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Department of Mathematics

Lecture Notes

Calculus & Linear Algebra (18MAT11)

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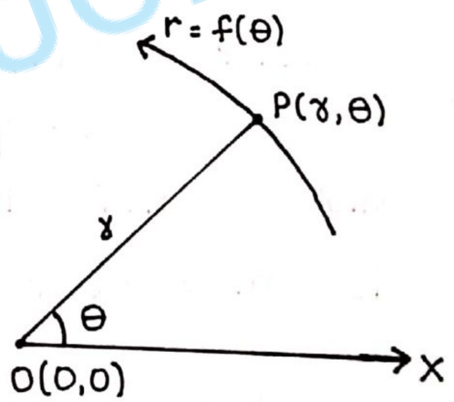
CALCULUS AND LINEAR ALGEBRA

MODULE - 01

POLAR CURVES AND EVALUATES

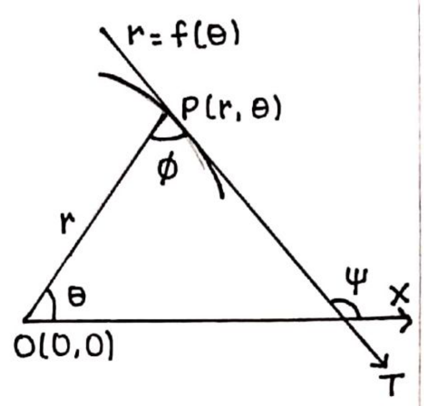
Introduction :-

Let $\overrightarrow{OX}$ be the intial line or intial ray, there is any point on the plane $P(r,\theta)$ with the radius of Vector $OP = r$ and $\angle XOP = \theta$ and $r = f(\theta)$ be a polar curve at the point $P(r,\theta)$ Here, the Coordinates of P is called as the Polar Coordinates.



A. ANGLE BETWEEN RADIUS VECTOR AND TANGENT TO THE POLAR CURVE

Let $\overrightarrow{OX}$ be the intial line $P(r,\theta)$ be a point in the plane and $\overrightarrow{OP} = r$ be the radius of vector and Let $r = f(\theta)$ be a polar curve at the point 'P', 'T' be the tangent to the curve $r = f(\theta)$ at P and making angle with the intial line ψ



let ϕ be the angle between radius of vector and tangent to the curve $r = f(\theta)$ at P

W.K.T

$$\psi = \phi + \theta$$

$$\Rightarrow \tan\psi = \tan(\phi + \theta)$$

$$\tan\psi = \frac{\tan\phi + \tan\theta}{1 - \tan\phi\tan\theta} \longrightarrow (1)$$

also W.K.T The slope of the tangent T is $m = \tan\psi = \frac{dy}{dx} \longrightarrow (2)$

let $x = r\cos\theta$ and $y = r\sin\theta$

differentiate x and y w r t θ.

$$\therefore \frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

$$(2) \Rightarrow \tan\psi = \frac{dy}{dx} \quad \frac{dy/d\theta}{dx/d\theta}$$

$$\Rightarrow \tan\psi = \frac{dr/d\theta \ \sin\theta + r\cos\theta}{dr/d\theta \ \cos\theta - r\sin\theta}$$

$$= \frac{\dfrac{\frac{dr}{d\theta}\sin\theta}{\frac{dr}{d\theta}\cos\theta} + \dfrac{r\cos\theta}{\frac{dr}{d\theta}\cos\theta}}{1 - \dfrac{r\sin\theta}{\frac{dr}{d\theta}\cos\theta}}$$

$$\Rightarrow \tan\psi = \tan\theta + r\frac{d\theta}{dr} \Big/ 1 - r\frac{d\theta}{dr}\tan\theta$$

$$\frac{\tan\theta + \tan\phi}{1 - \tan\phi\tan\theta} = \frac{\tan\theta + r\frac{d\theta}{dr}}{1 - r\frac{d\theta}{dr}\tan\theta}$$

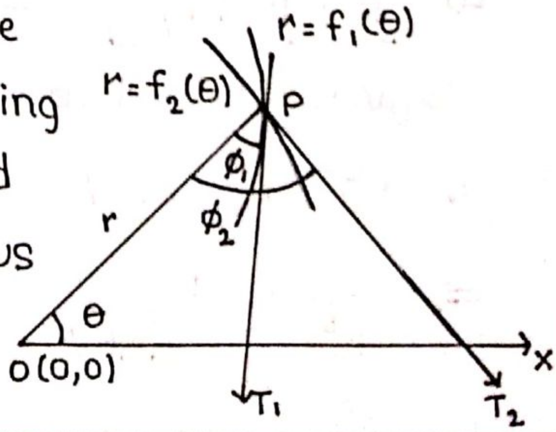
$$\boxed{\tan\phi = r\frac{d\theta}{dr}}$$

$$\phi = \tan^{-1} r\frac{d\theta}{dr}$$

Interms of, $\cot\phi = \frac{1}{r}\frac{dr}{d\theta}$

B. ANGLE BETWEEN TWO POLAR CURVES

Let $r = f_1(\theta)$, $r = f_2(\theta)$ be the given two polor curves having the tangents T_1 and T_2 and making an angles of radius of vector ϕ_1 and ϕ_2 then the angle between the

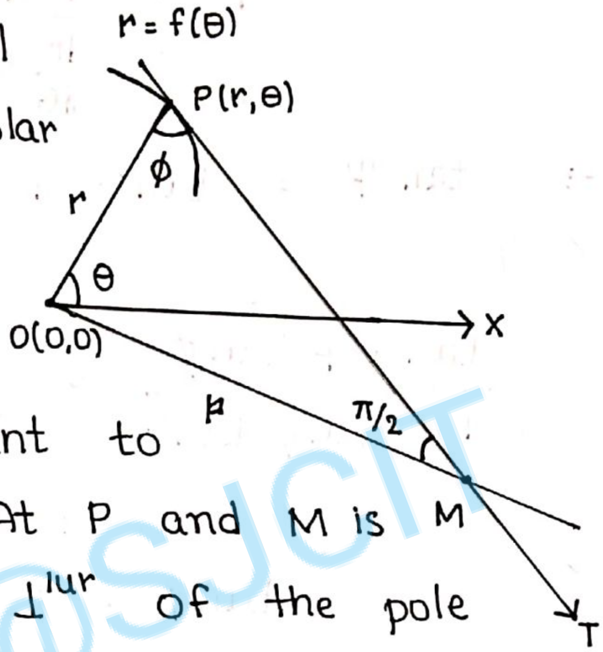


given two polar curves is the angle between there two tangent $= |\phi_2 - \phi_1|$

WITH USUAL NOTATION, PROVE THAT

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Let $\overrightarrow{OX}$ be the intial line $r = f(\theta)$ be a polar curve at $P(r,\theta)$ and $OP = r$ be the radius of Vector.



Let T be the tangent to the curve $r = f(\theta)$. At P and M is the foot of the $\perp^{lur}$ of the pole $O(0,0)$ having the perpendicular distance $OM = p$ from the ΔOPM we have $\underline{|OMP} = 90^\circ$

$$\therefore \sin\phi = \frac{OM}{OP}$$

$$\Rightarrow \sin\phi = \frac{p}{r}$$

$$\Rightarrow \boxed{p = r\sin\phi} \longrightarrow (1)$$

Square on both Sides

$$\Rightarrow p^2 = r^2 \sin^2\phi$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\phi}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \phi$$

$$\boxed{\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2 \phi\right)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \left(\frac{1}{r}\left(\frac{dr}{d\theta}\right)\right)^2\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

$$\boxed{\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

I. Find the angle between the radius vector and tangent for the following polar curves.

① $r = a(1-\cos\theta)$

Given,

$r = a(1-\cos\theta) \longrightarrow ①$

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = a(0 + \sin\theta)$$

$$\frac{dr}{d\theta} = a \sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a \sin\theta}{r}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\cancel{a} \sin\theta}{\cancel{a}(1-\cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{\sin\theta}{(1-\cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{2\sin\theta/2 \cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow \cot\phi = \cot\theta/2$$

$$\boxed{\phi = \theta/2}$$

When, $\theta = \frac{\pi}{3}$

$$\phi = \frac{\pi/3}{2}$$

$$\phi = \frac{\pi}{6} = 30^\circ$$

(2) $r = a(1+\cos\theta)$

Given

$r = a(1+\cos\theta) \longrightarrow (1)$

differentiate (1) w.r.t. θ

$$\frac{dr}{d\theta} = a(0 - \sin\theta)$$

$$\frac{dr}{d\theta} = a(-\sin\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{r}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{(1+\cos\theta)}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$\Rightarrow \cot\phi = -\tan\theta/2$

$\Rightarrow \cot\phi = \cot(\pi/2 + \theta/2)$

$$\phi = \frac{\pi}{2} + \frac{\theta}{2} \quad //$$

③ $r = a(1 + \sin\theta)$

Given

$r = a(1 + \sin\theta) \longrightarrow ①$

differentiate ① w.r.t. θ

$$① \Rightarrow \frac{dr}{d\theta} = a(0 + \cos\theta)$$

$$\Rightarrow \frac{dr}{d\theta} = a\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{a(1+\sin\theta)}$$

$$\Rightarrow \cot\phi = \frac{\cos\theta}{1+\sin\theta}$$

$$\Rightarrow \cot\phi = \frac{\cos^2\theta/2 - \sin^2\theta/2}{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{(\cos\theta/2 + \sin\theta/2)(\cos\theta/2 - \sin\theta/2)}{(\cos\theta/2 + \sin\theta/2)^2}$$

$$= \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$$

$$= \frac{1 - \tan \theta/2}{1 + \sin\theta/2 / \cos\theta/2}$$

$$= \frac{1 - \tan \theta/2}{1 + \tan \theta/2}$$

$$= \frac{\tan(\pi/4) - \tan(\theta/2)}{1 + \tan(\pi/4)\tan(\theta/2)}$$

$$\Rightarrow \cot\phi = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \cot\phi = \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\Rightarrow \phi = \frac{\pi}{2} - \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow \phi = \frac{\pi}{4} + \frac{\theta}{2} \;//$$

II. Show that the following pairs of curves intersect each other orthogonally.

①. $r = a(1 + \cos\theta)$, $r = b(1 - \cos\theta)$

Given

$r = a(1 + \cos\theta) \longrightarrow$ ①

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{r}$$

$$\Rightarrow \cot\phi_1 = \frac{-a\sin\theta}{a(1+\cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{-\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi_1 = -\tan\theta/2$$

$$\cot\phi = \cot\left(\frac{\pi}{2}+\frac{\theta}{2}\right)$$

$$\therefore \phi_1 = \frac{\pi}{2}+\frac{\theta}{2}$$

Similarlly differentiate ② w.r.t.θ

$$\frac{dr}{d\theta} = b\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{b\sin\theta}{r}$$

$$\Rightarrow \cot\phi_2 = \frac{b\sin\theta}{b(1-\cos\theta)}$$

$$\Rightarrow \cot\phi_2 = \frac{\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \cot\phi_2 = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\therefore \cot\phi_2 = \theta/2$$

$$\therefore |\phi_1-\phi_2| = \left|\frac{\pi}{2}+\frac{\theta}{2}-\frac{\theta}{2}\right| = \frac{\pi}{2} //$$

$\therefore$ the Given Curves are intersecting orthogo-nally.

② $r = a(1+\sin\theta)$, $r = a(1-\sin\theta)$

Given

$r = a(1+\sin\theta) \longrightarrow ①$

differentiate ① w.r.t. θ

$$\frac{dr}{d\theta} = a\cos\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a\cos\theta}{r}$$

$$\Rightarrow \cot\phi_1 = \frac{a\cos\theta}{a(1+\sin\theta)}$$

$$\Rightarrow \cot\phi_1 = \frac{\cos\theta}{(1+\sin\theta)}$$

$$\Rightarrow \cot\phi_1 = \frac{\cos^2\theta/2 - \sin^2\theta/2}{\cos^2\theta/2 + \sin^2\theta/2 + 2\sin\theta/2\cos\theta/2}$$

$$= \frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2}$$

$$= \frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$\therefore \cot\phi_1 = \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$$

$$\Rightarrow \cot\phi_1 = \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right]$$

$$\phi_1 = \frac{\pi}{4} + \frac{\theta}{2}$$

Similarlly differentiate ② w.r.t. θ

$$② \Rightarrow \frac{dr}{d\theta} = -a\cos\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a\cos\theta}{a(1-\sin\theta)}$$

$$\Rightarrow \cot\phi_2 = -\left[\frac{\cos^2\theta/2 - \sin^2\theta/2}{(\cos\theta/2 - \sin\theta/2)^2}\right]$$

$$\Rightarrow \cot\phi_2 = -\left[\frac{\cos\theta/2 + \sin\theta/2}{\cos\theta/2 - \sin\theta/2}\right]$$

$$= -\left[\frac{1+\tan\theta/2}{1-\tan\theta/2}\right]$$

$$= -\tan\left[\pi/4 + \theta/2\right]$$

$$\cot\phi_2 = \cot\left[\frac{\pi}{2} + \left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$

$$\therefore \phi_2 = \frac{\pi}{2} + \frac{\pi}{4} + \frac{\theta}{2}$$

$$\therefore |\phi_1 - \phi_2| = \left|\frac{\pi}{4} + \frac{\theta}{2} - \frac{\pi}{2} - \frac{\pi}{4} - \frac{\theta}{2}\right| = \frac{\pi}{2} //$$

$\therefore$ The Given curves are intersecting orthogonally

③ $r^n = a^n(\cos n\theta)$, $r^n = b^n(\sin n\theta)$

Given

$r^n = a^n(\cos n\theta) \longrightarrow$ ①

differentiate ① w r t θ

$$\Rightarrow n r^{n-1}\frac{dr}{d\theta} = -a^n(\sin n\theta)$$

$$\Rightarrow \frac{r^n}{r}\frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a^n\sin n\theta}{r^n}$$

$$\cot\phi_1 = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\cot\phi_1 = -\tan\theta$$

$$\cot\phi_1 = \cot\left(\frac{\pi}{2} + n\theta\right)$$

$$\phi_1 = \frac{\pi}{2} + n\theta$$

Similarlly differentiate ② w.r.t θ

$$② \Rightarrow n r^{n-1} \frac{dr}{d\theta} = b^n n\cos\theta$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = b^n \cos n\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{b^n \cos n\theta}{r^n}$$

$$\Rightarrow \cot\phi_2 = \frac{b^n \cos n\theta}{r^n}$$

$$\cot\phi_2 = \cot n\theta$$

$$\phi_2 = n\theta$$

$$\therefore \; |\phi_1 - \phi_2| = \left|\frac{\pi}{2} + n\theta - n\theta\right| = \pi/2 \;//$$

$\therefore$ the Given curves are intesecting orthogonally

III. Find the angle between the Given pairs of Curves.

① $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$

Given :-

$$r = \sin\theta + \cos\theta \longrightarrow ①$$

$r = 2\sin\theta \longrightarrow (2)$

differentiate equation (1) w. r. t. θ

$$(1) \Rightarrow \frac{dr}{d\theta} = \cos\theta - \sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\cos\theta - \sin\theta}{\sin\theta + \cos\theta}$$

$$= \frac{1 - \tan\theta}{1 + \tan\theta}$$

$$= \frac{\tan(\pi/4) - \tan\theta}{\tan(\pi/4) + \tan\theta}$$

$$\Rightarrow \cot\phi_1 = \tan(\pi/4 - \theta)$$

$$\Rightarrow \cot\phi_1 = \cot\left[\pi/2 - [\pi/4 - \theta]\right]$$

$$\Rightarrow \phi_1 = \frac{\pi}{2} - \frac{\pi}{4} + \theta$$

III^{ly} $(2) \Rightarrow \frac{dr}{d\theta} = 2\cos\theta$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{r}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{2\cos\theta}{2\sin\theta}$$

$$\Rightarrow \cot\phi_2 = \cot\theta$$

$$\Rightarrow \phi_2 = \theta$$

$$\therefore |\phi_1 - \phi_2| = \left| \frac{\pi}{2} - \frac{\pi}{4} + \theta - \theta \right|$$

$$\Rightarrow |\phi_1 - \phi_2| = \pi/4 \; //$$

$\therefore$ The Given pairs of Curves are intersecting at $\frac{\pi}{4}$ $[45^\circ]$

(2) $r = a \log\theta$ and $r = \frac{a}{\log\theta}$

Given

$$r = a \log\theta \longrightarrow (1)$$

$$r = \frac{a}{\log\theta} \longrightarrow (2)$$

differentiate equation (1) and (2) w.r.t. θ

$$(1) \Rightarrow \frac{dr}{d\theta} = a \cdot \left(\frac{1}{\theta}\right)$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{a}{\theta}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{\theta}{a}$$

$$\Rightarrow r \frac{d\theta}{dr} = r \frac{\theta}{a}$$

$$\Rightarrow \tan\phi_1 = \frac{(a \log\theta)\theta}{a}$$

$$\Rightarrow \tan\phi_1 = \theta \log\theta \longrightarrow (3)$$

III'ly differentiate equation (2) w.r.t θ

$$(2) \Rightarrow \frac{dr}{d\theta} \log\theta + \frac{r}{\theta} = 0$$

$$\Rightarrow \log\theta \frac{dr}{d\theta} = -\frac{r}{\theta} = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = -\frac{1}{\theta\log\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = -\theta\log\theta$$

$$\Rightarrow \tan\phi_2 = -\theta\log\theta \longrightarrow (4)$$

W.K.T

$$\tan(\phi_1-\phi_2) = \frac{\tan\phi_1 - \tan\phi_2}{1+\tan\phi_1\tan\phi_2}$$

$$\Rightarrow \tan(\phi_1-\phi_2) = \frac{\theta\log\theta + \theta\log\theta}{1+(\theta\log\theta)(-\theta\log\theta)}$$

$$\Rightarrow \tan(\phi_1-\phi_2) = \frac{2\theta\log\theta}{1-\theta^2(\log\theta)^2} \longrightarrow (5)$$

$\therefore$ from equation (1) and (2)

$$a\log\theta = \frac{a}{\log\theta}$$

$$\Rightarrow (\log\theta)^2 = 1$$

$$\Rightarrow \log_e\theta = 1$$

$$\Rightarrow \theta = e^1 = e$$

$$\therefore (5) \Rightarrow \tan(\phi_1-\phi_2) = \frac{2e}{1-e^2}$$

$$\therefore |\phi_1-\phi_2| = \tan^{-1}\left(\frac{2e}{1-e^2}\right) = 2\tan^{-1}e \,//$$

$\therefore$ The Given two Curves or pair of curves are intersecting at $2\tan^{-1}e$.

③ $r^2 \sin 2\theta = 4$ and $r^2 = 16 \sin 2\theta$

Given

$r^2 \sin 2\theta = 4 \longrightarrow ① \longleftarrow \quad r^2 = \frac{4}{\sin 2\theta}$

$r^2 = 16 \sin 2\theta \longrightarrow ②$

$\therefore$ from equation ① and ②

$$\frac{4}{\sin 2\theta} = 16 \sin 2\theta$$

$\Rightarrow 4 \sin^2 2\theta = 1$

$\Rightarrow \sin^2 2\theta = \frac{1}{4}$

$\Rightarrow \sin 2\theta = \frac{1}{4}$

$\Rightarrow \sin 2\theta = \frac{1}{2}$

$\Rightarrow 2\theta = \sin^{-1}(1/2)$

$\Rightarrow 2\theta = \frac{\pi}{6}$

$\Rightarrow \theta = \frac{\pi}{12}$

differentiate equation ① and ② w.r.t. θ

① $\Rightarrow 2r \frac{dr}{d\theta} \sin 2\theta + r^2 \cos(2\theta) . 2 = 0$

$\Rightarrow r\frac{dr}{d\theta} . \sin 2\theta + r^2 \cos 2\theta = 0$

$\Rightarrow r\frac{dr}{d\theta} . \sin 2\theta = -r^2 \cos 2\theta$

$\Rightarrow r\dfrac{dr}{d\theta} = \dfrac{-r^2\cos 2\theta}{\sin\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\cot 2\theta$

$\Rightarrow \cot\phi_1 = \cot(-2\theta)$

$\Rightarrow \phi_1 = -2\theta$

(2) $\Rightarrow 2r \cdot \dfrac{dr}{d\theta} = 16(\cos 2\theta)(2)$

$\Rightarrow r\dfrac{dr}{d\theta} = 16\cos 2\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{16\cos 2\theta}{r^2}$

$\Rightarrow \cot\phi_2 = \dfrac{16\cos 2\theta}{16\sin 2\theta}$

$\Rightarrow \cot\phi_2 = \cot 2\theta$

$\Rightarrow \phi_2 = 2\theta$

$\therefore |\phi_1 - \phi_2| = |-2\theta - 2\theta| = 4\theta$

$\therefore |\phi_1 - \phi_2| = 4\left(\dfrac{\pi}{12}\right) = \dfrac{\pi}{3}$ //

$\therefore$ The Give pair of curves are intersecting at $\dfrac{\pi}{3}$

(4) $r = a(1-\cos\theta)$ and $r = 2a\cos\theta$

Given,

$r = a(1-\cos\theta) \longrightarrow$ (1)

$r = 2a\cos\theta \longrightarrow (2)$

differentiate equation (1) and (2) w.r.t. θ

(1) $\Rightarrow \dfrac{dr}{d\theta} = a(0+\sin\theta)$

$\Rightarrow \dfrac{dr}{d\theta} = a\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a\sin\theta}{r}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{a\sin\theta}{a(1-\cos\theta)}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{(1-\cos\theta)}$

$\Rightarrow \cot\phi_1 = \dfrac{2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$

$\Rightarrow \cot\phi_1 = \dfrac{\cos\theta/2}{\sin\theta/2}$

$\Rightarrow \cot\phi_1 = \cot\theta/2 \qquad \therefore \phi_1 = \theta/2$

IIIly (2) d. w.r.t θ

$\dfrac{dr}{d\theta} = -2a\sin\theta$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{-2a\sin\theta}{2a\cos\theta}$

$\Rightarrow \dfrac{1}{r}\dfrac{dr}{d\theta} = -\tan\theta$

$\Rightarrow \cot \phi_2 = -\tan\theta$

$\Rightarrow \cot \phi_2 = \cot(\pi/2 + \theta)$

$$\phi_2 = \frac{\pi}{2} + \theta$$

$$\therefore |\phi_1 - \phi_2| = \left|\frac{\theta}{2} - \frac{\pi}{2} - \theta\right|$$

$$|\phi_1 - \phi_2| = \left|\frac{\pi}{2} + \frac{\theta}{2}\right|$$

From ① and ②

$a(1-\cos\theta) = 2a\cos\theta$

$\Rightarrow (1-\cos\theta) = 2\cos\theta$

$\Rightarrow (1-\cos\theta) = 2\cos\theta$

$\Rightarrow 3\cos\theta = 1$

$\Rightarrow \cos\theta = 1/3$

$\Rightarrow \theta = \cos^{-1}(1/3)$

$\therefore |\phi_1 - \phi_2| = |\pi/2 + 1/2 \cos^{-1}(1/3)|$ //

$\therefore$ The Given two curves or pair of curves are intersecting at $|\pi/2 + 1/2 \cos^{-1}(1/3)|$

⑤ $r = \dfrac{a}{1+\cos\theta}$ and $r = \dfrac{b}{1-\cos\theta}$

Given,

$r = \dfrac{a}{1+\cos\theta}$ or $r(1+\cos\theta) = a \longrightarrow$ ①

① d w. r t θ

$$(1+\cos\theta)\frac{dr}{d\theta} - r\sin\theta = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi_1 = \frac{2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi_1 = \tan\theta/2$$

$$\Rightarrow \cot\phi_1 = \cot(\pi/2 - \theta/2)$$

$$\therefore \phi_1 = (\pi/2 - \theta/2)$$

Similarly differentiate ② w. r t. θ

$$r = \frac{b}{1-\cos\theta} \quad \text{or} \quad r(1-\cos\theta) = b \longrightarrow ②$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta} = -\cot\theta/2$$

$$\Rightarrow \cot\phi_2 = -\frac{\theta}{2}$$

$$\therefore |\phi_2 - \phi_1| = \left|-\frac{\theta}{2} - \frac{\pi}{2} + \frac{\theta}{2}\right| = \frac{\pi}{2}$$

$\therefore$ The Given pair of curves are intersect-ing at the angle of $\frac{\pi}{2}$ or 90°

IV. Find the pedal equation For the Following polar curves.

① $r = ae^{\theta\cot\alpha}$

Let,

$r = ae^{\theta\cot\alpha} \longrightarrow ①$

① equation differentiate w.r.t. θ

① $\Rightarrow \frac{dr}{d\theta} = a \cdot e^{\theta\cot\alpha} \cdot \cot\alpha$

$\Rightarrow \frac{dr}{d\theta} = (a\cot\alpha)e^{\theta\cot\alpha}$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{(a\cot\alpha)e^{\theta\cot\alpha}}{r}$

$\Rightarrow \cot\phi = \frac{(a\cot\alpha)e^{\theta\cot\alpha}}{ae^{\theta\cot\alpha}}$

$\Rightarrow \cot\phi = \cot\alpha$

$\Rightarrow \phi = \alpha$

W.K.T the pedal equation

$p = r\sin\phi$

$$\boxed{\Rightarrow p = r\sin\alpha}$$

② $r^n = a^n\cos n\theta$

Given,

$r^n = a^n\cos n\theta \longrightarrow ①$

① equation differentiate w.r.t. θ

① $\Rightarrow nr^{n-1}\frac{dr}{d\theta} = a^n(-\sin n\theta)n$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = -a^n \sin n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{r^n}$$

$$\Rightarrow \cot\phi = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\Rightarrow \cot\phi = -\tan n\theta$$

W. K. T The pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\phi\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left[1 + \tan^2 n\theta\right]$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \sec^2 n\theta$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \cos^2 n\theta}$$

$$\Rightarrow p^2 = r^2 \cos^2 n\theta$$

$$\Rightarrow p = r \cos n\theta$$

$$\Rightarrow p = r\left(\frac{r^n}{a^n}\right) \quad (\because ①)$$

$$\Rightarrow p = \frac{r^n+1}{a^n}$$

$$\boxed{\Rightarrow a^n p = r^{n+1}}$$

③ $\frac{2a}{r} = 1 + \cos\theta$

Given.

$$\frac{2a}{r} = 1+\cos\theta \quad \text{or} \quad 2a = r(1+\cos\theta) \longrightarrow ①$$

② equation differentiate w.r.t. θ

$$① \Rightarrow (1+\cos\theta)\frac{dr}{d\theta} + r(-\sin\theta) = 0$$

$$\Rightarrow (1+\cos\theta)\frac{dr}{d\theta} - r\sin\theta = 0$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1+\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{2\sin\theta/2\,\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi = \tan\theta/2$$

$\therefore$ w.k.T The pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1+\tan^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(\sec^2\theta/2)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\cos^2\theta/2}$$

$$\Rightarrow p^2 = r^2\cos^2\theta/2$$

$$\Rightarrow p^2 = r^2\left(\frac{1+\cos\theta}{2}\right)$$

$$\Rightarrow 2p^2 = r^2(1+\cos\theta)$$

$$\Rightarrow \quad 2p^2 = r^2\left(\frac{2a}{r}\right)$$

$$\boxed{\Rightarrow \quad p^2 = ar}$$

(4) $r^m = a^m(\cos m\theta + \sin m\theta)$

Given

$$r^m = a^m[\cos m\theta + \sin m\theta] \longrightarrow (1)$$

(1) differentiate w.r.t. θ

$$(1) \Rightarrow \quad m r^{m-1}\frac{dr}{d\theta} = a^m[-\sin m\theta(m) + \cos m\theta(m)]$$

$$\Rightarrow \frac{r^m}{r}\frac{dr}{d\theta} = \frac{a^m(\cos m\theta - \sin m\theta)}{r^m}$$

$$\Rightarrow \cot\phi = \frac{a^m(\cos m\theta - \sin m\theta)}{a^m(\cos m\theta + \sin m\theta)}$$

$$\Rightarrow \cot\phi = \frac{\cos m\theta - \sin m\theta}{\cos m\theta + \sin m\theta}$$

$$\therefore \quad 1 + \cot^2\phi = 1 + \frac{(\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{(\cos m\theta + \sin m\theta)^2 + (\cos m\theta - \sin m\theta)^2}{(\cos m\theta + \sin m\theta)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{2(\cos^2 m\theta + \sin^2 m\theta)}{\left(\frac{r^m}{a^m}\right)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{2}{r^{2m}/a^{2m}}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{2a^{2m}}{r^{2m}}$$

$\therefore$ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\cdot\frac{2a^{2m}}{r^{2m}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{2a^{2m}}{r^{2m+2}}$$

$$\boxed{\Rightarrow 2a^{2m}p^2 = r^{2m+2}}$$

5. $\frac{l}{r} = 1 + e\cos\theta$

Given,

$$\frac{l}{r} = 1 + e\cos\theta \longrightarrow (1)$$

differentiate w r t θ

$$(1) \Rightarrow (1+e\cos\theta)\frac{dr}{d\theta} + r(0 - e\sin\theta) = 0$$

$$\Rightarrow 1 + e\cos\theta\frac{dr}{d\theta} = re\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{e\sin\theta}{1+e\cos\theta} = \cot\phi$$

$$\therefore 1+\cot^2\phi = 1 + \frac{e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{(1+e\cos\theta)^2 + e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{1+2e\cos\theta + e^2\cos^2\theta + e^2\sin^2\theta}{(1+e\cos\theta)^2}$$

$$= \frac{1 + 2e\cos\theta + e^2(1)}{(1+e\cos\theta)^2}$$

$$= \frac{1 + 2e\cos\theta + e^2}{(1+e\cos\theta)^2}$$

$$\Rightarrow 1 + \cot^2\phi = \frac{1 + e^2 + 2\left(\frac{l}{r} - 1\right)}{(l/r)^2}$$

$$= \frac{1 + e^2 + \frac{2l}{r} - 2}{l^2/r^2}$$

$$= \frac{e^2 + \frac{2l}{r} - 1}{l^2/r^2}$$

$$1 + \cot^2\phi = \frac{r^2(e^2 + 2l/r - 1)}{l^2}$$

$\therefore$ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \cdot r^2 \left(\frac{e^2 + 2l/r - 1}{l^2}\right)$$

$$\boxed{\Rightarrow \frac{1}{p^2} = \frac{e^2 + 2l/r - 1}{l^2}} \; //$$

Radius of Curvature

Generally the Curvature of any curve Can be denoted by 'k' and the reciprocal of the curvature will be called as the radius of curvature and it will be denoted as,

$$\rho = \frac{1}{K}$$

Note:-

1. the radius of Curvature can be evaluated in the Cartisian form by the following formula.

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}, \quad y_2 \neq 0$$

where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$ at any point P

In other way $\rho = \frac{(1+x_1^2)^{3/2}}{x_2}$, $x_2 \neq 0$

where $x_1 = \frac{dx}{dy}$, $x_2 = \frac{d^2x}{dy^2}$ at any point P

2. The radius of curvature for the polar curve can be evaluated as,

$\rho = r\frac{dr}{dp}$, where $f(p, r, c)$ is the pedal equation.

V

1. Find the radius of curvature of the curve $y = a\log(\sec(x/a))$ at any point

Given,

$y = a\log(\sec(x/a))$

differentiate ① w.r.t. x

$$① \Rightarrow \frac{dy}{dx} = a \cdot \frac{1}{\sec(x/a)} \sec(x/a)\tan(x/a) \cdot 1/a$$

$$\Rightarrow \frac{dy}{dx} = \tan(x/a) = y_1 \longrightarrow (2)$$

differentiate (2) w.r.t. x

$$(2) \Rightarrow \frac{d^2y}{dx^2} = \sec^2(x/a) \cdot 1/a$$

$$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{1}{a}\sec^2(x/a)$$

$\therefore$ W.K.T

$$\rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$\Rightarrow \rho = \frac{(1+\tan^2(x/a))^{3/2}}{1/a \sec^2(x/a)}$$

$$\Rightarrow \rho = \frac{a(\sec^2(x/a))^{3/2}}{\sec^2(x/a)}$$

$$\Rightarrow \rho = \frac{a \sec^3(x/a)}{\sec^2(x/a)}$$

$$\boxed{\Rightarrow \rho = a\sec(x/a)}$$

2. Find the radius of the curvature of the curve $x^3 + y^3 = 3axy$ at the point $P(3a/2, 3a/2)$.

Given,

$$x^3 + y^3 = 3axy \longrightarrow (1)$$

differentiate w r t x

$$(1) \Rightarrow 3x^2 + 3y^2\frac{dy}{dx} = 3a\left[1 \cdot y + x\frac{dy}{dx}\right]$$

$$\Rightarrow x^2 + y^2 y_1 = a[y + x y_1]$$

$$\Rightarrow x^2 + y^2 y_1 - ay - axy_1 = 0$$

$$\Rightarrow (y^2 - ax) y_1 + (x^2 - ay) = 0$$

$$\Rightarrow (y^2 - ax) y_1 + (x^2 - ay) = 0$$

$$\Rightarrow (y^2 - ax) y_1 = ay - x^2$$

$$\Rightarrow y_1 = \frac{ay - x^2}{y^2 - ax} \longrightarrow (2)$$

$$\therefore (y_1)_p = \frac{a\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -\frac{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)} = -1$$

differentiate (2) w.r.t x

$$(2) \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{(y^2 - ax)(ay_1 - 2x) - (ay - x^2)(2yy_1 - a)}{(y^2 - ax)^2}$$

$$(y_2)_p = \frac{\left[\left(\frac{3a}{2}\right)^2 - 4\left(\frac{3a}{2}\right)\right] a(-1) - 2\left(\frac{3a}{2}\right) - 4\left(\frac{3a}{2}\right) - \left(\frac{3a}{2}\right)^2 \left[2\left(\frac{3a}{2}\right)(-1) - a\right]}{\left(\frac{3a}{2}\right)^2 - a\left(\frac{3a}{2}\right)^2}$$

$$\Rightarrow y_2 = \frac{\left(\frac{3a^2}{4}\right)(-4a) - \left(-\frac{3a^2}{4}\right)(-4a)}{\left(\frac{3a^2}{4}\right)^2}$$

$$\Rightarrow y_2 = \frac{-3a^3 - 3a^3}{\frac{9a^4}{16}}$$

$$\Rightarrow y_2 = \frac{-6a^3 \times 16}{9a^4} = \frac{-32}{3a}$$

$$\therefore \rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right| = \left| \frac{(1+(-1)^2)^{3/2}}{-32/3a} \right|$$

$$\Rightarrow \rho = \frac{2^{3/2}}{32/3a}$$

$$\Rightarrow \rho = \frac{2 \cdot 2^{1/2} \times 3a}{32}$$

$$\Rightarrow \rho = \frac{3\sqrt{2}\,a}{16}$$

$$\Rightarrow \rho = \frac{3\sqrt{2} \cdot a}{8 \times \sqrt{2} \times \sqrt{2}}$$

$$\boxed{\Rightarrow \rho = \frac{3a}{8\sqrt{2}}}$$

③ Find the radius of curvature of the curve $a^2y = x^2 - a^2$ at the point where the curve meets at x-axis

Given curve $a^2y = x^2 - a^2$ and the curve meet the x axis at the point 'P' and its y coordinate becomes zero.

$\therefore$ the required point $P(a, 0)$

differentiate equation w r t x

$a^2y = x^2 - a^2 \longrightarrow ①$

$$① \Rightarrow a^2 y_1 = 2x - 0$$

$$\Rightarrow y_1 = \frac{2x}{a^2} \longrightarrow ②$$

$$\therefore (y_1)_p = \frac{2(a)}{a^2} = \frac{2}{a} \longrightarrow ③$$

differentiate ③ w.r.t x

$$③ \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{2}{a^2}$$

$$(y_2)_p = \frac{2}{a^2} \neq 0$$

$$\rho = \left| \frac{(1 - y_1^2)^{3/2}}{y_2} \right|$$

$$\rho = \frac{\left(1 + (2/a)^2\right)^{3/2}}{2/a^2}$$

$$\Rightarrow \rho = \frac{(1 + 4/a^2)^{3/2}}{2/a^3}$$

$$\Rightarrow \rho = \frac{a^2 \left(\frac{a^2+4}{a^2}\right)^{3/2}}{2}$$

$$\Rightarrow \rho = \frac{\frac{a^2 (a^2+4)^{3/2}}{a^3}}{2}$$

$$\Rightarrow \boxed{\rho = \frac{(a^2+4)^{3/2}}{2a}}$$

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④. Find the radius of the curvature of the curve $a^2y = x^3 - a^3$ and meets at x axis.

Given,

$$a^2y = x^3 - a^3 \longrightarrow ①$$

differentiate w r t. x

$$① \Rightarrow a^2y_1 = 3x^2 \longrightarrow ②$$

or

$$y_1 = \frac{3x^2}{a^2} \longrightarrow ②$$

$$\therefore (y_1)_p = \frac{3(a^2)}{a^2} = 3$$

differentiate ② w. r t x

$$② \Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{6x}{a^3}$$

$$(y_2)_p = \frac{6(a^2)}{a^3} = \frac{6}{a}$$

$$\Rightarrow \rho = \left| \frac{(1+y_1^2)^{3/2}}{y_2} \right|$$

$$\Rightarrow \rho = \frac{(1+3^2)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{(1+9)^{3/2}}{6/a}$$

$$\Rightarrow \rho = \frac{a \cdot 10^{3/2}}{6}$$

$$\Rightarrow \rho = \frac{a.10.10^{1/2}}{6}$$

$$\Rightarrow \boxed{\rho = \frac{5\sqrt{10}.a}{3}}$$

(5) Find the radius of the curvature of the curve $x^2y = a(x^2+y^2)$ at Point $P(-2a, 2a)$

Given,

$$x^2y = a(x^2+y^2) \longrightarrow (1)$$

differentiate (1) w.r.t y

$$(1) \Rightarrow x^2(1) + y\,2x\frac{dx}{dy} = a\left(2x\frac{dx}{dy} + 2y\right)$$

$$\Rightarrow x^2 + 2xy.x_1 = 2ax_1 + 2ay$$

$$\Rightarrow 2xy.x_1 - 2ax.x_1 = 2ay - x^2$$

$$\Rightarrow 2xx_1(y-a) = 2ay - x^2$$

$$\Rightarrow x_1 = \frac{2ay-x^2}{2x(y-a)} = \frac{2ay-x^2}{2xy-2ax}$$

$$\therefore (x_1)_P = \frac{2a(2a)-(-2a)^2}{2(-2a)[2a-a]}$$

$$\Rightarrow (x_1)_P = 0$$

differentiate (2) w.r.t y

$$(2) \Rightarrow x_2 = \frac{d^2x}{dy^2} = \frac{(2xy-2ax)(2a-2x.x_1)-(2a-2x.x_1) - (2ay-x^2)[2x+xx_1y-2ax_1]}{[2x(y-a)]^2}$$

$$\therefore (x_2)_p = \frac{[-8a^2+4a^2][2a-0]-0}{[-4a(a)]^2}$$

$$\Rightarrow x_2 = \frac{-8a^3}{16a^4}$$

$$\Rightarrow x_2 = \frac{-1}{2a} \neq 0$$

$$\therefore \rho = \left|\frac{(1+x_1^2)^{3/2}}{x_2}\right| = \left|\frac{(1+0)^{3/2}}{-1/2a}\right| = 2a$$ //

⑥ Find the radius of the curvature of the polar curve $r = a(1+\cos\theta)$

Given,

$r = a(1+\cos\theta) \longrightarrow ①$

differentiate ① w r t θ

① $\Rightarrow \frac{dr}{d\theta} = a(0-\sin\theta) = -a\sin\theta$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-a\sin\theta}{a(1+\cos\theta)}$$

$$\Rightarrow \cot\phi = \frac{-2\sin\theta/2\cos\theta/2}{2\cos^2\theta/2}$$

$$\Rightarrow \cot\phi = -\tan(\theta/2)$$

W.K.T the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \tan^2 \theta/2\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \sec^2 \theta/2$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \cos^2 \theta/2}$$

$$\Rightarrow p^2 = r^2 \cos^2 \theta/2$$

$$\Rightarrow p^2 = r^2\left(\frac{1+\cos\theta}{2}\right)$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left(\frac{r}{a}\right)$$

$$\Rightarrow 2ap^2 = r^3 \longrightarrow (2)$$

differentiate (2) w r t p

$$(2) \Rightarrow 2a \cdot 2p = 3r^2 \frac{dr}{dp}$$

$$\Rightarrow 4ap = 3r^2 \frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{4ap}{3r^2}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{4ap}{3r^2}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{4ap}{3r}$$

$$\Rightarrow \rho = \frac{4ap}{3r}$$

$$\Rightarrow \rho^2 = \frac{16a^2p^2}{9r^2}$$

$$\Rightarrow \rho^2 = \frac{16a^2}{9} \cdot \frac{1}{r^2}\left(\frac{r^3}{2a}\right)$$

$$\Rightarrow \rho^2 = \left(\frac{8a}{9}\right) r$$

$$\boxed{\Rightarrow \rho^2 \propto r}$$

⑦ Show that for the curve $r(1-\cos\theta) = 2a$ or $\frac{2a}{r} = (1-\cos\theta)$, and ρ^2 varies as r^3 $[\rho^2 \propto r^3]$

Given,

$$r(1-\cos\theta) = 2a \longrightarrow ①$$

differentiate ① w.r.t θ

$$① \Rightarrow (1-\cos\theta)\frac{dr}{d\theta} + r(0+\sin\theta) = 0$$

$$\Rightarrow (1-\cos\theta)\frac{dr}{d\theta} = -r\sin\theta$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{-\sin\theta}{1-\cos\theta}$$

$$\Rightarrow \cot\phi = \frac{-2\sin\theta/2\,\cos\theta/2}{2\sin^2\theta/2}$$

$$\Rightarrow \cot\phi = -\cot(\theta/2)$$

∴ W.K.T the pedal eqⁿ.

$$\frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\phi\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(1 + \cot^2\theta/2\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}\left(\operatorname{cosec}^2\theta/2\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2\sin^2\theta/2}$$

$$\Rightarrow p^2 = r^2\sin^2\theta/2$$

$$\Rightarrow p^2 = r^2\left(\frac{1-\cos\theta}{2}\right)$$

$$\Rightarrow p^2 = \frac{r^2}{2}\left(\frac{2a}{r}\right)$$

$$\Rightarrow p^2 = ar \longrightarrow (2)$$

differentiate (2) w.r.t. p

$$(2) \Rightarrow 2p = a\frac{dr}{dp}$$

$$\Rightarrow \frac{dr}{dp} = \frac{2p}{a}$$

$$\Rightarrow r\frac{dr}{dp} = \frac{2rp}{a}$$

$$\Rightarrow \rho = \frac{2rp}{a}$$

$$\Rightarrow \rho^2 = \frac{4r^2p^2}{a^2}$$

$$\Rightarrow \rho^2 = \frac{4r^2}{a^2}(ar)$$

$$\Rightarrow \rho^2 = \left(\frac{4}{a}\right) r^3$$

$$\boxed{\Rightarrow \rho^2 \propto r^3}$$

8. For the cardiant, $r = a(1-\cos\theta)$, show that $\frac{\rho^2}{r}$ is constant

Given,

$$r = a(1-\cos\theta) \rightarrow (1)$$

differentiate (1) w.r.t θ

$$\frac{dr}{d\theta} = \sin\theta . a$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{a \, 2\sin\theta/2 \cos\theta/2}{a . 2\sin^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \cot\theta/2$$

$$\cot\phi = \cot\theta/2$$

$$\phi = \theta/2$$

$\therefore$ W.K.T the pedal equation

$$p = r\sin(\theta/2)$$

$$p^2 = r^2\sin^2(\theta/2)$$

$$p^2 = r^2\left(\frac{1-\cos\theta}{2}\right)$$

$$\Rightarrow \quad p^2 = \frac{r}{2}\left(\frac{r}{a}\right)$$

$$\Rightarrow \quad p^2 = \frac{r^3}{2a}$$

then differentiate w.r.t p

$$2p = \frac{3r^2}{2a} \frac{dr}{dp}$$

$$r\frac{dr}{dp} = \frac{2p}{\frac{3r}{2a}}$$

$$\Rightarrow \rho = \frac{2p \times 2a}{3r}$$

$$\Rightarrow \rho^2 = \frac{16a}{9r^2}\left(\frac{r^3}{2a}\right)$$

$$\Rightarrow \boxed{\frac{\rho^2}{r} = \frac{8a}{9}} \quad //$$

(9) Find the radius of curvature for the curve $\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right)$ at any point on it

Given,

$$\theta = \frac{\sqrt{r^2-a^2}}{a} - \cos^{-1}\left(\frac{a}{r}\right) \longrightarrow (1)$$

differentiate (1) w.r.t. r

$$(1) \Rightarrow \frac{d\theta}{dr} = \frac{1}{a}\frac{1}{2\sqrt{r^2-a^2}}(2r) - \left(\frac{-1}{\sqrt{1-(a/r)^2}}\right)\frac{d}{dr}\left(\frac{a}{r}\right)$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{r}{a\sqrt{r^2-a^2}} + \frac{1}{\sqrt{1-a^2/r^2}}\, a\left(\frac{-1}{r^2}\right)$$

$$= \frac{r}{a\sqrt{r^2-a^2}} - \frac{ar}{\sqrt{r^2-a^2}}\left(\frac{1}{r^2}\right)$$

$$= \frac{r}{a\sqrt{r^2-a^2}} - \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{d\theta}{dr} = \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r}{a} - \frac{a}{r}\right]$$

$$= \frac{1}{\sqrt{r^2-a^2}}\left[\frac{r^2-a^2}{ar}\right]$$

$$= \frac{1}{\sqrt{r^2-a^2}}\,\frac{\sqrt{r^2-a^2}\,\sqrt{r^2-a^2}}{ar}$$

$$\frac{d\theta}{dr} = \frac{\sqrt{r^2-a^2}}{ar}$$

$$\Rightarrow \frac{dr}{d\theta} = \frac{ar}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot\phi = \frac{a}{\sqrt{r^2-a^2}}$$

$$\Rightarrow \cot\phi = \frac{a^2}{r^2-a^2}$$

$$\Rightarrow 1+\cot^2\phi = \frac{1+a^2}{r^2-a^2} = \frac{r^2-a^2+a^2}{r^2-a^2} = \frac{r^2}{r^2-a^2}$$

$\therefore$ the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\frac{r^2}{(r^2-a^2)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2-a^2}$$

$$\Rightarrow p^2 = r^2-a^2 \longrightarrow (2)$$

differentiate (2) w.r.t. p

$$(2) \Rightarrow 2p = 2r\frac{dr}{dp}$$

$$\Rightarrow r\frac{dr}{dp} = p$$

$$\boxed{\Rightarrow \rho = \sqrt{r^2-a^2}}$$

(10) Find the radius of curvature $r^n = a^n(\sin n\theta)$

Given,

$$r^n = a^n(\sin n\theta) \longrightarrow (1)$$

differentiate w r t r

$$n r^{n-1} \frac{dr}{d\theta} = a^n (\cos n\theta)$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\cot \phi = \cot n\theta$$

$\therefore$ The pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2} (1 + \cot^2 \phi)$$

$$\frac{1}{p^2} = \frac{\operatorname{cosec}^2 n\theta}{r^2}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2 \sin^2 n\theta}$$

$$\Rightarrow p^2 = r^2 \sin^2 n\theta$$

$$\Rightarrow p^2 = r \sin n\theta$$

$$\Rightarrow p^2 = r \frac{r^n}{a^n} = \frac{r^{n+1}}{a^n}$$

$$\Rightarrow a^n p = r^{n+1} \longrightarrow (2)$$

differentiate (2) w r t p

$$(2) \Rightarrow a^n . 1 = (n+1) r^n \frac{dr}{dp}$$

$$\Rightarrow r^n \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r . r^{n-1} \frac{dr}{dp} = \frac{a^n}{n+1}$$

$$\Rightarrow r\frac{dr}{dp} = \left(\frac{a^n}{n+1}\right)\cdot\left(\frac{1}{r^{n-1}}\right)$$

$$\Rightarrow \rho = \left(\frac{a^n}{n+1}\right)\left(\frac{1}{r^{n-1}}\right)$$

$$\boxed{\Rightarrow \rho \propto \frac{1}{r^{n-1}}}$$

(11) Find the radius of Curvature $r^n = a^n(\cos n\theta)$

Given,

$r^n = a^n(\cos n\theta) \longrightarrow (1)$

differentiate (1) w.r.t r

$$(1) \Rightarrow \quad n\, r^{n-1}\frac{dr}{d\theta} = -a^n .\, n \sin^n\theta$$

$$\frac{1}{r}\frac{r^n}{r}\frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{a^n \cos n\theta}$$

$$\cot\phi = -\tan n\theta$$

$$\cot\phi = \cot(\pi/2 + n\theta)$$

$$\phi = \frac{\pi}{2} + n\theta$$

$\therefore$ W K T the pedal equation

$$\Rightarrow \quad p = r \sin\phi$$

$$\Rightarrow \quad p = r \sin\left(\pi/2 + n\theta\right)$$

$$\Rightarrow \quad p = r \cos n\theta$$

$$\Rightarrow p = r\left(\frac{r^n}{a^n}\right)$$

$$\Rightarrow \quad p = \frac{r^{n+1}}{a^n}$$

$$\boxed{\Rightarrow \quad a^n p = r^{n+1}} \longrightarrow (2)$$

Then differentiate w r t p

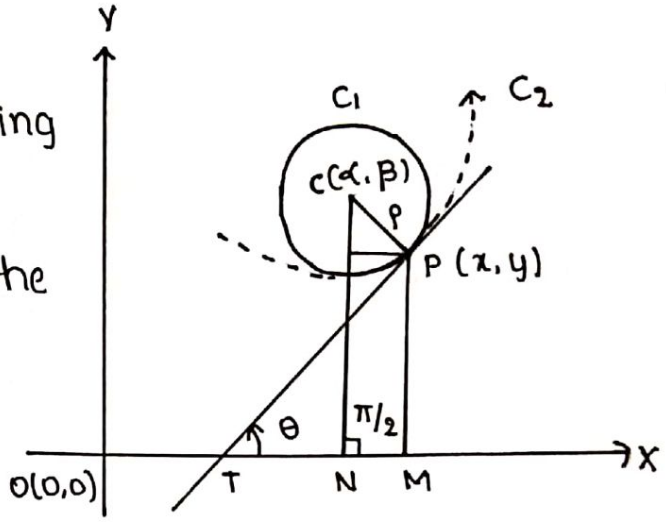
$$a^n = (n+1) r^n \frac{dr}{dp}$$

$$\frac{a^n}{(n+1)}\left(\frac{1}{r^{n-1}}\right) = \rho$$

$$\boxed{\Rightarrow \rho \propto \frac{1}{r^{n-1}}}$$

//

The Centre of Curvature, Invaluates and Evaluates



Let C_1 and C_2 be the two smooth curves passing through the same point $P(x, y)$ the centre of the curve C_1 is $C(\alpha, \beta)$ called as the centre of curvature, and it's radius ρ is called circle curvature the curve C_2 is called as the evaluate of the invaluate C_1. the evaluate of an invaluate can be derived by the locus of the centre of the curvature $C(\alpha, \beta)$.

The centre of curvature $C(\alpha, \beta)$ can be evaluated by using the following formulas

$$\alpha = x - \rho \sin \psi$$

$$\beta = y + \rho \cos \psi$$

$$\Rightarrow \alpha = \frac{x - y_1(1 + y_1^2)}{y_2}, \quad \beta = \frac{y + (1 + y_1^2)}{y_2}$$

where, $y_1 = \frac{dy}{dx}$, $y_2 = \frac{dy^2}{dx^2}$

VIP ①. Show that the evaluate of parabola, $y^2 = 4ax$ is $27ay^2 = 4(x-2a)^3$

Given.

$y^2 = 4ax \longrightarrow ①$

$\Rightarrow y = 2a^{1/2} x^{1/2}$

differentiate ① w.r.t. x

① $\Rightarrow 2y \frac{dy}{dx} = 4a$

$\Rightarrow y \frac{dy}{dx} = 2a$

$\Rightarrow \frac{dy}{dx} = y_1 = \frac{2a}{y} \longrightarrow ②$

differentiate ② w.r.t. x

② $\Rightarrow \frac{d^2y}{dx^2} = y_2 = \frac{y(a) \; 2ay_1}{y^2}$

$\Rightarrow y_2 = \frac{-2ay_1}{y^2}$

$\Rightarrow y_2 = \frac{-2a(2a/y)}{y^2}$

$\Rightarrow y_2 = \frac{-4a^2}{y^3}$

$\Rightarrow y_2 = \frac{-4a^2}{(2a^{1/2} \cdot x^{1/2})^3}$

$\Rightarrow y_2 = \frac{-4a^3}{8a^{3/2} x^{3/2}}$

$\Rightarrow y_2 = \frac{-a^{1/2}}{2x^{3/2}}$

$$\therefore \quad y_1 = \frac{2a}{2a^{1/3}\, x^{1/2}}$$

$$\Rightarrow y_1 = \frac{a^{1 \times 1/2}}{x^{1/2}}$$

$\therefore$ W.K.T

$$\alpha = x - \frac{y_1 (1 + y_1^2)}{y_2}$$

$$\Rightarrow \alpha = x - \frac{\left(\frac{a^{1/2}}{x^{1/2}}\right)(1 + a/x)}{\frac{-a^{1/2}}{2x^{3/2}}}$$

$$\Rightarrow \alpha = \frac{x - 2a^{1/2}\, x^{3/2}\left(\frac{x+a}{x}\right)}{-a^{1/2}\, x^{1/2}}$$

$$\Rightarrow \alpha = \frac{x + 2x \cdot x^{1/2}\left(\frac{x+a}{x}\right)}{x^{1/2}}$$

$$\Rightarrow \alpha = x + 2(x + a)$$

$$\Rightarrow \alpha = 3x + 2a$$

$$\Rightarrow 3x = \alpha - 2a$$

$$\Rightarrow x = \frac{(\alpha - 2a)}{3}$$

$$\Rightarrow x^3 = \frac{(\alpha - 2a)^3}{27} \longrightarrow (3)$$

$$\Rightarrow \beta = \frac{y + (1 + y_1)^2}{y_2} = \frac{2a^{1/2}\, x^{1/2} + \left(1 + \frac{a}{x}\right)}{\frac{-a^{1/2}}{2x^{3/2}}}$$

$$\Rightarrow \beta = 2a^{1/2}x^{1/2} - \frac{2x^{3/2}\left(x+\frac{a}{x}\right)}{a^{1/2}}$$

$$\Rightarrow \beta = 2a^{1/2}x^{1/2} - \frac{2x x^{1/2}\left(\frac{x+a}{x}\right)}{a^{1/2}}$$

$$\Rightarrow \beta = 2a^{1/2}x^{1/2} - \frac{2x^{1/2}(x+a)}{a^{1/2}}$$

$$\Rightarrow \beta = \frac{1}{a^{1/2}}\left| 2ax^{1/2} - 2x^{3/2} - 2ax^{1/2} \right|$$

$$\Rightarrow \beta = \frac{-2x^{3/2}}{a^{1/2}}$$

$$\Rightarrow -2x^{3/2} = \beta a^{1/2}$$

$$\Rightarrow x^{3/2} = \frac{-\beta^{a/2}}{2}$$

$$\Rightarrow (x^{3/2})^2 = \left[\frac{-\beta a^{1/2}}{2}\right]^2$$

$$\Rightarrow x^3 = \frac{\beta^2 a}{4} \longrightarrow (4)$$

From (3) and (4)

$$\frac{(\alpha-2a)^3}{27} = \frac{\beta^2 a}{4}$$

$$\Rightarrow 4(\alpha-2a)^3 = 27a\beta^2 \longrightarrow (5)$$

$\therefore$ the locus of equation (5) is

$$\boxed{4(x-2a)^3 = 27ay^2}$$

② Show that evaluate of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$

Given.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow ①$$

Let $x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \frac{dx}{d\theta} = -a\sin\theta , \frac{dy}{d\theta} = b\cos\theta$$

$$y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta \longrightarrow ②$$

differentiate ② w r t x

$$\Rightarrow y_2 = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow y_2 = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$\Rightarrow y_2 = \frac{d}{d\theta}\left(-\frac{b}{a}\cot\theta\right)\cdot\frac{1}{dx/d\theta}$$

$$\Rightarrow y_2 = \frac{-b}{a}(-\operatorname{cosec}^2\theta)\cdot\frac{1}{-a\sin\theta}$$

$$\Rightarrow y_2 = -\frac{b}{a^2}\frac{1}{\sin^2\theta}\cdot\frac{1}{\sin\theta}$$

$$\Rightarrow y_2 = \frac{-b}{a^2\sin^2\theta}$$

$\therefore$ W.K.T

$$\alpha = \frac{x - y_1(1+y_1^2)}{y_2}$$

$$\alpha = \frac{a\cos\theta - (-b/a \cot\theta)\left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{-b/a^2 \sin^3\theta}$$

$$\Rightarrow \alpha = \frac{a\cos\theta - \frac{b}{a}\frac{\cos\theta}{\sin\theta}\left[1 + \frac{b^2}{a^2}\frac{\cos^2\theta}{\sin^2\theta}\right]}{\frac{b}{a^2\sin^3\theta}}$$

$$\Rightarrow \alpha = a\cos\theta - (a\sin^3\theta)\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{a^2\sin^2\theta + b^2\cos^2\theta}{a^2\sin^2\theta}\right)$$

$$\Rightarrow \alpha = a\cos\theta - \frac{\cos\theta}{a}\left[a^2(1-\cos^2\theta) + b^2\cos^2\theta)\right]$$

$$\Rightarrow \alpha = a\cos\theta - \frac{\cos\theta}{a}\left[a^2 - a^2\cos^2\theta + b^2\cos^2\theta\right]$$

$$\Rightarrow \alpha = a\cos\theta - a\cos\theta + \left[a\cos^3\theta - \frac{b^2}{a}\cos^3\theta\right]$$

$$\Rightarrow \alpha = \left(a - \frac{b^2}{a}\right)\cos^3\theta$$

$$\Rightarrow \alpha = \left(\frac{a^2 - b^2}{a}\right)\cos^3\theta$$

$$\Rightarrow \cos^3\theta = \frac{a\alpha}{a^2 - b^2}$$

$$\Rightarrow (\cos^3\theta)^{2/3} = (a\alpha)^{2/3} / (a^2-b^2)^{2/3}$$

$$\Rightarrow \cos^2\theta = \frac{(a\alpha)^{2/3}}{(a^2-b^2)^{2/3}} \longrightarrow ③$$

$$\beta = \frac{y + (1+y_1^2)}{y_2}$$

$$\Rightarrow \beta = \frac{b\sin\theta + \left(1 + \frac{b^2}{a^2}\cot^2\theta\right)}{-b/a^2 \sin^3\theta}$$

$$\Rightarrow \beta = b\sin\theta - \frac{a^2\sin^3\theta}{b} - b\sin\theta + b\sin^3\theta$$

$$\Rightarrow \beta = \left(b - \frac{a^2}{b}\right)\sin^3\theta$$

$$\Rightarrow \beta = \left(\frac{b^2-a^2}{b}\right)\sin^3\theta$$

$$\Rightarrow \sin^3\theta = \frac{b\beta}{b^2-a^2}$$

$$\Rightarrow \sin^3\theta = \frac{b\beta}{-(a^2-b^2)}$$

$$\Rightarrow \sin^6\theta = \frac{(b\beta)^2}{(a^2-b^2)^{2/3}}$$

$$\Rightarrow \sin^2\theta = \frac{(b\beta)^{2/3}}{(a^2-b^2)^{2/3}} \longrightarrow ④$$

$$③ + ④ = \cos^2\theta + \sin^2\theta = \frac{(a\alpha)^{2/3}}{(a^2-b^2)^{2/3}} + \frac{(b\beta)^{2/3}}{(a^2-b^2)^{2/3}}$$

$$\Rightarrow (a\alpha)^{2/3} + (b\beta)^{2/3} = (a^2 - b^2)^{2/3}$$

$\therefore$ the locus of (α, β)

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3} \ //$$

③ Show that the evaluate of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \ , \quad (ax)^{2/3} - (by)^{2/3} = (a^2 + b^2)^{2/3}$$

Given,

$$x = a \sec\theta \ , \quad y = b\tan\theta$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \longrightarrow ①$$

$$\frac{dx}{d\theta} = a \sec\theta \tan\theta \ , \quad \frac{dy}{d\theta} = b\sec^2\theta$$

$$\therefore y_1 = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{b\sec^2\theta}{a\sec\theta\tan\theta} = \frac{b\sec\theta}{a\tan\theta}$$

$$y_1 = \frac{b}{a} \operatorname{cosec}\theta$$

$$y_2 = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{d\theta}\left(\frac{dy}{dx}\right)\frac{d\theta}{dx}$$

$$= \frac{d}{d\theta}\left(\frac{b}{a}\operatorname{cosec}\theta\right)\frac{1}{dx/d\theta}$$

$$= \frac{b}{a}\left(-\operatorname{cosec}\theta \cdot \cot\theta\right) \cdot \frac{1}{a\sec\theta\tan\theta}$$

$$= \frac{-b}{a^2} \frac{1}{\sin\theta} \frac{\cos\theta}{\sin\theta} \cdot \cos\theta \cdot \frac{\cos\theta}{\sin\theta}$$

$$y_2 = \frac{-b}{a^2} \frac{\cos^3\theta}{\sin^3\theta}$$

$$\therefore \alpha = x - \frac{y_1(1+y_1^2)}{y_2}$$

$$\alpha = \frac{a\sec\theta - \left(\frac{b}{a}\operatorname{cosec}\theta\right)\left[1 + \frac{b^2}{a^2}\operatorname{cosec}^2\theta\right]}{-\frac{b}{a^2}\frac{\cos^3\theta}{\sin^3\theta}}$$

$$\alpha = a\sec\theta + \frac{b}{a} \frac{1}{\sin\theta} \frac{a^2\tan^3\theta}{b\cos^3\theta}\left[1 + \frac{b^2}{a^2} \cdot \frac{1}{\sin^2\theta}\right]$$

$$\alpha = a\sec\theta + \frac{a\sin^2\theta}{\cos^3\theta}\left[\frac{a^2\sin^2\theta + b^2}{a^2\sin^2\theta}\right]$$

$$\alpha = a\sec\theta + \frac{1}{a\cos^3\theta}\left[a^2\sin^2\theta + b^2\right]$$

$$\alpha = a\sec\theta + \frac{a\sin^2\theta}{\cos^3\theta} + \frac{b^2}{a\cos^3\theta}$$

$$\alpha = a\sec\theta + a\tan^2\theta\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$\alpha = a\sec\theta + a\sec\theta(\sec^2\theta - 1) + \frac{b^2}{a}\sec^3\theta$$

$$\alpha = a\sec\theta + a\sec^3\theta - a\sec\theta + \frac{b^2}{a}\sec^3\theta$$

$$\alpha = \left(a + \frac{b^2}{a}\right)\sec^3\theta$$

$$\alpha = \left(\frac{a^2+b^2}{a}\right)\sec^3\theta$$

$$\sec^3\theta = \frac{a\alpha}{a^2+b^2}$$

$$\sec^2\theta = \frac{(a\alpha)^{2/3}}{(a^2+b^2)^{2/3}} \longrightarrow (4)$$

Similarlly $\beta = y + \frac{(1+y_1^2)}{y_2}$

$$\tan^2\theta = \frac{(b\beta)^{3/2}}{(a^2+b^2)^{2/3}} \longrightarrow (5)$$

$$(4) + (5) = 1 = \frac{(a\alpha)^{2/3} - (b\beta)^{2/3}}{(a^2+b^2)^{2/3}}$$

$$(a^2+b^2)^{2/3} = (ax)^{2/3} - (by)^{2/3} \quad //$$

A. Find the pedal equation.

① $r^n = a^n \sin n\theta$

Given,

$$r^n = a^n \sin n\theta \longrightarrow ①$$

① equation differentiate w.r.t. θ

$$① \Rightarrow n r^{n-1} \frac{dr}{d\theta} = a^n \cos n\theta . n$$

$$\Rightarrow \frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\Rightarrow \frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\Rightarrow \cot\phi = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\Rightarrow \cot\phi = \cot n\theta$$

$$\Rightarrow \phi = n\theta$$

We know that, the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2\phi)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 n\theta)$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 n\theta$$

$$\Rightarrow p^2 = r^2 \sin^2 n\theta$$

$$\Rightarrow p = r \sin n\theta$$

$$\Rightarrow p = r\left(\frac{r^n}{a^n}\right)$$

$$\boxed{\Rightarrow p = \frac{r^{n+1}}{a^n}}$$

B. Find the pedal equation for the following polar curves.

① . $r^m = a^m \cos m\theta + b^m \sin m\theta$

Given.

$$r^m = a^m \cos m\theta + b^m \sin m\theta \longrightarrow ①$$

differentiate ① w.r.t θ

$$m r^{m-1} \frac{dr}{d\theta} = -\sin m\theta \cdot m a^m + \cos m\theta \cdot m b^m$$

$$\frac{r^m}{r} \frac{dr}{d\theta} = b^m \cos m\theta - a^m \sin m\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$\cot \phi = \frac{b^m \cos m\theta - a^m \sin m\theta}{a^m \cos m\theta + b^m \sin m\theta}$$

$$1 + \cot^2 \phi = 1 + \frac{(b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$= \frac{(a^m \cos m\theta + b^m \sin m\theta)^2 + (b^m \cos m\theta - a^m \sin m\theta)^2}{(a^m \cos m\theta + b^m \sin m\theta)^2}$$

$$1 + \cot^2 \phi = \frac{2(a^{2m} + b^{2m})}{r^{2m}}$$

∴ W.K.T the pedal equation

$$\frac{1}{p^2} = \frac{1}{r^2}(1 + \cot^2 \phi)$$

$$\frac{1}{p^2} = \frac{1}{r^2}\left(\frac{a^{2m} + b^{2m}}{r^{2m}}\right)$$

$$\Rightarrow \frac{1}{p^2} = \frac{a^{2m}+b^{2m}}{r^{2+2m}}$$

$$\boxed{\Rightarrow r^{2m+2} = p^2\left(a^{2m}+b^{2m}\right)}$$