Module 2 :

→ Properties of Integers :

- Mathematical Induction
- Recursive Definitions.

→ Fundamental Principles of counting :

- The Rules of sum and Product
- Permutations
- combinations
- Binomial and Multinomial theorems
- Combinations with repetitions .

Mathematical Induction

The method of mathematical Induction is based on a Principle called the "Induction Principle". This Principle is based on another Principle known as "well-ordering Principle".

Note :- z^+ denotes the set of all positive integers.

Well-ordering Principle :-

QP "Every non-empty subset of z^+ contains a smallest (least) element"

(OR)

"The set of all +ve integers is well ordered".

Induction Principle :- (Principle of Mathematical Induction)

QP Let $S(n)$ denote an open statement that involves a +ve integer 'n'. Suppose that the foll. cond^ns hold :

(i) $S(1)$ is true.

(ii) If whenever $S(k)$ is true for some particular, arbitarily chosen $k \in z^*$, then $S(k+1)$ is true.

Then $S(n)$ is true for all $n \in z^+$.

Method of Mathematical Induction :-

Suppose we wish to prove that certain statement $S(n)$ is true, $\forall\, n \geq 1$. The method of proving this using principle of Induction is called the Method of Mathematical Induction. There are 2 steps in this method :-

1) Basic step : verify that $S(n)$ is true for $n=1$.

2) Induction step : Assuming $S(k)$ is true, where $k \geq 1$, show that $S(k+1)$ is true.

Note :-

1) without basic step, one cannot write the proof induction.

2) Suppose we require to prove a statement $s(n)$ for $n \geq n_0$, where n_0 is an integer. In such situations verifying $s(n_0)$ is true is the basic step, and in the Induction step, we take $K \geq n_0$.

Alternate form of Induction Principle :-

Let $s(n)$ denote a statement that involves a +ve integer 'n'. Suppose that

(i) for some +ve integers $n_0, n_0+1, n_0+2 \ldots n_0+p$, the statements $s(n_0), s(n_0+1), \ldots, s(n_0+p)$ are true

and (ii) if $s(n_0), s(n_0+1), \ldots s(K)$ are true where $K \geq n_0+p$, then $s(K+1)$ is true.

Then $s(n)$ is true $\forall\ n \geq n_0$.

Problems

1) Prove by M.I that, $\forall\ n \geq 1$, $1+2+3+4+\dots+n = \frac{1}{2}n(n+1)$

Soln :- Let $S(n)$: $1+2+3+\dots+n = \frac{1}{2}n(n+1)$

Basic step :- $S(1)$ is the statement

$1 = \frac{1}{2}\cdot 1\,(2)$ which is true.

Thus the statement $S(n)$ is verified for $n=1$.

Induction step :- Assume that $S(n)$ is true for $n=k$ where $k \geq 1$.

ie assume that $S(k)$: $1+2+3+\dots+k = \frac{1}{2}k(k+1)$ is true

adding $(k+1)$ to both sides, we get

$1+2+3+\dots+k+k+1 = \frac{1}{2}k(k+1)+(k+1)$

$= \frac{1}{2}(k+1)\left[k+2\right]$

$= \frac{1}{2}(k+1)(k+2)$

This is the statement $S(k+1)$.

Thus $S(k+1)$ is true, whenever $S(k)$ is true for $k \geq 1$.

Hence ~~the~~ ~~proof~~ by M.I, $S(n)$ is true $\forall\ n \geq 1$.

2) Prove by M.I, that $1^2+3^2+5^2+\dots+(2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$ $\forall\ n \geq 1$.

QP

Soln :- Let $S(n)$: $1^2+3^2+5^2+\dots+(2n-1)^2 = \frac{1}{3}n(2n-1)(2n+1)$.

Basic step : $S(1)$: $1^2 = \frac{1}{3}(1)(1)(3)$ which is true.

Induction step : Assume that $S(n)$ is true for $n=k$, where $k \geq 1$. ie $1^2+3^2+5^2+\dots+(2k-1)^2 = \frac{1}{3}k(2k-1)(2k+1)$

adding $(2k+1)^2$ on b.s

$[2(k+1)-1]^2$
$=(2k+2-1)^2$
$=(2k+1)^2$

$1^2+3^2+5^2+\dots+(2k-1)^2+(2k+1)^2 = \frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2$

$= \frac{1}{3}(2k+1)\left[k(2k-1)+3(2k+1)\right]$

$2k^2+3k+2k+3$

$k(2k+3)+1(2k+3)$

$(2k+3)(k+1)$

$= \frac{1}{3}(2k+1)\left[2k^2-k+6k+3\right]$

$= \frac{1}{3}(2k+1)(2k^2+5k+3)$.

$= \frac{1}{3}(2k+1)(k+1)(2k+3)$

This is the statement $S(k+1)$.

Thus $S(k+1)$ is true whenever $S(k)$ is true, where $k \geq 1$.

Hence by M.I, $S(n)$ is true $\forall\, n \geq 1$.

3) Prove by M.I, $1\cdot3+2\cdot4+3\cdot5+\dots+n(n+2) = \frac{n(n+1)(2n+7)}{6}$

Soln :- Let $S(n)$ denote the given statement.

Basic step :- $S(1)$: $1\cdot3 = \frac{1(\cancel{2})(\cancel{9})}{\cancel{6}}$ which is true.

Induction step : Assume that $S(k)$ is true, where $k \geq 1$.

ie $1\cdot3+2\cdot4+3\cdot5+\dots+k(k+2) = \frac{k(k+1)(2k+7)}{6}$

adding $(k+1)(k+3)$ on b.s, we get

$1\cdot3+2\cdot4+3\cdot5+\dots+k(k+2)+(k+1)(k+3) = \frac{k(k+1)(2k+7)}{6}+(k+1)(k+3)$

$= \frac{(k+1)}{6}\left[k(2k+7)+6(k+3)\right]$

$2k^2+9k+4k+18$

$k(2k+9)+2(2k+9)$

$(2k+9)(k+2)$

$= \frac{(k+1)}{6}\left[2k^2+13k+18\right]$

$= \frac{(k+1)(k+2)(2k+9)}{6}$

This is the statement $S(k+1)$.

This is the statement $S(k+1)$ is true whenever $S(k)$ is true, where $k \geq 1$

$\therefore$ $S(k+1)$ is true whenever $S(k)$ is true, where $k \geq 1$

Hence by M.I, $S(n)$ is true $\forall\, n \geq 1$.

Do 11^{th}, then 12^{th} prob & 13^{th} prob

4) Prove by M.I, for every +ve integer n, 5 divides $n^5 - n$.

SP

Soln :

Note :- Ex :- 5 divides 30 means $\frac{30}{5}$.

(or) 30 is a multiple of 5

ie $30 = 5 \times m$ (Here $m = 6$) for some +ve integer m

Let $S(n)$: 5 divides $n^5 - n$.

Basic step :- $S(1)$: 5 divides $1^5 - 1 = 0$. (ie $\frac{0}{5} = 0$ ✓)

which is true.

Induction Step : Assume that $S(n)$ is true for $n = k$ where $k \geq 1$.

ie for $k \geq 1$, 5 divides $k^5 - k$.

ie $k^5 - k$ is a multiple of 5

(or) $\boxed{k^5 - k = 5m}$, for some +ve integer 'm'.

$$\therefore (k+1)^5 - (k+1) = (k+1)^2 (k+1)^2 (k+1) - (k+1)$$
$$= (k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + \cancel{1}) - (k + \cancel{1})$$
$$= (k^5 - k) + 5(k^4 + 2k^3 + 2k^2 + k)$$
$$= 5m + 5(k^4 + 2k^3 + 2k^2 + k)$$
$$= 5(m + k^4 + 2k^3 + 2k^2 + k)$$

$\Rightarrow (k+1)^5 - (k+1)$ is a multiple of 5

ie 5 divides $(k+1)^5 - (k+1)$. This is precisely (exactly / just) the statement $S(k+1)$.

Thus $S(k+1)$ is true whenever $S(k)$ is true for $k \geq 1$.

$\therefore$ By M.I, $S(n)$ is true $\forall\, n \geq 1$.

5) P.T $4n < (n^2 - 7)$ $\forall$ +ve integers $n \geq 6$.

SP

Soln :- Let $S(n)$: $4n < (n^2 - 7)$.

To prove that $S(n)$ is true $\forall$ +ve integers $n \geq n_0$, where $n_0 = 6$.

Basic step : $S(6) : (4\times 6) < (6^2-7)$

i.e. $24 < 29$ which is true.

$\therefore$ $S(n)$ is true for $n = n_0 = 6$.

Induction step : Assume that $S(n)$ is true for $n=k$ where $k \geq 6$. i.e. $4k < (k^2-7)$ for $k \geq 6$.

we want $(k+1)^2 - 7$

$k^2+1+2k-7$

$k^2+(2k+1)-7$

add 4 on b.s

then $4(k+1) = 4k+4 < (k^2-7)+4$

$< (k^2-7)+(2k+1)$

because when $k \geq 6$, we have $2k \geq 12 \Rightarrow 2k+1 \geq 13 > 4$ (or) $k>5 \Rightarrow 2k>10$, $2k+1>11>4$

$\therefore$ $4(k+1) < (k^2+2k+1)-7$

i.e. $4(k+1) < (k+1)^2-7$. This is the statement $S(k+1)$

$\Rightarrow$ $S(k+1)$ is true, whenever $S(k)$ is true $\forall\, k \geq 6$.

Thus $S(n)$ is true $\forall\, n \geq 6$.

6) Show that $2^n > n^2$ $\forall$ the integers 'n' greater than 4.

Soln :- To prove $S(n) : 2^n > n^2$ is true $\forall\, n \geq 5$.

Basic step : for $n=5$, $S(n)$ gives

$S(5) : 2^5 > 5^2$ (i.e. $32 > 25$) which is true.

Induction step : Assume that $S(n)$ is true for $n=k$, where $k \geq 5$. i.e. $2^k > k^2$ for $k \geq 5$.

$\times$plying by 2 on b.s

$2.2^k = 2^{k+1} > 2k^2$

Since $k>4$, we have $2k^2 = k^2+k^2 = k^2+(k\times k)$

$> k^2+4k$

$> k^2+2k+2k$

$> k^2+2k+1 = (k+1)^2$

∴ thus $2^{k+1} > (k+1)^2$ for $k>4$.

This is precisely the statement $S(k+1)$.

Thus $S(k+1)$ is true whenever $S(k)$ is true $\forall\ k \geq 5$.

Hence by M.I, $S(n)$ is true $\forall\ n > 4$.

7) By M.I, Prove that $n! \geq 2^{n-1}\ \forall\ n \geq 1$.

Soln:- Let $S(n)$ be the given statement.

Basic step: for $n=1$, $S(1)$: $1! \geq 2^{1-1}$

$1 \geq 1$ which is true.

Induction step: Assume that $S(n)$ is true for $n=k$, where $k \geq 1$. ie $k! \geq 2^{k-1}$ for $k \geq 1$.

ie $2^{k-1} \leq k!$.

$\boxed{\cancel{2^k = 2^{k+1-1} = 2}}$ xplying by 2 on b.s

$$2^k = 2 \cdot 2^{k-1} \leq 2 \cdot k!$$

$$\leq (k+1) \cdot k! \quad \text{because } (k+1) \geq 2 \text{ for } k \geq 1$$

$$= (k+1)!$$

ie $(k+1)! \geq 2^k$, which is the statement $S(k+1)$.

Thus $S(k+1)$ is true whenever $S(k)$ is true for $k \geq 1$.

Hence by M.I, $S(n)$ is true $\forall\ n \geq 1$.

8) For all $n \in Z^+$, show that if $n \geq 14$, then n can be written as a sum of 3's and / or 8's.

P.T.O.

Soln: To prove:

$S(n)$: n can be written as a sum of 3's or 8's is true for all integers $n \geq 14$.

Basic step: we have $14 = (3+3)+8$.

$\Rightarrow S(14)$ is true.

Induction step: Assume that $S(n)$ is true for $n=K$ where $K \geq 14$.

then $K = (3+3+3+\ldots)+(8+8+\ldots)$

suppose this representation of K has r no of 3's and s no of 8's.

Since $K \geq 14$, we must have $r \geq 2$, $s \geq 1$.

$$\therefore \quad K+1 = \{(\underbrace{3+3+3+\ldots}_{r})+(\underbrace{8+8+\ldots}_{s})\}+1$$

$$= (\underbrace{3+3+3+\ldots}_{r})+(\underbrace{8+8+\ldots}_{s-1})+8+1$$

$$= (\underbrace{3+3+3+\ldots}_{r})+(\underbrace{8+8+\ldots}_{s-1})+(3+3+3)$$

$$= (\underbrace{3+3+3+\ldots}_{r+3})+(\underbrace{8+8+\ldots}_{s-1})$$

$\Rightarrow (K+1)$ is a sum of 3's & 8's. ie $(K+1)$ is true.

Hence by H.I, $S(n)$ is true for all +ve integers $n \geq 14$

Q) By H.I, Prove that, for every +ve integer n, the no $A_n = 5^n + 2 \cdot 3^{n-1} + 1$ is a multiple of 8.

Soln:- Let $S(n)$ be the given statement.

Basic step: $S(1)$: $5^1 + 2 \cdot 3^0 + 1 = 8$ is a multiple of 8.

$\therefore$ for $n=1$, A_n is ~~(a multiple of 8)~~ true.

Induction step : Assume that A_n is a multiple of 8 for $n = k \geq 1$.

ie $A_k = 5^k + 2 \cdot 3^{k-1} + 1$ is a multiple of 8.

we find,

$$A_{k+1} - A_k = (5^{k+1} + 2 \cdot 3^{k} + 1) - (5^k + 2 \cdot 3^{k-1} + 1)$$
$$= (5^k \cdot 5 - 5^k) + 2(3^k - 3^{k-1})$$
$$= 5^k(5-1) + 2 \cdot 3^{k-1}(3-1)$$
$$= 4 \cdot 5^k + 4 \cdot 3^{k-1}$$
$$= 4(5^k + 3^{k-1})$$

5 & 3 are odd $\Rightarrow$ 5^k & 3^{k-1} are also odd.

$\Rightarrow$ $5^k + 3^{k-1}$ is Even.

$\Rightarrow$ $4(5^k + 3^{k-1})$ is a multiple of 8.

ie $A_{k+1} - A_k$ " ——————— "

Since A_k is a multiple of 8 (by assumption), A_{k+1} is also a multiple of 8.

if A_n is a multiple of 8 for $n = k$, then

Thus A_n is a multiple of 8 for $n = k+1$.

Hence by M.I, for every +ve integer n, $A_n = 5^n + 2 \cdot 3^{n-1} + 1$ is a multiple of 8.

10) Prove by mathematical Induction that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each +ve integer n.

QP

Soln :- Let $A_n = 6^{n+2} + 7^{2n+1}$

Basic step : $A_1 = 6^3 + 7^3 = 559$ is divisible by 43.

$\therefore$ A_n is divisible by 43 for $n = 1$.

Induction step : Assume that A_n is divisible by 43 for $n = k \geq 1$. ie $A_k = 6^{k+2} + 7^{2k+1}$

$\therefore A_{k+1} = 6^{k+3} + 7^{2(k+1)+1}$

$= \cancel{6^{k+3}} \cancel{+} = 6^{k+3} + 7^{2k+3}$

$= 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 7^2$

$= 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 49$

$= 6^{k+2} \cdot 6 + 7^{2k+1} (6 + 43)$

$a(b+c) = ab + bc$

$= 6^{k+2} \cdot 6 + 7^{2k+1} \cdot 6 + 7^{2k+1} \cdot 43$

$= 6 \left[6^{k+2} + 7^{2k+1} \right] + 43 \cdot 7^{2k+1}$

$= 6 A_k + 43 \cdot 7^{2k+1}$

By assumption, A_k is divisible by $43 \Rightarrow 6 A_k$ is also divisible by 43, and $43 \cdot 7^{2k+1}$ is divisible by 43

$\therefore 6A_k + 43 \cdot 7^{2k+1}$ is divisible by 43

$\Rightarrow A_{k+1}$ is divisible by 43.

Thus if $S(k)$ is true, then $S(k+1)$ is true for $k \geq 1$.

Hence by M.I, $S(n)$ is true $\forall\, n \geq 1$.

11) Prove $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1} \quad \forall\, n \in Z$.

Q8

ie $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} = \frac{n}{n+1}$

Soln :- Let $S(n)$ be the given statement.

Basic step : $S(1) : \frac{1}{1(2)} = \frac{1}{1+1} \Rightarrow \frac{1}{2} = \frac{1}{2}$ which is true

PTO

Induction step : Assume $S(k)$ is true for $k \geq 1$. (6)

ie $\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{k(k+1)} = \frac{k}{k+1}$ is true.

adding $\frac{1}{(k+1)(k+2)}$ on b.s

$$\frac{1}{1.2} + \frac{1}{2.3} + \dots + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right]$$

$$= \frac{1}{(k+1)}\left[\frac{k^2+2k+1}{k+2}\right]$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2}$$

which is precisely the statement $S(k+1)$.

Thus $S(k+1)$ is true, whenever $S(k)$ is true.

Hence by H.I, $S(n)$ is true $\forall n \geq 1$.

12) Establish the following by H.I

$$\sum_{i=1}^{n} i\,2^i = 2 + (n-1)\,2^{n+1}$$

Soln :- Let $S(n)$ be the given statement.

Basic step : $S(1)$; $1 \times 2^1 = 2 + (1-1)\,2^2$

$2 = 2$ which is true.

Induction step : Assume that the result is true for $n = k$, where $k \geq 1$.

ie $S(k)$: $\sum_{i=1}^{k} i\,2^i = 2 + (k-1)\,2^{k+1}$ is true.

ie $1\times 2^1 + 2\times 2^2 + 3\times 2^3 + 4\times 2^4 + \dots + k\times 2^k = 2 + (k-1)\,2^{k+1}$

add $(k+1)\times 2^{k+1}$ on b.s

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$$1\times 2^1 + 2\times 2^2 + 3\times 2^3 + \dots + k\times 2^k + (k+1)\times 2^{(k+1)}$$

$$= 2 + (k-1)2^{k+1} + (k+1)\times 2^{(k+1)}$$

$$= 2 + 2^{k+1}\left[k-\cancel{1}+k+\cancel{1}\right]$$

$$= 2 + 2^{k+1}\cdot 2k$$

$$= 2 + k\cdot 2^{(k+2)}$$

which is precisely the statement $S(k+1)$.

Thus $S(k+1)$ is true whenever $S(k)$ is true, $\forall\ k \geq 1$.

Hence $S(n)$ is true $\forall\ n \geq 1$.

HW

1) P.T $1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{1}{6}n(2n+1)(n+1)$

2) P.T $1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

QP 3) P.T $2^n < n!$ $\forall$ +ve integers $n > 3$.

4) For every integer n, 3 divides $n^3 - n$.

QP 5) For all $n \in Z^+$, S.T if $n \geq 24$ then n can be written as a sum of 5's and 7's.

13) Let $H_1 = 1$, $H_2 = 1 + \frac{1}{2}$, $H_3 = 1 + \frac{1}{2} + \frac{1}{3}$, ... $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

QP Prove that $\sum_{i=1}^{n} H_i = (n+1)H_n - n$, $\forall\ n \geq 1$.

Soln :- Let $S(n)$: $\sum_{i=1}^{n} H_i = (n+1)H_n - n$. $\forall\ n \geq 1$.

$\Rightarrow H_1 + H_2 + H_3 + \dots + H_n = (n-1)H_n - n$.

Basic step : $S(1)$: $H_1 = (1+1)H_1 - 1$.

$1 = 2(1) - 1$.

$1 = 1$.

$\therefore$ $S(1)$ is true.

Induction step : Assume $S(k)$ is true $\forall\ k \geq 1$.

ie $S(k)$: $\sum_{i=1}^{k} H_i = (k+1)H_k - k$ is true $\forall\ k \geq 1$.

ie $S(k)$: $H_1 + H_2 + \dots + H_k = (k+1)H_k - k$ is true $\forall\ k \geq 1$.

adding H_{K+1} on b.s.

$$H_1+H_2+\dots+H_K+H_{K+1} = (K+1)H_K - K + H_{K+1}$$

$$= (K+1)\left\{H_{K+1} - \frac{1}{K+1}\right\} - K + H_{K+1}$$

$\left[\because H_K = H_{K+1} - \frac{1}{(K+1)} \text{ from the def}^n \text{ of } H_n\right]$

$H_K = 1+\frac{1}{2}+\dots+\frac{1}{K}$

$H_{K+1} = 1+\frac{1}{2}+\dots+\frac{1}{K}+\frac{1}{K+1}$

$H_{K+1} = H_K + \frac{1}{K+1}$

$$= (K+1)H_{K+1} - 1 - K + H_{K+1}$$

$$= (K+1+1)H_{K+1} - (K+1)$$

$$= (K+2)H_{K+1} - (K+1).$$

this is precisely the statement $S(K+1)$.

By M.I, $S(n)$ is true $\forall n \geq 1$.

Recursive Definitions

Consider $a_1, a_2, \ldots a_n$, a sequence in which a_1 is the first term, a_2 is the 2^{nd} term ...... a_n is n^{th} term. The element a_n is called the general term of the sequence denoted by $\{a_n\}$. If the first term is denoted by a_0, then n^{th} term is denoted by a_{n-1} and this sequence is denoted by $\{a_{n-1}\}$.

For describing a sequence, 2 methods are commonly used:

(i) Explicit method (ii) Recursive method.

In Explicit method, the general term of the sequence is explicitly indicated.

Ex :- $E = \{a_n\}$, where $a_n = 2n$, $n \in Z^+$.

In Recursive method, first few terms of the sequence are explicitly indicated and the general term is specified through a rule (formula) which indicates how to obtain new terms of the sequence from the terms already known.

Ex :- fibonacci numbers $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ $\forall n \geq 2$.

Note :- 1) A recursive definition of a sequence should contain 2 parts : in the 1st part, a few terms of the sequence are indicated explicitly and in the 2^{nd} part, the rule to obtain new terms for sequence is indicated.

2) Recursive rule/formula : is a formula for a_n in terms of a_{n-1}, a_{n-2} etc (or) a formula for a_{n+1} in terms of a_n, a_{n-1} etc.

P.T.O.

Problems

1) obtain a recursive def^n for the sequence $\{a_n\}$ in each of the following cases:

i) $a_n = 6^n$ ii) $a_n = n(n+3)$

iii) $a_n = (n+1)!$ iv) $a_n = 2-(-1)^n$.

Sol^n:- i) $a_n = 6^n$.

$a_1 = 6^1 = 6$.

$a_2 = 6^2 = 36 = 6\times 6 = 6\times a_1$

$a_3 = 6^3 = 216 = 6\times 36 = 6\times a_2$

$a_4 = 6^4 = 1296 = 6\times 216 = 6\times a_3$

⋮

$a_1 = 6$; $\boxed{a_n = 6\,a_{n-1}}$ $\forall\, n\geq 2$. is the recursive def^n.

→ same as $a_{n+1} = 6a_n$ $\forall\, n\geq 1$

ii) $a_n = n(n+3)$.

$a_1 = 1(1+3) = 1\times 4 = 4$

$a_2 = 2(2+3) = 2\times 5 = 10 = a_1+6 = a_1+(2\times 3)$

$a_3 = 3(3+3) = 3\times 6 = 18 = a_2+8 = a_2+(2\times 4)$

$a_4 = 4(4+3) = 4\times 7 = 28 = a_3+10 = a_3+(2\times 5)$

⋮

$a_n = a_{n-1} + 2(n+1)$ $\forall\, n\geq 2$.

Thus $a_1 = 4$, $a_n = a_{n-1}+2(n+1)$ $\forall\, n\geq 2$ is the recursive def^n.

iii) $a_n = (n+1)!$.

$a_1 = (1+1)! = 2! = 2$.

$a_2 = (2+1)! = 3! = 6 = a_1\times 3$

$a_3 = (3+1)! = 4! = 24 = a_2\times 4$

$a_4 = (4+1)! = 5! = 120 = a_3\times 5$

⋮

$a_n = a_{n-1}\times(n+1)$ $\forall\, n\geq 2$.

Thus $a_1 = 2$, $a_n = (n+1)\,a_{n-1}$ $\forall\, n\geq 2$ is the recursive def^n.

iv) $a_n = 2-(-1)^n$.

$a_1 = 2-(-1)^1 = 2+1 = 3$.

$a_2 = 2-(-1)^2 = 2-1 = 1 = 3-2 = a_1-(-1)^2\times 2$

$a_3 = 2-(-1)^3 = 2+1 = 3 = 1+2 = a_2-(-1)^3\times 2$

$a_4 = 2-(-1)^4 = 2-1 = 1 = 3-2 = a_3-(-1)^4\times 2$

⋮

$a_1 = 3$, $a_n = a_{n-1} - 2(-1)^n$ $\forall\, n\geq 2$. is the recursive def^n

2) Find an explicit def^n of the sequence defined recursively by $a_1 = 7$, $a_n = 2a_{n-1} + 1$ for $n \geq 2$.

Soln:-

$$a_n = 2a_{n-1} + 1 = 2^1 a_{n-1} + (2^1 - 1)$$
$$= 2\{2a_{n-2} + 1\} + 1 = 4a_{n-2} + 3 = 2^2 a_{n-2} + (2^2 - 1)$$
$$= 4\{2a_{n-3} + 1\} + 3 = 8a_{n-3} + 7 = 2^3 a_{n-3} + (2^3 - 1)$$
$$= 8\{2a_{n-4} + 1\} + 7 = 16a_{n-4} + 15 = 2^4 a_{n-4} + (2^4 - 1).$$
$$\vdots$$

taking $a_{n-4} = a_1$ $\Rightarrow n-4 = 1$ $\Rightarrow n = 5$ $\Rightarrow n-1 = 5-1 = 4$

$2^{n-1} a_1$

$$\therefore a_n = 2^{n-1} \cdot a_1 + 2^{n-1} - 1$$
$$= 2^{n-1}\{a_1 + 1\} - 1$$
$$= 2^{n-1}\{7 + 1\} - 1$$
$$= 2^{n-1}(8) - 1$$
$$= 2^{n-1} \cdot 2^3 - 1 = \boxed{2^{n+2} - 1}$$
$$= \boxed{4 \times 2^n - 1}$$

$a_1 = a_n - (n-$

Thus $a_n = 4 \times 2^n - 1$ is an explicit def^n for the given sequence.

3) A sequence $\{a_n\}$ is defined recursively by $a_1 = 4$, $a_n = a_{n-1} + n$ $\forall\ n \geq 2$. find a_n in explicit form.

Soln:-

$$a_n = a_{n-1} + n.$$
$$= \{a_{n-2} + (n-1)\} + n$$
$$= a_{n-3} + (n-2) + (n-1) + n.$$
$$= a_{n-4} + (n-3) + (n-2) + (n-1) + n$$
$$\vdots$$

taking $a_{n-4} = a_1$ $\Rightarrow n-4 = 1 \Rightarrow n =$

$$= a_1 + 2 + 3 + 4 + \dots + n.$$
$$= (a_1 - 1) + (1 + 2 + 3 + \dots + n)$$
$$= (a_1 - 1) + \frac{n(n+1)}{2}$$

$\boxed{a_n = 3 + \frac{n(n+1)}{2}}$ is the explicit formula for a_n.

P.T.O.

4) A sequence $\{a_n\}$ is defined recursively by $a_0 = 1,\ a_1 = 1,\ a_2 = 1$ and $a_n = a_{n-1} + a_{n-3}\ \forall\ n \geq 3$.

Prove that $a_{n+2} \geq (\sqrt{2})^n\ \forall\ n \geq 0$.

Soln :- Let $S(n) : a_{n+2} \geq (\sqrt{2})^n$

Basic step : $S(0) : a_2 \geq (\sqrt{2})^0$

$1 \geq 1$ — True

$S(1) : a_3 \geq (\sqrt{2})^1$

$2 \geq \sqrt{2}$ — True

$S(2) : a_4 \geq (\sqrt{2})^2$

$2 \geq 2$

($a_3 = ?$
$a_n = a_{n-1} + a_{n-3}$
$a_3 = a_2 + a_0 = 1+1=2$)
$a_4 = a_3 + a_1 = 1+1 = 2$

Thus $S(n)$ is true for $n = 0, 1, 2$.

Induction step : Assume $S(K)$ is true.

ie $S(K) : a_{K+2} \geq (\sqrt{2})^K\ \forall\ K \geq 0$ is true. $\rightarrow$ (1)

TPT : $a_{K+3} \geq (\sqrt{2})^{K+1}$

Consider $a_{K+3} = a_{K+2} + a_K$ (using the defn of a_n)

$\geq (\sqrt{2})^K + (\sqrt{2})^{K-2}$ (using (1))

$\geq (\sqrt{2})^{K-2}\left[(\sqrt{2})^2 + 1\right]$

$\geq (\sqrt{2})^{K-2}\,[3]$

$\geq (\sqrt{2})^{K+1-1-2}\,(3)$

$\geq (\sqrt{2})^{(K+1)} \cdot (\sqrt{2})^{-3} \cdot (3)$

$\geq (\sqrt{2})^{(K+1)} \dfrac{3}{2\sqrt{2}}$

$\geq (\sqrt{2})^{(K+1)}$. $\left(\because \dfrac{3}{2\sqrt{2}} \geq 1\right)$

Thus $a_{K+3} \geq (\sqrt{2})^{K+1}$

Hence by M.I, required result is true $\forall\ n \geq 0$.

5) For the sequence $\{a_n\}$ defined recursively by $a_1 = 8,\ a_2 = 22,\ a_n = 4(a_{n-1} - a_{n-2})$ for $n \geq 3$, prove that $a_n = (5+3n)\,2^{n-1}$ for $n \geq 1$.

Sol^n :- $S(n)$: $a_n = (5+3n)\, 2^{n-1} \quad \forall\, n \geq 1$.

Basic step : $S(1)$: $a_1 = (5+3)\, 2^0 = 8$.

$S(2)$: $a_2 = (5+6)\, 2 = 22$

$S(n)$ is true for $n = 1, 2$.

Induction step : Assume $S(K) : a_K = (5+3K)\, 2^{K-1}$ is true $\forall\, K \geq 1$. $\longrightarrow$ (1)

TPT : $a_{K+1} = \{5 + 3(K+1)\}\, 2^{K+1-1}$

ie $a_{K+1} = \{8 + 3K\}\, 2^K$.

Consider $a_{K+1} = 4\,(a_K - a_{K-1})$ (using the def^n of a_n)

$$a_{K+1} = 4\left[(5+3K)\,2^{K-1} - (5+3(K-1))\,2^{K-2}\right] \quad \text{(using (1))}$$

$$= 4\left[5\,(2^{K-1} - 2^{K-2}) + 3\{K \cdot 2^{K-1} - (K-1)\,2^{K-2}\}\right]$$

$$= 4\left[5 \cdot 2^{K-2}(2-1) + 3 \cdot 2^{K-2}\{2K - K + 1\}\right]$$

$$= 4\left[5 \cdot 2^{K-2} + 3 \cdot 2^{K-2}(K+1)\right]$$

$$= 4 \cdot 2^{K-2}\left[5 + 3K + 3\right]$$

$$= \cancel{4} \cdot 2^K \cdot \cancel{2^{-2}}\left[8 + 3K\right]$$

$$= 2^K\,(8 + 3K)$$ is the statement $S(K+1)$.

Thus by M.I, required result is true $\forall\, n \geq 1$.

6) A sequence $\{C_n\}$ is defined recursively by $C_n = 3C_{n-1} - 2C_{n-2}$ $\forall\, n \geq 3$, with $C_1 = 5$ and $C_2 = 3$. S.T $C_n = -2^n + 7$.

Sol^n :- Let $S(n)$: $C_n = -2^n + 7$. $\forall\, n \geq 1$.

Basic step : $S(1)$: $C_1 = -2^1 + 7$

$5 = 5$ – true.

$S(2)$: $C_2 = -2^2 + 7$

$3 = 3$ – true

$\therefore$ $S(n)$ is true for $n = 1, 2$.

4) Induction step : Assume $S(k) : C_k = -2^k + 7$ is true $\forall k \geq 1$

8) TPT : $S(k+1) : C_{k+1} = -2^{k+1} + 7$.

Sol

Consider $C_{k+1} = 3C_k - 2C_{k-1}$ (by the def^n of C_n).

$$= 3(-2^k + 7) - 2(-2^{k-1} + 7)$$

$$= (-3 \cdot 2^k + 2 \cdot 2^{k-1}) + 21 - 14$$

$$= 2^{k-1}(-3 \cdot 2 + 2) + 7$$

$$= -4 \cdot 2^{k-1} + 7$$

$$= -2^2 \cdot 2^{k-1} + 7$$

$$= -2^{k+1} + 7$$

which is exactly the statement $S(k+1)$.

Thus by M.I, $S(n)$ is true $\forall n \geq 1$.

7) The Fibonacci numbers are defined by $f_0 = 0$, $f_1 = 1$ and

Q) $f_n = f_{n-1} + f_{n-2}$ $\forall n \geq 2$. Prove that $\sum_{i=0}^{n} F_i^2 = f_n \times f_{n+1}$

Soln :- Let $S(n) : \sum_{i=0}^{n} F_i^2 = F_n \times F_{n+1}$

Basic step : $S(0) : F_0^2 = F_0 \times F_1$

$$0^2 = 0 \times 1$$

$$0 = 0$$

$S(1) : F_1^2 = F_1 \times F_2$

$$1^2 = 1 \times (F_1 + F_0)$$

$$1 = 1 \times (1 + 0)$$

$$1 = 1$$

$S(n)$ is true for $n = 0, 1$.

Induction step : Assume $S(k)$ is true $\forall k \geq 1$.

ie $S(k) : \sum_{i=0}^{k} F_i^2 = F_k \times F_{k+1}$ $\forall k \geq 1$. $\rightarrow (1)$

TPT : $S(k+1) : \sum_{i=0}^{k+1} F_i^2 = F_{k+1} \times F_{k+2}$

Consider $\sum_{i=0}^{k+1} F_i^2 = F_0^2 + F_1^2 + \ldots + F_k^2 + F_{k+1}^2$

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$$\sum_{i=0}^{k+1} F_i^2 = \sum_{i=0}^{k} F_i^2 + F_{k+1}^2$$

$$= (F_k \times F_{k+1}) + F_{k+1}^2 \quad (\text{using the eq}^n \text{ 1})$$

$$= F_{k+1}(F_k + F_{k+1})$$

$$= F_{k+1}(F_{k+2}) \quad (\because F_n = F_{n-1} + F_{n-2})$$

$$= F_{k+1} \times F_{k+2}$$

this is exactly the statement $s(k+1)$.

Thus by M.I, $S(n)$ is true $\forall n \geq 0$.

8) For the fibonacci sequence, $F_0, F_1, F_2 \dots$ prove that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

Soln:- Let $S(n): F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$

Basic step: $S(0): F_0 = \frac{1}{\sqrt{5}}[1-1] = 0.$

$$S(1): F_1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\cancel{\frac{1}{2}} + \frac{\sqrt{5}}{2} - \cancel{\frac{1}{2}} + \frac{\sqrt{5}}{2}\right]$$

$$= \frac{1}{\sqrt{5}} \cdot \frac{2\sqrt{5}}{2} = 1.$$

Thus $S(n)$ is true for $n = 0, 1$.

Induction step: Assume $S(k)$ is true $\forall k \geq 1$.

ie $S(k): F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] \quad \forall k \geq 1.$

TPT: $S(k+1)$ is true ie $F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right].$

Consider $F_{k+1} = F_k + F_{k-1}$ (by defn of fibonacci no's)

$$\Rightarrow F_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1+\sqrt{5}}{2} + 1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left\{\frac{1-\sqrt{5}}{2} + 1\right\}\right]$$

$$f_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right]$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{1+\sqrt{5}}{2}\right)^{2} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right]$$

$$\left[\because \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+5+2\sqrt{5}}{4} = \frac{6+2\sqrt{5}}{4} = \frac{2(3+\sqrt{5})}{4} = \frac{3+\sqrt{5}}{2}\right]$$

$$\therefore f_{k+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right]$$

Thus $S(K+1)$ is true whenever $S(K)$ is true.

Thus by M.I, the result is true $\forall\, n \geq 0$.

9) Let f_n denotes the fibonacci numbers. Prove that

QP

$$\sum_{i=1}^{n} \frac{f_{i-1}}{2^i} = 1 - \frac{f_{n+2}}{2^n}$$

Soln:- Let $S(n) : \sum_{i=1}^{n} \frac{f_{i-1}}{2^i} = 1 - \frac{f_{n+2}}{2^n}$.

Basic step: $n=1$; $S(1) : \frac{f_0}{2^1} = 1 - \frac{f_3}{2^1}$

we have $f_0 = 0$, $f_1 = 1$, $f_2 = f_1 + f_0 = 1$; $f_3 = f_2 + f_1 = 1+1 = 2$.

$\therefore S(1) : \frac{0}{2} = 1 - \frac{2}{2} \Rightarrow 0 = 0$

$S(n)$ is true for $n=1$.

$\therefore S(n)$ is true $\forall\, K \geq 1$.

Induction step: Assume $S(K)$ is true

ie $S(K) : \sum_{i=1}^{K} \frac{f_{i-1}}{2^i} = 1 - \frac{f_{K+2}}{2^K}$. $\rightarrow (1)$

TPT: $S(K+1) : \sum_{i=1}^{K+1} \frac{f_{i-1}}{2^i} = 1 - \frac{f_{K+3}}{2^{K+1}}$

Consider $\sum_{i=1}^{K+1} \frac{f_{i-1}}{2^i} = \underbrace{\frac{f_0}{2} + \frac{f_1}{2^2} + \frac{f_2}{2^3} + \dots + \frac{f_{K-1}}{2^K}} + \frac{f_K}{2^{K+1}}$

$$= \sum_{i=1}^{K} \frac{f_{i-1}}{2^i} + \frac{f_K}{2^{K+1}}$$

$$\Rightarrow \sum_{i=1}^{k+1} \frac{f_{i-1}}{2^i} = \left(1 - \frac{f_{k+2}}{2^k}\right) + \frac{f_k}{2^{k+1}} \quad (\text{using } ①)$$

$$= 1 - \frac{1}{2^k}\left[f_{k+2} - \frac{f_k}{2}\right]$$

$$= 1 - \frac{1}{2^k}\left[\frac{2f_{k+2} - f_k}{2}\right]$$

$$= 1 - \frac{1}{2^{k+1}}\left[f_{k+2} + f_{k+2} - f_k\right]$$

but $f_{k+2} = f_{k+1} + f_k$.

$\Rightarrow f_{k+2} - f_k = f_{k+1}$

$$\therefore \sum_{i=1}^{k+1} \frac{f_{i-1}}{2^i} = 1 - \frac{1}{2^{k+1}}\left[f_{k+2} + f_{k+1}\right]$$

$$= 1 - \frac{1}{2^{k+1}} f_{k+3} \quad \left(\because f_n = f_{n-1} + f_{n-2} \quad f_{k+3} = f_{k+2} + f_{k+1}\right)$$

this is the statement $S(k+1)$.

Thus by M.I, the required result is true $\forall\, n \geq 1$.

10) If f_i's are fibonacci numbers defined recursively by $f_0 = 0$, $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$ and L_i's are Lucas numbers defined recursively as $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ $\forall\, n \geq 2$, then P.T $L_n = f_{n-1} + f_{n+1}$ $\forall\, n \geq 1$.

QP

Soln :- Let $S(n)$: $L_n = f_{n-1} + f_{n+1}$ $\forall\, n \geq 1$.

Basic Step : $S(1)$: $L_1 = f_0 + f_2$ $\quad (\because f_2 = f_1 + f_0 = 1 + 0 = 1)$

$1 = 0 + 1 = 1$

$\therefore S(n)$ is true for $n = 1$.

Induction step : Assume $S(k)$ is true $\forall\, k \geq 1$.

i.e. $L_k = f_{k-1} + f_{k+1}$ $\forall\, k \geq 1 \rightarrow ①$

TPT : $S(k+1)$: $L_{k+1} = f_k + f_{k+2}$

Consider $L_{k+1} = L_k + L_{k-1}$ (by the def^n of L_n)

$\Rightarrow L_{k+1} = (F_{k-1} + F_{k+1}) + (F_{k-2} + F_k)$ (by the assumption made)

$= (F_{k-1} + F_{k-2}) + (F_{k+1} + F_k)$

$= F_k + F_{k+2}$ (by def^n of F_n)

this is the statement $S(k+1)$.

Thus by m.I, the required result is true $\forall\, n \geq 1$

11) The Ackerman's numbers are defined recursively for $m, n \in \mathbb{N}$ as follows:

$A_{0,n} = n+1$ for $n \geq 0 \rightarrow$ ①

$A_{m,0} = A_{m-1,1}$ for $m > 0 \rightarrow$ ②

$A_{m,n} = A_{m-1,p}$ where $p = A_{m,n-1}$ for $m, n > 0 \rightarrow$ ③

Prove that $A_{1,n} = n+2$ $\forall\, n \in \mathbb{N}$.

Soln :- Let $S(n)$: $A_{1,n} = n+2$ $\forall\, n \in \mathbb{N}$.

Basic step : $S(1)$: $A_{1,1} = 1+2 \Rightarrow RHS = 3$.

~~consider LHS : $A_{1,1} = A_{0,1}$ (by ②)~~
~~$A_{1,1} = 1+1 = 2$ (by ①)~~

consider LHS : $A_{1,1} = A_{0,p}$ where $p = A_{1,0} = A_{0,1} = 1+1 = 2$ (by ② & ①)

$\therefore A_{1,1} = A_{0,2}$

$= 2+1$ (by ①)

$LHS = 3$.

$\therefore S(1)$ is true.

Induction step : Assume $S(k)$ is true $\forall\, k \geq 1$.

ie $A_{1,k} = k+2$ $\forall\, k \geq 1 \rightarrow$ ④

TPT : $S(k+1)$: $A_{1,k+1} = k+3$ is true.

consider $A_{1,k+1} = A_{0,p}$ (using ③) where $p = A_{1,k} = k+2$ (using ④)

$= A_{0,k+2}$

$= k+2+1$ (using ①)

$= k+3$ is the statement $S(k+1)$, in conclusion

Principles of Counting

Rules of Sum and Product:

→ **The Sum Rule:** If T_1 and T_2 are 2 tasks such that T_1 can be performed in 'm' different ways, T_2 can be performed in 'n' different ways and if these 2 tasks cannot be performed simultaneously, then one of the task T_1 or T_2 can be performed in $m+n$ ways.

Ex:- Suppose a library has 12 books on Maths, 10 books on physics, 16 books on computers. Suppose the student wish to choose one of these books to study, then the no. of ways he can choose the book is $10+12+16 = 38$.

→ **The Product Rule:** If T_1 and T_2 are 2 tasks such that T_1 can be performed in 'm' different ways and for each of these ways, T_2 can be performed in 'n' different ways, then both the tasks can be performed in $m \times n$ different ways.

Ex:- Suppose a person has 8 shirts and 5 ties. Then the no. of ways of choosing a shirt and a tie is $8 \times 5 = 40$.

Problems :-

1) A license plate consists of two English letters followed by 4 digits. If repetitions are allowed, how many of the plates have only vowels and even digits?

Soln :- Each of the first 2 positions can be filled in 5 ways (with 5 vowels) & each of the remaining 4 places can be filled in 5 ways (with digits 0, 2, 4, 6, 8)

$\therefore$ No of possible license plates of the given type is

$(5\times5)\times(5\times5\times5\times5) = 5^6 = 15,625$.

2) Cars of a particular company come in 4 models, 12 colors, 3 engine sizes, & 2 transmission types.

(a) How many distinct cars of this company can be manufactured?

(b) Of these, how many have the same color?

Soln :- (a) The no of distinct cars that can be manufactured is $4\times12\times3\times2 = 288$.

(b) For any chosen color, no of distinct cars that can be manufactured is

$$4\times3\times2 = 24$$

3) Find the no of 3-digit even no's with no repeated digits.

Soln :- Consider the no in the form xyz, where each of x, y, z represents a digit.

Since xyz has to be even $\Rightarrow$ z has to be 0, 2, 4, 6 (or) 8.

If $z=0$ then x has 9 choices & y has 8 choices.
↳ 1 choice only

If $z = 2, 4, 6$ (or) 8 then x has 8 choices & y has 8 choices
↳ 4 choices

[Note that y can take 0, x cannot take 0.]

$\therefore$ The desired number is $(9\times8\times1)+(8\times8\times4)$

$= 72 + 256$

$= 328$

4) A bit is either 0 (or) 1. A byte is a sequence of 8 bits. Find

(i) the no of bytes

(ii) the no of bytes that begin with 11 & end with 11

(iii) the no of bytes that begin with 11 & do not end with 11.

(iv) the no of bytes that begin with 11 (or) end with 11.

Soln :-

(i) Since each byte contains 8 bits & each bit is a 0 (or) 1 (two choices), the no of bytes is $2^8 = 256$.

[∵ ↓ ↓ ↓ - - - - - ↓ constitutes a byte
0 (or) 1
$2 \times 2 \times \cdots \times 2_{(8 \text{ times})} = 2^8$]

(ii) In a byte beggining & ending with 11, there are 4 empty positions. These can be filled in $2^4 = 16$ ways.

∴ There are 16 bytes which begin & end with 11.

[11 _ _ _ _ 11
↓ ↓ ↓ ↓
0 (or) 1
$2 \times 2 \times 2 \times 2 = 2^4$]

(iii) In a byte beginning with 11, there are 6 empty positions. These can be filled in $2^6 = 64$ ways.

Thus there are 64 bytes that begin with 11.

Thus there are 16 bytes that begin & end with 11,

Now, Since there are 16 bytes that begin with 11, but do not end with the no of bytes that begin with 11 is $64 - 16 = 48$.

(iv) No of bytes that end with 11 is 64 (by result (iii))

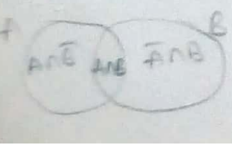
No of bytes that begin & end with 11 is 16.

Also no of bytes that begin (or) end with 11 is

∴ no of bytes that begin (or) end with 11 is $64 + 64 - 16 = 112$. [∵ for any 2 finite sets A & B, $|A \cup B| = |A| + |B| - |A \cap B|$]

(iii) Begin with 11, but donot end with 11 = Begin with 11 − Begin & end with 11

(iv) Begin with 11 (or) end with 11 = Begin with 11 + end with 11 − Begin & end with 11



(iii) => To find $A \cap \bar{B}$

(iv) => To find $A \cup B$

A : Begin with 11
B : End with 11
$A \cap B$: Begin & end with 11
$A \cup B$: Begin (or) 11 —— "

5) Find the number of proper divisors of 441,000 → are all divisors excluding 1 & itself ②(a)

Soln:- we have $441000 = 2^3 3^2 5^3 7^2$

2	441000
2	220500
2	110250
3	55125
3	18375
5	6125
5	1225
5	245
7	49
7	7
	1

∴ Every divisor of $n = 441000$ must be of the form $d = 2^p 3^q 5^r 7^s$, where $0 \le p \le 3$, $0 \le q \le 2$, $0 \le r \le 3$, $0 \le s \le 2$.

∴ p can be chosen in 4 ways, q in 3 ways, r in 4 ways and s in 3 ways.

Thus total no of divisors $= 4 \times 3 \times 4 \times 3 = 144$.

but total no of proper divisors $= 144 - 2 = 142$

[-2 ∵ a no is always divisible by 1 and by itself, so leaving these 2 choices]

6) License plates consists of 2 letters followed by 4 digits. In how many ways can this be done if

(i) No letter or digit can be repeated

(ii) Repititions are allowed.

(iii) only vowels & Even digits can be used (with repetitions)

Solns:- (i) $\underset{\text{letters}}{\_\_\ \_\_}$, $\underset{\text{digits}}{\_\_\ \_\_\ \_\_\ \_\_}$.

Since there are 26 alphabets (letters) in English, the 1st place ~~can be fitted~~ has 26 choices, 2nd place has 25 choices (since no repetition is allowed).

~~∴ Total no of ways it can be done is $26 \times 25 = 650$ ways~~

~~||ly~~ |||ly the digits are 10 in number (0, 1, 2 . . . 9)

∴ 1st place can be filled in 10 ways, 2nd in 9 ways, 3rd in 8 ways, 4th in 7 ways.

∴ Total no of ways this can be done is $26 \times 25 \times 10 \times 9 \times 8 \times 7$

$= 32,76,000$

PTO

(ii) $(26 \times 26) \times (10 \times 10 \times 10 \times 10) = 26^2 \times 10^4 = 6760000$

$\because$ repetitions are allowed.

(iii) there are 5 vowels to be used & Even digits $\Rightarrow$

0, 2, 4, 6, 8 (5 choices)

$\therefore (5 \times 5) \times (5 \times 5 \times 5 \times 5 \times \cancel{5}) = 5^{\cancel{7}\,6} = \cancel{78125}\ 15,625$

7) find the total no of the integers that can be formed from the digits 1, 2, 3, 4 if no digit is repeated in any one integer.

Soln :- Let S_1, S_2, S_3, S_4 denote the no of integers containing 1 digit, 2 digits, 3 digits, ~~3~~, 4 digits resp.

$S_1 \rightarrow$ 1 digits only $\Rightarrow$ 4 choices (1 or 2 or 3 or 4)

There are 4 integers containing exactly 1 digit $\therefore \boxed{S_1 = 4}$

A 2 digit integer without repetition can be chosen in $4 \times 3 = 12$ ways. ie $\boxed{S_2 = 12}$

_ _ : 4 choices, 3 choices

A 3 digit integer without repetition can be chosen in $4 \times 3 \times 2 = 24$ ways ie $\boxed{S_3 = 24}$

_ _ _ : 4, 3, 2

lly $S_4 = 4 \times 3 \times 2 \times 1 \Rightarrow \boxed{S_4 = 24}$

Thus the required no is $S_1 + S_2 + S_3 + S_4$

$= 4 + 12 + 24 + 24$

$= 64.$

8) Determine the no of 6 digit integers (no leading 0's) in which (i) no digit is repeated (ii) No digit is repeated & it is Even (iii) No digit is repeated & is divisible by 5.

Soln :- (i) _ _ _ _ _ _ $\quad 9 \times 9 \times 8 \times 7 \times 6 \times 5 =$

No zero, Can be zero

There are 2 possibilities

(ii) if last place is 0 $\Rightarrow$ _ _ _ _ _ 0 (1 choice)

Case(1)

$9 \times 8 \times 7 \times 6 \times 5 \times 1 = S_1$ (say)

Case(2) :- if last (unit) Place is not 0, i.e. unit place takes 2, 4, 6, 8 (4 choices), then total no. of choices = $8 \times 8 \times 7 \times 6 \times 5 \times 4 = S_2$ (say) — — — — — $\underset{4\text{ choices}}{\overset{2/4/6/8}{\downarrow}}$ (2)(b)

Thus req no. $= S_1 + S_2 =$

(iii) Case (1) : If unit place is zero,

— — — — — $\underset{1\text{ choice}}{\overset{0}{\downarrow}}$ $9 \times 8 \times 7 \times 6 \times 5 \times 1 = S_3$

Case (2) :- If unit place is not zero, i.e. ~~only~~ 5.

— — — — — $\underset{1\text{ choice}}{\overset{5}{\downarrow}}$ $8 \times 8 \times 7 \times 6 \times 5 \times 1 = S_4$

No zero, No 5 ; can be zero

$\therefore$ req no. $= S_3 + S_4 =$

Permutations :- Linear arrangement of distinct objects is called Permutations.

Number of arranging 'r' distinct objects out of 'n' objects is denoted by $P(n,r)$ or ^nP_r

and $P(n,r) = n(n-1)(n-2)\ldots(n-(r-1))$

$$= \frac{n!}{(n-r)!}$$

Note :- 1) No. of different arrangement of 'n' objects taken all 'n' at a time $= n!$. This is the no. of permutations of 'n' distinct objects.

2) Suppose we wish to find the no. of permutations that can be formed from a collection of 'n' objects of which n_1 are of one type, n_2 are of II type, n_3 are of III type, ... n_k are of k^{th} type and $n_1+n_2+\ldots+n_k = n$, then no. of permutations of 'n' objects taken all at a time $= \dfrac{n!}{n_1!\, n_2! \ldots n_k!}$

3) Circular arrangement of n different objects is $(n-1)!$.

Problems :- Order → 1, 9, 11, 7, 8, 2, 3, 5, 4, 6

1) 5 Men and 4 women sit in a row such that the women occupy the even places. How many such arrangements are Possible?

Soln :. 5 men be seated in 5! ways and 4 women be placed in even places in 4! ways.

∴ Total no. of arrangements is $5! \times 4! = 120 \times 24 = 2880$.

$\Rightarrow n(n-1) = 90$

$\Rightarrow n^2 - n - 90 = 0$

$\Rightarrow n^2 - 10n + 9n - 90 = 0$

$\Rightarrow n(n-10) + 9(n-10) = 0$

$(n-10)(n+9) = 0$

$n = 10, -9$

$\therefore \boxed{n = 10}$

(ii) $P(n,3) = 3\, P(n,2)$

$$\frac{n!}{(n-3)!} = 3 \cdot \frac{n!}{(n-2)!}$$

$$\frac{(n-2)!}{(n-3)!} = 3$$

$$\frac{(n-2)!}{(n-3)(n-2)!} = 3$$

$$\frac{1}{(n-3)} = 3$$

$$\frac{(n-2)(n-3)!}{(n-3)!} = 3$$

$\Rightarrow n-2 = 3 \Rightarrow \boxed{n = 5}$

---

2) How many different strings (sequences) of length 4 can be formed using the letters of the word FLOWER?

Soln :- FLOWER has 6 letters, all of which are distinct.

The required no of strings is the no of permutations of these six letters chosen 4 at a time.

No is $P(6,4) = \dfrac{6!}{(6-4)!} = \dfrac{6!}{2!} = \dfrac{6\times5\times4\times3\times2!}{2!}$

$= 360$.

3) Find the no of distinguishable permutations of the letters in the following words:

(i) BASIC (ii) CALCULUS (iii) MATHEMATICS

(iv) STRUCTURES (v) ENGINEERING

Soln :- (i) The word BASIC has 5 letters $(n=5)$, all letters being distinct.

$\therefore$ No of Permutations of BASIC is $5! = 120$.

(ii) The word CALCULUS has 8 letters $(n=8)$, of which 2 are C, 2 are L, 2 are U, and 1 each are A & S.

$\therefore$ Required No of permutations is $\dfrac{8!}{2!\,2!\,2!\,1!\,1!} = 5040$

(iii) The word "Mathematics" has 11 letters, in which
$(n=11)$

M = 2 H = 1 C = 1
A = 2 E = 1 S = 1
T = 2 I = 1

$\therefore$ No of permutations is $\dfrac{11!}{2! \times 2!\, 2!\,(1!)^5} = 4{,}989{,}600$

(iv) STRUCTURES has 10 letters, where
$(n=10)$

S = 2, T = 2, R = 2, U = 2, C = 1, E = 1,

$\therefore$ No of permutations is $\dfrac{10!}{2!\,2!\,2!\,2!\,1!\,1!}$

$= 226{,}800$

(v) ENGINEERING has 11 letters, where
$(n=11)$

E = 3, N = 3, G = 2, I = 2, R = 1

$\therefore$ No of Permutations is $\dfrac{11!}{3!\,3!\,2!\,2!\,1!} = 2{,}77{,}200$

4) Find the no of permutations of the letters of the word MASSASAUGA. In how many of these, all four A's are together? How many of them begin with S?

Soln:- (i) Given word has 10 letters of which 4 are A, 3 are S & 1 each are M, U, G.

$\therefore$ required no of permutations is $\dfrac{10!}{4!\,3!\,1!\,1!\,1!}$

$= 25{,}200.$

(ii) If in a permutation, all A's are to be together, we treat all of A's as one single letter say X. Then the letters to be permuted read (AAAA), S, S, S, M, U, G, which are 7 in no
X

$\therefore$ No of Permutations is $\dfrac{7!}{1!\,3!\,1!\,1!\,1!} = 840.$

(iii) For permutations beginning with S, there occur nine empty positions to fill, where two are S, 4 are A, 1 each are M, U, G.

$\therefore$ No of such permutations $= \dfrac{9!}{2!\,4!\,(1!)^3} = 7560$.

5) Find the no of permutations of the letters of the word MISSISSIPPI. How many of these begin with an I? How many of these begin & end with a S? ~~How many of these have no consecutive S's?~~

Soln :- (i) Given word has 11 letters, in which there are 4 I's, 4 S's, 2 P's, 1 M.

$\therefore$ No of permutations is $\dfrac{11!}{4!\,4!\,2!\,1!} = 34{,}650$

(ii) For permutations beginning with I, there occur 10 empty positions to fill, where 3 are I, 4 are S, 2 are P, 1 M.

$\therefore$ No of such permutations is $\dfrac{10!}{3!\,4!\,2!\,1!} = 12{,}600$

(iii) For permutations beginning & ending with S, there occur 9 empty positions to fill, where 2 are S, 4 are I, 2 are P, 1 M.

$\therefore$ No of such permutations $= \dfrac{9!}{2!\,4!\,2!\,1!} = 3780$.

~~(iv) Let all 4 S be treated as a single letter X, so that there are 1 M, 4 I, 2P & 1 X, totalling to 8, There can be arranged in~~

6) (a) How many arrangements are there for all letters in the word SOCIOLOGICAL?

(b) In how many of these arrangements (i) A & G are adjacent (ii) all the vowels are adjacent.

~~$\dfrac{8!}{1!\,4!\,2!\,1!} = 840$ ways. Also the 4 S's can be arranged among themselves in 4! ways. $\therefore$ req. no $= 840 \times 4! = 20{,}160$.~~

Soln :- (a) Given word has 12 letters, of which 3 are o, 2 each are C, I, L & 1 each are S, A, G. (5)

$\therefore$ no of arrangements of these letters is

$$\frac{12!}{3!\,2!\,2!\,2!\,(1!)^3} = 99,79,200.$$

(b) (i) If in an arrangement, A & G are to be adjacent, we treat A & G together as a single letter say X, such that there are 3 o's, 2 c's, 2 I's, 2 L's, 1 S & 1 X, totalling to 11 letters. These can be arranged in

$$\frac{11!}{3!\,2!\,2!\,2!\,1!\,1!} = 8,31,600 \text{ ways}$$

Also, the letters A & G can be arranged among themselves in 2 ways [$\because$ A & G $\rightarrow$ total 2, each is 1, 1 $\Rightarrow \frac{2!}{1!\,1!} = 2$]

$\therefore$ total no of arrangements in this case is

$8,31,600 \times 2 = 1,663,200$

(ii) If in an arrangement, all the vowels are to be adjacent, we treat all the vowels present in the given word (namely A, o, I) as a single letter, say Y, such that there are 2 c's, 2 L's, 1 S, 1 G, 1 Y, totalling to 7 letters. These can be arranged in

$$\frac{7!}{2!\,2!\,1!\,1!\,1!} = 1,260 \text{ ways.}$$

Also, since the given word contains 3 o's, 2 I's & 1 A, totalling to 6 the letters o, I, A (clubbed as Y) can be arranged among themselves in $\frac{6!}{3!\,2!\,1!} = 60$ ways.

∴ Total $\underline{no}$ of arrangements in this Case is

$1260 \times 60 = 75,600$

7) How many six digit numbers can one make using the digits 1, 3, 3, 7, 7, 8 ?

Soln :- Here $n = 6$.

The given digits are 1, 3, 3, 7, 7, 8 in which there are 1 1's, 2 3's, 2 7's, 1 8.

1 1's → 1 type, 2 3's → 2^{nd} type, 2 7's → 3^{rd} type, 1 8 → 4^{th} type

$\underline{No}$ of six digits $\underline{no's}$ using given digits is $\dfrac{6!}{1!\,2!\,2!\,1!}$

$= \dfrac{6 \times 4 \times 5 \times 3 \times 2}{4}$

$= 30 \times 6$

$= 180$.

8) How many +ve integers 'n' can we form using the digits 3, 4, 4, 5, 5, 6, 7 if we want 'n' to exceed 50,00,000

Soln :- Let $n = x_1 x_2 x_3 x_4 x_5 x_6 x_7$

where $x_1, x_2 \ldots x_7$ are the given digits.

Since n exceeds 50,00,000, x_1 can take 5 (or) 6 (or) 7.

If $x_1 = 5$, then $x_2 \ldots x_7$ is an arrangement of the remaining 6 digits which contains 2 4's & one each are 3, 5, 6, 7.

∴ $\underline{No}$ of such arrangements is $\dfrac{6!}{2!\,1!\,1!\,1!\,1!} = 360$.

If $x_1 = 6$, then $x_2 \ldots x_7$ is an arrangement of the remaining 6 digits which contains 2 4's, 2 5's, and one each are 3, 7.

∴ $\underline{No}$ of such arrangements $= \dfrac{6!}{2!\,2!\,1!\,1!} = 180$.

If $x_1 = 7$, then $x_2 \ldots x_7$ is an arrangement of the remaining 6 digits which contains 2 4's, 2 5's, 1 each are 3, 6.

$\therefore$ No of such arrangements is $\frac{6!}{2!\,2!\,1!\,1!} = 180$.

$\therefore$ By Sum Rule, the no of n's of the desired type is

$360 + 180 + 180 = 720$.

9) In how many ways can 6 men & 6 women be seated in a row

(i) if any person may sit next to each other?

(ii) if men & women must occupy alternate seats?

Soln:- (i) If any person may sit next to each other, no distinction be made b/w men & women in their seating.

$\therefore$ Number of ways they can be seated is $12!$ $[\because (6+6)!]$

$= 47,90,01,600$

(ii) If men & women are to occupy alternate seats,

6 men can be seated in $6!$ ways in odd places &

6 women " ______ " ______ even places.

$\therefore$ No of ways in which men occupy ~~even~~ odd places & women occupy even places is $6! \times 6! = 518400$.

$|||^{ly}$ No of ways in which men occupy even places & women occupy odd places is $6! \times 6! = 518400$

$\therefore$ Total no of ways men & women occupy alternate seats is $518400 + 518400 = 10,36,800$.

10) For non-negative integers n & r, if $n+1 > r$, prove that

$$P(n+1, r) = \frac{(n+1)}{(n+1-r)} P(n,r).$$

Soln :- we have $P(n,r) = \frac{n!}{(n-r)!}$ & $P(n+1,r) = \frac{(n+1)!}{(n+1-r)!}$

$$\therefore \frac{P(n+1,r)}{P(n,r)} = \frac{(n+1)!}{(n+1-r)!} \times \frac{(n-r)!}{n!}$$

$$= \frac{(n+1)!}{n!} \times \frac{(n-r)!}{(n+1-r)!}$$

$$= \frac{(n+1).n!}{n!} \times \frac{(n-r)!}{(n+1-r)(n-r)!}$$

$$= \frac{(n+1)}{(n+1-r)}$$

Thus $P(n+1,r) = \left(\frac{n+1}{n+1-r}\right) P(n,r)$.

11) 4 different Mathematics books, 5 different CO books & 2 different GTC books are to be arranged in a shelf. How many different arrangements are possible if
(a) the books in each particular subject must all be together?
(b) only the mathematics books must be together?

Soln :- (a) The mathematics books can be arranged among themselves in $4!$ ways, the CO books can be arranged among themselves in $5!$ ways. the GTC books can be arranged among themselves in $2!$ ways and the 3 groups in $3!$ ways.

$\therefore$ No of possible arrangements $= 4! \times 5! \times 2! \times 3! = 34,560$

(b) Since the Mathematics books has to be together, Consider the 4 mathematics books as one single book. Then we have 8 books, which can be arranged in $8!$ ways. But the mathematics books can be arranged among themselves in $4!$ ways.

$\therefore$ the required no of arrangements is $8! \times 4! = 9,67,680$

Combinations: Selecting a set of 'r' objects from a set of 'n' objects $(n \geq r)$ without regard to order is called a combination of 'r' objects.

The total no. of combination of 'r' different objects that can be selected from 'n' different objects is denoted by $C(n, r)$ or ${}^nC_r$ and is given by
$$ {}^nC_r = \frac{n!}{(n-r)!\, r!} $$

Note :- ${}^nC_r = {}^nC_{n-r}$.

Problems :-

1) How many committees of 5 members with one chair person can be selected from 12 persons?

Soln :- 1 chair person out of 12 can be selected in 12 ways. Remaining 4 members out of 11 persons can be selected in ${}^{11}C_4$ ways.

$\therefore$ Req. no. of committees $= 12 \times {}^{11}C_4$
$= 3960$.

2) A certain Q.P contains 3 parts A, B, C with 4 questions in Part A, 5 in Part B, 6 in Part C. It is required to answer 7 questions selecting atleast 2 from each part. In how many different ways can a student select his questions for answering?

Soln :-

	No. of questions it contains	No. of questions chosen case(1)	case(2)	case(3)
A	4	2	2	3
B	5	2	3	2
C	6	3	2	2
No. of ways of selection		${}^4C_2 \times {}^5C_2 \times {}^6C_3$ (S_1)	${}^4C_2 \times {}^5C_3 \times {}^6C_2$ (S_2)	${}^4C_3 \times {}^5C_2 \times {}^6C_2$ (S_3)

$\therefore$ Total no. of ways $= S_1 + S_2 + S_3 = 1200 + 900 + 600 = 2700$.

3) Find the no. of committees of 5 that can be selected from 7 men and 5 women, if the committee is to consist of atleast 1 man and atleast 1 woman?

Soln :-

Men	Women	No. of committees of 5.
1	4	$^7C_1 \times {}^5C_4 = 35$
2	3	$^7C_2 \times {}^5C_3 = 210$
3	2	$^7C_3 \times {}^5C_2 = 350$
4	1	$^7C_4 \times {}^5C_1 = 175$.

$\therefore$ Req. no. of committees containing atleast 1 man and 1 woman is

$35 + 210 + 350 + 175 = 770$.

4) A woman has 11 close relatives and she wishes to invite 5 of them to dinner. In how many ways can she invite them in the following situations :

i) there is no restriction on the choice

ii) 2 particular persons will not attend separately.

iii) 2 " ____ " ____ " together.

Soln :- i) Since there is no restriction on the choice, 5 out of 11 can be invited in $^{11}C_5$ ways $= 462$ ways.

ii) Let A and B be 2 particular persons. The following cases arises:

a) If both A and B are invited, then no. of ways of selection is $^9C_3 = 84$ ways.

b) If both A and B are not invited, then no. of ways of selection is $^9C_5 = 126$ ways.

$\therefore$ Total no. of ways $= 84 + 126 = 210$.

iii) The following cases arises:

a) If A is invited and B is not invited, then no. of ways of selection is $^9C_4 = 126$ ways.

b) If B is invited and A is not invited, then no. of ways of selection is $^9C_4 = 126$ ways.

c) If both A and B are not invited, then no. of ways of selection is $^9C_5 = 126$ ways.

∴ Total no. of ways $= 126 + 126 + 126$
$= 378.$

5) Find the no. of arrangements of all the letters of the word TALLAHASSEE. How many of these arrangements have no adjacent A's?

Soln :- i) Given word has 11 letters, in which there are 3 A's, 2 each are L's, S's, E's and 1 each are T and H.

∴ No. of such arrangements is $\frac{11!}{3!(2!)^3(1!)^2} = 8,31,600.$

ii) If the A's are disgarded, the remaining 8 letters can be arrayed in $\frac{8!}{(2!)^3(1!)^2} = 5040$ ways.

In each of these arrangements, there are 9 possible locations for the 3 A's. $[\nearrow T \nearrow L \nearrow L \nearrow H \nearrow S \nearrow S \nearrow E \nearrow E \nearrow]$ These location can be chosen in $^9C_3$ ways.

∴ No. of arrangements having no adjacent A's is $5040 \times {^9C_3}$
$= 4,23,360.$

6) Find how many distinct 4-digit integers one can make from the digits 1, 3, 3, 7, 7, 8 ?

Soln :- The following 3 cases arises :

i) 4-digit integers containing all distinct no.'s (1, 3, 7, 8) can be done in $4!$ ways $= 24$ ways.

ii) 4-digit integers containing 2 pair of identical no.'s (3, 3, 7, 7) can be done in $\frac{4!}{2!2!} = 6$ ways.

P.T.O.

iii) 4-digit integers containing 1 pair of identical no.'s are :

3, 3, 1, 7 (or) 3, 3, 1, 8 (or) 3, 3, 7, 8 (or)

7, 7, 1, 3 (or) 7, 7, 1, 8 (or) 7, 7, 3, 8.

which are 6 in no.

This can be done in $\frac{4!}{2!\,1!\,1!} \times 6 = 72$ ways.

$\therefore$ Total no. of ways $= 24 + 6 + 72 = 102$.

7) A man has 15 close friends of whom 6 are women. In how many ways can he invite 3 or more of these friends to a party if he wants the same no. of men (including himself) as women?

soln :- out of 15 friends, 6 are women and 9 are men.

women	Men		No. of ways
2	1	(+ himself in all cases)	$6_{C_2} \times 9_{C_1} = 135$
3	2		$6_{C_3} \times 9_{C_2} = 720$
4	3		$6_{C_4} \times 9_{C_3} = 1260$
5	4		$6_{C_5} \times 9_{C_4} = 756$
6	5		$6_{C_6} \times 9_{C_5} = 126$

$\therefore$ Total no. of ways $= 2997$.

8) How many bytes contain i) exactly two 1's ii) exactly four 1's iii) exactly 6 1's iv) atleast six 1's.

soln :- 1 byte = 8 bits and each bit is a 0 or 1.

i) Two 1's can be chosen in 8_{C_2} ways and each of the remaining 6 locations are filled by zeroes only (1 choice).

$\therefore$ No. of bytes containing exactly two 1's $= 8_{C_2} \times 1^6 = 28$.

_ _ 1 _ _ _ 1 _

ii) $8_{C_4} \times 1^4 = 70$.

iii) $8_{C_6} \times 1^2 = 28$.

iv) atleast 6 1's can be chosen in the following ways:

6 1's $\Rightarrow$ $^8C_6$

7 1's $\Rightarrow$ $^8C_7$

8 1's $\Rightarrow$ $^8C_8$

$\therefore$ No. of bytes with atleast 6 1's is

$$^8C_6 + {}^8C_7 + {}^8C_8 = 37.$$

9) Prove the following identities:

i) $C(n, r-1) + C(n, r) = C(n+1, r)$.

ii) $C(m, 2) + C(n, 2) = C(m+n, 2) - mn$.

Soln:- i) $C(n, r-1) + C(n, r) = {}^nC_{r-1} + {}^nC_r$

$$= \frac{n!}{(n-r+1)!\,(r-1)!} + \frac{n!}{(n-r)!\,r!}$$

$$= \frac{n!}{(n-r+1)(n-r)!\,(r-1)!} + \frac{n!}{(n-r)!\,r(r-1)!}$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{1}{n-r+1} + \frac{1}{r}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!}\left[\frac{\cancel{r} + n - \cancel{r} + 1}{r(n-r+1)}\right]$$

$$= \frac{n!}{(n-r)!\,(r-1)!} \times \frac{(n+1)}{r(n-r+1)}$$

$$= \frac{(n+1)!}{r!\,(n-r+1)!} = C(n+1, r).$$

ii) $C(m,2) + C(n,2) = \dfrac{m!}{(m-2)!\,2!} + \dfrac{n!}{(n-2)!\,2!}$

$$= \frac{m(m-1)\cancel{(m-2)!}}{2\,\cancel{(m-2)!}} + \frac{n(n-1)\cancel{(n-2)!}}{2\,\cancel{(n-2)!}}$$

$$= \frac{1}{2}\left[m(m-1) + n(n-1)\right]$$

$$= \frac{1}{2}\left[m^2 + n^2 - m - n\right]$$

$$= \frac{1}{2}\left[(m+n)^2 - 2mn - m - n\right]$$

$$= \frac{1}{2}\left[(m+n)^2 - (m+n) - 2mn\right]$$

$$= \frac{1}{2}\left[(m+n)(m+n-1)\right] - mn$$

$$= \frac{1}{2}\left[\frac{(m+n)(m+n-1)(m+n-2)!}{(m+n-2)!}\right] - mn$$

$$= \frac{1}{2!}\frac{(m+n)!}{(m+n-2)!} - mn$$

$$= c(m+n, 2) - mn.$$

Binomial and Multinomial theorem :

A basic property of $C(n,r)$ is that, it is the coefficient of $x^{n-r}y^r$ in the expansion of the expression $(x+y)^n$, where x & y are any two real no's. In other words, the expansion of $(x+y)^n$ in powers of x & y is as follows

$$(x+y)^n = x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \dots + {}^nC_{n-1} x y^{n-1} + y^n$$

$$= \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r \longrightarrow (1).$$

Since ${}^nC_r = {}^nC_{n-r}$;

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^r y^{n-r} \longrightarrow (2).$$

Thus ${}^nC_r$ is the coefficient of $x^r y^{n-r}$ in the expansion of $(x+y)^n$.

${}^nC_r$ is also denoted by $\binom{n}{r}$

$$\therefore \; 1 \Rightarrow (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$$

$$2 \Rightarrow (x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

$$\Bigg\} \; 3$$

Expression 3 is called Binomial theorem for a +ve integral index 'n'.

The no's $\binom{n}{r}$ for $r = 0,1,2\dots n$, namely $\binom{n}{0}, \binom{n}{1}, \binom{n}{2} \dots \binom{n}{n}$ in the above result are called the Binomial Coefficients.

The generalization of Binomial Theorem is known as

Multinomial Theorem :

Statement :— For +ve integers n & K, the coefficient of $x_1^{n_1} x_2^{n_2} \dots x_K^{n_K}$ in the expansion of $(x_1 + x_2 + \dots + x_K)^n$ is

$$\frac{n!}{n_1!\, n_2! \dots n_K!}$$ where each $n_i \le n$ & $n_1 + n_2 + \dots + n_K = n$.

Note :-

1) The general term in the expansion of $(x_1+x_2+\dots+x_k)^n$ is

$$\frac{n!}{n_1!\,n_2!\dots n_k!}\, x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$ (Alternate statement of Multinomial theorem)

3) The expression $\frac{n!}{n_1!\,n_2!\dots n_k!}$ is also written as $\binom{n}{n_1, n_2, \dots n_k}$

2) Multinomial theorem can also be stated as

$$(x_1+x_2+\dots+x_k)^n = \sum_{n_i} \binom{n}{n_1, n_2, \dots n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}$$

Problems :-

1) Prove the following identities for a +ve integer n:

(i) $\binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n} = 2^n$

(ii) $\binom{n}{0}-\binom{n}{1}+\binom{n}{2}\mp\dots+(-1)^n\binom{n}{n} = 0.$

Soln :- The Binomial theorem for a +ve integral index n is given by $(x+y)^n = \sum_{r=0}^{n}\binom{n}{r} x^r y^{n-r}.$

(i) Taking $x=1;\ y=1$, we get

$$2^n = \sum_{r=0}^{n}\binom{n}{r} 1^r\, 1^{n-r} = \sum_{r=0}^{n}\binom{n}{r}$$

ie $2^n = \binom{n}{0}+\binom{n}{1}+\dots+\binom{n}{n}$

(ii) Taking $x=-1;\ y=1$, we get

$$0 = \sum_{r=0}^{n}\binom{n}{r}(-1)^r\, 1^{n-r} = \sum_{r=0}^{n}(-1)^r\binom{n}{r}$$

$$\Rightarrow\ 0 = \binom{n}{0}-\binom{n}{1}+\binom{n}{2}\mp\dots+(-1)^n\binom{n}{n}.$$

P.T.O

2) Compute the following:—

(i) $\binom{12}{5,3,2,2}$ (ii) $\binom{8}{4,2,2,0}$ (iii) $\binom{10}{5,3,2,2}$

Soln :- (i) $\binom{12}{5,3,2,2} = \dfrac{12!}{5!\,3!\,2!\,2!} = 166,320$

(ii) $\binom{8}{4,2,2,0} = \dfrac{8!}{4!\,2!\,2!} = 420.$

(iii) Meaningless since $5+3+2+2 = 12 > n = 10$

3) Find the sum of all coefficients in the expansion of
(a) $(x+y)^n$ (b) $(x_1+x_2+\dots+x_k)^n$

Soln :— (a) we have $(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}$

Taking $x=1;\ y=1$, we get

$$2^n = \sum_{r=0}^{n} \binom{n}{r} 1^r\, 1^{n-r} = \sum_{r=0}^{n} \binom{n}{r}$$

i.e. $2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$

Thus, the sum of all the coefficients in the expansion of $(x+y)^n$ is 2^n.

(b) By Multinomial theorem, we have

$$\sum_{n_i} \binom{n}{n_1, n_2, \dots n_k} x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} = (x_1+x_2+\dots+x_k)^n$$

Taking $x_1 = 1,\ x_2 = 1 \dots x_k = 1$, we get

$$\sum_{n_i} \binom{n}{n_1, n_2 \dots n_k} = (\underbrace{1+1+\dots+1}_{k \text{ times}})^n = k^n.$$

Thus, the sum of all the coefficients in the expansion of $(x_1+x_2+\dots+x_k)^n$ is k^n.

4) Obtain the sum of all the coefficients in the expansion of (i) $(x+y)^{10}$ (ii) $(x+y+z)^{12}$ (iii) $(x+y+z+w)^5$ (iv) $(2s-3t+5u+6v-11w+3x+2y)^{10}$

Soln :- (i) From result (a) of above problem, the sum of the coefficients in the expansion of $(x+y)^{10}$ is 2^{10}.

(ii) From result (b) of above problem, the sum of the coefficients in the expansion of $(x+y+z)^{12}$ is 3^{12}.

(iii) From result (b) again, sum of coeff in the expansion of $(x+y+z+w)^5$ is 4^5.

(iv) By virtue of the multinomial theorem, we have

$$(2s-3t+5u+6v-11w+3x+2y)^{10} =$$

$$= \sum_{n_i} \binom{10}{n_1, n_2 \ldots n_7} (2s)^{n_1} (-3t)^{n_2} (5u)^{n_3} (6v)^{n_4} (-11w)^{n_5} (3x)^{n_6} (2y)^{n_7}$$

$$= \sum_{n_i} \binom{10}{n_1, n_2 \ldots n_7} 2^{n_1} (-3)^{n_2} 5^{n_3} 6^{n_4} (-11)^{n_5} 3^{n_6} 2^{n_7} \times s^{n_1} (t)^{n_2} u^{n_3} v^{n_4} w^{n_5} x^{n_6} y^{n_7}$$

Taking $s=1$, $t=1$, $u=1$, $v=1$, $w=1$, $x=1$, $y=1$ we get

$$(2-3+5+6-11+3+2)^{10} = \sum_{n_i} \binom{10}{n_1, n_2, \ldots n_7} 2^{n_1} (-3)^{n_2} 5^{n_3} 6^{n_4} (-11)^{n_5} 3^{n_6} 2^{n_7}$$

The sum in the RHS of the above expression is the sum of the coefficients in the expansion of $(2s-3t+5u+6v-11w+3x+2y)^{10}$.

This sum $= (2-3+5+6-11+3+2)^{10}$

$= 4^{10}$

5) Find the coefficient of

(i) x^3y^9 in the expansion of $(x+2y)^{12}$.

(ii) x^0 in the expansion of $\left(3x^2 - \frac{2}{x}\right)^{15}$

(iii) x^{12} " ————— " $x^3(1-2x)^{10}$

(iv) x^k " ————— " $(1+x+x^2)(1+x)^n$; where n is a +ve integer & $0 \le k \le n+2$.

Soln - (i) By Binomial theorem,

$$(x+2y)^{12} = \sum_{r=0}^{12} \binom{12}{r} x^r (2y)^{12-r}$$

$$= \sum_{r=0}^{12} \binom{12}{r} 2^{12-r} \cdot x^r \cdot y^{12-r}$$

Taking $r=3$ in the above expansion, the coefficient of $x^3 y^9$ is $\binom{12}{3} 2^9 = \frac{12!}{9!\,3!} \cdot 2^9 = 1,12,640$

(ii) By Binomial theorem,

$$\left(3x^2 - \frac{2}{x}\right)^{15} = \sum_{r=0}^{15} \binom{15}{r} (3x^2)^r \cdot \left(\frac{-2}{x}\right)^{15-r}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r \cdot (-2)^{15-r} \cdot x^{2r} \cdot \left(\frac{1}{x}\right)^{15-r}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r \cdot (-2)^{15-r} \cdot x^{2r} \cdot x^{r-15}$$

$$= \sum_{r=0}^{15} \binom{15}{r} 3^r \cdot (-2)^{15-r} \cdot x^{3r-15}$$

Taking $r=5$ in the above expansion, the coefficient of x^0 is $\binom{15}{5} 3^5 \cdot (-2)^{10} = {}^{15}C_5 \cdot 3^5 \cdot (-2)^{10}$

$= 74,72,42,496$

(iii) By Binomial theorem,

$$(1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} 1^{10-r} \cdot (-2x)^r$$

$$\therefore \quad x^3 (1-2x)^{10} = \sum_{r=0}^{10} \binom{10}{r} \cdot (-2)^r \cdot x^{r+3}$$

Taking $r=9$ in the above expansion, we get the coeff of x^{12} as $\binom{10}{9} (-2)^9 = {}^{10}C_9 \times (-2)^9$

$= -5,120.$

(iv) Using Binomial theorem,

$$(1+x)^n = \sum_{r=0}^{n} \binom{n}{r} 1^{n-r} \cdot x^r$$

$$\therefore (1+x+x^2)(1+x)^n = (1+x+x^2) \sum_{r=0}^{n} \binom{n}{r} x^r$$

$$= \sum_{r=0}^{n} \binom{n}{r} x^r + \sum_{r=0}^{n} \binom{n}{r} x^{r+1} + \sum_{r=0}^{n} \binom{n}{r} x^{r+2}$$

Coeff of x^k is obtained by taking $r=k$ in 1^{st} sum, $r=k-1$ in 2^{nd} sum & $r=k-2$ in 3^{rd} sum.

ie coeff of x^k is

$$\binom{n}{k} + \binom{n}{k-1} + \binom{n}{k-2}.$$

Say; for Ex:- coeff of x^{15} is $\binom{n}{15} + \binom{n}{14} + \binom{n}{13}$.

6) Determine the coefficient of the following :-

(i) xyz^5 in the expansion of $(x+y+z)^7$

(ii) $x^2y^2z^3$ " ——— " $(3x-2y-4z)^7$

(iii) x^4y^4 " ——— " $(2x^3-3xy^2+z^2)^6$

(iv) xyz^{-2} " ——— " $(x-2y+3z^{-1})^4$

(v) $a^2b^3c^2d^5$ " ——— " $(a+2b-3c+2d+5)^{16}$.

Soln:- (i) By Multinomial theorem, we have that the General term in the expansion of $(x+y+z)^7$ is

$$\binom{7}{n_1, n_2, n_3} x^{n_1} y^{n_2} z^{n_3}$$

for $n_1 = 1$, $n_2 = 1$, $n_3 = 5$, we have

$$\binom{7}{1,1,5} xyz^5 = \frac{7!}{1!\,1!\,5!} xyz^5 = 42\,xyz^5$$

This shows that the required coefficient of xyz^5 is 42.

(ii) The general term in the expansion of $(3x-2y-4z)^7$ (17)

is $\binom{7}{n_1,n_2,n_3} (3x)^{n_1} (-2y)^{n_2} (-4z)^{n_3}$.

for $n_1 = 2$; $n_2 = 2$, $n_3 = 3$, we have

$\binom{7}{2,2,3} (3x)^2 (-2y)^2 (-4z)^3$

$= \binom{7}{2,2,3} 3^2 . (-2)^2 . (-4)^3 \; x^2 y^2 z^3$

$\therefore$ coeff of $x^2y^2z^3$ is $\binom{7}{2,2,3} \times 9 \times 4 \times (-64)$

$= -2304 \times \frac{7!}{2!\,2!\,3!} = -4,83,840$

(iii) The general term in the expansion of $(2x^3 - 3xy^2 + z^2)^6$ is

$\binom{6}{n_1,n_2,n_3} (2x^3)^{n_1} (-3xy^2)^{n_2} (z^2)^{n_3}$

$A = \binom{6}{n_1,n_2,n_3} 2^{n_1} (-3)^{n_2} . x^{3n_1} . x^{n_2} . y^{2n_2} . z^{2n_3}$

$= \binom{6}{n_1,n_2,n_3} 2^{n_1} (-3)^{n_2} . x^{3n_1+n_2} . y^{2n_2} . z^{2n_3}$

$3n_1+2=11$
$3n_1=9$
$n_1=3$

for $n_3 = 0$; $n_2 = 2$; $n_1 = 3$, we have

$\binom{6}{3,2,0} 2^3 (-3)^2 . x^{11} y^4$

$\therefore$ coeff of $x^{11}y^4$ is $2^3 (-3)^2 . \binom{6}{3,2,0}$

$= 8 \times 9 \times \frac{6!}{3!\,2!\,0!} = 4320$

(iv) The general term in the expansion of $(x - 2y + 3z^{-1})^4$ is

$\binom{4}{n_1,n_2,n_3} x^{n_1} (-2y)^{n_2} (3z^{-1})^{n_3}$

$= \binom{4}{n_1,n_2,n_3} (-2)^{n_2} 3^{n_3} . x^{n_1} . y^{n_2} . z^{-n_3}$

for $n_1 = 1$, $n_2 = 1$, $n_3 = 2$, we have

$\binom{4}{1,1,2} (-2)^1 \cdot 3^2 \cdot xyz^{-2}$

$\therefore$ coeff of xyz^{-2} is $-2 \cdot 9 \times \binom{4}{1,1,2}$

$= -18 \times \dfrac{4!}{1!\,1!\,2!} = -216$

(v) The general term in the expansion of $(a+2b-3c+2d+5)^{16}$ is $\binom{16}{n_1, n_2, n_3, n_4, n_5} a^{n_1} (2b)^{n_2} (-3c)^{n_3} (2d)^{n_4} \, 5^{n_5}$

for $n_1 = 2$; $n_2 = 3$, $n_3 = 2$, $n_4 = 5$; $n_5 = 16 - (2+3+2+5)$ $\left[\because n_1 + \dots + n_5 = n \Rightarrow n_5 = n - (n_1 + \dots + n_4)\right]$

ie $n_5 = 4$, we have

$\binom{16}{2,3,2,5,4} 2^3 \cdot (-3)^2 \cdot 2^5 \cdot 5^4 \; a^2 b^3 c^2 d^5$

$\therefore$ coeff of $a^2 b^3 c^2 d^5$ is $8 \times 9 \times 2^5 \cdot 5^4 \times \binom{16}{2,3,2,5,4}$

$= 1440000 \times \dfrac{16!}{2!\,3!\,2!\,5!\,4!}$

H.W

Find the coefficient of :

1) $x^5 y^2$ in the expansion of $(2x-3y)^7$
2) $x^3 z^4$ " ______ " $(x+y+z)^7$
3) $x^3 y^3 z^2$ " ______ " $(2x-3y+5z)^8$
4) $w^3 x^2 y z^2$ " ______ " $(2w-x+3y-2z)^8$
5) $x_1^2 x_3 x_4^3 x_5^4$ " ______ " $(x_1+x_2+x_3+x_4+x_5)^{10}$.

Combinations with Repetitions :

Suppose we have to select a combination of r-objects, with repetitions, from a set of n-distinct objects, where $r \leq n$, then the no of such selections is given by

$$C(n+r-1, r) = \frac{(n+r-1)!}{r!\,(n-1)!} = C(r+n-1, n-1)$$

ie $C(n+r-1, r) = C(r+n-1, n-1)$ represents the no of $\rightarrow (1)$

Combinations of n distinct objects, taken 'r' at a time, with repetitions allowed.

Note :- 1) Eqn (1) represents the no of ways in which r identical objects can be distributed among n distinct containers.

2) Eqn (1) also represents the no of non-negative integer ($0, 1, 2, 3 \ldots$) ie $x_i \geq 0 \ \forall \ i = 1, 2 \ldots n$ Soln's of the Eqn $x_1 + x_2 + \ldots + x_n = r$.

[A non-negative integer soln of the eqn $x_1 + x_2 + \ldots + x_n = r$ is an n-tuple $(x_1, x_2, \ldots x_n)$, where $x_1, x_2 \ldots x_n$ are non-negative integers whose sum is r]

$n \geq r$ — comb^n — out of n, we select r

$r \geq n$ or $n \geq r$ — comb^n with rept $\rightarrow$ dist in places to find int sol

Problems :-

1) In how many ways can we distribute 10 identical marbles among 6 distinct containers?

places $\rightarrow n$
dist. item $\rightarrow r$.

Soln:- Here $r = 10$; $n = 6$.

$\therefore$ Required no is $C(6+10-1, 10) = C(15, 10)$

$$= \frac{15!}{10!\,5!} = 3003.$$

2) Find the no of ways of placing 8 identical balls in 5 numbered boxes?

Soln:- Here $r = 8$; $n = 5$

Required no is $C(5+8-1, 8) = C(12, 8) = \frac{12!}{8!\,4!} = 495.$

3) A bag contains coins of 7 different denominations, with atleast 1 dozen coins in each denomination. In how many ways can we select a dozen coins from the bag?

Soln:. Here $r = 12$; $n = 7$.

The no of ways making a selection of dozen coins from the bag is $C(7+12-1, 12) = C(18, 12) = \frac{18!}{12!\,6!} = 18,564.$

4) Find the no of non-negative integer solutions of the eqn
$x_1 + x_2 + x_3 + x_4 + x_5 = 8$.

Soln:. The required no is $C(5+8-1, 8) = C(12, 8)$

[Since here $r = 8$; $n = 5$] $= \frac{12!}{8!\,4!} = 495.$

5) Find the no of non-negative integer soln of the eqn
$x_1 + x_2 + x_3 + x_4 = 7$

Soln:. Here $n = 4$; $r = 7$.

Required no is $C(4+7-1, 7) = C(10, 7) = \frac{10!}{7!\,3!}$

$=$

6) Find the no of distinct terms in the expansion of
$(x_1 + x_2 + x_3 + x_4 + x_5)^{16}$

Soln:- The general term in the expansion of $(x_1 + x_2 + x_3 + x_4 + x_5)^{16}$ is of the form $\binom{16}{n_1, n_2, n_3, n_4, n_5} x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$

where each n_i is a non-negative integer & the sum of these n_i's is 16.

$\therefore$ No of distinct terms in the expansion = the no of non-negative integer solutions of the eqn $n_1 + n_2 + n_3 + n_4 + n_5 = 16$

$\therefore$ Required no is $C(5+16-1, 16) = C(20, 16) = 4845.$

7) Find the no of non-negative integer soln's of the inequality $x_1 + x_2 + \dots + x_6 < 10$. (19)

Soln:- Given $x_1 + x_2 + \dots + x_6 < 10$. [ex: $2+3<10$. $\Rightarrow 2+3 = 9-4$]

$\Rightarrow x_1 + x_2 + \dots + x_6 = 9 - x_7$,

so that x_7 is a non-negative integer.

$\therefore$ the required no is the no of non-negative soln's of the eqn $x_1 + x_2 + \dots + x_7 = 9$.

This no is $C(7+9-1, 9) = (15, 9) = 5005$.

8) Find the no of distinct terms in the expansion of $(w+x+y+z)^{10}$

Soln:. The general term in the expansion of $(w+x+y+z)^{10}$ is $\binom{10}{n_1, n_2, n_3, n_4} w^{n_1} x^{n_2} y^{n_3} z^{n_4}$

where each n_i is a non-negative integer & the sum of these n_i's is 10.

$\therefore$ no of distinct terms in the expansion = no of non-negative integer soln of the eqn $n_1 + n_2 + n_3 + n_4 = 10$

This no = $C(4+10-1, 10) = C(13, 10) = 286$.

9) Find the no. of non-negative integer soln of the inequality $x_1 + x_2 + \dots + x_5 \le 19$

Soln:. Given $x_1 + x_2 + \dots + x_5 \le 19$.

$\Rightarrow x_1 + x_2 + \dots + x_5 = 19 - x_6$, so that x_6 is non-negative

$\therefore$ Required no = no of non-negative integer solns of the eqn $x_1 + x_2 + \dots + x_5 + x_6 = 19$

This no is $C(6+19-1, 19) = C(24, 19)$

$= 42,504$.

10) Find the no. of integer solutions of $x_1 + x_2 + \dots + x_5 = 30$ where $x_1 \geq 2$, $x_2 \geq 3$, $x_3 \geq 4$, $x_4 \geq 2$, $x_5 \geq 0$.
$\hookrightarrow (1)$

Soln :- Let $y_1 = x_1 - 2$; $y_2 = x_2 - 3$; $y_3 = x_3 - 4$

$y_4 = x_4 - 2$; $y_5 = x_5$.

then $y_1, y_2 \dots y_5$ are all non-negative integers.

Given Eqn now becomes,

$$(y_1+2) + (y_2+3) + (y_3+4) + (y_4+2) + y_5 = 30$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 30 - 11$$

$y_i \geq 0$ for $i = 1, 2 \dots 5$.

$$\Rightarrow y_1 + \dots + y_5 = 19. \longrightarrow (2)$$

No of non-negative integer soln of (2) is

$$C(5+19-1, 19) = C(24-1, 19) = C(23, 19)$$
$$= 8855$$

11) How many integer soln's are there to $x_1 + x_2 + \dots + x_5 = 20$ where $x_i \geq 2$?

HW

Soln :- Given $x_1 \geq 2$; $x_2 \geq 2 \dots$ $x_5 \geq 2$.

Let $y_1 = x_1 - 2$; $y_2 = x_2 - 2 \dots$ $y_5 = x_5 - 2$

$\therefore$ Given Eqn becomes

$$y_1 + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 = 20.$$

$$\Rightarrow y_1 + y_2 + \dots + y_5 = 10.$$

$\therefore$ Required no $= C(5+10-1, 10) = C(14, 10) = 1001$.

12) In how many ways can we distribute 12 identical pencils to 5 children so that every child gets atleast 1 pencil ?

HW

Soln :- First let us distribute one pencil to each child. So that Remaining 7 pencils are to be distributed to 5 children. This can be done in $C(5+7-1, 7) = C(11, 7)$

$= 330$ ways.

$n = 5$,
$r = 7$

13) In how many ways can 10 identical ~~products~~ pens be distributed among 5 children in the foll cases:-

(i) There are no restrictions.

(ii) Each child gets atleast one pen

(iii) The youngest child gets atleast 2 pens.

Soln:- (i) Since there are No restrictions in ~~select~~ distributing 10 pens to 5 children, no of ways of distribution is $C(5+10-1, 10) = C(14, 10) = 1001$

(ii) First we distribute one pen to each child. Then the remaining 5 pens are to be distributed among 5 children in $C(5+5-1, 5) = C(9, 5) = 126$ ways.

(iii) First we give 2 pens to the youngest child, then the remaining 8 pens are to be distributed among 5 children. This can be done in
$$C(5+8-1, 8) = C(12, 8) = 495 \text{ ways.}$$

14) In how many ways can one distribute eight identical balls into 4 distinct containers so that

(i) No container is empty?

(ii) the 4th container gets an odd no of balls?

Soln:- (i) First we distribute one ball to each container. Then the remaining 4 balls is to be distributed among 4 containers & this can be done in
$$C(4+4-1, 4) = C(7, 4) = 35 \text{ ways}$$

(ii) If the 4th container has to get an odd no of balls, we have to put 1 (or) 3 (or) 5 (or) 7 balls into it.

Case (i) :- Suppose we put 1 ball to 4th container, then the remaining 7 balls ~~is to~~ can be distributed into the remaining 3 containers in $C(3+7-1, 7) = C(9, 7)$ ways.

Case (ii) :- Suppose we put 3 balls to 4th container, then the remaining 5 balls can be distributed into the remaining

3 containers in $C(3+5-1, 5) = C(7,5)$ ways.

Case (iii) :- Suppose we put 5 balls to 4^{th} container, then the remaining 3 balls can be distributed into the remaining 3 containers in $C(3+3-1, 3) = C(5,3)$ ways.

Case (iv) :- $C(3+1-1, 1) = C(3,1)$ ways.

$\therefore$ Total no of ways of distributing the given balls with the given condn is

$$C(9,7) + C(7,5) + C(5,3) + C(3,1) = 70.$$

15) Find the no of ways of distributing 7 identical pens & 7 identical pencils to 5 children so that each gets atleast 1 pen & atleast 1 pencil.

Soln :- First let us distribute 1 pen & 1 pencil to each child, then the remaining 2 pens can be distributed to 5 children in $C(5+2-1, 2) = C(6,2)$ ways.

Illy the remaining 2 pencils can be distributed to 5 children in $C(5+2-1, 2) = C(6,2)$ ways.

$\therefore$ no of ways of distributing the given sets under the given condn is $C(6,2) \times C(6,2)$

$$= 225$$

16) Find the no of ways of giving 10 identical gift boxes to 6 persons A, B, C, D, E, F in such a way that the total no of boxes given to A & B together does not exceed 4.

Soln: suppose 'r' boxes are given to A & B of the 10 boxes, together, then $0 \le r \le 4$.

$\therefore$ no of ways of giving 'r' boxes to A & B is

$$C(2+r-1, r) = C(r+1, r) = \frac{(r+1)!}{r!\,1!} = \frac{(r+1)\,r!}{r!} = r+1 \quad \to (1).$$

P.T.O.

No. of ways in which remaining $(10-r)$ boxes can be given to C, D, E, F is

$C(4+(10-r)-1, (10-r)) = C(13-r, 10-r)$ $\rightarrow (2)$.

Consequently, the no. of ways in which 'r' boxes may be given to A & B and $(10-r)$ boxes to C, D, E, F is

$(r+1) \times C(13-r, 10-r)$.

Since $0 \le r \le 4$, Total no. of ways in which the boxes may be given is,

$$\sum_{r=0}^{4} (r+1) \times C(13-r, 10-r).$$

17) A total amount of Rs 1500 is to be distributed to 3 poor students A, B, C of a class. In how many ways the distribution can be made in multiples of Rs 100 ? if

(i) everyone of these must get atleast Rs 300 ?

(ii) A must get atleast Rs 500, & B & C must get atleast Rs 400 each ?

Soln:- Taking Rs 100 as a unit, there are 15 units for distribution, among 3 students A, B, C.

Case (i) : Each of the 3 students must get atleast 3 units.

$\therefore$ let us first distribute 3 units to each A, B, C. then the remaining 6 units are to be distributed to A, B, C, and this can be done in $C(3+6-1, 6) = C(8,6) = 28$ ways.

Case (ii) :- A must get atleast 5 units & B & C must get atleast 4 units each.

$\therefore$ let us first distribute 5 units to A & 4 units each to B & C, then the remaining 2 units are to be distributed to A, B, C, This can be done in $C(3+2-1, 2) = C(4,2)$ $= 6$ ways.

18) A total of Rs 10,000 is to be distributed to 4 persons A, B, C, D in multiples of Rs 1000. In how many ways can the distribution be done (i) if there is no restriction (ii) if everyone of these persons should recieve atleast Rs 1000 ? (iii) If everyone should recieve atleast Rs 1000 & A in particular should recieve atleast Rs 5000 ?

Soln :- Taking Rs 1000 as one unit, there are 10 units for distribution, among 4 persons.

Case (i) :- Since there is no restriction, 10 units can be distributed among 4 persons in

$$C(4+10-1, 10) = C(13, 10) = 286 \text{ ways.}$$

Case (ii) :- Each of the 4 students must get atleast 1 unit. Let us first distribute 1 unit to each A, B, C, D. then the remaining 6 units are to be distributed among 4 persons. This can be done in $C(4+6-1, 6) = C(9, 6) = 84$ ways.

Case (iii) :- A should recieve atleast 5 units & each of B, C, D should recieve atleast 1 unit. ~~then the~~ Let us first distribute 5 units to A, and 1 unit each to B, C, D. then the remaining 2 units are to be distributed among 4 persons. This can be done in

$$C(4+2-1, 2) = C(5, 2) = 10 \text{ ways.}$$

19) A message is made up of 12 different symbols & is to be transmitted through a communication channel. In addition to the 12 symbols, the transmitter will also send a total of 45 blank spaces b/w the symbols, with atleast 3 spaces b/w each pair of consecutive symbols. In how many ways can the transmitter send such a message ?

P.T.O.

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Soln:- The 12 symbols can be arranged in $12!$ ways. For each of these arrangements, there are 11 positions b/w the 12 symbols. Since there must be atleast 3 spaces b/w consecutive ~~successive~~ symbols, we can make use of 33 blank spaces out of 45 ~~33 of the 45 spaces will be used up~~. The remaining $(45-33)=$ 12 blank spaces can be accomodated in 11 positions. This can be done in $C(11+12-1, 12)$ $= C(22, 12)$ ways.

$\therefore$ No of ways the transmitter can send such a message is $C(22,12) \times 12! = \frac{22!}{10!\,12!} \times 12! = \frac{22!}{10!}$
$= 3.0974 \times 10^{14}$.

20) Six distinct symbols are transmitted through a communica-tion channel. A total of 12 blanks are to be inserted b/w the symbols with atleast 2 blanks b/w every pair of symbols. In how many ways can the symbols and the blanks be arranged?

HW

Soln:- The 6 symbols can be arranged in $6!$ ways. For each of these arrangements, there are 5 positions b/w the 6 sym-bols. Since there must be atleast 2 blanks b/w every pair of symbols, 10 of the 12 blanks will be used up. The remaining 2 blanks are to be accomadated in 5 posi-tions. This can be done in $C(5+2-1, 2) = C(6,2)$ ways

$\therefore$ No of ways the symbols & the blanks can be arrang-ed is $C(6,2) \times 6! = 10,800$.