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Department of Mathematics
Lecture Notes
Calculus & Linear Algebra (18MAT11)

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CALCULUS AND LINEAR ALGEBRA

MODULE -03

INTEGRAL CALCULUS

Note:-

$$\int x^n dx = \frac{x^{n+1}}{n+1}$$

$$\int \sin x dx = -\cos x$$

$$\int \cos x dx = \sin x$$

$$\int e^{ax} dx = \frac{e^{ax}}{a}$$

$$\int u \cdot v dx = u \int v dx - \int (u' \int v dx) dx$$

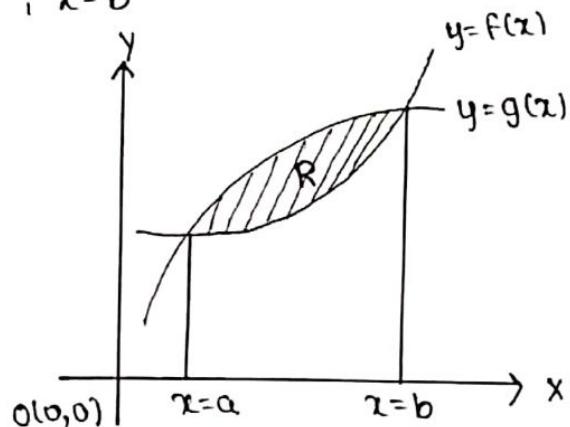
Multiple Integrals:

1. Integral $\int_a^b f(x) dx$ can be described as the

length of a curve $y=f(x)$ from $x=a$ to $x=b$

also it is called the line integral.

2. Integral $\int_a^b \int_{y=f(x)}^{g(x)} \phi(x, y) dy dx$ can be described as region of surface bounded between $y=f(x)$, $y=g(x)$ and $x=a$, $x=b$



3. In Integral,

$$I = \int_{x=a}^b \int_{y=f(x)}^{g(x)} \int_{z=h(x,y)}^{h_2(x,y)} \phi(x, y, z) dz dy dx \text{ used to}$$

calculate the volume between the mentioned boundaries

1. Evaluate $\int_0^1 \int_0^x (x^2 + y^2) dy dx$

$$\begin{aligned} \text{Let } I &= \int_0^1 \int_0^x (x^2 + y^2) dy dx \\ &= \int_{x=0}^1 \left[\int_{y=0}^x (x^2 + y^2) dy \right] dx \end{aligned}$$

$$= \int_{x=0}^1 \left[x^2 y + \frac{y^3}{3} \right]_{y=0}^x dx$$

$$= \int_{x=0}^1 \left[x^3 + \frac{x^3}{3} \right] dx$$

$$= \int_0^1 \frac{4x^3}{3} dx$$

$$= \frac{4}{3} \int_0^1 x^3 dx$$

$$= \frac{4}{3} \left[\frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{3} [x^4]_0^1$$

$$= \frac{1}{3} [1 - 0]$$

$$I = \frac{1}{3}$$

2. Evaluate $\int_0^1 \int_x^{\sqrt{x}} xy dy dx$

$$\text{Let } I = \int_0^1 \int_x^{\sqrt{x}} xy dy dx$$

$$= \int_{x=0}^1 \int_{y=x}^{\sqrt{x}} xy dy dx$$

$$= \int_{x=0}^1 x \int_{y=x}^{\sqrt{x}} y dy dx$$

$$= \int_{x=0}^1 x \int_{y=x}^{\sqrt{x}} y dy dx$$

$$= \int_{x=0}^1 x \left[\frac{y^2}{2} \right]_{\sqrt{x}}^{\sqrt{x}} dx$$

$$= \frac{1}{2} \int_{x=0}^1 x [y^2]_{\sqrt{x}}^{\sqrt{x}} dx$$

$$= \frac{1}{2} \int_{x=0}^1 x(x-x^2) dx$$

$$= \frac{1}{2} \int_{x=0}^1 (x^2 - x^3) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - \frac{1}{4} \right]$$

$$= \frac{1}{2} \left[\frac{4-3}{12} \right]$$

$$I = \frac{1}{24}$$

3. Evaluate $\int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dz dy dx$

$$I = \int_{-c}^c \int_{-b}^b \int_{-a}^a (x^2 + y^2 + z^2) dz dy dx$$

$$= \int_{x=-c}^c \int_{y=-b}^b \left[x^2 z + y^2 z + \frac{z^3}{3} \right]_{z=-a}^a dy dx$$

$$= \int_{x=-c}^c \int_{y=-b}^b \left\{ \left[ax^2 + ay^2 + \frac{a^3}{3} \right] - \left[-ax^2 - ay^2 - \frac{a^3}{3} \right] \right\} dy dx$$

$$= \int_{x=-c}^c \int_{y=-b}^b \left[2ax^2 + 2ay^2 + \frac{2a^3}{3} \right] dy dx$$

$$= 2a \int_{x=-c}^c \int_{y=-b}^b \left[x^2 + y^2 + \frac{a^2}{3} \right] dy dx$$

$$= 2a \int_{x=-c}^c \left[x^2 y + \frac{y^3}{3} + \frac{a^2 y}{3} \right]_{y=-b}^b dx$$

$$\begin{aligned}
&= 2a \int_{x=-c}^c \left\{ \left[bx^2 + \frac{b^3}{3} + \frac{ba^2}{3} \right] - \left[-\frac{bx^2}{1} - \frac{b^3}{3} - \frac{ba^2}{3} \right] \right\} dx \\
&= 2a \int_{x=-c}^c \left[2bx^2 + \frac{2b^3}{3} + \frac{2ba^2}{3} \right] dx \\
&= 4ab \int_{x=-c}^c \left[x^2 + \frac{b^2}{3} + \frac{a^2}{3} \right] dx \\
&= 4ab \left[\frac{x^3}{3} + \frac{b^2 x}{3} + \frac{a^2 x}{3} \right]_{-c}^c \\
&= 4ab \left\{ \left[\frac{c^3}{3} + \frac{cb^2}{3} + \frac{ca^2}{3} \right] - \left[-\frac{c^3}{3} - \frac{cb^2}{3} - \frac{ca^2}{3} \right] \right\} \\
&= 4ab \left[\frac{ac^3}{3} + \frac{2cb^2}{3} + \frac{2ca^2}{3} \right] \\
&= 8abc \left[\frac{a^2}{3} + \frac{b^2}{3} + \frac{c^2}{3} \right] \\
&= \frac{8}{3} abc [a^2 + b^2 + c^2]
\end{aligned}$$

4. Evaluate $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dy dz dx$

$$I = \int_{z=-1}^1 \int_{x=0}^z \int_{x-z}^{x+z} (x+y+z) dy dz dx$$

$$I = \int_{z=-1}^1 \int_{x=0}^z xy + \frac{y^2}{2} + yz \int_{u=x-z}^{x+z} du dz$$

$$\begin{aligned}
&= \int_{z=-1}^1 \int_{x=0}^z \left\{ x(x+z) + \frac{(x+z)^2}{2} + z(x+z) \right\} - \left[x(x-z) + \frac{(x-z)^2}{2} + z(x-z) \right] dx dz
\end{aligned}$$

$$\begin{aligned}
&= \int_{z=-1}^1 \int_{x=0}^z \left\{ x(x+z-x+z) + \frac{1}{2} [(x+z)^2 - (x-z)^2 + z(x+z) - x(z-x)] \right\} dx dz
\end{aligned}$$

$$= \int_{z=-1}^1 \int_{x=0}^z (4zx + 2z^2) dz dx$$

$$= \int_{z=-1}^1 \left[4z \frac{x^2}{2} + 2z^2 x \right]_{x=0}^z dz$$

$$= \int_{z=-1}^1 (2z^3 + 2z^3) dz$$

$$= \int_{z=-1}^1 4z^3 dz$$

$$= 4 \left[\frac{z^4}{4} \right]_{-1}^1$$

$$= [z^4]_{-1}^1$$

$$= (1^4) - (-1)^4$$

$$= 1 - 1$$

$$\boxed{I = 0}$$

5. Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz dy dx$

$$I = \int_{x=0}^a \int_{y=0}^x e^{x+y} e^z dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x e^{x+y} \int_{z=0}^{x+y} e^z dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x e^{x+y} [e^z]_{z=0}^{x+y} dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x e^{x+y} [e^{x+y} - 1] dy dx$$

$$= \int_{x=0}^a \int_{y=0}^x [e^{2x+2y} - e^{x+y}] dy dx$$

$$\begin{aligned}
&= \int_{x=0}^a e^{2x} \int_{y=0}^x e^{2y} dy dx - \int_{x=0}^a e^x \int_{y=0}^x e^y dy dx \\
&= \int_{x=0}^a e^{2x} \int_{y=0}^x e^{2y} dy dx - \int_{x=0}^a e^x \int_{y=0}^x e^y dy dx \\
&= \frac{1}{2} \int_{x=0}^a e^{2x} \left[\frac{e^{2y}}{2} \right]_0^x dx - \int_{x=0}^a e^x [e^y]_0^x dx \\
&= \frac{1}{2} \int_0^a (e^{4x} - e^{2x}) dx - \int_0^a (e^{2x} - e^x) dx \\
&= \frac{1}{2} \left[\frac{e^{4x}}{4} - \frac{e^{2x}}{2} \right]_0^a - \left[\frac{e^{2x}}{2} - \frac{e^x}{1} \right]_0^a \\
&= \frac{1}{2} \left\{ \left[\frac{e^{4a}}{4} - \frac{e^{2a}}{2} \right] \cdot \left[\frac{1}{4} - \frac{1}{2} \right] \right\} - \left\{ \left[\frac{e^{2a}}{2} - e^a \right] - \left[\frac{1}{2} - 1 \right] \right\} \\
&= \frac{1}{2} \left\{ \frac{e^{4a}}{4} - \frac{e^{2a}}{2} + \frac{1}{4} \right\} - \left\{ \frac{e^{2a}}{2} - e^a + \frac{1}{2} \right\} \\
&= \frac{e^{4a}}{8} + \frac{e^{2a}}{4} + \frac{1}{8} - \frac{e^{2a}}{2} + e^a - \frac{1}{2} \\
&= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^a \cdot \frac{3}{8}
\end{aligned}$$

$I = \frac{1}{8} [e^{4a} - 6e^{2a} + 8e^a - 3]$

G. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dz dy dx$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dz dy dx$$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{(\sqrt{a^2-x^2-y^2})^2 - z^2} dz dy dx$$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{(\sqrt{a^2-x^2-y^2})^2 - z^2} dz dy dx$$

$$\text{Let } K = \sqrt{a^2 - x^2 - y^2}$$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^K \frac{1}{\sqrt{K^2-z^2}} dz dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(z/K) \right]_0^K dy dx$$

$$= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin^{-1}(0) \right] dy dx$$

$$= \frac{\pi}{2} \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} 1 dy dx$$

$$= \frac{\pi}{2} \int_{x=0}^a \sqrt{a^2-x^2} dx$$

$$\text{Let } x = a \sin \theta \Rightarrow \theta = \sin^{-1}(x/a)$$

$$dx = a \cos \theta d\theta$$

$$\text{U.L} \Rightarrow x = a \Rightarrow \theta = \frac{\pi}{2}$$

$$\text{L.L} \Rightarrow x = 0 \Rightarrow \theta = 0$$

$$\therefore I = \frac{\pi}{2} \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} a \cos \theta \cdot a \cos \theta d\theta$$

$$\begin{aligned}
 &= \frac{\pi a^2}{2} \int_0^{\pi/2} \cos^2 \theta d\theta \\
 &= \frac{\pi a^2}{2} \int_0^{\pi/2} \left(\frac{1 + \cos 2\theta}{2} \right) d\theta \\
 &= \frac{\pi a^2}{4} \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\
 &= \frac{\pi a^2}{4} \left(\frac{\pi}{2} + 0 \right) - (0+0)
 \end{aligned}$$

$$I = \frac{\pi^2 a^2}{8}$$

7. Evaluate $\int_0^1 \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dz dy dx$

$$I = \int_0^1 \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}} dz dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{(\sqrt{a^2-x^2-y^2})^2 - z^2} dz dy dx$$

$$\text{let } k = \sqrt{a^2-x^2-y^2}$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^k \frac{1}{\sqrt{k^2-z^2}} dz dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} \left[\sin^{-1}(z/k) \right]_{z=0}^k dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} [\sin^{-1}(1) - \sin^{-1}(0)] dy dx$$

$$= \frac{\pi}{2} \int_{x=0}^1 \int_{y=0}^{\sqrt{a^2-x^2}} 1 \, dy \, dx$$

$$= \frac{\pi}{2} \int_{x=0}^1 [y]_0^{\sqrt{a^2-x^2}} \, dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{a^2-x^2} \, dx$$

$$\text{Let } x = a \sin \theta \Rightarrow \theta = \sin^{-1}(x/a)$$

$$dx = a \cos \theta d\theta$$

$$\text{UL : } x = 1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\text{LL : } x = 0 \Rightarrow \theta = 0$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sqrt{1-\sin^2 \theta} \cdot \cos \theta \, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \cos \theta \cos \theta \, d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \left(1 - \frac{\cos \theta}{2} \right) \, d\theta$$

$$= \frac{\pi}{2} \left[0 + \frac{\sin 2\theta}{2} \right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \left[\frac{\pi}{2} + 0 \right] - (0+0)$$

$$I = \frac{\pi^2}{8}$$

8. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} xyz \, dz \, dy \, dx$$

$$\begin{aligned}
&= \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} xy \left[\frac{z^2}{2} \right]_0^{\sqrt{a^2-x^2-y^2}} dx dy \\
&= \frac{1}{2} \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} xy (a^2 - x^2 - y^2) dy dx \\
&= \frac{1}{2} \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} (a^2 xy - x^3 y) dy dx \\
&= \frac{1}{2} \int_{x=0}^a \left[a^2 x \cdot \frac{y^2}{2} - x^3 \frac{y^2}{2} - \frac{x^4}{4} \right]_0^{\sqrt{a^2-x^2}} dx \\
&= \frac{1}{2} \int_{x=0}^a \left[\frac{a^2 x}{2} (a^2 - x^2) - \frac{x^3}{2} (a^2 - x^2) - \frac{x^4}{4} (a^2 - x^2) \right] dx \\
&= \frac{1}{2} \int_{x=0}^a (a^2 - x^2) \left(\frac{a^2 x}{2} - \frac{x^3}{2} \cdot x \frac{(a^2 - x^2)}{4} \right) dx \\
&= \frac{1}{2} \int_{x=0}^a \frac{(a^2 - x^2)}{4} \left[2a^2 x - 2x^3 - a^2 x + x^5 \right] dx \\
&= \frac{1}{8} \int_0^a (a^2 - x^2) (a^2 x - x^3) dx \\
&= \frac{1}{2} \int_0^a (a^4 x - a^2 x^3 - a^2 x^3 + x^5) dx \\
&= \frac{1}{8} \int_0^a [x^5 - 2a^2 x^3 + a^4 x] dx \\
&= \frac{1}{8} \left\{ \frac{x^6}{6} - \frac{2a^2 x^4}{4} + a^4 \frac{x^2}{2} \right\}_0^a \\
&= \frac{1}{8} \left\{ \frac{a^6}{6} - \frac{a^6}{2} + \frac{a^6}{2} \right\} \Rightarrow J = \frac{a^6}{48}
\end{aligned}$$

\Rightarrow change of order of Integration:

1. Evaluate $\int_0^1 \int_{\sqrt{x}}^{\sqrt[4]{x}} xy \, dy \, dx$ by change of order of Integration.

$$\Rightarrow I = \int_0^1 \int_x^{\sqrt[4]{x}} xy \, dy \, dx$$

here x varies as $x=0, x=1 \rightarrow ①$
 and then y varies as $y=x$
 and $y=\sqrt[4]{x} \Rightarrow y^4=x \rightarrow ②$

From ① & ②

$$x = \sqrt[4]{x}$$

$$\Rightarrow x^2 = x$$

$$\Rightarrow x^2 - x = 0$$

$$\Rightarrow x(x-1) = 0$$

$$\Rightarrow x=0, x=1$$

$$\Rightarrow y=0, y=1$$

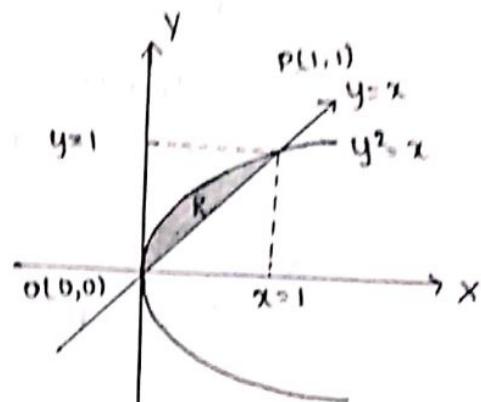
\therefore By the change of order of Integral

$$I = \int_{y=0}^1 \int_{x=y^2}^{\sqrt[4]{y}} xy \, dx \, dy$$

$$= \int_{y=0}^1 y \int_{x=y^2}^{\sqrt[4]{y}} x \, dx \, dy$$

$$= \int_{y=0}^1 y \left(\frac{x^2}{2} \right) \Big|_{y^2}^{\sqrt[4]{y}} \, dy$$

$$= \int_{y=0}^1 y \left(\frac{y^2}{2} - \frac{y^4}{2} \right) \, dy$$



$$= \frac{1}{2} \int_0^1 (y^3 - y^5) dy$$

$$= \frac{1}{2} \left[\frac{y^4}{4} - \frac{y^6}{6} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{4} - \frac{1}{6} \right]$$

$$= \frac{1}{2} \left[\frac{6-4}{24} \right]$$

$$= \frac{1}{2} \times \frac{2}{24}$$

$$I = \frac{1}{24} \text{ Sq units.}$$

2. Evaluate by change of order of Integration.

$$\int_0^{4a} \int_x^{2\sqrt{ax}} x^2 dy dx, a > 0.$$

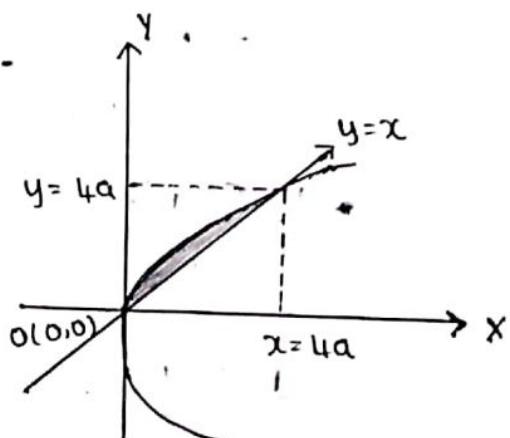
$$I = \int_{x=0}^{4a} \int_{y=x}^{2\sqrt{ax}} x^2 dy dx \rightarrow ①$$

Here, $x=0, x=4a$

$$y=x, y=2\sqrt{ax}, \Rightarrow y^2=4ax$$

$\rightarrow ①$

$\rightarrow ②$



From ① and ②

$$x = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 4ax$$

$$\Rightarrow x^2 - 4ax = 0$$

$$\Rightarrow x(x - 4a) = 0$$

$$\Rightarrow x=0, x=4a$$

$$\Rightarrow y=0, y=4a$$

$$I = \int_{y=0}^{4a} \int_{x=y}^{y^2/4a} x^2 dx dy$$

$$= \int_{y=0}^{4a} \left[\frac{x^3}{3} \right]_{y^2/4a}^{y^2/4a} dy$$

$$= \frac{1}{3} \int_{y=0}^{4a} \left[\left(\frac{y^2}{4a} \right)^3 - y^3 \right] dy$$

$$= \frac{1}{3} \int_{y=0}^{4a} \left[\frac{y^6}{64a^3} - y^3 \right] dy$$

$$= \frac{1}{3} \left[\frac{y^7}{7 \times 64a^3} - \frac{y^4}{4} \right]_0^{4a}$$

$$= \frac{1}{3} \left[\frac{(4a)^7}{7 \times 64a^3} - \frac{(4a)^4}{4} \right]$$

$$= \frac{1}{3} \left[\frac{4^7 \times a^7}{4^3 \times 7 \times a^3} - \frac{(4)^4 (a^4)}{4} \right]$$

$$= \frac{1}{3} \left[\frac{(4^5) a^4 - 7(4^4) (a^4)}{28} \right]$$

$$= \frac{1}{3} \left[\frac{256 a^4 - 1792 \cdot a^4}{28} \right]$$

$$= \frac{1}{3} \cdot a^4 \left[\frac{256 - 1792}{28} \right]$$

$$= \frac{a^4 [-1536]}{28 \times 3}$$

$$= \frac{a^4 (-1536)}{84}$$

$$= a^4 = 18.28$$

$$= \frac{1}{3} \left[\frac{(4^5) a^4 - 7(4^4) a^4}{28} \right]$$

$$= \frac{1}{3} [4^4] \left[\frac{4a^4 - 7a^4}{28} \right]$$

$$= \frac{1}{3} [4^4] \left[\frac{3a^4}{28} \right]$$

$$= \frac{16 \times 16 \times a^4}{28}$$

$$I = \frac{64 a^4}{7} \text{ sq units}$$

3. Evaluate $\int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$ by changing the order of integration.

$$\text{let } I = \int_0^{4a} \int_{x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$$

$$I = \int_{x=0}^{4a} \int_{y=x^2/4a}^{2\sqrt{ax}} xy \, dy \, dx$$

$$x=0, x=4a \text{ and } y=x^2/4a$$

$$x=0, x=4a \text{ and } y=\frac{x^2}{4a}$$

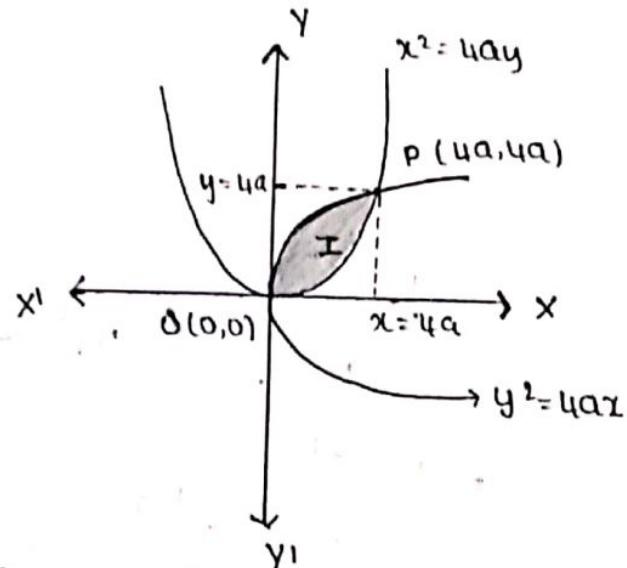
$$\Rightarrow x^2 = 4ay, y = 2\sqrt{ax} \Rightarrow y^2 = 4a \rightarrow ②$$

From ① and ②

$$\frac{x^2}{4a} = 2\sqrt{ax}$$

$$\Rightarrow x^2 = 8a\sqrt{ax}$$

$$x^4 = 64a^2(ax)$$



$$\Rightarrow x^4 - 64a^3x = 0$$

$$\Rightarrow x(x^3 - 64a^3) = 0$$

$$\Rightarrow x = 0, x^3 = 64a^3$$

$$\Rightarrow x^3 = (4a)^3$$

$$\Rightarrow x = 4a$$

$$\text{where } x = 0 \Rightarrow y = 0$$

$$x = 4a \Rightarrow y = 4a$$

∴ The change of order of integration we have

$$I = \int_{y=0}^{4a} \int_{x=y^2/4a}^{2\sqrt{ay}} xy \, dx \, dy$$

$$= \int_{y=0}^{4a} y \int_{y^2/4a}^{2\sqrt{ay}} x \, dx \, dy$$

$$= \int_{y=0}^{4a} y \left[\frac{x^2}{2} \right]_{y^2/4a}^{2\sqrt{ay}} \, dy$$

$$= \frac{1}{2} \int_{y=0}^{4a} y \left(y [2(\sqrt{ay})^2 - (\frac{y^2}{4a})^2] \right) \, dy$$

$$= \frac{1}{2} \int_0^{4a} y \left[4ay - \frac{y^4}{(6a^2)} \right] \, dy$$

$$= \frac{1}{2} \left\{ \frac{4ay^3}{3} - \frac{y^6}{6a^2} \right\}_0^{4a}$$

$$= \frac{1}{2} \left\{ \frac{4a}{3} y^3 - \frac{y^6}{96a^2} \right\}_0^{4a}$$

$$= \frac{1}{2} \left[\frac{4a}{3} (4a)^3 - \frac{(4a)^6}{96a^2} \right]$$

$$= \frac{(4a)^4}{2} \left[\frac{1}{3} - \frac{16a^3}{96a^2} \right]$$

$$= \frac{(4a)^4}{2} \left[\frac{1}{3} - \frac{1}{6} \right]$$

$$= \frac{(4a)^4}{2} \times \frac{1}{6}$$

$$= \frac{(4a)^4}{12}$$

$$= \frac{4^4 \times a^4}{12}$$

$$= \frac{4 \times 4 \times 4 \times 4 \times a^4}{12}$$

$$I = \frac{64a^4}{3}$$

4. Evaluate $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y} dy dx$

$$I = \int_{x=0}^\infty \int_{y=x}^\infty \frac{e^{-y}}{y} dy dx$$

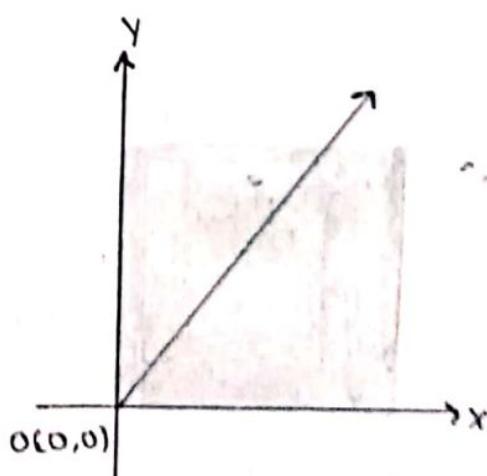
Here x varies from

0 to ∞ and y varies from x to ∞

∴ By the change of order of integration.

$$I = \int_{x=0}^\infty \int_{y=x}^\infty \frac{e^{-y}}{y} dy dx$$

$$I = \int_{y=0}^\infty \int_{x=0}^y \left[\frac{e^{-y}}{y} \right] dx dy$$



$$\begin{aligned}
 &= \int_{y=0}^{\infty} \frac{e^{-y}}{y} \int_{x=0}^y dx dy \\
 &= \int_{y=0}^{\infty} \frac{e^{-y}}{y} [x]_0^y dy \\
 &= \int_{y=0}^{\infty} e^{-y} dy \\
 &= \left[\frac{e^{-y}}{-1} \right]_0^{\infty} \\
 &= [e^{-y}]_0^{\infty} \\
 &= -[e^{-\infty} - e^0] \\
 &= -[0 - 1] \\
 &= -(-1) \\
 \Rightarrow I &= 1
 \end{aligned}$$

5. Evaluate $\int_0^1 \int_{x^2}^{2-x} xy dy dx$ using change of order of integration.

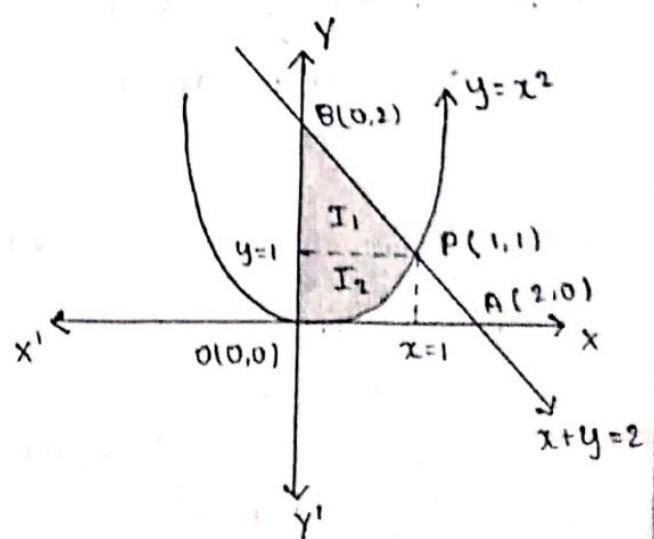
$$\text{Let } I = \int_{x=0}^1 \int_{y=x^2}^{2-x} xy dy dx$$

x varies from 0 to 1 and
 y varies from $y = x^2$

$$\Rightarrow y + x = 2$$

or

$$x + y = 2$$



$$x+y=2$$

$$\frac{x}{2} + \frac{y}{2} = 2$$

By change of order of integration the boundary region written as in

$$I = \int_{y=0}^1 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^2 \int_{x=0}^{2-y} xy \, dx \, dy$$

$$I = I_1 + I_2$$

$$I_1 = \int_{y=0}^1 \int_{x=0}^{\sqrt{y}} xy \, dx \, dy$$

$$= \int_{y=0}^1 y \left[\frac{x^2}{2} \right]_0^{\sqrt{y}} \, dy$$

$$= \frac{1}{2} \int_0^1 y^2 \, dy$$

$$= \frac{1}{2} \left[\frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{1}{3} - 0 \right]$$

$$I_1 = \frac{1}{6}$$

$$I_2 = \int_{y=1}^2 \int_{x=0}^{2-y} xy \, dx \, dy$$

$$= \int_{y=1}^2 y \left[\frac{x^2}{2} \right]_0^{2-y} \, dy$$

$$= \frac{1}{2} \int_{y=1}^2 y (2-y)^2 \, dy$$

$$= \frac{1}{2} \int_{y=1}^{y=0} y (y^2 - 4y + 4) \, dy$$

$$\begin{aligned}
&= \frac{1}{2} \int_{y=1}^2 (y^3 - 4y^2 + 4y) dy \\
&= \frac{1}{2} \left[\frac{y^4}{4} - \frac{4y^3}{3} + \frac{4y^2}{2} \right]_1^2 \\
&= \frac{1}{2} \left\{ \left[\frac{16}{4} - \frac{32}{3} + \frac{16}{2} \right] - \left[\frac{1}{4} - \frac{4}{3} + \frac{4}{2} \right] \right\} \\
&= \frac{1}{2} \left[4 - \frac{32}{3} + 8 - \frac{1}{4} + \frac{4}{3} - 2 \right] \\
&= \frac{1}{2} \left[10 - \frac{28}{3} - \frac{1}{4} \right] \\
&= \frac{1}{2} \left[\frac{120 - 1125 - 3}{12} \right] \\
&= \frac{1}{2} \left[\frac{5}{12} \right]
\end{aligned}$$

$$\Rightarrow I_2 = \frac{5}{24}$$

$$I = I_1 + I_2$$

$$I = \frac{1}{6} + \frac{5}{24}$$

$$I = \frac{4+5}{24} = \frac{9}{24} = \frac{3}{8} //$$

6. Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy$ by change into polar co-ordination.

$$\text{Let } I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \rightarrow ①$$

$$\Rightarrow I = \int_{x=0}^\infty \int_{y=0}^\infty e^{-(x^2+y^2)} dy dx$$

Let $x = r \cos \theta, y = r \sin \theta$

$$\Rightarrow dx dy = r dr d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta \rightarrow ②$$

$$\text{Let } r^2 = t$$

$$\Rightarrow 2r dr = dt$$

$$\Rightarrow r dr = \frac{1}{2} dt$$

$$\therefore \text{UL : } r = \infty \Rightarrow t = \infty$$

$$\text{LL : } r = 0 \Rightarrow t = 0$$

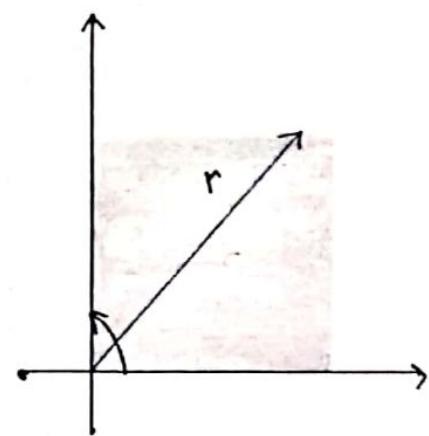
$$I = \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} \frac{1}{2} dt d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \int_{t=0}^{\infty} e^{-t} dt d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} \left[\frac{e^{-t}}{-1} \right]_{t=0}^{\infty} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} [e^{-\infty} - e^0] d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} [0 - 1] d\theta$$



$$= \frac{1}{2} \int_{\theta=0}^{\pi/2} 1 d\theta$$

$$= \frac{1}{2} [\theta]_0^{\pi/2}$$

$$= \frac{1}{2} \left[\frac{\pi}{2} - 0 \right]$$

$$\Rightarrow I = \frac{\pi}{4}$$

7. Evaluate $\int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-(x^2+y^2)} dy dx = \frac{\pi}{4}$

$$I = \int_{x=0}^{\infty} \int_{y=0}^{\infty} e^{-x^2} \cdot e^{-y^2} dy dx = \frac{\pi}{4} \rightarrow ②$$

$$= \int_{x=0}^{\infty} e^{-x^2} dx \int_{y=0}^{\infty} e^{-y^2} dy = \frac{\pi}{4} \rightarrow ③$$

$$\text{Let } x = y$$

$$③ \Rightarrow \int_{x=0}^{\infty} e^{-x^2} dx \int_{x=0}^{\infty} e^{-x^2} dx = \frac{\pi}{4}$$

$$\Rightarrow \left[\int_{x=0}^{\infty} e^{-x^2} dx \right]^2 = \frac{\pi}{4}$$

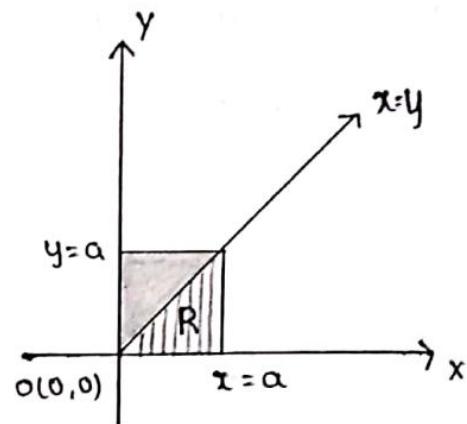
$$\Rightarrow \int_0^{\infty} e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$$

8. Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2} dx dy$

$$I = \int_0^a \int_{y=0}^a \frac{x}{x^2+y^2} dx dy$$

$$I = \int_{y=0}^a \int_{x=y}^a \frac{x}{x^2+y^2} dx dy$$

∴ By change the order of integration, we have.



$$I = \int_{x=0}^a \int_{y=0}^x \frac{x}{x^2+y^2} dy dx$$

$$= \int_{x=0}^a x \int_{y=0}^x \frac{1}{x^2+y^2} dy dx$$

$$= \int_{x=0}^a x \left[\tan^{-1}(y/x) \right]_{y=0}^x dx$$

$$= \int_{x=0}^a \left[\tan^{-1}(y/x) \right]_{y=0}^x dx$$

$$= \int_{x=0}^a \left[\pi/4 - 0 \right] dx$$

$$= \frac{\pi}{4} \int_{x=0}^a 1 dx$$

$$= \frac{\pi}{4} [x]_0^a$$

$$I = \frac{\pi a}{4}$$

q. Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} y \sqrt{x^2+y^2} dx dy$ by change of polar coordinates

Here x varies from 0 to $\sqrt{a^2 - y^2}$

$$\Rightarrow x = \sqrt{a^2 - y^2}$$

$$x^2 = a^2 - y^2$$

$$x^2 + y^2 = a^2$$

y varies from 0 to a

let $x = r \cos \theta, y = r \sin \theta$

$$\Rightarrow dx dy = r dr d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^a r \sin \theta \cdot r \cdot r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^a r^3 \sin \theta dr d\theta$$

$$\Rightarrow I = \int_{\theta=0}^{\pi/2} \sin \theta d\theta \int_{r=\theta}^a r^3 dr$$

$$= -[\cos \theta]_0^{\pi/2} \left[\frac{r^4}{4} \right]_0^a$$

$$= -[\cos \pi/2 - \cos 0] \left[\frac{a^4}{4} \right]$$

$$= (0-1) \frac{a^4}{4}$$

$$I = \frac{a^4}{4}$$

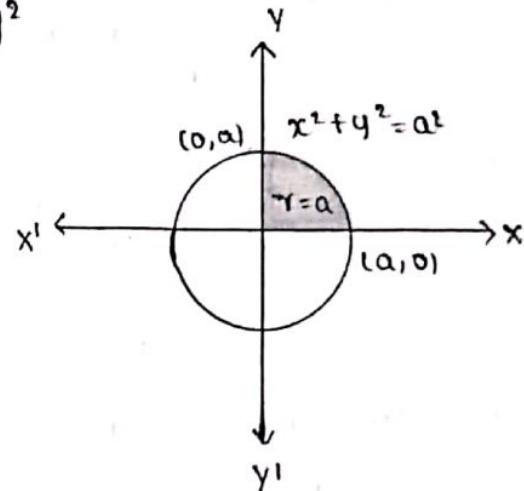
APPLICATIONS

1. Find the area between the parabolas.

$$y^2 = 4ax, x^2 = 4ay$$

The area between the given two parabolas

$y^2 = 4ax, x^2 = 4ay$ can be evaluated as



$$A = \iint dx dy$$

$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}}$$

$$x=0 \quad y = \frac{x^2}{4a}$$

$$= \int_{x=0}^{4a} (y) \int_{x^2/4a}^{2\sqrt{ax}} dx$$

$$= \int_0^{4a} \left[2\sqrt{ax} - \frac{x^2}{4a} \right] dx$$

$$= 2\sqrt{a} \int_0^{4a} \sqrt{x} dx - \frac{1}{4a} \int_0^{4a} x^2 dx$$

$$= 2\sqrt{a} \int_0^{4a} \sqrt{x} dx - \frac{1}{4a} \int_0^{4a} x^2 dx$$

$$= 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{4a} - \frac{1}{4a} \left[\frac{x^3}{3} \right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3} \left[x^{3/2} \right]_0^{4a} - \frac{1}{12a} \left[x^3 \right]_0^{4a}$$

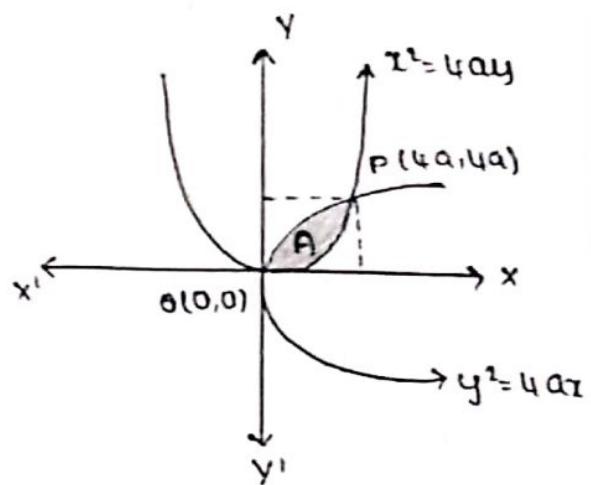
$$= \frac{4\sqrt{a}}{3} (4a)^{3/2} - \frac{1}{12a} (4a)^3$$

$$= \frac{4\sqrt{a}}{3} (4a)2\sqrt{a} \cdot \frac{1}{12a} 4ax 4a^2$$

$$= \frac{32a^2}{3} - \frac{16a^2}{3}$$

$$\Rightarrow A = \frac{16a^2}{3} \text{ Sq units.}$$

$$\Rightarrow \int_0^a \int_0^{\sqrt{ax-y^2}} (x^2+y^2) dx dy$$



$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} (x^2+y^2) dx dy \Rightarrow ①$$

x varies from $x=0$ to $x=\sqrt{a^2-y^2}$

$$x = \sqrt{a^2-y^2}$$

$$x^2 = a^2 - y^2$$

$$x^2 + y^2 = a^2$$

It is a circle having centre $(0,0)$
with radius 'a' lies from 0 to a

$$\text{let } x = r \cos \theta$$

$$y = r \sin \theta$$

$$dx dy = r dr d\theta$$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^a r^2 (\cos^2 \theta + \sin^2 \theta) r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^a r^2 dr d\theta$$

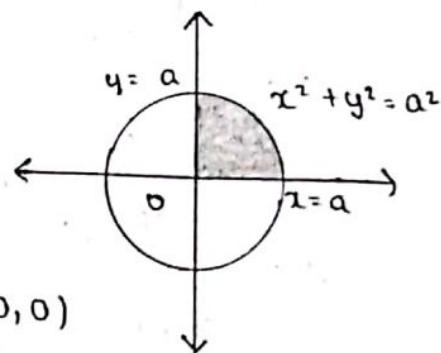
$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^4}{4} \right]_0^a d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} a^4 d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} 1 \cdot d\theta$$

$$= \frac{a^4}{4} \int_0^{\pi/2} 1 \cdot d\theta$$

$$= \frac{a^4}{4} [1]_0^{\pi/2}$$



$$I = \frac{a^4}{4} \left[\frac{\pi}{2} - 0 \right]$$

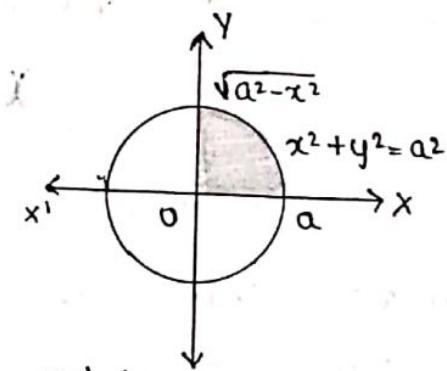
$$I = \frac{\pi a^4}{8}$$

Q. Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2} dx dy$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} y^2 \sqrt{x^2+y^2} dx dy$$

x varies from $x=0$, to $x=a$

y varies from $y=0$ to $\sqrt{a^2-x^2}$



Let $x = r \cos \theta$, $y = r \sin \theta$, $dx dy = r dr d\theta$

$$I = \int_{\theta=0}^{\pi/2} \int_0^a y^2 \sqrt{x^2+y^2} dx dy$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^2 \sin^2 \theta \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} \cdot r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^2 \sin^2 \theta \cdot r \cdot r dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_0^a r^4 \sin^2 \theta dr d\theta$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \sin^2 \theta \int_0^a r^4 dr d\theta$$

$$= \int_{\theta=0}^{\pi/2} -2 \sin \theta \cos \theta \left[\frac{r^5}{5} \right]_0^a d\theta$$

$$= \left[\frac{r^5}{5} \right]_0^a \left[\frac{\pi}{2} - 0 \right]$$

$$= \left[\frac{a^5}{5} - 0 \right] \left[\frac{1}{2} \cdot \frac{\pi}{2} \right]$$

$$I = \frac{\pi a^5}{20} \text{ Sq. units}$$

3. Evaluate change of order of integral $\int_{-a}^a \int_0^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dy dx$

$$\text{Let, } I = \int_{x=-a}^a \int_{y=0}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2} dy dx$$

$$\text{Let } x = r \cos \theta, y = r \sin \theta$$

then x varies from $x = 0$ to a

y varies from $y = 0$ to $y = \sqrt{a^2 - x^2}$

$$\Rightarrow y = \sqrt{a^2 - x^2}$$

$$x^2 + y^2 = a^2$$

$$I = \int_0^\pi \int_0^a \sqrt{x^2+y^2} dy dx$$

$$= \int_0^\pi \int_0^a \sqrt{r^2 \sin^2 \theta + r^2 \cos^2 \theta} r dr d\theta$$

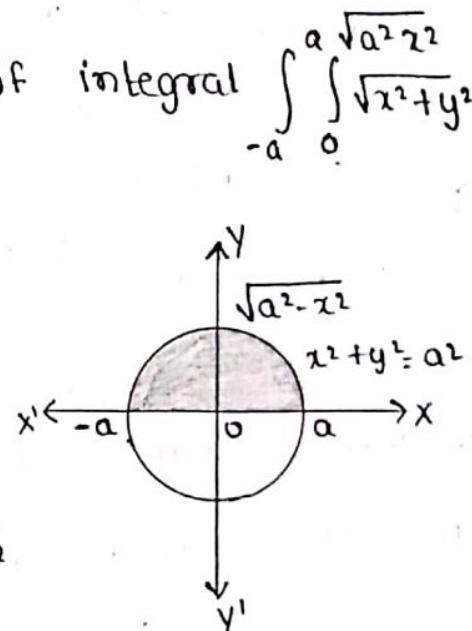
$$= \int_{\theta=0}^\pi \int_0^a r^2 r dr d\theta$$

$$= \int_0^\pi \int_0^a r^3 dr d\theta$$

$$= \int_0^\pi \left[\frac{r^4}{3} \right]_0^a d\theta$$

$$= \frac{a^3}{3} [\pi - 0]$$

$$I = \frac{\pi a^3}{3}$$



- (4). use double integration to find the area of a ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and hence find the area of the circle $\frac{x^2}{a^2} + \frac{y^2}{b^2} = a^2$.
- \Rightarrow Given the ellipse $x^2 + y^2 = a^2 \rightarrow ①$
- Covered the area with the co-ordinate axis as shown in the diagram.
- \therefore The area of the ellipse is

$$A = A_1 + A_2 + A_3 + A_4$$

$$A = 4 \int \int dx dy$$

$$\Rightarrow A = 4 \int_{x=0}^a \int_{y=0}^{b/a \sqrt{a^2 - x^2}} 1 \cdot dy dx$$

$$A = 4 \int_0^a [y]_{0}^{\frac{b}{a} \sqrt{a^2 - x^2}} dx$$

$$= 4 \int_0^a \frac{b}{a} \sqrt{a^2 - x^2} dx$$

$$A = \frac{4b}{a} \int_0^a \sqrt{a^2 - x^2} dx \rightarrow ②$$

$$\text{Let } x = a \sin \theta \Rightarrow \theta = \sin^{-1}(x/a)$$

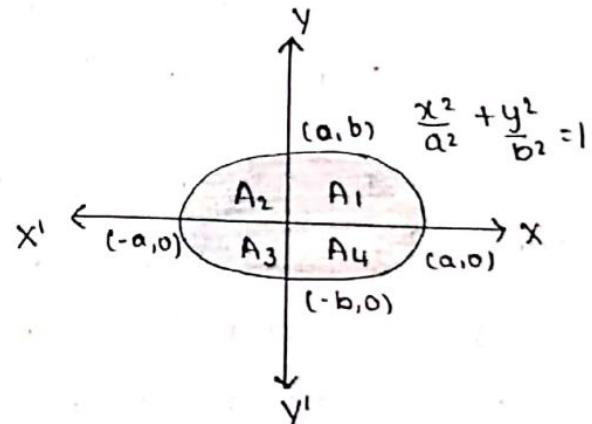
$$\Rightarrow dx = a \cos \theta d\theta$$

$$\text{U.L : } x=a \Rightarrow \theta = \frac{\pi}{2}$$

$$\text{L.L : } x=0 \Rightarrow \theta = 0$$

$$\therefore A = \frac{4b}{a} \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta$$

$$= \frac{4b}{a} \int_0^{\pi/2} a \sqrt{1 - \sin^2 \theta} \cdot a \cos \theta d\theta$$



$$\begin{aligned}
 &= 4ab \int_0^{\pi/2} a \cos^2 \theta d\theta \\
 &= 4ab \int_0^{\pi/2} \left[\frac{1 + \cos 2\theta}{2} \right] d\theta \\
 &= 2ab \int_0^{\pi/2} (1 + \cos 2\theta) d\theta \\
 &= 2ab \left[\theta + \frac{\sin 2\theta}{2} \right]_0^{\pi/2} \\
 &= 2ab \left[\frac{\pi}{2} + 0 \right] \\
 \Rightarrow A &= 2ab \left(\frac{\pi}{2} \right)
 \end{aligned}$$

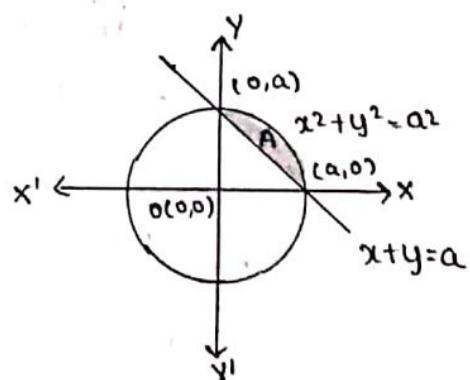
$$A = \pi ab \text{ Sq units}$$

when $b=a$, then the given ellipse becomes a circle $x^2+y^2=a^2$ and its area is

$$A = \pi \times a \times a = \pi a^2 \text{ Sq units}$$

5. Find the area by double integration between the circle $x^2+y^2=a^2$ and to the straight line $x+y=a$
- \rightarrow Given that the straight line cuts the x axis at $(a,0)$ and y -axis at $(0,a)$
- \therefore The area between the circle $x^2+y^2=a^2$ and the straight line $x+y=a$ is $A=\iint dx dy$

$$\begin{aligned}
 A &= \int_0^a \int_{y=a-x}^{\sqrt{a^2-x^2}} 1 dy dx \\
 &= \int_0^a [y]_{y=a-x}^{\sqrt{a^2-x^2}} dx \\
 &= \int_0^a (\sqrt{a^2-x^2} - (a-x)) dx
 \end{aligned}$$



$$A = \int_{x=0}^a \sqrt{a^2 - x^2} dx - \int_0^a (a - x) dx$$

Let $x = a \sin \theta \Rightarrow \theta = \sin^{-1}(x/a)$

$$\therefore dx = a \cos \theta d\theta$$

$$UL : x=a \Rightarrow \theta = \frac{\pi}{2}$$

$$LL : x=0 \Rightarrow \theta = 0$$

$$\begin{aligned}\therefore A &= \int_0^{\pi/2} \sqrt{a^2 - a^2 \sin^2 \theta} \cdot a \cos \theta d\theta - \left[ax - \frac{x^2}{2} \right]_0^a \\ &= \int_0^{\pi/2} a^2 \cos^2 \theta d\theta - \left\{ \left(a^2 - \frac{a^2}{2} \right) - (0-0) \right\} \\ &= a^2 \int_0^{\pi/2} \cos^2 \theta d\theta - \frac{a^2}{2} \\ &= a^2 \left(\frac{2-1}{2} \right) \frac{\pi}{2} - \frac{a^2}{2}\end{aligned}$$

$$A = \frac{\pi a^2}{4} - \frac{a^2}{2} \text{ Sq units.}$$

VOLUME OF TETRAHYDRAL

1. Find the volume of tetrahedral bounded by the planes $x=0, y=0, z=0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

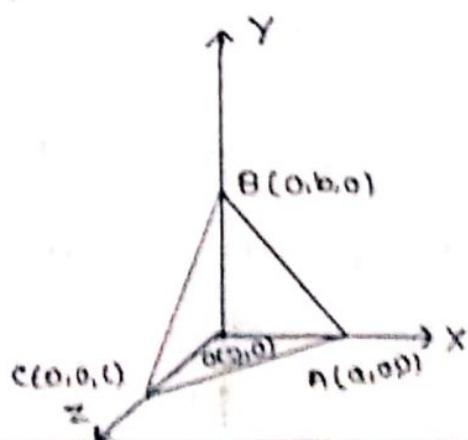
\Rightarrow Given tetrahedral bounded by

the plan $x=0, y=0, z=0$

$$\text{and } \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

\therefore The volume $V = \iiint dv$

$$= \int_{x=0}^a \int_{y=0}^{ba(a-x)} \int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} 1 dz dy dx$$



$$\begin{aligned}
&= \int_{x=0}^a \int_{y=0}^{b/a(a-x)} \int_{z=0}^{c[1-\frac{x}{a}-\frac{y}{b}]} dz dy dx \\
&= \int_{x=0}^a \int_{y=0}^{b/a(a-x)} c \left[1 - \frac{x}{a} - \frac{y}{b} \right] dy dx \\
&= c \int_{x=0}^a \left[y - \frac{xy}{a} - \frac{y^2}{2b} \right] \int_{y=0}^{b/a(a-x)} dx \\
&= c \int_{x=0}^a \left[\frac{b}{a}(a-x) - \frac{x}{a} \cdot \frac{b}{a}(a-x) - \frac{1}{2b} \frac{b^2}{a^2} (a-x^2) \right] dx \\
&= c \int_0^a \left[\frac{b}{a}(a-x) - \frac{b}{a^2} x(a-x) - \frac{b}{2a^2} (a-x^2) \right] dx \\
&= c \int_0^a \frac{b}{a^2} (a-x)(a-x) - \frac{1}{2} (a-x) dx \\
&= c \int_0^a \frac{b}{a^2} (a-x)^2 \left[1 - \frac{1}{2} \right] dx \\
&= c \int_0^a \frac{b}{2a^2} (a-x^2) dx \\
&= \frac{bc}{2a^2} \int_0^a (x^2 - 2ax + a^2) dx \\
&= \frac{bc}{2a^2} \left[\frac{x^3}{3} - ax^2 + a^2 x \right]_0^a \\
&= \frac{bc}{2a^2} \left[\frac{a^3}{3} - a^3 + a^3 \right] \\
&= V = \frac{bc}{2a^2} \times \frac{a^3}{3} \\
&\Rightarrow V = \frac{abc}{6} \text{ cubic meter.}
\end{aligned}$$

- Q. Find the volume of the tetrahedral bounded by the planes, $x=0, y=0, z=0$ and $x+ay+3z=6$.
 \Rightarrow Given that the volume of tetrahedral bounded by planes

$x=0, y=0, z=0$ and $x+2y+3z=6$

$$V = \iiint 1 \, dv$$

$$= \int_{x=0}^6 \int_{y=0}^{1/2(6-x)} \int_{z=0}^{1/3(6-x-2y)} 1 \cdot dz \, dy \, dx$$

$$= \int_{x=0}^6 \int_{y=0}^{1/2(6-x)} [z] \Big|_{z=0}^{1/3(6-x-2y)} \Big|_y^1$$

$$= \frac{1}{3} \int_{x=0}^6 [6y - xy - 2y] \, dy \, dx$$

$$= \frac{1}{3} \int_{x=0}^6 [6x - xy - y^2] \Big|_{y=0}^{1/2(6-x)}$$

$$= \frac{1}{3} \int_0^6 (6-x) y - y^2 \int_{y=0}^{1/2(6-x)} dx$$

$$= \frac{1}{3} \int_0^6 \left[\frac{(6-x)(6-x)}{2} - \frac{(6-x)^2}{4} \right] dx$$

$$= \frac{1}{3} \int_0^6 \left[\frac{(6-x)^2}{2} - \frac{(6-x)^2}{4} \right] dx$$

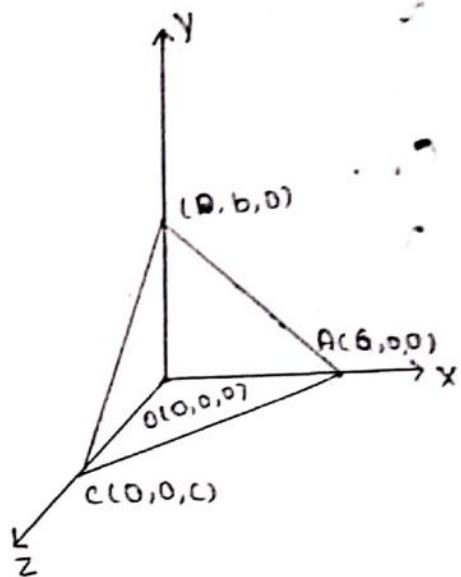
$$= \frac{1}{3} \int_0^6 \left(\frac{1}{2} - \frac{1}{4} \right) (6-x)^2 dx$$

$$= \frac{1}{3} \int_0^6 \frac{1}{4} (6-x)^2 dx$$

$$= \frac{1}{12} \int_0^6 (x^2 - 12x + 36) dx$$

$$= \frac{1}{12} \left[\frac{x^3}{3} - 6x^2 + 36x \right]_0^6$$

$$= \frac{1}{12} \left[\frac{216}{3} - 216 + 216 \right] = \frac{216}{36} \Rightarrow V = 6 \text{ cubic units}$$



3. Find the volume of a tetrahedral bounded by the plane $x=0, y=0, z=0$ and $x+y+z=1$

Given that, the volume of the tetrahedral

$x=0, y=0, z=0$ and $x+y+z=1$

$$= \int_{x=0}^1 \int_{y=0}^{(1-x)} \int_{z=0}^{(1-x-y)} 1 dz dy dx$$

$$= \int_{x=0}^1 \int_{y=0}^{(1-x)} [z]_0^{(1-x-y)} dy dx$$

$$= \int_{x=0}^1 \int_0^{(1-x)} (1-x-y) dy dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2} \right]_0^{(1-x)}$$

$$= \int_0^1 \left[(1-x) - x(1-x) - \frac{(1-x)^2}{2} \right] dx$$

$$= \frac{1}{2} \int_0^1 (1-x) \left[(1-x) - \frac{1}{2}(1-x) \right] dx$$

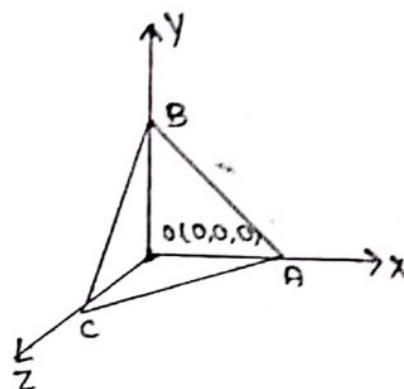
$$= \int_0^1 (1-x)^2 \left[1 - \frac{1}{2} \right] dx$$

$$= \frac{1}{2} \int_0^1 (1-2x+x^2) dx$$

$$= \frac{1}{2} \left[x - \frac{2x^2}{2} + \frac{x^3}{3} \right]_0^1$$

$$= \frac{1}{2} \left[1 - \frac{2(1)^2}{2} + \frac{1^3}{3} \right]$$

$$= \frac{1}{2} \left[1 - 0 - \frac{1}{3} \right] = \frac{1}{6} \text{ cubic units.}$$



4. Find the area bounded by $\theta = 0$, $\theta = \pi$ of a Cardiac $r = a(1 + \cos\theta)$

$$A = \iint dx dy$$

$$= \iint r dr d\theta$$

$$= \int_0^a \int_{r=0}^{a(1+\cos\theta)} r dr d\theta$$

$$= \int_{\theta=0}^{\pi} \left[\frac{r^2}{2} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{1}{2} \int_{\theta=0}^{\pi} a^2 (1 + \cos\theta)^2 d\theta$$

$$= \frac{a^2}{2} \int_{\theta=0}^{\pi} (1 + \cos\theta)^2 d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} [1 + 2\cos\theta + \cos^2\theta] d\theta$$

$$= \frac{a^2}{2} \int_0^{\pi} [1 + 2\cos\theta + \frac{1+\cos\theta}{2}] d\theta$$

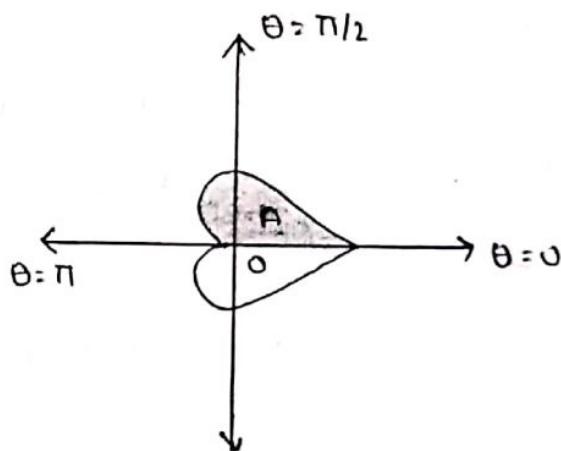
$$= \frac{a^2}{4} \int_0^{\pi} [2 + 4\cos\theta + 1 + \cos 2\theta] d\theta$$

$$= \frac{a^2}{4} \int_0^{\pi} [\cos 2\theta + 4\cos\theta + 3] d\theta$$

$$= \frac{a^2}{4} \left[\frac{\sin 2\theta}{2} + 4\sin\theta + 3\theta \right]_0^{\pi}$$

$$= \frac{a^2}{4} [(0 + 0 + 3\pi) - (0 + 0 + 0)]$$

$$A = \frac{3a^2\pi}{4} \text{ Sq units}$$

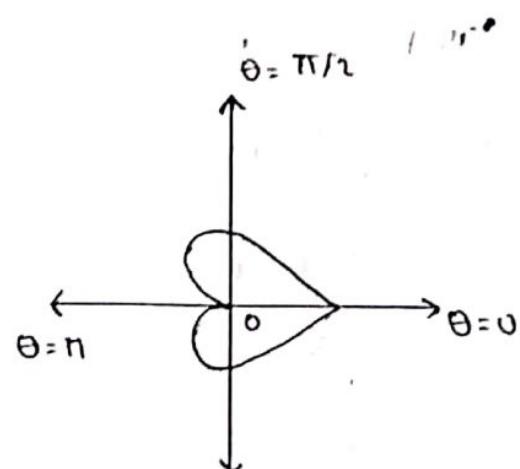


5. Find the volume of the cardiac $r = a(1 + \cos\theta)$ above the initial line

W.R.T

The volume of the cardiac

$r = a(1 + \cos\theta)$ above the initial line



$$V = \iint 2\pi r^2 \sin\theta dr d\theta$$

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta dr d\theta$$

$$= 2\pi \int_{\theta=0}^{\pi} \sin\theta \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} d\theta$$

$$= \frac{2\pi}{3} \int_{\theta=0}^{\pi} a^3 (1 + \cos\theta)^3 \sin\theta d\theta$$

$$= \frac{2\pi a^3}{3} \int_{\theta=0}^{\pi} (1 + \cos\theta)^3 \sin\theta d\theta$$

$$\text{Let } 1 + \cos\theta = t$$

$$\Rightarrow -\sin\theta d\theta = dt$$

$$\Rightarrow \sin\theta d\theta = -dt$$

$$\text{UL : } \theta = \pi \Rightarrow t = 1 + \cos(\pi) = 1 - 1 = 0$$

$$\text{LL : } \theta = 0 \Rightarrow t = 1 + \cos\theta = 1 + 1 = 2$$

$$V = \frac{2\pi a^3}{3} \int_{\frac{\pi}{2}}^0 t^3 (-dt)$$

$$= \frac{2\pi a^3}{3} \int_0^2 t^3 dt$$

$$= \frac{2\pi a^3}{3} \left[\frac{t^4}{4} \right]_0^2$$

$$= \frac{2\pi a^3}{4 \times 3} \times (24 - 0)$$

$$= \frac{2\pi a^3}{4 \times 3} \times 2 \times 2 \times 2 \times 2$$

$$A = \frac{8\pi a^3}{3} \text{ cubic units}$$

\Rightarrow Beta - Gamma functions

Definition :- For any $(m, n) > 0$, then the improper integral can be defined as $B(m, n) \int_0^\infty z^{n-1} (1-z)^{m-1} dz \rightarrow ①$ is called the Beta function in m and n .

$$\text{Let, } z = \sin^2 \theta \Rightarrow \theta = \sin^{-1}(\sqrt{z})$$

$$\Rightarrow dz = 2 \sin \theta \cos \theta d\theta$$

$$\text{UL : } z=1 \Rightarrow \theta = \frac{\pi}{2}$$

$$\text{LL : } z=0 \Rightarrow \theta = 0$$

$$① \Rightarrow B(m, n) = \int_0^{\pi/2} (\sin^2 \theta)^{m-1} \cdot (1 - \sin^2 \theta)^{n-1} 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cdot \cos^{2n-1} \theta \cdot \sin \theta \cos \theta d\theta$$

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m+2} \theta \cdot \cos^{2n+2} \theta d\theta$$

$$\Rightarrow B(m, n) = 2 \int_0^{\pi/2} \sin^{2m+1} \theta \cdot \cos^{2n+1} \theta d\theta \rightarrow ②$$

$$\text{Let } p = 2m+1, q = 2n+1$$

$$\Rightarrow m = \frac{p-1}{2}, n = \frac{q-1}{2}$$

$$② \Rightarrow B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) = 2 \int_0^{\pi/2} \sin^p \theta \cdot \cos^q \theta d\theta$$

$$\Rightarrow \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta = \frac{1}{2} B\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

Gamma function: for every $n > 0$ gamma function can be defined as

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx \rightarrow ①$$

$$\text{let } x = y^2$$

$$\Rightarrow dx = 2y dy$$

$$\text{UL: } x = \infty \Rightarrow y = \infty$$

$$\text{LL: } x = 0 \Rightarrow y = 0$$

$$\therefore ① \Rightarrow \Gamma n = \int_0^\infty e^{-y^2} (y^2)^{n-1} 2y dy$$

$$= 2 \int_0^\infty e^{-y^2} y^{2n-2+1} dy$$

$$\Rightarrow \Gamma n = 2 \int_0^\infty e^{-y^2} y^{2n-1} dy$$

$$\Rightarrow \Gamma n = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx \quad (\because y = x)$$

RELATION BETWEEN BETA AND GAMMA FUNCTION

We know that,

$$B(m, n) = 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^{2m-1} \theta \cos^{2n-1} \theta d\theta \rightarrow ①$$

$$\Gamma n = 2 \int_0^\infty e^{-x^2} x^{2n-1} dx \rightarrow ②$$

$$\Gamma n = 2 \int_0^\infty e^{-y^2} y^{2n-1} dy \rightarrow ③$$

$$\Gamma{m+n} = 2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr \rightarrow ④$$

$$\text{Let } \Gamma{m} \Gamma{n} = 2 \int_0^{\infty} e^{-x^2} x^{2m-1} y^{2n-1} dx dy \rightarrow ⑤$$

$$\text{Let } x = r \cos\theta, y = r \sin\theta$$

$$\Rightarrow dx dy = r dr d\theta$$

$$⑤ \Rightarrow \Gamma{m} \Gamma{n} = 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} (r \cos\theta)^{2m-1} (r \sin\theta)^{2n-1} r dr d\theta$$

$$= 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r^{2m-1 + 2n-1 + 1} \sin^{2n-1}\theta \cos^{2m-1}\theta dr d\theta$$

$$= 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \sin^{2n-1}\theta \cos^{2m-1}\theta dr d\theta$$

$$= 4 \int_{r=0}^{\infty} \int_{\theta=0}^{\pi/2} e^{-r^2} r^{2(m+n)-1} \sin^{2n-1}\theta \cos^{2n-1}\theta dr d\theta$$

$$\Gamma{m} \Gamma{n} = \left[2 \int_0^{\infty} e^{-r^2} r^{2(m+n)-1} dr \right] \cdot \left[2 \int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta d\theta \right]$$

$$\Gamma{m} \Gamma{n} = (\Gamma{m+n}) B(m, n)$$

$$B(m, n) = \frac{\Gamma{m} \Gamma{n}}{\Gamma{m+n}}$$

Note :-

$$① B(m, n) = B(n, m)$$

$$② \Gamma{n+1} = n \Gamma{n} = n!$$

$$③ \Gamma{1} = 1$$

$$④ \Gamma{1/2} = \sqrt{\pi}$$

$$⑤ \Gamma{n} \Gamma{1-n} = \frac{\pi}{\sin(n\pi)}$$

① Show that $\Gamma(1/2) = \sqrt{\pi}$ using Beta - gamma function.

WKT

$$\Gamma_m = 2 \int_0^\infty e^{-x^2} x^{2m-1} dx \rightarrow ①$$

$$\Gamma_n = 2 \int_0^\infty e^{-y^2} y^{2n-1} dy \rightarrow ②$$

$$\therefore \Gamma_m \Gamma_n = [2 \int_0^\infty e^{-x^2} x^{2n-1} dx] [2 \int_0^\infty e^{-y^2} y^{2m-1} dy]$$

$$\Rightarrow \Gamma_m \Gamma_n = \int_0^\infty \int_0^\infty e^{-x^2} e^{-y^2} x^{2n-1} y^{2m-1} dx dy \rightarrow ③$$

$$\text{Let } m = \frac{1}{2}, n = \frac{1}{2}$$

$$③ \Rightarrow \Gamma(1/2) \Gamma(1/2) = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} x^0 y^0 dx dy$$

$$\Rightarrow (\Gamma(1/2))^2 = 4 \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx dy \rightarrow ④$$

$$\text{Let } x = r \cos \theta, y = r \sin \theta$$

$$\Rightarrow dx dy = r dr d\theta$$

$$\therefore ④ (\Gamma(1/2))^2 = 4 \int_{r=0}^\infty \int_{\theta=0}^{\pi/2} e^{-r^2} r dr d\theta$$

$$= 4 \int_{r=0}^\infty r e^{-r^2} dr [\theta]_0^{\pi/2}$$

$$= \frac{4\pi}{2} \int_{r=0}^\infty r e^{-r^2} dr$$

$$= \pi \int_0^\infty (2r)_e^{-r^2} dr$$

$$\text{Let } r^2 = t$$

$$\Rightarrow 2r dr = dt$$

$$(\Gamma(1/2))^2 = \pi \int_0^\infty e^{-t} dt$$

$$\begin{aligned}
 (\sqrt{\gamma_2})^2 &= \pi \left(\frac{e^{-t}}{-1} \right)^{\infty} \\
 &= -\pi [e^{-t}]_0^{\infty} \\
 &= \pi [e^{-\infty} - 1] \\
 &= -\pi (0 - 1) \\
 \Rightarrow (\sqrt{\gamma_2})^2 &= \pi
 \end{aligned}$$

$$\sqrt{\gamma_2} = \sqrt{\pi} //$$

2. Show that $\int_0^2 (4-x^2)^{3/2} dx = 3\pi$

$$I = \int_0^2 (4-x^2)^{3/2} dx$$

$$\text{Let } x = 2\sin\theta \Rightarrow \theta = \sin^{-1}(x/2).$$

$$\Rightarrow dx = 2\cos\theta d\theta$$

$$\text{UL : } x=2 \Rightarrow \theta = \sin^{-1}(1) = \frac{\pi}{2}$$

$$\text{LL : } x=0 \Rightarrow \theta = \sin^{-1}(0) = 0$$

$$\therefore I = \int_0^{\pi/2} (4-4\sin^2\theta)^{3/2} \cdot 2\cos\theta d\theta$$

$$= 2 \int_0^{\pi/2} (4)^{3/2} (\cos^2\theta)^{3/2} \cos\theta d\theta$$

$$= 2 \int_0^{\pi/2} (2^2)^{3/2} \cos^3\theta \cos\theta d\theta$$

$$= 2 \int_0^{\pi/2} (2^2)^{3/2} \cos^3\theta \cos\theta d\theta$$

$$I = 16 \int_0^{\pi/2} \cos^4\theta d\theta$$

$$\therefore \int_0^{\pi/2} \cos^4\theta d\theta = \int_0^{\pi/2} \sin^2\theta \cos^4\theta d\theta = \int_0^{\pi/2} \sin^2\theta \cos^4\theta d\theta$$

$$\therefore P = 0, q = 4$$

$$\int_0^{\pi/2} \cos^4 \theta d\theta = \frac{1}{2} \beta \left(\frac{p+1}{2}, \frac{q+1}{2} \right)$$

$$= \frac{1}{2} \beta \left(\frac{1}{2}, \frac{5}{2} \right)$$

$$= \frac{1}{2} \frac{\Gamma(1/2) \Gamma(5/2)}{\Gamma(1/2 + 5/2)}$$

$$= \frac{1}{2} \frac{\Gamma(1/2) (5/2 - 1) \Gamma(5/2 - 1)}{\Gamma(3)}$$

$$= \frac{1}{2} \frac{\Gamma(1/2) (3/2) \Gamma(3/2)}{2!}$$

$$= \frac{1}{2} \frac{\Gamma(1/2) (3/2) (3/2 - 1) \Gamma(3/2 - 1)}{2}$$

$$= \frac{1}{2} \frac{\Gamma(1/2) (3/2) (3/2 - 1) \Gamma(3/2 - 1)}{2}$$

$$= \frac{3}{16} \sqrt{\pi} \sqrt{\pi}$$

$$\int_0^{\pi/2} \cos^4 \theta d\theta = \frac{3\pi}{16}$$

$$\therefore I = 16 \times \frac{3\pi}{16}$$

$$\Rightarrow I = 3\pi$$

3. Show that $\int_0^{\pi/2} \sqrt{\sin \theta} d\theta \times \int_0^{\pi/2} \frac{1}{\sqrt{\sin \theta}} d\theta$

$$I_1 \times I_2$$

$$I_1 = \int_0^{\pi/2} \sqrt{\sin \theta} d\theta$$

$$= \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta$$

$$P = 1/2, q = 0$$

$$\begin{aligned} I_1 &= \frac{1}{2} B \left(\frac{p+1}{2}, \frac{q+1}{2} \right) \\ &= \frac{1}{2} B \left(\frac{1/2+1}{2}, \frac{0+1}{2} \right) \\ &= \frac{1}{2} B \left(3/4, 1/2 \right) \\ &= \frac{1}{2} \frac{\sqrt{3/4} \sqrt{1/2}}{\sqrt{3/4 + 1/2}} \\ &= \frac{1}{2} \frac{\sqrt{3/4} \sqrt{1/2}}{\sqrt{5/4}} \\ &= \frac{1}{2} \frac{\sqrt{3/4} \sqrt{1/2}}{\frac{1}{4} \sqrt{1/4}} \\ \Rightarrow I_1 &= 2 \frac{\sqrt{3/4} \sqrt{1/2}}{\sqrt{1/4}} \end{aligned}$$

$$\begin{aligned} \therefore I_2 &= \int_0^{\pi/2} \frac{1}{\sqrt{\sin \theta}} d\theta \\ &= \int_0^{\pi/2} \frac{1}{\sin^{1/2} \theta} d\theta \\ &= \int_0^{\pi/2} \sin^{-1/2} \theta \cos^0 \theta d\theta \end{aligned}$$

$$P = -1/2, q = 0$$

$$\begin{aligned} I_2 &= \frac{1}{2} B \left(-\frac{1/2+1}{2}, \frac{0+1}{2} \right) \\ &= \frac{1}{2} B \left(1/4, 1/2 \right) \end{aligned}$$

$$= \frac{1}{2} \frac{\sqrt{1/4} \sqrt{1/2}}{\sqrt{1/4 + 1/2}}$$

$$\Rightarrow I_2 = \frac{1/2 \sqrt{1/4} \sqrt{1/2}}{\sqrt{3/4}}$$

$$I = 2 \frac{\sqrt{2/4} \sqrt{1/2}}{\sqrt{1/4}} \cdot 1/2 \frac{\sqrt{1/4} \sqrt{1/2}}{\sqrt{3/4}}$$

$$\therefore I = \sqrt{1/2} \sqrt{1/2}$$

$$I = \sqrt{\pi} \sqrt{\pi}$$

$$I = \pi$$

4. Evaluate $\int_0^1 x^{3/2} (1-x)^{1/2} dx$ by expressing in terms of Beta - gamma function.

$$\text{Let } I = \int_0^1 x^{3/2} (1-x)^{1/2} dx$$

$$\text{Let } x = \sin^2 \theta \Rightarrow \theta = \sin^{-1}(\sqrt{x})$$

$$dx = 2 \sin \theta \cos \theta d\theta$$

$$\text{UL : } x=1 \Rightarrow \theta = \pi/2$$

$$\text{LL : } x=0 \Rightarrow \theta = 0$$

$$I = \int_0^{\pi/2} (\sin^3 \theta)^{3/2} (1 - \sin^2 \theta)^{1/2} 2 \sin \theta \cos \theta d\theta$$

$$= 2 \int_0^{\pi/2} \sin^3 \theta \cos \theta \sin \theta \cos \theta d\theta$$

$$I = \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta$$

$$= \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta \cdot \int_0^{\pi/2} \sin^p \theta \cos^q \theta d\theta$$

$$\therefore p=4, q=2$$

$$\begin{aligned}
 \int_0^{\pi/2} \sin^4 \theta \cos^2 \theta d\theta &= \frac{1}{2} B\left(\frac{5}{2}, \frac{3}{2}\right) \\
 &= \frac{1}{2} \frac{\sqrt{5/2} \sqrt{3/2}}{\sqrt{4}} \\
 &= \frac{1}{2} \frac{3/2 \cdot 1/2 \sqrt{1/2} \cdot 1/2 \sqrt{1/2}}{3!} \\
 &= \frac{1}{2} \frac{3/2 \cdot 1/2 \sqrt{1/2} \cdot 1/2 \sqrt{1/2}}{6} \\
 &= \frac{1}{2^5} \sqrt{\pi} \sqrt{\pi} \\
 &= \frac{\pi}{2^5} \\
 \therefore I &= \frac{\pi}{16}
 \end{aligned}$$

5. Evaluate $\int_0^4 x^{3/2} (4-x)^{5/2} dx$ by using Beta-gamma function, show that $B(m,n) = \int_0^\infty \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\text{Let } I = \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$\text{then, } x = \frac{1}{y}, \quad y = \frac{1}{x}$$

$$dx = \frac{1}{y^2} dy$$

$$\text{U.L : } x = \infty \Rightarrow y = 0$$

$$\text{LL : } x = 1 \Rightarrow y = 1$$

$$\therefore I = \int_0^1 \frac{(1/y)^{m-1}}{(1+y)^{m+n}} \left(-\frac{1}{y^2} \right) dy$$

$$= \int_0^1 \frac{\frac{1}{y^{m-1}} \cdot \frac{1}{y^2}}{\frac{(1+y)^{m+n}}{y^{(m+n)}}} dy$$

$$= \int_0^1 \frac{\frac{1}{y^{m-1}}}{\frac{(1+y)^{m+n}}{y^{m+n}}} dy$$

$$= \int_0^1 \frac{y^{m+n}}{y^{m+1} (1+y)^{m+n}} dy$$

$$= \int_0^1 \frac{y^m y^n}{y^m y^1 (1+y)^{m+n}} dy$$

$$I = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}} dy$$

$$I = \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\therefore \beta(m, n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}} dx //$$

6. Show that $\int_0^\infty \frac{e^{-x^2} dx}{\sqrt{x}} \times \int_0^\infty \sqrt{x} e^{-x^2} dx = \frac{\pi}{2\sqrt{2}}$

$$I = \int_0^\infty e^{-x^2} \sqrt{x} dx \quad \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$$

$$\Rightarrow I = I_1 \times I_2$$

$$I_1 = \int_0^{\infty} e^{-x^2} \sqrt{x} dx \Rightarrow \int_0^{\infty} e^{-x^2} x^{1/2} dx \\ = \int_0^{\infty} e^{-x^2} x^{2n-1} dx$$

$$\therefore 2m-1 = \frac{1}{2} \Rightarrow 2m = \frac{3}{2} \Rightarrow m = \frac{3}{4}$$

$$I_1 = \frac{\sqrt{n}}{2} = \frac{\sqrt{3/4}}{2}$$

$$I_2 = \int_0^{\infty} e^{-x^2}/\sqrt{x} dx \Rightarrow \int_0^{\infty} e^{-x^2} x^{-1/2} dx \Rightarrow \int_0^{\infty} e^{-x^2} x^{2m-1} dx$$

$$\therefore 2m-1 = -\frac{1}{2} \Rightarrow 2m = 1 - \frac{1}{2} \Rightarrow 2m = \frac{1}{2} \Rightarrow m = \frac{1}{4}$$

$$I_2 = \frac{\sqrt{1/4}}{2}$$

$$I = I_1 \times I_2$$

$$I = \frac{\sqrt{3/4}}{2} \times \frac{\sqrt{1/4}}{2} \Rightarrow \frac{1}{4} \sqrt{3/4} \sqrt{1/4} \Rightarrow \frac{1}{4} \sqrt{1 - 1/4} = \frac{1}{4} \sqrt{1/4} \\ \Rightarrow \frac{1}{4} \frac{\pi}{\sin \pi}$$

$$I = \frac{1}{4} \frac{\pi}{\sin \frac{\pi}{4}}$$

$$I = \frac{1}{4} \frac{\pi}{\frac{1}{\sqrt{2}}} \\ = \frac{\sqrt{2}}{4} \pi \Rightarrow \frac{\pi}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}} \text{ hence proved.}$$