Module - 2

FOURIER SERIES

Introduction:-

Expressing the function $f(x)$ in the combinations of constants and trignometric ratios in any other interval $(0,\pi)$, $(0,2\pi)$ ... is called the Fourier series.

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + \dots + a_n\cos nx + \dots + b_1\sin x + b_2\sin 2x + b_3\sin 3x + \dots + b_n\sin nx + \dots$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx)$$

where a_0 is called the constant.

a_n, b_n are called fourier coefficients.

Fourier Series Expansion of f(x) over the period 2π:-

The Fourier series Expansion of $f(x)$ over the interval $(c, c+2\pi)$ can be denoted as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n\cos nx + b_n\sin nx) \rightarrow (1)$$

where $a_0 = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\,dx$

$$a_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{c}^{c+2\pi} f(x)\sin nx\,dx$$

This is also called direchlet's Property.

If $c=0$, then the constant and the fourier coefficients of $f(x)$ can be evaluated as $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx$.

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

If $c=-\pi$, then the fourier series coefficients of $f(x)$ in the interval $(-\pi,\pi)$ can be evaluated as

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$$

Note:-

① Suppose $f(x)$ is the continuous function in the interval $[0,2\pi]$ then $f(x)$ is an even function when $f(2\pi-x)=f(x)$ and $f(x)$ is an add function when $f(2\pi-x)=-f(x)$.

② If $f(x)$ is a discontinuous function over the interval $[0,2\pi]$ $f(x)=\begin{cases}\phi(x), & 0\le x\le \pi\\ \psi(x), & \pi\le x<2\pi\end{cases}$, then

$f(x)$ is an even function when $\phi(2\pi-x)=\psi(x)$ and $f(x)$ is an odd function when $\phi(2\pi-x)=-\psi(x)$.

③ Suppose $f(x)$ is the continuous function in the interval $[-\pi,\pi]$, then

$f(x)$ is an even function when $f(-x)=f(x)$ and $f(x)$ is an odd function when $f(-x)=-f(x)$.

④ Suppose $f(x)$ is a discontinuous function in the interval $[-\pi,\pi]$, that is $f(x)=\begin{cases}\phi(x), & -\pi\le x<0\\ \psi(x), & 0\le x<\pi\end{cases}$, then

$f(x)$ is an even function when $\phi(-x)=\psi(x)$ and

$f(x)$ is an even function when $\phi(-x) = \psi(x)$ and $f(x)$ is an odd function when $\phi(-x) = -\psi(x)$.

Even and odd functions:-

WKT the fourier series expansion of $f(x)$ over the period 2π is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$, then

1. If $f(x)$ is an even function, then $b_n = 0$, hence the fourier series of $f(x)$ can be expanded as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$ is also called the half-range cosine series in the $[0,\pi]$ for which

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

2. If $f(x)$ is an odd function, then a_0 and a_n values will be zero, hence the fourier series expansion of $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ is also called the fourier half-range sin series in the $[0,\pi]$ for which $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$

Important Results:-

1. $\sin(n\pi) = 0$, $\sin(2n\pi) = 0$, $\forall\ n \in \mathbb{Z}$
2. $\cos(n\pi) = (-1)^n$, $\cos(2n\pi) = 1$, $\forall\ n \in \mathbb{Z}$
3. $\sin\left(n+\frac{1}{2}\right)\pi = (-1)^n$, $\forall\ n \in \mathbb{Z}$
4. $\cos\left(n+\frac{1}{2}\right)\pi = 0$, $\forall\ n \in \mathbb{Z}$
5. $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} (-1)^{\frac{n-1}{2}}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$
6. $\cos\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2}, & \text{if } n \text{ is even} \end{cases}$

① Obtain the fourier series of $f(x) = \frac{\pi - x}{2}$ in $[0, 2\pi]$ and hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

Given:- $f(x) = \frac{\pi - x}{2}$, $x \in [0, 2\pi]$

$$\therefore f(2\pi - x) = \frac{\pi - (2\pi - x)}{2}$$

$$= \frac{\pi - 2\pi + x}{2}$$

$$= \frac{-\pi + x}{2}$$

$$\therefore f(2\pi - x) = -f(x)$$

$\therefore$ $f(x)$ is an odd function in $[0, 2\pi]$

$\therefore a_0 = 0$ & $a_n = 0$

$\therefore$ The fourier series of

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow ①$$

when $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} \left(\frac{\pi - x}{2}\right) \sin nx \, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} (\pi - x) \sin nx \, dx$$

$$= \frac{1}{\pi}\left\{\left[(\pi - x)\int_0^{\pi} \sin nx \, dx - \int_0^{\pi} (-1)\left[\int \sin nx \, dx\right] dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi - x)\cos nx\right]_0^{\pi} - \frac{1}{n}\int_0^{\pi} \cos nx \, dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi - x)\cos nx\right]_0^{\pi} - \frac{1}{n^2}\left[\sin nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}[0 - \pi] - \frac{1}{n^2}[0 - 0]\right\}$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}\right]$$

$$\Rightarrow \boxed{b_n = \frac{1}{n}}$$

$$① \Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx$$

$$\Rightarrow \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \rightarrow ②$$

$$② \Rightarrow \frac{\pi - \pi/2}{2} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow \frac{\pi}{4} = \frac{1}{1}\sin\left(\frac{\pi}{2}\right) + \frac{1}{2}\sin(\pi) + \frac{1}{3}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{4}\sin(2\pi) + \frac{1}{5}\sin\left(\frac{5\pi}{2}\right) + \dots$$

$$\Rightarrow \frac{\pi}{4} = 1 + 0 - \frac{1}{3} + 0 + \frac{1}{5} - \frac{1}{7} + \dots$$

$$\Rightarrow \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$$

② Obtain a series of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in $[0, 2\pi]$, Hence deduce that

1. $\frac{\pi^2}{6} = \frac{1}{1} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

2. $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\Rightarrow f(x) = \left(\frac{\pi - x}{2}\right)^2, \quad x \in (0, 2\pi)$$

$$\Rightarrow f(x) = \left(\frac{x - \pi}{2}\right)^2 = \frac{1}{4}(x - \pi)^2$$

$$\therefore f(2\pi - x) = \frac{1}{4}(2\pi - x - \pi)^2$$

$$= \frac{1}{4}(\pi - x)^2$$

$$= \frac{1}{4}(x - \pi)^2$$

$$\therefore f(2\pi - x) = f(x)$$

$\therefore f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{4}(x - \pi)^2\,dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} (x^2 - 2\pi x + \pi^2)\,dx$$

$$= \frac{1}{2\pi}\left[\frac{x^3}{3} - \frac{2\pi x^2}{2} + \pi^2 x\right]_0^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^3}{3} - \pi^3 + \pi^3\right]$$

$$= \frac{1}{2\pi} \times \frac{\pi^3}{3}$$

$$\therefore \boxed{a_0 = \frac{\pi^2}{6}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{4}(x-\pi)^2 \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{\pi} (x-\pi)^2 \cos nx \, dx$$

$$= \frac{1}{2\pi}\left\{(x-\pi)^2 \int_0^{\pi} \cos nx \, dx - \int_0^{\pi}\left[2(x-\pi)\int \cos nx \, dx\right]dx\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{1}{n}\left[(x-\pi)^2 \sin nx\right]_0^{\pi} - \frac{2}{n}\int_0^{\pi}\left[(x-\pi)\sin nx \, dx\right]\right\}$$

$$= \frac{1}{2\pi}\left[0 - \frac{2}{n}\int_0^{\pi}(x-\pi)\sin nx \, dx\right]$$

$$= -\frac{1}{n\pi}\int_0^{\pi}(x-\pi)\sin nx \, dx$$

$$= -\frac{1}{n\pi}\left\{(x-\pi)\int_0^{\pi}\sin nx \, dx - \int_0^{\pi}\left[1 \cdot \int \sin nx \, dx\right]dx\right\}$$

$$= -\frac{1}{n\pi}\left\{-\frac{1}{n}\left[(x-\pi)\cos nx\right]_0^{\pi} + \left[\frac{1}{n^2}\sin nx\right]_0^{\pi}\right\}$$

$$= \frac{1}{n\pi}\left\{-\frac{1}{n}\left[0+\pi\right] + \frac{1}{n^2}(0)\right\}$$

$$= \frac{1}{n\pi}\left[\frac{-\pi}{n}\right]$$

$$\boxed{a_n = \frac{1}{n^2}}$$

$$① \Rightarrow \frac{1}{4}(x-\pi)^2 = \frac{\pi^2/6}{2} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$\Rightarrow \frac{1}{4}(x-\pi)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx \rightarrow ②$$

when $x = 0$

$$② \Rightarrow \frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2} \rightarrow ①$$

$$② \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{4} - \frac{\pi^2}{12} = \frac{3\pi^2 - \pi^2}{12} = \frac{\pi^2}{6}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{\pi^2}{6}$$

ii) when $x = \pi$

$$② \Rightarrow 0 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots \frac{\pi^2}{12}$$

③ Find the fourier series expansion of $f(x) = (\pi - x)^2$ in $[0, 2\pi]$.

Given:- $f(x) = (\pi - x)^2$, $x \in [0, 2\pi]$

$$\Rightarrow f(x) = (x-\pi)^2$$

$$\therefore f(2\pi - x) = (2\pi - x - \pi)^2$$

$$= (\pi - x)^2$$

$$\therefore f(2\pi - x) = (x-\pi)^2$$

$\therefore$ $f(x)$ is an even function.

$$\Rightarrow b_n = 0$$

$$\Rightarrow f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x-\pi)^2\,dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (x^2 + \pi^2 - 2x\pi)\,dx$$

$$= \frac{2}{\pi} \left(\frac{x^3}{3} + \pi^2 x - \frac{2x^2\pi}{2} \right)_0^{\pi}$$

$$= \frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ \cos nx.\ dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(x-\pi)^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{(x-\pi)^2\int_0^{\pi}\cos nx\, dx - \int_0^{\pi}\left[2(x-\pi)\int \cos nx\, dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\left[(x-\pi)^2 \sin nx\right]_0^{\pi}\frac{1}{n} - \frac{2}{n}\ (x-\pi)\sin nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{-0 + \frac{2}{n^2}\left[(x-\pi)\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{2}{n^2}(-\pi)\right\}$$

$$\boxed{a_n = \frac{4}{n^2}}$$

$$① \Rightarrow (x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx$$

$$(x-\pi)^2 = \frac{2\pi^2}{6} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx \rightarrow ②$$

when $x = 0$

$$② \Rightarrow \pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = \pi^2 - \frac{\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2} = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{\pi^2}{6}$$

when $x = \pi$

$$② \Rightarrow 0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n 4}{n^2} = \frac{-\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$\Rightarrow -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} \dots \frac{-\pi^2}{12}$$

$\Rightarrow \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \ldots \frac{\pi^2}{12}$

(4) Obtain the fourier series of $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ in $[0, 2\pi]$.

Given: $f(x) = \begin{cases} x, & 0 \le x < \pi \\ 2\pi - x, & \pi \le x < 2\pi, \ x \in [0, 2\pi] \end{cases}$

hence $\phi(x) = x \quad \psi(x) = 2\pi - x$

Let $\phi(2\pi - x) = 2\pi - x = \psi(x)$

$\therefore f(x)$ is an even function.

$\Rightarrow b_n = 0$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$

$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$

$a_0 = \frac{2}{\pi} \int_0^{\pi} x\,dx$

$a_0 = \frac{2}{\pi} \left[\frac{x^2}{2}\right]_0^{\pi}$

$a_0 = \frac{1}{\pi}(\pi^2 - 0)$

$\therefore \boxed{a_0 = \pi}$

$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$

$a_n = \frac{2}{\pi} \int_0^{\pi} x \cos nx\,dx$

$a_n = \frac{2}{\pi} \left\{ x \int_0^{\pi} \cos nx\,dx - \int_0^{\pi} [1 \cdot \int \cos nx\,dx]\,dx \right.$

$a_n = \frac{2}{\pi} \left\{ \frac{1}{n} (x \sin nx)_0^{\pi} + \frac{1}{n^2} (\cos nx)_0^{\pi} \right\}$

$a_n = \frac{2}{\pi} \left\{ \frac{1}{n}(0 - 0) + \frac{1}{n^2} (\cos n\pi - 1) \right\}$

$a_n = \frac{2}{\pi n^2} [(-1)^n - 1]$

$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \forall n = 1, 3, 5 \ldots \\ 0, & \forall n = 2, 4, 6 \ldots \end{cases}$

$$① \Rightarrow f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{\pi n^2} \cos nx$$

⑤ Expand the function $f(x) = x(2\pi - x)$ in fourier series over the limits $[0, 2\pi]$ hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Given $f(x) = x(2\pi - x)$, $x \in [0, 2\pi]$

$$\Rightarrow f(2\pi - x) = (2\pi - x)(2\pi - 2\pi + x)$$

$$= x(2\pi - x)$$

$\therefore f(x)$ is even function.

$$\Rightarrow b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow ①$$

$$\Rightarrow a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x(2\pi - x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} [2\pi x - x^2]\,dx$$

$$= \frac{2}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left(\pi^3 - \frac{\pi^3}{3}\right)$$

$$= \frac{2}{\pi}\left(\frac{2\pi^3}{3}\right)$$

$$\boxed{a_0 = \frac{4\pi^2}{3}}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} (2\pi x - x^2) \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\left[(2\pi x - x^2)\int_0^{\pi} \cos nx\,dx - \int_0^{\pi}\left[(2\pi - 2x)\int \cos nx\,dx\right]dx\right\}\right.$$

$$= \frac{2}{\pi}\left[\frac{1}{n}(2\pi x - x^2)\sin nx\right]_0^{\pi} - \frac{2}{n^2}(\cos nx)_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{1}{n}\int_0^{\pi}[(2\pi - 2x)\sin nx]\,dx\right.$$

$$= \frac{2}{\pi}\left[\frac{2}{n}\int_0^{\pi}(\pi - x)\sin nx\right]dx$$

$$= \frac{2}{\pi}\left[\frac{2}{n}(\pi - x)\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{n^2}\pi\right]$$

$$\boxed{a_n = -\frac{4}{n^2}}$$

$$① \Rightarrow x(2\pi - x) = \frac{4\pi^2/3}{2} + \sum_{n=1}^{\infty}\left(\frac{-4}{n^2}\right)\cos nx$$

$$x(2\pi - x) = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx \rightarrow ②$$

when $x = 0$

$$② \Rightarrow \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx = \frac{2\pi^2}{3}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2\pi^2}{12}$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{\pi^2}{6}$$

when $x = \pi$

$$② \Rightarrow \pi^2 = \frac{2\pi^2}{3} - \sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi$$

$$\sum_{n=1}^{\infty}\frac{4}{n^2}\cos n\pi = \frac{2\pi^2}{3} - \pi^2$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2} = \frac{-\pi^2}{12}$$

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots \frac{-\pi^2}{12}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \cdots \frac{\pi^2}{12}$$

⑥ Find the fourier series of $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi. \end{cases}$

Given: $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ x, & 0 \le x < \pi \end{cases}$

$\therefore f(x)$: $y(x)$ either even no. (or) odd.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx] \rightarrow ①$$

$$\therefore a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx \right]$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx \right]$$

$$= \frac{1}{\pi} \left[\frac{x^2}{2} \right]_0^{\pi}$$

$$\Rightarrow a_0 = \frac{1}{\pi} \left(\frac{\pi^2}{2} \right)$$

$$\Rightarrow \boxed{a_0 = \frac{\pi}{2}}$$

$$\therefore a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\,dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} f(x) \cos nx\,dx + \int_{0}^{\pi} f(x) \cos nx\,dx \right]$$

$$= \frac{1}{\pi} \int_{0}^{\pi} x \cos nx\,dx$$

$$= \frac{1}{\pi} \left[x \int_{0}^{\pi} \cos nx\,dx - \int_{0}^{\pi} [1 . \int \cos nx\,dx]\,dx \right]$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (x \sin nx)_0^{\pi} + \frac{1}{n^2} (\cos nx)_0^{\pi} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{1}{n} (0-0) + \frac{1}{n^2} (\cos n\pi - 1) \right\}$$

$$\Rightarrow a_n = \frac{1}{\pi} \left\{ \frac{1}{n^2} (-1)^n - 1 \right\}$$

$$\Rightarrow b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\int_{0}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{x\int_{0}^{\pi}\sin nx\,dx - \int_{0}^{\pi}\left[1\cdot\int \sin nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(x\cos nx)_{0}^{\pi} + \frac{1}{n^2}(\sin nx)_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(\pi\cos n\pi - 0) + \frac{1}{n^2}(0-0)\right\}$$

$$\Rightarrow b_n = \frac{-\pi\cos n\pi}{n\pi}$$

$$\Rightarrow b_n = -\frac{(-1)^n}{n}$$

$$\Rightarrow b_n = \frac{(-1)^{n+1}}{n}$$

$$① \Rightarrow f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty}\left[\frac{1}{n^2}(-1)^n - 1\ \cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$$

⑦ Obtain the fourier series of $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

hence deduce the series

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Given $f(x) = \begin{cases} 1+\frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1-\frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

$\therefore \phi(x) = 1 + \frac{2x}{\pi}, \quad \psi(x) = 1 - \frac{2x}{\pi}$

$\phi(-x) = 1 - \frac{2x}{\pi} = \psi(x)$

$\therefore f(x)$ is an even function

$\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx \rightarrow ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}(0-0)$$

$$\boxed{a_0 = 0}$$

$$\therefore\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\, dx$$

$$= \frac{2}{\pi}\left\{\left(1 - \frac{2x}{\pi}\right)\int_0^{\pi}\cos nx\,dx - \int_0^{\pi}\left[\left(\frac{-2}{\pi}\right)\int \cos nx\,dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}\left(1 - \frac{2x}{\pi}\right)\sin nx - \int_0^{\pi}\frac{2}{n^2\pi}\,(\cos nx)_0^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n^2\pi}(\cos n\pi - 1)\right\}$$

$$= \frac{4}{n^2\pi^2}\left\{(-1)^n - 1\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$\therefore f(x) = \frac{0}{2} + \sum_{n=1}^{\infty}\frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos nx$$

$$\frac{4}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos nx = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases} \rightarrow (2)$$

when $x = 0$

$$(2) \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = 1$$

$$\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\frac{2}{1^2} + 0 + \frac{2}{3^2} + 0 + \frac{2}{5^2} + 0 + \dots \; \frac{\pi^2}{4}$$

$$2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+0+\right]\cdots\ \frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}-\cdots\ \frac{\pi^2}{8}$$

(5) Obtain the fourier series $f(x)=|x|$ in the interval $[-\pi,\pi]$ and hence deduce the series $\frac{1}{1^2}+\frac{1}{3^2}+\cdots+\frac{\pi^2}{8}$.

Given $f(x)=|x| = \begin{cases} -x, & -\pi\le x\le 0 \\ x, & 0\le x\le \pi \end{cases}$

$\therefore\ \phi(x)=-x \qquad \psi(x)=x$

$\phi(-x)=-(-x)=x=\psi(x)$

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=0}^{\infty}a_n\cos nx \rightarrow (1)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\,dx$$

$$=\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$$

$$\boxed{a_0=\pi}$$

$$\therefore\ a_n=\frac{2}{\pi}\int_0^{\pi}f(x)\cos nx\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx$$

$$=\frac{2}{\pi}\left\{x\int_0^{\pi}\cos nx\,dx-\int_0^{\pi}\left[1\cdot\int\cos nx\,dx\right]dx\right\}$$

$$=\frac{2}{\pi}\left\{\frac{1}{n}[x\sin nx]_0^{\pi}+\frac{1}{n^2}(\cos nx)_0^{\pi}\right\}$$

$$=\frac{2}{\pi n^2}[\cos n\pi-1]$$

$$\therefore\ a_n=\frac{2}{\pi n^2}[(-1)^n-1]$$

$$\therefore (1) \Rightarrow f(x)=\frac{\pi}{2}+\sum_{n=1}^{\infty}\frac{2}{\pi n^2}[(-1)^n-1]\cos nx$$

$$\frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\left(\frac{(-1)^n-1}{n^2}\right)\cos nx = \begin{cases}-x, & -\pi\le x\le 0\\ x, & 0\le x\le \pi\end{cases} \rightarrow (2)$$

when $x=0$

$$(2) \Rightarrow \frac{\pi}{2}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}=0$$

$$\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n^2}=\frac{-\pi^2}{4}$$

$$\frac{-2}{1^2}+0-\frac{2}{3^2}+0-\frac{2}{5^2}+\cdots \quad \frac{-\pi^2}{4}$$

$$-2\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right]=\frac{-\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{\pi^2}{8}$$

(9) Find the fourier series of $f(x)=\sqrt{1-\cos x}$ in the interval $[-\pi,\pi]$ (or) $[0,2\pi]$ and hence deduce $\sum_{n=1}^{\infty}\frac{1}{4n^2-1}=\frac{1}{2}$.

Let $f(x)=\sqrt{1-\cos x}=\sqrt{2\sin^2\left(\frac{x}{2}\right)}=\sqrt{2}\sin\left(\frac{x}{2}\right), x\in[-\pi,\pi]$

$$\therefore f(-x)=\sqrt{1-\cos(-x)}$$
$$=\sqrt{1-\cos x}$$
$$=f(x)$$

$\therefore f(x)$ is an even function.

$b_n=0$.

$$\therefore f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx \rightarrow (1)$$

$$a_0=\frac{2}{\pi}\int_0^{\pi}f(x)\,dx$$

$$=\frac{2}{\pi}\int_0^{\pi}\sqrt{2}\sin\left(\frac{x}{2}\right)dx$$

$$=\frac{2\sqrt{2}}{\pi}\left[-\frac{\cos(x/2)}{1/2}\right]_0^{\pi}$$

$\therefore f(-x) = \sqrt{1-\cos(-x)} = \sqrt{1-\cos x} = f(x)$

$\therefore f(x)$ is an even function.

$\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \longrightarrow ①$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin\left(\frac{x}{2}\right)dx$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{-\cos(x/2)}{1/2}\right]_0^{\pi}$$

$$= -\frac{4\sqrt{2}}{\pi}\left[\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$= -\frac{4\sqrt{2}}{\pi}\left[\cos\frac{\pi}{2} - \cos 0\right]$$

$$= -\frac{4\sqrt{2}}{\pi}[0-1]$$

$$\therefore \boxed{a_0 = \frac{4\sqrt{2}}{\pi}}$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \sqrt{2}\sin(x/2)\cos nx\,dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi} 2\cos nx \sin\left(\frac{x}{2}\right)dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(nx+\frac{x}{2}\right) - \sin\left(nx-\frac{x}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\int_0^{\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)\right]dx$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{\cos(n+1/2)x}{n+1/2} + \frac{\cos(n-1/2)x}{n-1/2}\right]_0^{\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left\{(0-0) - \left[-\frac{1}{n+1/2} + \frac{1}{n-1/2}\right]\right\}$$

$$= \frac{-\sqrt{2}}{\pi}\left[\frac{1}{n-1/2} - \frac{1}{n+1/2}\right]$$

$$= \frac{-\sqrt{2}}{\pi}\left[\frac{1}{\frac{2n-1}{2}} - \frac{1}{\frac{2n+1}{2}}\right]$$

$$= \frac{-2\sqrt{2}}{\pi}\left[\frac{1}{2n-1} - \frac{1}{2n+1}\right]$$

$$= -2\sqrt{2}\left[\frac{2n+1-2n+1}{4n^2-1}\right]$$

$$\therefore a_n = -\frac{4\sqrt{2}}{\pi}\left[\frac{1}{4n^2-1}\right]$$

$$① \Rightarrow \sqrt{1-\cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos nx$$

$x = 0$

$$② \Rightarrow 0 = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}$$

$$\frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}$$

$$\sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{2\sqrt{2}}{\pi}\cdot\frac{\pi}{4\sqrt{2}} = \frac{1}{2}$$

⑩ Find the fourier series $f(x) = x - x^2$, $x \in [-\pi, \pi]$

Given $f(x) = x - x^2$, $x \in [-\pi, \pi]$

$\therefore f(x)$ is neither even nor odd.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos nx + b_n \sin nx\right] \rightarrow ①$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi}(x - x^2)\,dx$$

$$= \frac{1}{\pi}\left(\frac{x^2}{2} - \frac{x^3}{3}\right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2} - \frac{\pi^3}{3}\right] - \left[\frac{\pi^2}{2} + \frac{\pi^3}{3}\right]\right\}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi^3}{3}\right]$$

$$\boxed{a_0 = \frac{-2}{3}\pi^2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{(x-x^2)\int_{-\pi}^{\pi}\cos nx - \int_{-\pi}^{\pi}(1-2x)\int\cos nx\,dx\right\}dx$$

$$= \frac{1}{\pi}\left[(1-2x)\int_{-\pi}^{\pi}\sin nx\,dx + \frac{(1-2x)}{n}\int_{-\pi}^{\pi}\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left[(1-2x)\ [\sin x\right]_{-\pi}^{\pi} - \left[\frac{(1-2x)\cos nx}{n^2}\right]_{-\pi}^{\pi}$$

$$= -\frac{1}{n\pi}\left\{-\frac{1}{n}\left[(1-2\pi)\cos n\pi - (1+2\pi)\cos n\pi\right] - 0\right\}$$

$$= \frac{1}{n^2\pi}\left\{(1-2\pi-1-2\pi)\cos n\pi\right\}$$

$$= \frac{1}{n^2\pi^2}\left[-4\pi\cos n\pi\right]$$

$$= \frac{-4}{n^2}\cos n\pi$$

$$= \frac{-4}{n^2}(-1)^n$$

$$\boxed{a_n = \frac{4}{n^2}(-1)^{n+1}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x-x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\left[(x-x^2)\int_{-\pi}^{\pi}\sin nx\,dx - \int_{-\pi}^{\pi}(1-2x)\int\sin nx\,dx\right]dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(x-x^2)\cos nx\right]_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left[(\pi-\pi^2)\cos n\pi - (-\pi-\pi^2)\cos n\pi\right] + \frac{1}{n}\int_{-\pi}^{\pi}(1-2x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\frac{1}{n}(2\pi\cos n\pi) + \frac{1}{n}I_1\right\}$$

$$b_n = \frac{1}{\pi}\left[-\frac{2\pi\cos n\pi}{n} + \frac{1}{n}I_1\right]$$

$$\therefore I_1 = \int_{-\pi}^{\pi}(1-2x)\cos nx\,dx$$

$$= \frac{1}{n}\left[(1-2x)\sin nx\right]_{-\pi}^{\pi} - \frac{2}{n^2}(\cos nx)_{-\pi}^{\pi}$$

$$\Rightarrow I_1 = 0$$

$$\therefore b_n = \frac{-2\pi\cos n\pi}{n\pi}$$

$$= -\frac{2}{n}\cos n\pi$$

$$= -\frac{2}{n}(-1)^n$$

$$\therefore b_n = \frac{2}{n}(-1)^{n+1}$$

$$① \Rightarrow (x-x^2) = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}(-1)^{n+1}\cos nx + \frac{2}{n}(-1)^{n+1}\sin nx\right] \rightarrow ②$$

when $x=0$

$$② \Rightarrow 0 = \frac{-\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^{n+1}$$

$$4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{3}$$

$$4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots\right] = \frac{\pi^2}{3}$$

$$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots = \frac{\pi^2}{12}$$

11. Find the fourier series $f(x) = x^2$ $[0, \pi]$ or $f(x) = x^2$ $[-\pi, \pi]$. (11)

Given $f(x) = x^2, x \in [0, \pi]$

WKT

The fourier half-range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi}$$

$$\boxed{a_0 = \frac{2}{3}\pi^2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \frac{2}{\pi}\left\{x^2\int_0^{\pi}\cos nx\, dx - \int_0^{\pi}\left[2x\int\cos nx\, dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{\frac{1}{\pi}\left[x^2 \sin nx\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi}[x \sin nx]dx\right\}$$

$$= \frac{-4}{n\pi}\int_0^{\pi} x \sin nx\, dx$$

$$= \frac{-4}{n\pi}\left\{x\int_0^{\pi}\sin nx\, dx - \int_0^{\pi}\left[1 \cdot \int \sin nx\, dx\right]dx\right\}$$

$$= \frac{-4}{n\pi}\left\{-\frac{1}{n}\left[x \cos nx\right]_0^{\pi} + \frac{1}{n^2}(\sin nx)_0^{\pi}\right\}$$

$$= \frac{-4}{n\pi}\left\{-\frac{1}{n}\left[\pi \cos n\pi - 0\right]\right\}$$

$$= \frac{4}{n^2\pi}\left[\pi(-1)^n\right]$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$\therefore (1) \Rightarrow x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} (-1)^n \cos nx.$

(12) Find the fourier half range cosine & sin series of $f(x) = x(\pi - x)$ in the $[0, \pi]$.

Given $f(x) = x(\pi - x) = \pi x - x^2$, $x \in [0, \pi]$

$\therefore$ WKT

The fourier-half range cosine series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \rightarrow (1)$$

$$\therefore a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) dx$$

$$= \frac{2}{\pi} \left[\frac{\pi x^2}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{6} \right]$$

$$a_0 = \frac{2\pi^2}{6} = \frac{\pi^2}{3}$$

$$\therefore a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx \, dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \left\{ (\pi x - x^2) \int_0^{\pi} \cos nx \, dx - \int_0^{\pi} \left[(\pi - 2x) \int \cos nx \, dx \right] dx \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{1}{\pi} \left[(\pi x - x^2) \sin nx \right]_0^{\pi} - \frac{1}{n} \int_0^{\pi} (\pi - 2x) \sin nx \, dx \right\}$$

$$= -\frac{2}{n\pi} \int_0^{\pi} (\pi - 2x) \sin nx \, dx$$

$$= -\frac{2}{n\pi} \left\{ (\pi - 2x) \int_0^{\pi} \sin nx \, dx - \int_0^{\pi} \left[(-2) \int \sin nx \, dx \right] dx \right\}$$

$$= -\frac{2}{n\pi} \left\{ -\frac{1}{n} \left[(\pi - 2x) \cos nx \right]_0^{\pi} - \frac{2}{n^2} \left[\sin nx \right]_0^{\pi} \right\}$$

$$= -\frac{2}{n\pi} \left\{ -\frac{1}{n} \left[-\pi \cos n\pi - \pi \right] - \frac{2}{n^2} (0 - 0) \right\}$$

$$= -\frac{2}{n\pi}\left\{\frac{\pi}{n}(1+\cos n\pi)\right\}$$

$$a_n = \frac{-2}{n^2}\left[1+(-1)^n\right]$$

$$① \Rightarrow x(\pi - x) = \frac{\pi^2}{6} - 2\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n}{n^2}\right]\cos nx$$

WKT

The fourier half range sin series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \rightarrow ②$$

$$\therefore b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi x - x^2)\sin nx\, dx$$

$$= \frac{2}{\pi}\left\{(\pi x - x^2)\int_0^{\pi}\sin nx\, dx - \int_0^{\pi}\left[(\pi - 2x)\int \sin x\, dx\right]dx\right\}$$

$$= \frac{2}{\pi}\left\{-\frac{1}{n}\left[(\pi x - x^2)\cos nx\right]_0^{\pi} + \frac{1}{n}\int_0^{\pi}(\pi - 2x)\cos nx\, dx\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi}(\pi - 2x)\cos nx\, dx$$

$$= \frac{2}{n\pi}\left\{(\pi - 2x)\int_0^{\pi}\cos nx\, dx - \int_0^{\pi}\left[(-2)\int \cos nx\, dx\right]dx\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}\left[(\pi - 2x)\sin nx\, dx\right]_0^{\pi} - \frac{2}{n^2}\left[\cos nx\right]_0^{\pi}\right\}$$

$$= \frac{2}{n\pi}\left\{\frac{1}{n}(0-0) - \frac{2}{n^2}\left[\cos n\pi - \cos 0\right]\right\}$$

$$= -\frac{4}{n^3\pi}\left[(-1)^n - 1\right]$$

$$\therefore b_n = \frac{4}{n^3\pi}\left[1-(-1)^n\right]\sin nx.$$

(13) Expand the function $f(x) = x\sin x$ as a fourier series in the interval $-\pi \le x \le \pi$. Deduce that

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \frac{1}{7\cdot 9} + \dots = \frac{\pi-2}{4}.$$

For the given function $f(x) = x\sin x$, over the interval $[-\pi,\pi]$ which is an even function.

$$f(x) = x\sin x$$

$$f(-x) = (-x)\sin(-x) = x\sin x = f(x)$$

Hence, $b_n = 0$

$$\therefore a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\,dx$$

$$= \frac{2}{\pi}\left[x(-\cos x) - (1)(-\sin x)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi\right]$$

$$\boxed{a_0 = 2}$$

$$\therefore a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - 1\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right]$$

$$\cos(n\pi - \pi) = \cos(\pi - n\pi) = -\cos n\pi$$

$$\cos(n\pi + \pi) = \cos(\pi + n\pi) = -\cos n\pi$$

$$\sin(n+1)\pi = \sin(\pi + n\pi) = -\sin n\pi = 0$$

$$\sin(n-1)\pi = -\sin(\pi - n\pi) = -\sin n\pi = 0$$

$$= \frac{\cos(n-1)\pi}{n-1} - \frac{\cos(n+1)\pi}{n+1}$$

$$= \frac{-\cos n\pi}{n-1} - \frac{\cos n\pi}{n+1}$$

$$a_n = \cos n\pi \left[\frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \cos n\pi \left[\frac{n-1-n-1}{n^2-1}\right] \quad n \neq 1$$

$$a_n = \frac{-2(-1)^n}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1} \quad n \neq 1$$

when $n = 1$ in ① we get

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx = \frac{1}{\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{2}\cos 2\pi + 0 - 0 - 0\right] = -\frac{1}{2}$$

Therefore, $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$x \sin x = \frac{2}{2} + \left(\frac{-1}{2}\right)\cos x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n^2-1} \cos nx$$

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1} + \dots\right]$$

Put $x = \frac{\pi}{2}$ we get

$$\frac{\pi}{2} = 1 - 2\left[\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + \dots\right]$$

$$\frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} + \dots\right)$$

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \dots\dots$$

Fourier series having the period $2l$:-

The fourier series of $f(x) = [c, c+2l]$, over the period $2l$ is defined as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Note :-

① If $c=0$, we can define $f(x)$ in the interval $[0, 2l]$

for $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right]$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

② If $c = -\pi$, we can define $f(x)$ in the interval $[-\pi, \pi]$

for $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

③ If $f(x)$ is an even function in the intervals $[0, 2l]$ or $[-l, l]$, then $b_n = 0$, we have

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$ and it also called the

Fourier half-range cosine series in $[0,l]$.

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

④ If $f(x)$ is an odd function in $[0,2l]$ or $[-l,l]$, then $a_0 = 0$, $a_n = 0$, we have

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ and it is also called the Fourier half-range sin series in $[0,l]$.

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

⑤ If $f(x)$ is continuous in $[0,2l]$, then $f(x)$ is even when $f(2l-x) = f(x)$, $f(x)$ is odd then, $f(2l-x) = -f(x)$.

⑥ $f(x) = \begin{cases} \phi(x), & 0 \le x \le l \\ \psi(x), & l \le x \le 2l \end{cases}$, then

$\phi(2l-x) = \psi(x)$ is even function

$\phi(2l-x) = -\psi(x)$ is odd function

⑦ If $f(x)$ is continuous $[-l,l]$, then $f(x)$ is even $f(-x) = f(x)$ and odd when $f(-x) = -f(x)$.

⑧ $f(x) = \begin{cases} \phi(x), & -l \le x \le 0 \\ \psi(x), & 0 \le x \le l \end{cases}$ then

$\phi(-x) = \psi(x)$ is even function

$\phi(-x) = -\psi(x)$ is odd function.

① Find the Fourier series of $f(x) = x(2-x)$ in the interval $(0,2)$ hence deduce $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Given $f(x) = x(2-x)$, $x \in (0,2)$

$l = 1$

$\therefore f(2-x) = (2-x)[2-(2-x)]$

$\Rightarrow f(2-x) = (2-x)[2-2+x]$

$\Rightarrow f(2-x) = x(2-x) = f(x)$

$\therefore$ $f(x)$ is an even function.

$\Rightarrow b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x(2-x)\,dx$$

$$= 2\left[x^2 - \frac{x^3}{3}\right]_0^{l=1}$$

$$= 2\left[1 - \frac{1}{3}\right]$$

$$= 2\left[\frac{2}{3}\right]$$

$$\boxed{a_0 = \frac{4}{3}}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$= 2\int_0^1 (2x-x^2)\cos(n\pi x)\,dx$$

$$= 2\left\{(2x-x^2)\int_0^1 \cos(n\pi x)dx - \int_0^1\left[(2-2x)\int \cos(n\pi x)\,dx\right]dx\right\}$$

$$= 2\left\{\frac{1}{n\pi}\left[(2x-x^2)\sin(n\pi x)\right]_0^1 - \frac{2}{n\pi}\int_0^1 (1-x)\sin(n\pi x)\,dx\right\}$$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$

$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$= \frac{2}{1}\int_0^1 \pi x\,dx$

$= 2\pi \int_0^1 x\,dx$

$= 2\pi \left[\frac{x^2}{2}\right]_0^1$

$= 2\pi \left[\frac{1}{2} - 0\right]$

$\boxed{a_0 = \pi}$

$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$\therefore a_n = \frac{2}{l}\int_0^1 f(x)\cos(n\pi x)\,dx$

$= 2\int_0^1 f(x)\cos(n\pi x)\,dx$

$= 2\pi\int_0^1 x\cos(n\pi x)\,dx$

$= 2\pi\left\{x\int_0^1 \cos(n\pi x)\,dx - \int_0^1\left[1.\int\cos(n\pi x)\,dx\right]dx\right\}$

$= 2\pi\left\{\frac{1}{n\pi}\left[x\sin(n\pi x)\right]_0^1 + \frac{1}{n^2\pi^2}\left[\cos(n\pi x)\right]_0^1\right\}$

$= \frac{2\pi}{n^2\pi^2}\left[\cos n\pi - 1\right]$

$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$

$\therefore ① \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right]\cos(n\pi x) = \begin{cases} \pi x, & 0 \le x < 1 \\ \pi(2-x), & 1 \le x < 2 \end{cases} \rightarrow ②$

when $x = 0$

$② \Rightarrow \frac{\pi}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = 0$

$\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2} = \frac{-\pi}{2}$

$$= -\frac{4}{n\pi}\int_0^1 (1-x)\ \sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi}\int_0^1 (x-1)\ \sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi}\left\{(x-1)\int_0^1 \sin(n\pi x)\,dx - \int_0^1\left[1.\int \sin(n\pi x)\,dx\right]dx\right\}$$

$$= \frac{4}{n\pi}\left\{-\frac{1}{n\pi}\left[(x-1)\cos(n\pi x)\right]_0^1 + \frac{1}{n^2\pi^2}\left[\sin(n\pi x)\right]_0^1\right.$$

$$= \frac{-4}{n^2\pi^2}\left[0+1\right]$$

$$\boxed{a_n = \frac{-4}{n^2\pi^2}}$$

$\therefore$ ① $\Rightarrow x(2-x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x) \rightarrow$ ②

when $x=0$

$$0 = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{2}{3}\times\frac{\pi^2}{4}$$

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots = \frac{\pi^2}{6}.$$

② Find the fourier series of $f(x) = \begin{cases}\pi x, & 0\le x<1\\ \pi(2-x), & 1\le x<2\end{cases}$

Given $f(x) = \begin{cases}\pi x, & 0\le x<1\\ \pi(2-x), & 1\le x<2\end{cases}$

$f(x) = \begin{cases}\pi x, & 0\le x<l\\ \pi(2-x), & l\le x<2l\end{cases}$ $\qquad 2l=1$ (sic) $\quad \therefore l=1$

$\therefore \phi(x) = \pi x \qquad \psi(x) = \pi(2-x)$

$\therefore \phi(2-x) = \pi(2-x) = \psi(x)$

$\therefore f(x)$ is an even function. $\Rightarrow b_n = 0$

$$\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} = \frac{-\pi^2}{4}$$

$$\Rightarrow \frac{-\pi^2}{4} = \frac{(-2)}{1^2} + \frac{(-2)}{3^2} + \frac{(-2)}{5^2} + \cdots$$

$$\Rightarrow \frac{-\pi^2}{4} = (-2)\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$$\Rightarrow \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

③ Find the fourier series of $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$

Given $f(x) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases}$

$\therefore \phi(x) = l + x \qquad \psi(x) = l - x$

$\therefore \phi(-x) = l - x = \psi(x).$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \longrightarrow ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (l-x)\,dx$$

$$= \frac{2}{l}\left[lx - \frac{x^2}{2}\right]_0^l$$

$$= \frac{2}{l}\left[l^2 - \frac{l^2}{2}\right]$$

$$= \frac{2}{l}\left[\frac{l^2}{2}\right]$$

$$\boxed{a_0 = l}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l (l-x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{(l-x)\int_0^l \cos\left(\frac{n\pi x}{l}\right)dx - \int_0^l\left[(-1)\int \cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{2}{l}\left\{\frac{l}{n\pi}\left[(l-x)\sin\left(\frac{n\pi x}{l}\right)\right]_0^l - \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l\right\}$$

$$= \frac{2}{l}\left\{0 - \frac{l^2}{n^2\pi^2}(\cos n\pi - 1)\right\}$$

$$= \frac{2l^2}{l n^2 \pi^2}\left((-1)^n - 1\right)$$

$$= \frac{-2l}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$\boxed{a_n = \frac{2l}{n^2\pi^2}\left(1 - (-1)^n\right)}$$

$$① \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1-(-1)^n}{n^2}\right]\cos\left(\frac{n\pi x}{l}\right) = \begin{cases} l+x, & -l \le x \le 0 \\ l-x, & 0 \le x \le l \end{cases} \rightarrow ②$$

when $x = 0$

$$② \Rightarrow \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = l$$

$$\Rightarrow \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = l - \frac{l}{2} = \frac{l}{2}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{l}{2} \times \frac{\pi^2}{2l} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \dots = \frac{\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$$

④ Find the fourier series of $f(x) = |x|$ in the interval $(-l, l)$.

$$f(x) = |x| = \begin{cases} -x, & -l \le x \le 0 \\ x, & 0 \le x \le l \end{cases}$$

$\phi(x) = -x \qquad \psi(x) = x$

$\phi(-x) = -(-x) = x = \psi(x) \Rightarrow b_n = 0.$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x\,dx$$

$$= \frac{2}{l}\left(\frac{x^2}{2}\right)_0^l$$

$$= \frac{2}{l}\left(\frac{l^2}{2}\right)$$

$$\boxed{a_0 = l}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left\{x\int_0^l \cos\left(\frac{n\pi x}{l}\right) - \int_0^l\left[1 \cdot \int \cos\left(\frac{n\pi x}{l}\right)dx\right]dx\right\}$$

$$= \frac{2}{l}\left\{\frac{x}{n\pi}\left[x\sin\left(\frac{n\pi x}{l}\right)\right]_0^l + \frac{l^2}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_0^l\right.$$

$$= \frac{2l^2}{ln^2\pi^2}\left[\cos n\pi - 1\right]$$

$$a_n = \frac{2l^2}{ln^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore (1) \Rightarrow f(x) = \frac{l}{2} + \sum_{n=1}^{\infty}\frac{2l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\frac{l}{2} + \frac{2l}{n^2\pi^2}\sum_{n=1}^{\infty}\left[(-1)^n - 1\right]\cos\left(\frac{n\pi x}{l}\right) = \begin{cases} -x, & -l \le x \le 0 \\ x, & 0 \le x \le l \end{cases}$$

where $x = 0$

$$\frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] = 0$$

$$\sum_{n=1}^{\infty}\left[\frac{(-1)^n - 1}{n^2}\right] = -\frac{l}{2} \times \frac{\pi^2}{2l}$$

$$\left[\frac{-2}{1^2}\right] + \left[\frac{-2}{3^2}\right] + \left[\frac{-2}{5^2}\right] + \cdots = \frac{-\pi^2}{4}$$

$$-2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] = \frac{-\pi^2}{4}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

⑤ Find the fourier series of $f(x) = \begin{cases} 1+\frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1-\frac{4x}{3}, & 0 \le x \le \frac{3}{2} \end{cases}$

and hence deduce that the series

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

$$f(x) = \begin{cases} 1+\frac{4x}{3}, & -\frac{3}{2} \le x \le 0 \\ 1-\frac{4x}{3}, & 0 \le x \le \frac{3}{2} \end{cases} = \begin{cases} \phi(x), & -l \le x \le 0 \\ \psi(x), & 0 \le x \le l \end{cases}$$

$\therefore \phi(x) = 1 + \frac{4x}{3} \qquad \psi(x) = 1 - \frac{4x}{3} \qquad l = \frac{3}{2}$

$\therefore \phi(-x) = 1 - \frac{4x}{3} = \psi(x)$

$\therefore f(x)$ is an even function. $\Rightarrow b_n = 0$

$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{4}{3}\int_0^{3/2} f(x)\,dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)dx$$

$$= \frac{4}{3}\left[x - \frac{2x^2}{3}\right]_0^{3/2}$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{2}{3}\left(\frac{3}{2}\right)^2\right]$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{3}{2}\right]$$

$$\Rightarrow \boxed{a_0 = 0}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{4}{3}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{4}{3}\left\{\left(1 - \frac{4x}{3}\right)\int_0^{3/2}\cos\left(\frac{2n\pi x}{3}\right)dx - \int_0^{3/2}\left[\left(-\frac{4}{3}\right)\int\cos\left(\frac{2n\pi x}{3}\right)dx\right]dx\right\}$$

$$= \frac{4}{3}\left\{\frac{3}{2n\pi}\left[\left(1-\frac{4x}{3}\right)\sin\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2} - \frac{4}{3}\,\frac{3^2}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2}\right\}$$ (18)

$$= \frac{4}{3}\left\{0 - \frac{3}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]\right\}$$

$$= \frac{-4}{n^2\pi^2}\left[\cos n\pi - 1\right]$$

$$= \frac{-4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$\therefore a_n = \frac{4}{n^2\pi^2}\left(1-(-1)^n\right)$$

$$\therefore ① \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^2}\right)\cos\left(\frac{2n\pi x}{3}\right) = \begin{cases} 1+\frac{4x}{3}, & \frac{-3}{2}\le x\le 0 \\ 1-\frac{4x}{3}, & 0\le x\le 3/2 \end{cases} \rightarrow ②$$

when $x = 0$

$$② \Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = 1$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \cdots = \frac{\pi^2}{4}$$

$$\Rightarrow \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

⑥ Find the half range Cosine series of $f(x) = x(l-x)$ in the interval $[0,l]$.

Given: $f(x) = x(l-x)$, $x \in [0,l]$

WKT the fourier half range cosine series in $[0,l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \rightarrow ①$$

$$\therefore a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l (lx - x^2)\,dx$$

$$= \frac{2}{l}\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{\ell}\left[\frac{\ell^3}{2} - \frac{\ell^3}{3}\right]$$

$$a_0 = \frac{2}{\ell} \times \frac{\ell^3}{6}$$

$$\boxed{a_0 = \frac{\ell^2}{3}}$$

$$\therefore a_n = \frac{2}{\ell}\int_0^{\ell} a_n \cos\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell}\int_0^{\ell} (\ell x - x^2) \cos\left(\frac{n\pi x}{\ell}\right) dx$$

$$= \frac{2}{\ell}\left\{(\ell x - x^2)\int_0^{\ell} \cos\left(\frac{n\pi x}{\ell}\right) dx - \int_0^{\ell}\left[(\ell - 2x)\int \cos\left(\frac{n\pi x}{\ell}\right) dx\right] dx\right\}$$

$$= \frac{2}{\ell}\left\{\frac{\ell}{n\pi}(0-0) - \frac{\ell}{n\pi}\int_0^{\ell} (\ell - 2x) \sin\left(\frac{n\pi x}{\ell}\right) dx\right\}$$

$$= \frac{-2}{n\pi}\left\{(\ell - 2x)\int_0^{\ell} \sin\left(\frac{n\pi x}{\ell}\right) dx - \int_0^{\ell}\left[(-2)\int \sin\left(\frac{n\pi x}{\ell}\right) dx\right] dx\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{\ell}{n\pi}\left[(\ell - 2x)\cos\left(\frac{n\pi x}{\ell}\right)\right]_0^{\ell} - \frac{2\ell^2}{n^2\pi^2}\left(\sin\left(\frac{n\pi x}{\ell}\right)\right)_0^{\ell}\right\}$$

$$= \frac{-2}{n\pi}\left\{-\frac{\ell}{n\pi}\left[-\ell\cos n\pi - \ell\right] - 0\right\}$$

$$a_n = \frac{-2\ell^2}{n^2\pi^2}\left[(-1)^n + 1\right]$$

$$\therefore f(x) = \frac{\ell^2}{6} + \left(\frac{-2\ell^2}{\pi^2}\right)\sum_{n=1}^{\infty}\left(\frac{(-1)^n + 1}{n^2}\right)\cos\left(\frac{n\pi x}{\ell}\right)$$

⑦ Find the half-range cosine series $f(x) = (x-1)^2$ in the interval $[0,1]$.

Given: $f(x) = (x-1)^2$, $x \in [0,1]$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right) \to ①$$

$$\therefore a_0 = \frac{2}{\ell}\int_0^{\ell} f(x)\, dx$$

$$= \frac{2}{\ell}\int_0^{1} (x-1)^2\, dx$$

$$= 2\left[\frac{(x-1)^3}{3}\right]_0^1$$

$$\boxed{a_0 = \frac{2}{3}}$$

$$\therefore a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 f(x)\cos(n\pi x)dx$$

$$= 2\int_0^1 (x-1)^2\cos(n\pi x)\,dx$$

$$= 2\left\{(x-1)^2\int_0^1 \cos(n\pi x)dx - \int_0^1\left[2(x-1)\int\cos(n\pi x)dx\right]dx\right\}$$

$$= 2\left\{\frac{1}{n\pi}(x-1)^2\left[\sin(n\pi x)\right]_0^1 + \frac{2}{n^2\pi^2}\left[\left[\cos(n\pi x)\right]2(x-1)\right]_0^1\right\}$$

$$a_n = \frac{4}{n^2\pi^2}\left\{\left[(x-1)\cos(n\pi x)\right] + 0\right\} = \frac{4}{n^2\pi^2}\left\{[0+1]\right\} = \frac{4}{n^2\pi^2}$$

$$f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos(n\pi x)$$

Harmonic Fourier Series:

① Let $y = f(x)$ be a periodic function of the period 2π, then the fourier series of $f(x)$ in the harmonics can be expressed as $f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x) + \ldots$

where $\frac{a_0}{2}$ is called the constant term, (a_1, b_1) are called the first harmonic coefficients, (a_2, b_2) are called the second harmonic coefficients and which will be evaluated as

$$a_0 = \frac{2}{N}\sum y \qquad a_2 = \frac{2}{N}\sum y\cos 2x$$

$$a_1 = \frac{2}{N}\sum y\cos x \qquad b_2 = \frac{2}{N}\sum y\sin 2x$$

$$b_1 = \frac{2}{N}\sum y\sin x$$

Generally, $a_n = \frac{2}{N} \Sigma y \cos nx$, $\forall n = 1, 2, 3, \ldots$

$b_n = \frac{2}{N} \Sigma y \sin nx$, $\forall n = 1, 2, 3 \ldots$

where N is the number of terms given in the table which should always be even number.

② Suppose $y = f(x)$ be a periodic function over the period $2l$, then $f(x) = \frac{a_0}{2} + \left[a_1 \cos\left(\frac{\pi x}{l}\right) + b_1 \sin\left(\frac{\pi x}{l}\right)\right] + \left[a_2 \cos\left(\frac{2\pi x}{l}\right) + b_2 \sin\left(\frac{2\pi x}{l}\right)\right] + \ldots$, where $a_0 = \frac{2}{N} \Sigma y$ $\quad \because \theta = \frac{\pi x}{l}$

$a_1 = \frac{2}{N} \Sigma y \cos\theta$

$a_2 = \frac{2}{N} \Sigma y \sin\theta$

$a_n = \frac{2}{N} \Sigma y \cos n\theta$

$b_n = \frac{2}{N} \Sigma y \sin n\theta$.

① The following value of function y gives the displacement in inches of a certain machine part for rotation x of a flywheel. Expand y in terms of fourier series upto the second harmonic.

rotations	x	0	$\pi/6$	$2\pi/6$	$3\pi/6$	$4\pi/6$	$5\pi/6$	π
displacement	y	0	9.2	14.4	17.8	17.3	11.7	0

Sol: The fourier series of $y = f(x)$ upto the second harmonics is

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
$0°$	0	0	0	0	0
$30°$	9.2	7.9674	4.6000	4.6	7.9674
$60°$	14.4	7.2	12.4707	-7.2	12.4707
$90°$	17.8	0	17.8	-17.8	0
$120°$	17.3	-8.65	14.9822	-8.65	-14.9822
$150°$	11.7	-10.1325	5.85	5.85	-10.1325
Σ	70.40	-3.6151	55.7029	-23.20	-4.6766

$N = 6$

$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(70.4) = 23.46$

$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{6}(-3.6151) = -1.2050$

$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{6}(55.7029) = 18.5676$

$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{2}{6}(-23.20) = -7.7333$

$b_2 = \frac{2}{N}\Sigma y\sin 2x = \frac{2}{6}(-4.6766) = -1.5589$

$\therefore f(x) = \frac{a_0}{2} + (a_1\cos x + b_1\sin x) + (a_2\cos 2x + b_2\sin 2x)$

$= \frac{23.46}{2} + (-1.2050\cos x + 18.5676\sin x) + (-7.7333\cos 2x - 1.5589\sin 2x)$

$= 11.73 + (-1.2050\cos x + 18.5676\sin x) + (-7.7333\cos 2x - 1.5589\sin 2x)$

② Express y as a fourier series upto first harmonic for the given data

x	0	30	60	90	120	150	180	210	240	270	300	330
y	1.8	1.1	0.30	0.16	1.50	1.30	2.16	1.25	1.30	1.52	1.76	2.00

x	y	$y\cos x$	$y\sin x$
0	1.8	1.8	0
30	1.1	0.9526	0.55
60	0.30	0.15	0.2598
90	0.16	0	0.16
120	1.50	-0.75	1.2990
150	1.30	-1.1258	0.65
180	2.16	-2.16	0
210	1.25	-1.0825	-0.625
240	1.30	-0.65	-1.1258
270	1.52	0	-1.52
300	1.76	0.88	-1.5242
330	2.00	1.7320	-1
Σ	16.15	-0.2537	-2.8762

$N = 12$

$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{12}(16.15) = 2.6916$

$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{12}(-0.2537) = -0.0422$

$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{12}(-2.8762) = -0.47936$

$f(x) = \frac{2.6916}{2} + (-0.0422\cos x - 0.47936\sin x)$

$= 1.3458 + (-0.0422\cos x - 0.47936\sin x)$

③ Obtain the constant term and first sin, cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol: The given x varies as $0 \le x < 6$

$2l = 6$

$\therefore l = 3$

Scanned by CamScanner

∴ The fourier series expansion upto first harmonic is defined as $f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) \rightarrow ①$

x	y	$\theta = \frac{\pi x}{3}$	$y\cos\theta$	$y\sin\theta$
0	4	0	4	0
1	8	60°	4	6.9282
2	15	120°	-7.5	12.9903
3	7	180°	-7	0
4	6	240°	-3	-5.1961
5	2	300°	1	-1.7320
Σ	42	-	-8.5	12.9901

$$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(42) = 14$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-8.5) = -2.8333$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(12.9901) = 4.3300$$

$$\therefore f(x) = 7 - (2.8333)\cos\left(\frac{\pi x}{3}\right) + (4.3301)\sin\left(\frac{\pi x}{3}\right)$$

④ Obtain the constant term and the first sin & cosine terms in the fourier expansion of y from the following table.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Sol: The given x varies as $0 \le x < 6$

$2l = 6$

$\therefore l = 3$

∴ The fourier series expansion upto the first harmonic is defined as

$$f(x) = \frac{a_0}{2} + a_1 \cos\left(\frac{\pi x}{3}\right) + b_1 \sin\left(\frac{\pi x}{3}\right) \rightarrow ①$$

x	y	$\theta = \frac{\pi x}{3}$	$y\cos\theta$	$y\sin\theta$
0	9	0	9	0
1	18	60°	9	15.5884
2	24	120°	-12	20.7846
3	28	180°	-28	0
4	26	240°	-13	-22.5166
5	20	300°	10	-17.3205
Σ	125	-	-25	-3.4641

$$\therefore a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(125) = 41.667$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-25) = -8.3333$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(-3.4641) = -1.547$$

$$\therefore f(x) = \frac{41.667}{2} - 8.3333\cos\left(\frac{\pi x}{3}\right) - 1.547\sin\left(\frac{\pi x}{3}\right)$$

$$= 20.8335 - 8.3333\cos\left(\frac{\pi x}{3}\right) - 1.547\sin\left(\frac{\pi x}{3}\right)$$

⑤ The following table gives the variations of periodic current over a period T. Show by numerical analysis that there is a direct current part 0.75 Amp. The variable current and obtain the amplitude of first harmonic.

t(sec)	0	T/6	T/3	T/2	2T/3	5T/6	T
A	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Sol:- Given the period to a circuit $2l = T$

$l = T/2$

t	A	$\theta = \frac{\pi t}{l} = \frac{2\pi t}{T}$	$A\cos\theta$	$A\sin\theta$
0	1.98	0°	1.98	0
$T/6$	1.30	60°	0.65	1.1258
$T/3$	1.05	120°	-0.5250	0.9093
$T/2$	1.30	180°	-1.30	0
$\frac{2T}{3}$	-0.88	240°	0.44	0.7621
$\frac{5T}{6}$	-0.25	300°	-0.1250	0.2165
Σ	4.5	-	1.1200	3.0137

$$\therefore a_0 = \frac{2}{N} \Sigma A = \frac{2}{6}(4.5) = 1.5$$

$$a_1 = \frac{2}{N} \Sigma A\cos\theta = \frac{2}{6}(1.12) = 0.3733$$

$$b_1 = \frac{2}{N} \Sigma A\sin\theta = \frac{2}{6}(3.0137) = 1.0045$$

$$\therefore \text{Direct current} = \frac{a_0}{2} = \frac{1.5}{2} = 0.75 \text{ Amp}$$

$$\therefore \text{Amplitude} = \sqrt{a_1^2 + b_1^2} = \sqrt{(0.3733)^2 + (1.0045)^2}$$

$$= 1.0716$$