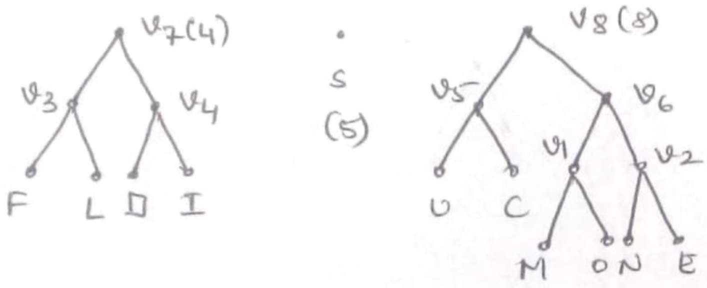
Also there is a blank space (□) b/w the 2 words of the message.

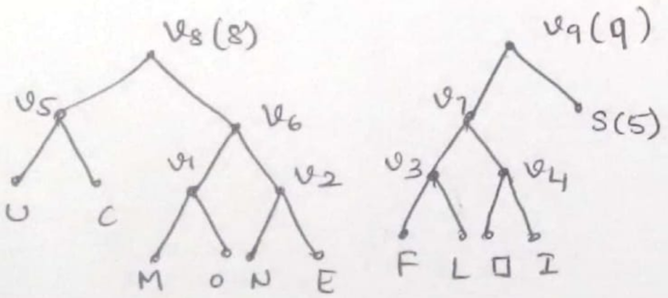
∴ Arranging these letters and □ in non-decreasing order of their weights (frequencies) in the form of isolated vertices is as shown below :

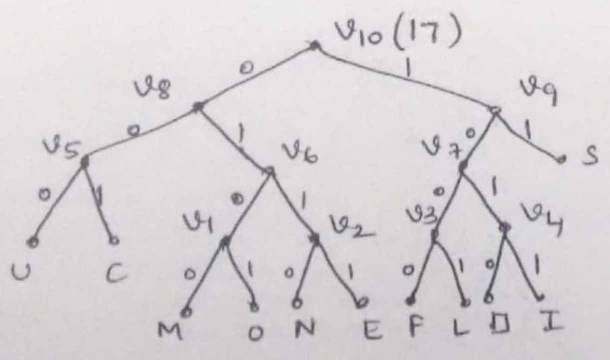












is the optimal tree.

M : 0100   I : 1011   S : 11   O : 0101   N : 0110

U : 000   C : 001   E : 0111   F : 1000   L : 1001   □ : 1010

Code : 0100101111111011010101101010110000010010111111111000 0001001

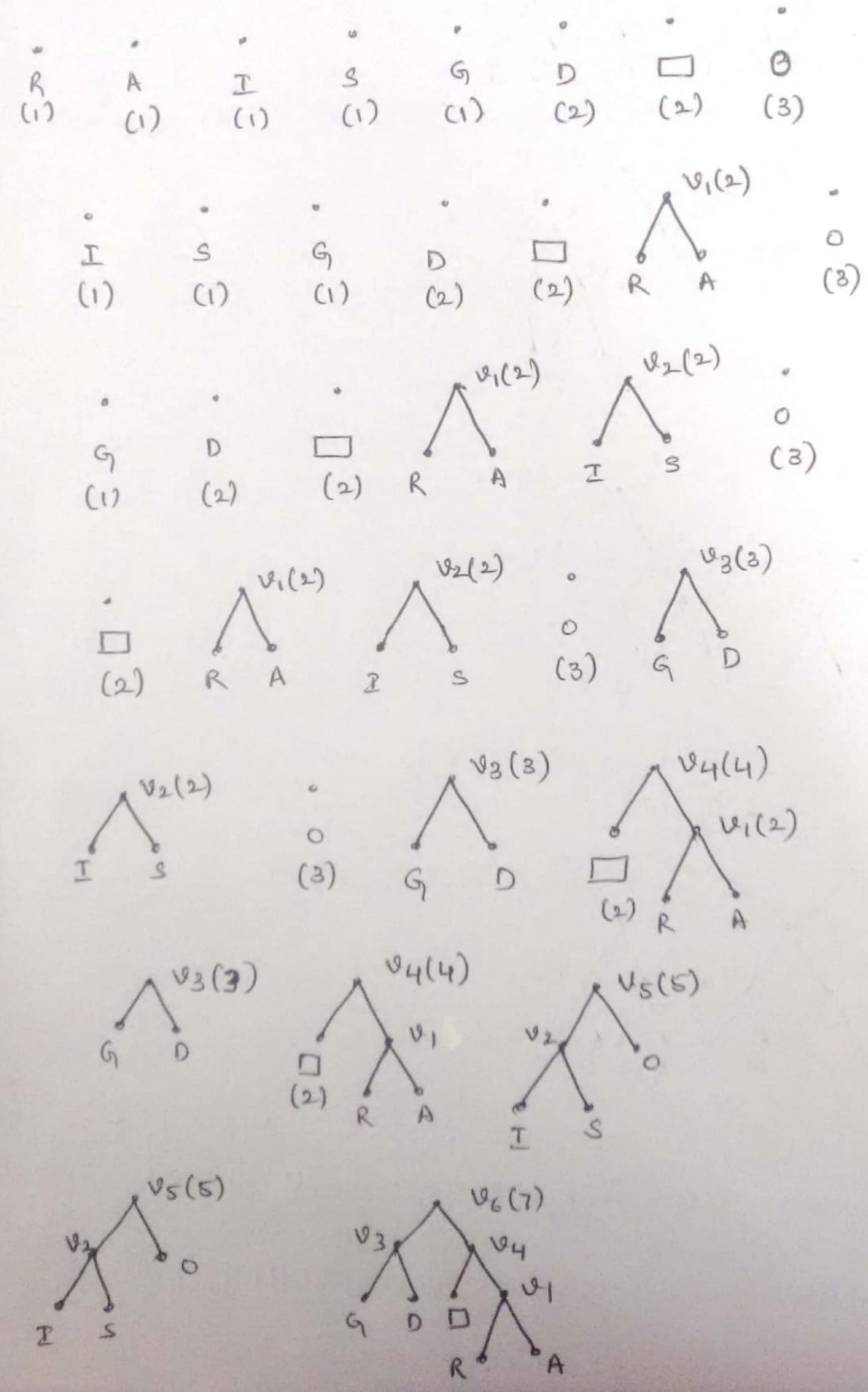
3) Obtain an optimal prefix code for the message

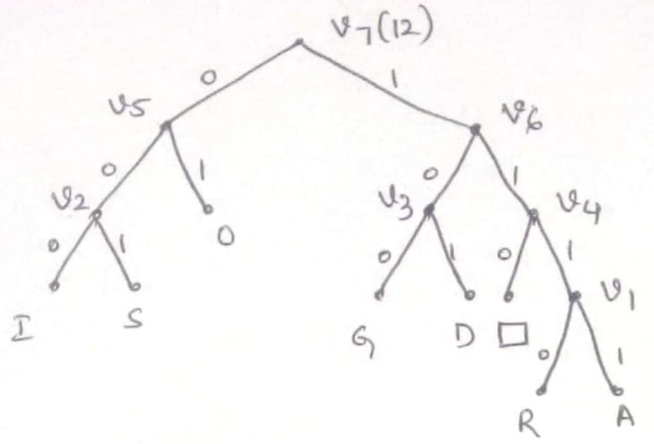
ROAD IS GOOD . Indicate the code .

Soln :- The given message consists of the letters R, O, A, D, I, S, G with frequencies 1, 3, 1, 2, 1, 1, 1 resp.

Also there are 2 blank spaces (□) b/w the words of the message.

∴ Arranging all these in non-decreasing order :





is the optimal tree.

R : 1110   O : 01   A : 1111   D : 101

□ : 110   I : 000   S : 001   G : 100

Code : 11100111111011110000001110100010101.

4) obtain an optimal prefix code for the symbols A, B, C, D, E, F, G, H, I, J that occur with frequencies : 78, 16, 30, 35, 125, 31, 20, 50, 80, 3.

QP

5) obtain an optimal prefix code for the letters of the following words and hence indicate the code :

QP i) ENGINEERING.

QP ii) FALL ON THE WALL.