Module-3

FOURIER TRANSFORMS

Definition :-

① Let $F(x)$ is a function defined on $[-\infty, \infty]$, then its Fourier transform can be defined as

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx = f(s)$$

where 'F' is called the fourier transform operator and 's' be the parameter either real (or) complex.

The inverse Fourier transform is defined as

$$F^{-1}[f(s)] = F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds.$$

② Let $F(x)$ be a function defined on $[0, \infty]$, then the Fourier cosine transform of $F(x)$ is defined as

$$F_c[F(x)] = \int_{0}^{\infty} F(x)\cos(sx)\,dx = f_c(s).$$

and its inverse fourier cosine transform is

$$F^{-1}[f_c(s)] = F(x) = \frac{2}{\pi}\int_{0}^{\infty} f_c(s)\cos(sx)\,ds.$$

Similarly, the Fourier sin transform of $F(x)$ is

$F_s[F(x)] = \int_{0}^{\infty} F(x)\sin(sx)\,dx = f_s(s)$ and its inverse sin transform is $F^{-1}[f_s(s)] = F(x) = \frac{2}{\pi}\int_{0}^{\infty} f_s(s)\sin(sx)\,ds.$

① Find the fourier transform of $e^{-a^2x^2}$, $a>0$ and hence show that the fourier transform of $e^{-x^2/2}$ is $\sqrt{2\pi}\, e^{-s^2/2}$.

Sol: Given $f(x) = e^{-a^2x^2}$, $a>0$

$\therefore$ WKT

$$\therefore F[f(x)] = \int_{-\infty}^{\infty} e^{isx} f(x)\,dx$$

$$= \int_{-\infty}^{\infty} e^{isx} e^{-a^2x^2}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-a^2x^2+isx}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-(a^2x^2-isx)}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[(ax)^2 - 2(ax)\left(\frac{is}{2a}\right) + \left(\frac{is}{2a}\right)^2 - \left(\frac{is}{2a}\right)^2\right]}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2 - \frac{i^2s^2}{4a^2}\right]}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left[\left(ax-\frac{is}{2a}\right)^2 + \frac{s^2}{4a^2}\right]}\,dx$$

$$= \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2} e^{-s^2/4a^2}\,dx$$

$$= e^{-s^2/4a^2} \int_{-\infty}^{\infty} e^{-\left(ax-\frac{is}{2a}\right)^2}\,dx \quad \rightarrow ①$$

Let $ax - \frac{is}{2a} = u$

$\Rightarrow a\,dx = du$

$\Rightarrow dx = \frac{1}{a}\,du$

UL: $x = \infty \Rightarrow u = \infty$

LL: $x = -\infty \Rightarrow u = -\infty$

$$\therefore F[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{a}\int_{-\infty}^{\infty} e^{-u^2}\,du$$

$$\Rightarrow F[e^{-a^2x^2}] = \frac{e^{-s^2/4a^2}}{a}\sqrt{\pi} \rightarrow ①$$

when $a = \frac{1}{2}$

$$① \Rightarrow F[e^{-x^2/2}] = \frac{\sqrt{\pi}}{1/\sqrt{2}}\, e^{-s^2/4(\frac{1}{2})} = \sqrt{\pi}\, e^{-s^2/2}$$

② Find the fourier transform of $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ and hence find $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

Sol:- Given $F(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$

$$\Rightarrow F(x) = \begin{cases} 1, & -a < x < a \\ 0, & x > a \end{cases}$$

$\therefore$ WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\,dx$$

$$= \int_{-\infty}^{a} e^{isx} F(x)\,dx + \int_{-a}^{a} e^{isx} F(x)\,dx + \int_{a}^{\infty} e^{isx} F(x)\,dx$$

$$= 0 + \int_{-a}^{a} e^{isx} \cdot dx + 0$$

$$\therefore F[F(x)] = \int_{-a}^{a} e^{isx}\,dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{1}{is}\left[e^{ias} - e^{-ias}\right]$$

$$= \frac{1}{is}\left[\cancel{\cos(as)} + i\sin(as) - \cancel{\cos(as)} + i\sin(as)\right]$$

$$= 2i \frac{\sin(as)}{is}$$

$$\Rightarrow f(s) = \frac{2\sin(as)}{s}$$

WKT The fourier inverse transform is

$$\therefore F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\frac{2\sin(as)}{s}ds = \begin{cases} 1, & -a < x < a \\ 0, & x > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)]\frac{\sin as}{s}ds = \pi\begin{cases} 1, & -a < x < a \\ 0, & x > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos(sx)\sin(as)}{s}ds - i\int_{-\infty}^{\infty}\frac{\sin(sx)\sin(as)}{s}ds = \pi\begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

$$\therefore \int_{-\infty}^{\infty}\frac{\cos(sx)\sin(as)}{s}ds = \pi\begin{cases} 1, & -a < x < a \\ 0, & x > a \end{cases} \rightarrow (1)$$

when $x = 0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty}\frac{\sin(as)}{s}ds = \pi$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin(as)}{s}ds = \pi$$

$$\Rightarrow 2\int_{0}^{\infty}\frac{\sin(as)}{s}ds = \pi$$

$$\Rightarrow \int_{0}^{\infty}\frac{\sin(as)}{s}ds = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty}\frac{\sin s}{s}ds = \frac{\pi}{2}$$ for $a = 1$.

$$\Rightarrow \int_{0}^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$$ for $s = x$.

③ Find the fourier transform $F(x) = \begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a, \end{cases}$

hence show that (i) $\int_0^\infty \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16}.$

Sol:- Given $F(x) = \begin{cases} a^2-x^2, & -a \le x \le a \\ 0, & x > a \end{cases}$

$\Rightarrow$ WKT $F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x) dx$

$= \int_{-\infty}^{a} e^{isx} F(x) dx + \int_{-a}^{a} e^{isx} F(x) dx + \int_{a}^{\infty} e^{isx} F(x) dx$

$= \int_{-a}^{a} (a^2-x^2) e^{isx} dx$

$= (a^2-x^2) \int_{-a}^{a} e^{isx} dx - \int_{-a}^{a} [-2x \int e^{isx} dx] dx$

$= \frac{1}{is} (0-0) + \frac{2}{is} \int_{-a}^{a} x e^{isx} dx$

$= \frac{2}{is} \left\{ x \int_{-a}^{a} e^{isx} dx - \int_{-a}^{a} [1 \cdot \int e^{isx} dx] dx \right\}$

$= \frac{2}{is} \left\{ \frac{1}{is} [x e^{isx}]_{-a}^{a} - \frac{1}{i^2 s^2} [e^{isx}]_{-a}^{a} \right\}$

$= \frac{2}{i^2 s^2} [x e^{isx}]_{-a}^{a} - \frac{2}{i^2 s^2} [e^{isx}]_{-a}^{a}$

$= -\frac{2}{s^2} [a e^{ias} + a e^{-ias}] + \frac{2}{is^3} [e^{ias} - e^{-ias}]$

$= -\frac{2a}{s^2} [e^{ias} + e^{-ias}] + \frac{2}{is^3} [e^{ias} - e^{-ias}]$

$= -\frac{2a}{s^2} [2\cos(as)] + \frac{2}{is^3} [2i\sin(as)]$

$$= \frac{-4a\cos(as)}{s^2} + \frac{4\sin(as)}{s^3}$$

$$= \frac{4\sin(as) - 4as\cos(as)}{s^3}$$

$$\Rightarrow f(s) = \frac{4}{s^3}\left[\sin(as) - as\cos(as)\right]$$

we know that

The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx}\cdot\frac{4}{s^3}\left[\sin(as) - as\cos(as)\,ds\right] = \begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\frac{[\sin(as) - as\cos(as)]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}[\cos(sx) - i\sin(sx)]\frac{[\sin(as) - as\cos(as)]}{s^3}\,ds = \frac{\pi}{2}\begin{cases} a^2-x^2, & |x| \le a \\ 0, & |x| > a \end{cases}$$

But $x = 0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty}\frac{\sin(as) - as\cos(as)}{s^3}\,ds = \frac{\pi}{2}a^2 \quad \rightarrow (2)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}, \text{ for } a = 1.$$

$$\Rightarrow 2\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} ds = \frac{\pi}{2}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} ds = \frac{\pi}{4}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4} \text{ for } s = x.$$

ii) Let $x = a/2$

$$① \Rightarrow \int_{-\infty}^{\infty} \frac{\sin(as) - s\cos as}{s^3} \cos\left(\frac{as}{2}\right) = \frac{\pi}{2}\left(a^2 - \frac{a^2}{4}\right) = \frac{3\pi a^2}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8} \text{ for } a = 1$$

$$\Rightarrow 2\int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^{\infty} \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{-3\pi}{16}$$

$$\Rightarrow \int_0^{\infty} \frac{x\sin x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = \frac{-3\pi}{16}$$

④ Find the fourier transform of $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

and hence find $\int_0^{\infty} \frac{\sin x}{x} dx$.

Given $F(x) = \begin{cases} 1, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

$$\Rightarrow F(x) = \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$\therefore$ WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)dx$$

$$= \int_{-\infty}^{-1} e^{isx} F(x)dx + \int_{-1}^{1} e^{isx} F(x)dx + \int_{1}^{\infty} e^{isx} F(x)dx$$

$$= 0 + \int_{-1}^{1} e^{isx} \cdot 1 dx + 0$$

$$\therefore F[F(x)] = \int_{-1}^{1} e^{isx} dx$$

$$= \left[\frac{e^{isx}}{is}\right]_{-1}^{1}$$

$$= \frac{1}{is}\left[e^{is} - e^{-is}\right]$$

$$= \frac{1}{is}\left[\cos(s) + i\sin(s) - \cos(s) + i\sin(s)\right]$$

$$= \frac{2i \sin(s)}{is}$$

$$\Rightarrow f(s) = \frac{2\sin(s)}{s}$$

WKT

The fourier inverse transform is

$$\therefore F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\, ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \frac{2\sin(s)}{s} ds = \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)] \frac{\sin(s)}{s} = \pi \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\cos(sx)\sin(s)}{s} ds - i\int_{-\infty}^{\infty} \frac{\sin(sx)\sin s}{s} ds = \begin{cases} \pi & 1, \ -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

$$\therefore \int_{-\infty}^{\infty} \frac{\cos(sx)\sin(s)}{s} ds = \pi \begin{cases} 1, & -1 < x < 1 \\ 0, & x > 1 \end{cases}$$

when $x = 0$

$\therefore$ ① $\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(s)}{s} ds = \pi(1)$

$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin(s)}{s} ds = \pi$

$\Rightarrow 2\int_{0}^{\infty} \frac{\sin(s)}{s} ds = \pi$

$\Rightarrow \int_{0}^{\infty} \frac{\sin(s)}{s} ds = \frac{\pi}{2}$

$\Rightarrow \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$ for $s = x$

⑤ Find the fourier transform $F(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1, \end{cases}$

hence show that

(i) $\int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3} dx = \frac{\pi}{4}$

(ii) $\int_{0}^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16}$

Given $F(x) = \begin{cases} 1-x^2, & -1 \leq x \leq 1 \\ 0, & x > 1 \end{cases}$

$\Rightarrow$ WKT

$$F[F(x)] = \int_{-\infty}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-\infty}^{-1} e^{isx} F(x)\, dx + \int_{-1}^{1} e^{isx} F(x)\, dx + \int_{1}^{\infty} e^{isx} F(x)\, dx$$

$$= \int_{-1}^{1} (1^2 - x^2)\, e^{isx}\, dx$$

$$= (1-x^2)\int_{-1}^{1} e^{isx}\, dx - \int_{-1}^{1} [-2x \int e^{isx} dx]\, dx$$

$$= \frac{1}{is}\left[(1-x^2)\, e^{isx}\right]_{-1}^{1} + \frac{2}{is}\int_{-1}^{1} x\, e^{isx}\, dx$$

$$= \frac{1}{is}(0-0) + \frac{2}{is}\int_{-1}^{1} x\, e^{isx}\, dx$$

$$= \frac{2}{is}\left\{ x\int_{-1}^{1} e^{isx}\, dx - \int_{-1}^{1} [1 \cdot \int e^{isx} dx]\, dx \right\}$$

$$= \frac{2}{is}\left\{ \frac{1}{is}\left[x e^{isx}\right]_{-1}^{1} - \frac{1}{i^2 s^2}\left[e^{isx}\right]_{-1}^{1} \right\}$$

$$= \frac{2}{i^2 s^2}\left[x e^{isx}\right]_{-1}^{1} - \frac{2}{i^2 s^2}\left[e^{isx}\right]_{-1}^{1}$$

$$= \frac{-2}{s^2}\left[e^{is} + e^{-is}\right] + \frac{2}{is^3}\left[e^{is} - e^{-is}\right]$$

$$= \frac{-2}{s^2}[2\cos(s)] + \frac{2}{is^3}[2i\sin(s)]$$

$$= \frac{-4\cos(s)}{s^2} + \frac{4\sin(s)}{s^3}$$

$$= \frac{4\sin(s) - 4s\cos(s)}{s^3}$$

$$\Rightarrow f(s) = \frac{4}{s^3}[\sin(s) - s\cos(s)]$$

WKT

The fourier inverse transform is

$$F^{-1}[f(s)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} f(s)\,ds = F(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-isx} \frac{4}{s^3}[\sin s - s\cos s]\,ds = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} e^{-isx}\left[\frac{\sin s - s\cos s}{s^3}\right] ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} [\cos(sx) - i\sin(sx)]\left[\frac{\sin(s) - s\cos s}{s^3}\right] ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos(sx)\,ds = \frac{\pi}{2}\begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \rightarrow (1)$$

But $x = 0$

$$\therefore (1) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2} \rightarrow (2)$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{2}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\,ds = \frac{\pi}{4}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\sin x - x\cos x}{x^3}\,dx = \frac{\pi}{4} \quad \text{for } s = x$$

(ii) Let $x = \frac{1}{2}$

$$(1) \Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right) ds = \frac{\pi}{2}\left(1^2 - \frac{1^2}{4}\right) = \frac{3\pi}{8}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow 2\int_{0}^{\infty} \frac{\sin s - s\cos s}{s^3}\cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{8}$$

$$\Rightarrow \int_0^\infty \frac{\sin s - s\cos s}{s^3} \cos\left(\frac{s}{2}\right) ds = \frac{3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{s\cos s - \sin s}{s^3} \cos\left(\frac{s}{2}\right) ds = -\frac{3\pi}{16}$$

$$\Rightarrow \int_0^\infty \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx = -\frac{3\pi}{16} \text{ for } s = x.$$

(6) Find the inverse fourier transform of e^{-s^2}.

Let $f(s) = e^{-s^2}$

$$\therefore F(x) = F^{-1}[f(s)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} f(s) ds$$

$$\Rightarrow F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-isx} e^{-s^2} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(s^2 + isx)} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left(s^2 + 2(s)\left(\frac{ix}{2}\right) + \left(\frac{ix}{2}\right)^2 - \left(\frac{ix}{2}\right)^2\right)} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(s + \frac{ix}{2}\right)^2 - \frac{i^2x^2}{4}\right]} ds$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\left[\left(s + \frac{ix}{2}\right)^2 + \frac{x^2}{4}\right]} ds$$

$$F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-\left(s + \frac{ix}{2}\right)^2} ds$$

Let $s + \frac{ix}{2} = u$

$\Rightarrow ds = du$

$$\therefore (1) \Rightarrow F(x) = \frac{e^{-x^2/4}}{2\pi} \int_{-\infty}^{\infty} e^{-u^2} du$$

$$\Rightarrow F(x) = \frac{e^{-x^2/4}}{2\pi} \sqrt{\pi}$$

$$\Rightarrow F(x) = \frac{1}{2\sqrt{\pi}\sqrt{\pi}} e^{-x^2/4} \sqrt{\pi}$$

$$\Rightarrow F(x) = \frac{1}{2\sqrt{\pi}} e^{-x^2/4}$$

⑦ Find the fourier cosine transform of $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

Sol: Given $F(x) = \begin{cases} x, & 0<x<1 \\ 2-x, & 1<x<2 \\ 0, & x>2 \end{cases}$

WKT the fourier cosine transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 F(x)\cos(sx)\,dx + \int_1^2 F(x)\cos(sx)\,dx + \int_2^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 x\cos(sx)\,dx + \int_1^2 (2-x)\cos(sx)\,dx + 0$$

$$\Rightarrow f_c(s) = \int_0^1 x\cos(sx)\,dx + \int_1^2 (2-x)\cos(sx)\,dx$$

$$\therefore \int_0^1 x\cos(sx)\,dx = x\int_0^1 \cos(sx)\,dx - \int_0^1 [1\int\cos(sx)\,dx]\,dx$$

$$= \frac{1}{s}[x\sin(sx)]_0^1 + \frac{1}{s^2}[\cos(sx)]_0^1$$

$$= \frac{1}{s}[\sin s - 0] + \frac{1}{s^2}[\cos s - \cos 0]$$

$$\Rightarrow \int_0^1 x\cos(sx)\,dx = \frac{1}{s}\sin s + \frac{1}{s^2}\cos s - \frac{1}{s^2}$$

$$\int_1^2 (2-x)\cos(sx)\,dx = (2-x)\int_1^2 \cos(sx)\,dx - \int_1^2 [(-1)\int\cos(sx)\,dx]\,dx$$

$$= \frac{1}{s}[(2-x)\sin(sx)]_1^2 - \frac{1}{s^2}[\cos(sx)]_1^2$$

$$= \frac{1}{s}[0 - \sin s] - \frac{1}{s^2}[\cos 2s - \cos s]$$

$$\Rightarrow \int_1^2 (2-x)\cos sx\,dx = \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 2s - \frac{1}{s}\sin s$$

$$\therefore (1) \Rightarrow f(s) = \frac{1}{s}\sin s + \frac{1}{s^2}\cos s - \frac{1}{s^2} + \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 2s - \frac{1}{s}\sin s$$

$$\Rightarrow f_c(s) = \frac{2}{s^2}\cos s - \frac{1}{s^2}\cos 2s - \frac{1}{s^2}$$

8) Find the fourier cosine transform of $F(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & x>4 \end{cases}$

Sol:- Given $F(x) = \begin{cases} 4x, & 0<x<1 \\ 4-x, & 1<x<4 \\ 0, & x>4 \end{cases}$

WKT The fourier cosine transform of $F(x)$ is

$$F_c[F(x)] = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^1 F(x)\cos(sx)\,dx + \int_1^4 F(x)\cos(sx)\,dx + \int_4^\infty F(x)\cos(sx)\,dx$$

$$= 4\int_0^1 x\cos(sx)\,dx + \int_1^4 (2-x)\cos(sx)\,dx + 0$$

$$\Rightarrow f_c(s) = \int_0^1 4x\cos(sx)\,dx + \int_1^4 (2-x)\cos(sx)\,dx$$

$$\therefore \int_0^1 4x\cos(sx)\,dx = 4x\int_0^1 \cos(sx)\,dx - \int_0^1 \left[4.\int \cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[4x\sin(sx)\right]_0^1 + \frac{4}{s^2}\left[\cos(sx)\right]_0^1$$

$$= \frac{4}{s}\left[\sin(s) - 0\right] + \frac{4}{s^2}\left[\cos s - \cos 0\right]$$

$$\Rightarrow \int_0^1 4x\cos(sx)\,dx = \frac{4}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2}$$

$$\int_1^4 (4-x)\cos(sx)\,dx = (4-x)\int_1^4 \cos(sx)\,dx - \int_1^4 \left[(-1)\int \cos(sx)\,dx\right]dx$$

$$= \frac{1}{s}\left[(4-x)\sin(sx)\right]_1^4 - \frac{1}{s^2}\left[\cos(sx)\right]_1^4$$

$$= \frac{1}{s}\left[0 - 3\sin s\right] - \frac{1}{s^2}\left[\cos 4s - \cos s\right]$$

$$\Rightarrow \int_1^4 (4-x)\cos(sx)\,dx = \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{1}{s}\sin s$$

$$\therefore (1) \Rightarrow f(s) = \frac{4}{s}\sin s + \frac{4}{s^2}\cos s - \frac{4}{s^2} + \frac{1}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{3}{s}\sin s$$

$$\Rightarrow f_c(s) = \frac{1}{s}\sin s + \frac{5}{s^2}\cos s - \frac{1}{s^2}\cos 4s - \frac{4}{s^2}$$

(9) Find the fourier sin transform of $e^{-|x|}$. Hence show that $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx = \frac{\pi}{2}e^{-m}$ for $m>0$.

$\therefore$ Given $F(x) = e^{-|x|}$

WKT $|x| = \begin{cases} x, & x>0 \\ -x, & x<0 \end{cases}$

$\therefore F(x) = \begin{cases} e^{-x}, & \text{for } x>0 \\ e^{x}, & \text{for } x<0 \end{cases}$

$\therefore$ WKT the fourier sin transform of $F(x)$ is

$$F_s[F(x)] = f_s(s) = \int_0^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^\infty e^{-x}\sin(sx)\,dx$$

$$\Rightarrow F_s[F(x)] = \left\{\frac{e^{-x}}{(-1)^2+s^2}\left[-1\sin(sx) - s\cos(sx)\right]\right\}_{x=0}^{\infty}$$

$$= \frac{-1}{1+s^2}\left[e^{-x}\left[\sin(sx) + s\cos(sx)\right]\right]_{x=0}^{\infty}$$

$$= \frac{-1}{1+s^2}\left\{0 - 1\left[0 + s(1)\right]\right\}$$

$$= \frac{-1}{1+s^2}(-s)$$

$$\Rightarrow f_s(s) = \frac{s}{s^2+1}$$

By the inverse Fourier sin Transform WKT

$$\frac{2}{\pi}\int_0^\infty f_s(s)\sin(sx)\,ds = F(x)$$

$$\Rightarrow \int_0^\infty \frac{s}{s^2+1}\sin(sx)\,ds = \frac{\pi}{2}F(x)$$

$$\Rightarrow \int_0^\infty \frac{s\sin(sx)}{s^2+1}\,ds = \frac{\pi}{2}e^{-|x|} \rightarrow (1)$$

$$(1) \Rightarrow \int_0^\infty \frac{s\sin(ms)}{s^2+1}\,ds = \frac{\pi}{2}e^{-m}$$

$$\Rightarrow \int_0^\infty \frac{x\sin(mx)}{x^2+1}\,dx = \frac{\pi}{2}e^{-m} \text{ for } s = x.$$

(10) Find the fourier sin and cosine transform of $F(x) = \begin{cases} x, & 0<x<2 \\ 0, & x>2 \end{cases}$

$$F(x) = \begin{cases} x, & 0<x<2 \\ 0, & x>2 \end{cases}$$

$\therefore$ WKT The fourier sin transform of $F(x)$ is

$$F_s[F(x)] = f_s(s) = \int_0^\infty F(x)\sin(sx)\,dx$$

$$= \int_0^2 x\sin(sx)\,dx + \int_2^\infty 0\sin(sx)\,dx$$

$$\Rightarrow F_s[F(x)] = \int_0^2 x\sin(sx)\,dx$$

$$= x\int_0^2 \sin(sx)\,dx - \int_0^2 [1.\int\sin(sx)\,dx]\,dx$$

$$\Rightarrow F_s[F(x)] = \int_0^2 x\sin(sx)\,dx$$

$$= x\int_0^2 \sin(sx)\,dx - \int_0^2 [1.\int\sin(sx)\,dx]\,dx$$

$$= \frac{1}{s}[x\cos(sx)]_0^2 + \frac{1}{s^2}[\sin(sx)]_0^2$$

$$= \frac{1}{s}[x\cos(sx)]_0^2 + \frac{1}{s^2}[\sin(sx)]_0^2$$

$$= \frac{1}{s}[2\cos(sx) - 0] + \frac{1}{s^2}[\sin 2s - 0]$$

$$\Rightarrow F_s[F(x)] = f_s(s) = -\frac{2\cos(sx)}{s} + \frac{1}{s^2}\sin 2s$$

$\therefore$ The fourier cosine transform of $F(x)$ is

$$F_c[F(x)] = f_c(s) = \int_0^\infty F(x)\cos(sx)\,dx$$

$$= \int_0^{2} x\cos(sx)\,dx + \int_2^\infty 0\,dx$$

$$\Rightarrow F_c[F(x)] = \int_0^2 x\cos(sx)\,dx$$

$$= x\int_0^2 \cos(sx)\,dx - \int_0^2 [1 \cdot \int \cos(sx)\,dx]\,dx$$

$$= \frac{1}{s}[x\sin(sx)]_0^2 + \frac{1}{s^2}[\cos(sx)]_0^2$$

$$= \frac{1}{s}[2\sin(2s) - 0] + \frac{1}{s^2}[\cos 2s - \cos(0)]$$

$$\Rightarrow F_c[F(x)] = f_c(s) = \left[\frac{2\sin(2s)}{2} + \frac{1}{s^2}\cos 2s - \frac{1}{s^2}\right]$$

Z-transforms And Difference Equations :-

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^4 + \ldots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots$$

Definition :-

Suppose $f(n)$ be a function in the variable n, such that the z-transform of $f(n)$ can be defined as

$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n} = F(z)$, where Z is called the z-transform operator.

Some important results:-

(1) $f(n) = 1$

$$\therefore Z[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= \frac{1}{z^0} + \frac{1}{z^1} + \frac{1}{z^2} + \frac{1}{z^3} + \dots$$

$$= 1 + \left(\frac{1}{z}\right) + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + \dots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$\Rightarrow \boxed{Z[1] = \frac{z}{z-1}}$$

(2) $f(n) = a^n$

$$\therefore Z[a^n] = \sum_{n=0}^{\infty} a^n \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \dots$$

$$= 1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \dots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left[\frac{z-a}{z}\right]^{-1}$$

$$\Rightarrow \boxed{Z[a^n] = \frac{z}{z-a}}$$

for $a = -1$

$\therefore Z[(-1)^n] = \dfrac{z}{z+1}$

③ $f(n) = n$

$\therefore Z[n] = \sum_{n=0}^{\infty} n \cdot z^{-n}$

$= 0 \cdot z^0 + 1 \cdot z^{-1} + 2 \cdot z^{-2} + 3 \cdot z^{-3} + 4 \cdot z^{-4} + \dots$

$= 1\left(\frac{1}{z}\right) + 2\left(\frac{1}{z}\right)^2 + 3\left(\frac{1}{z}\right)^3 + 4\left(\frac{1}{z}\right)^4 + \dots$

$= \frac{1}{z}\left[1 + 2\left(\frac{1}{z}\right) + 3\left(\frac{1}{z}\right)^2 + 4\left(\frac{1}{z}\right)^3 + \dots\right]$

$= \frac{1}{z}\left[1 - \frac{1}{z}\right]^{-2}$

$= \frac{1}{z}\left[\frac{z-1}{z}\right]^{-2}$

$= \frac{1}{z} \frac{z^2}{(z-1)^2}$

$\Rightarrow \boxed{Z[n] = \dfrac{z}{(z-1)^2}}$

④ $f(n) = n^2$

$\therefore$ WKT $Z[n^p] = -z \dfrac{d}{dz} Z[n^{p-1}] \rightarrow ①$

$\forall\ p = 2, 3, 4 \dots$

Let $p = 2$

$\therefore ① \Rightarrow Z[n^2] = -z \dfrac{d}{dz} Z[n]$

$= -z \dfrac{d}{dz}\left[\dfrac{z}{(z-1)^2}\right]$

$= -z\left[\dfrac{(z-1)^2(1) - z\,2(z-1)(1)}{(z-1)^4}\right]$

$$= -z\left[\frac{-z-1}{(z-1)^3}\right]$$

$$= \frac{z(z+1)}{(z-1)^3}$$

$$\Rightarrow \boxed{z[n^2] = \frac{z^2+z}{(z-1)^3}}$$

(5) z-transform of $\sin n\theta$ and $\cos n\theta$

Let $f(n) = e^{in\theta} = \cos n\theta + i\sin n\theta$

$$\therefore z[e^{in\theta}] = z[(e^{i\theta})^n]$$

$$= \frac{z}{(z-e^{i\theta})}$$

$$= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z[z-e^{-i\theta}]}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}$$

$$\Rightarrow z[\cos n\theta + i\sin n\theta] = \frac{z^2 - z[\cos\theta - i\sin\theta]}{z^2 - z(2\cos\theta) + 1}$$

$$\Rightarrow z[\cos n\theta] + iz[\sin n\theta] = \frac{(z^2 - z\cos\theta) + i(z\sin\theta)}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow z[\cos n\theta] + iz[\sin n\theta] = \left[\frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}\right] + i\left[\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}\right]$$

$$\therefore z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1} \qquad \therefore z[\sin n\theta] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

(6) WKT $\cosh\theta = \dfrac{e^{\theta} + e^{-\theta}}{2}$

$$\Rightarrow \cosh(n\theta) = \frac{e^{n\theta} + e^{-n\theta}}{2}$$

$$\Rightarrow Z[\cosh(n\theta)] = \frac{1}{2} Z[e^{n\theta} + e^{-n\theta}]$$

$$= \frac{1}{2}\left\{ Z[e^{n\theta}] + Z[e^{-n\theta}] \right\}$$

$$= \frac{1}{2}\left\{ Z[e^{\theta}]^{n} + Z[e^{-\theta}]^{n} \right\}$$

$$= \frac{1}{2}\left\{ \frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}} \right\}$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} + \frac{1}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\theta}+e^{-\theta})}{z^2-z(e^{\theta}+e^{-\theta})+1} \right]$$

$$\Rightarrow Z[\cosh n\theta] = \frac{z^2 - z\cosh\theta}{z^2 - 2z\cosh\theta + 1}$$

(2) WKT $\sinh\theta = \frac{e^{\theta}-e^{-\theta}}{2} \Rightarrow \sinh(n\theta) = \frac{e^{n\theta}-e^{-n\theta}}{2}$

$$\Rightarrow Z[\sinh(n\theta)] = \frac{1}{2} Z[e^{n\theta} - e^{-n\theta}]$$

$$= \frac{1}{2}\left\{ Z[e^{n\theta}] - Z[e^{-n\theta}] \right\}$$

$$= \frac{1}{2}\left\{ Z[e^{\theta}]^{n} - Z[e^{-\theta}]^{n} \right\}$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{1}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}-z+e^{\theta}}{(z-e^{\theta})(z-e^{-\theta})} \right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}-e^{-\theta}}{z^2-2z\cosh\theta+1} \right]$$

$$= \frac{z}{2}\left[\frac{2\sinh\theta}{z^2-2z\cosh\theta+1} \right]$$

$$\Rightarrow Z[\sinh n\theta] = \frac{z\sinh\theta}{z^2-2z\cosh\theta+1}$$

(8) if $z[f(n)] = F(z)$, then (i) $z[a^n f(n)] = F(z/a)$

(ii) $z[a^{-n} f(n)] = F(az)$ it is also called the damping room.

(1) Find the z-transform of $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$$= \cos\left(\frac{n\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{n\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)$$

$$\Rightarrow f(n) = \frac{1}{\sqrt{2}}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{\sqrt{2}}\sin\left(\frac{n\pi}{2}\right)$$

$$\Rightarrow z[f(n)] = \frac{1}{\sqrt{2}} z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \frac{1}{\sqrt{2}} z\left[\sin\left(\frac{n\pi}{2}\right)\right]$$

WKT $z[\cos n\theta] = \dfrac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$

$$\therefore z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1}$$

$$\therefore z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - 0}{z^2 - 2z(0) + 1}$$

$$\Rightarrow z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2 + 1}$$

$$z[\sin(n\theta)] = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2 - 2z\cos\left(\frac{\pi}{2}\right) + 1}$$

$$= \frac{z}{z^2 - 0 + 1}$$

$$\Rightarrow z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z}{z^2 + 1}$$

$\therefore$ ① $\Rightarrow F(z) = \frac{1}{\sqrt{2}} \frac{z^2}{z^2+1} - \frac{1}{\sqrt{2}} \frac{z}{z^2+1}$

$$\Rightarrow \boxed{F(z) = \frac{z^2 - z}{\sqrt{2}(z^2+1)}}$$

② Find the z-transform of $2n + \sin\left(\frac{n\pi}{4}\right) + 1$.

Let $f(n) = 2n + \sin\left(\frac{n\pi}{4}\right) + 1$

$\Rightarrow z[f(n)] = 2z[n] + z\left[\sin\left(\frac{n\pi}{4}\right)\right] + z[1] \rightarrow$ ①

WKT $z[n] = \frac{z}{(z-1)^2}$

$\Rightarrow z[\sin n\theta] = \frac{z \sin\theta}{z^2 - 2z\cos\theta + 1}$

$\Rightarrow z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z \sin(\pi/4)}{z^2 - 2z\cos(\pi/4) + 1}$

$= \frac{z(1/\sqrt{2})}{z^2 - 2z(1/\sqrt{2}) + 1}$

$\Rightarrow z[\sin(n\pi/4)] = \frac{z/\cancel{\sqrt{2}}}{\frac{\sqrt{2}z^2 - 2z + \sqrt{2}}{\cancel{\sqrt{2}}}}$

$\Rightarrow z\left[\sin\left(\frac{n\pi}{4}\right)\right] = \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$

$z[1] = \frac{z}{z-1}$

$\therefore$ ① $\Rightarrow F(z) = \frac{2z}{(z-1)^2} + \frac{z}{\sqrt{2}z^2 - 2z + \sqrt{2}} + \frac{z}{z-1}$

③ Find the z-transform (i) $\cos\left(\frac{n\pi}{2}+\theta\right)$

(ii) $3^n\left(\cos\left(\frac{n\pi}{4}\right)\right)$

Let $f(n) = \cos\left(\frac{n\pi}{2}+\theta\right)$

$\Rightarrow f(n) = \cos\left(\frac{n\pi}{2}\right)\cos\theta - \sin\left(\frac{n\pi}{2}\right)\sin\theta$

$\Rightarrow z[f(n)] = \cos\theta\, z\left[\cos\left(\frac{n\pi}{2}\right)\right] - \sin\theta\, z\left[\sin\left(\frac{n\pi}{2}\right)\right]$

WKT

$$z[\cos n\theta] = \frac{z^2 - z\cos\theta}{z^2-2z\cos\theta+1}$$

$$\Rightarrow z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2 - z\cos(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$\Rightarrow z\left[\cos\left(\frac{n\pi}{2}\right)\right] = \frac{z^2}{z^2+1}$$

$$\Rightarrow z[\sin(n\theta)] = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\Rightarrow z\left[\sin\left(\frac{n\pi}{2}\right)\right] = \frac{z\sin(\pi/2)}{z^2-2z\cos(\pi/2)+1}$$

$$= \frac{z}{z^2+1}$$

$$\therefore ① \Rightarrow F(z) = \frac{z^2\cos\theta}{z^2+1} - \frac{z\sin\theta}{z^2+1}$$

$$\Rightarrow F(z) = \frac{z^2\cos\theta - z\sin\theta}{z^2+1}$$

Let $f(n) = \cos\left(\frac{n\pi}{4}\right)$

$\Rightarrow z[f(n)] = z\left[\cos\left(\frac{n\pi}{4}\right)\right]$

$$\Rightarrow F(z) = \frac{z^2 - z\cos(\pi/4)}{z^2 - 2z\cos(\frac{\pi}{4}) + 1}$$

$$\Rightarrow F(z) = \frac{z^2 - z(\frac{1}{\sqrt{2}})}{z^2 - 2z(\frac{1}{\sqrt{2}}) + 1}$$

$$\Rightarrow F(z) = \frac{\sqrt{2}z^2 - z}{\sqrt{2}z^2 - 2z + \sqrt{2}}$$

WKT $\quad z[a^n f(n)] = F(\frac{z}{a})$

$$\Rightarrow z[3^n f(n)] = F(\frac{z}{3})$$

$$\Rightarrow z[3^n \cos(\frac{n\pi}{4})] = \frac{\sqrt{2}(\frac{z}{3})^2 - (\frac{z}{3})}{\sqrt{2}(\frac{z}{3})^2 - 2(\frac{z}{3}) + \sqrt{2}}$$

$$= \frac{\frac{\sqrt{2}z^2}{9} - \frac{z}{3}}{\frac{\sqrt{2}z^2}{9} - \frac{2z}{3} + \sqrt{2}}$$

$$= \frac{\sqrt{2}z^2 - 3z}{\sqrt{2}z^2 - 6z + 9\sqrt{2}}$$

(4) Find the z-transform of

(i) $a^n \sin n\theta$

(ii) $a^{-n} \cos n\theta$

(iii) $\sin(3n + \pi)$

(i) Let $f(n) = \sin n\theta$

$$\Rightarrow z[f(n)] = z[\sin n\theta]$$

$$\Rightarrow F(z) = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $z[a^n f(n)] = F\left(\frac{z}{a}\right)$

$$\Rightarrow z[a^n \sin(n\theta)] = \frac{\frac{z}{a}\sin\theta}{\left(\frac{z}{a}\right)^2 - 2\left(\frac{z}{a}\right)\cos\theta + 1}$$

$$= \frac{\frac{z}{a}\sin\theta}{\frac{z^2 - 2az\cos\theta + a^2}{a^2}}$$

$$= \frac{az\sin n\theta}{z^2 - 2az\cos\theta + a^2}$$

(ii) Let $f(n) = \cos n\theta$

$$\Rightarrow z[f(n)] = z[\cos n\theta]$$

$$\Rightarrow F(z) = \frac{z^2 - z\cos\theta}{z^2 - 2z\cos\theta + 1}$$

WKT $z[a^{-n} f(n)] = F(az)$

$$\Rightarrow z[a^n \cos(n\theta)] = \frac{a^2z^2 - az\cos\theta}{a^2z^2 - 3az\cos\theta + 1}$$

(iii) Let $f(n) = \sin(3n+5)$

$$\Rightarrow f(n) = \sin(3n)\cdot\cos 5 + \cos(3n)\cdot\sin 5$$

$$\Rightarrow z[f(n)] = \cos 5\cdot z[\sin(3n)] + \sin 5\cdot z[\cos(3n)]$$

$$\Rightarrow F(z) = \cos 5\cdot\left[\frac{z\sin 3}{z^2 - 2z\cos 3 + 1}\right] + \sin 5\left[\frac{z^2 - z\cos 3}{z^2 - 2z\cos 3 + 1}\right]$$

Inverse z - transforms :-

Suppose $F(z)$ be a z-transform of $f(x)$, then the inverse z-transform of $F(z)$ can be defined as

$$z^{-1}[F(z)] = f(n).$$

Standard Results :-

1) $z^{-1}\left[F\left(\frac{z}{a}\right)\right] = a^n f(n)$

2) $z^{-1}[F(az)] = a^{-n} f(n)$

3) $z^{-1}\left[\frac{z}{z-1}\right] = 1$

4) $z^{-1}\left[\frac{z}{z-a}\right] = a^n$

5) $z^{-1}\left[\frac{z}{(z-1)^2}\right] = n$

6) $z^{-1}\left[\frac{z^2+z}{(z-1)^3}\right] = n^2$

7) $z^{-1}\left[\frac{az}{(z-a)^2}\right] = a^n . n$

8) $z^{-1}\left[\frac{z}{z+1}\right] = (-1)^n$

9) $z^{-1}\left[\frac{z^2}{z^2+1}\right] = \cos\left(\frac{n\pi}{2}\right)$

10) $z^{-1}\left[\frac{z}{z^2+1}\right] = \sin\left(\frac{n\pi}{2}\right)$

① Find the inverse z-transform of the following :-

1. $\dfrac{z}{(z-2)(z-3)}$ 2. $\dfrac{z}{z^2+7z+10}$ 3. $\dfrac{3z^2+2z}{(5z-1)(5z+2)}$

4. $\dfrac{8z^2}{(2z-1)(4z-1)}$ 5. $\dfrac{2z^2+3z}{(z+2)(z-4)}$ 6. $\dfrac{z^3-20z}{(z-2)^3(z-4)}$ 7. $\dfrac{18z^2}{(2z-1)(4z+1)}$

① $\dfrac{z}{(z-2)(z-3)}$

Let $F(z) = \dfrac{z}{(z-2)(z-3)}$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{(z-2)(z-3)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{z-3} - \frac{1}{z-2}$$

$$\Rightarrow F(z) = \frac{z}{z-3} - \frac{z}{z-2}$$

$$\Rightarrow z^{-1}[F(z)] = z^{-1}\left[\frac{z}{z-3}\right] - z^{-1}\left[\frac{z}{z-2}\right]$$

$$\Rightarrow f(n) = 3^n - 2^n$$

(2) $$\frac{z}{z^2+7z+10}$$

$$F(z) = \frac{z}{z^2+7z+10}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{z^2+7z+10}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{z^2+2z+5z+10}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{(z+2)(z+5)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{1}{3}\frac{1}{z+2} - \frac{1}{3}\frac{1}{z+5}$$

$$\Rightarrow z^{-1}[F(z)] = \frac{1}{3}z^{-1}\left[\frac{z}{z-(-2)}\right] - \frac{1}{3}z^{-1}\left[\frac{z}{z-(-5)}\right]$$

$$\Rightarrow f(n) = \frac{1}{3}(-2)^n - \frac{1}{3}(-5)^n$$

$$\Rightarrow f(n) = \frac{1}{3}\left[(-2)^n - (-5)^n\right]$$

(3) $$\frac{3z^2+2z}{(5z-1)(5z+2)}$$

Let $$F(z) = \frac{3z^2+2z}{(5z-1)(5z+2)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{3z+2}{(5z-1)(5z+2)} \rightarrow (1)$$

$$\frac{3z+2}{(5z-1)(5z+2)} = \frac{A}{5z-1} + \frac{B}{5z+2}$$

$$\Rightarrow 3z+2 = A(5z+2) + B(5z-1) \rightarrow (2)$$

when $z = \frac{1}{5}$

② $\Rightarrow \frac{3}{5} + 2 = A(3)$

$\Rightarrow \frac{13}{5} = 3A$

$\Rightarrow A = \frac{13}{15}$

when $z = \frac{-2}{5}$

$\therefore$ ② $\Rightarrow 3\left(\frac{-2}{5}\right) + 2 = B[-2-1]$

$\Rightarrow \frac{-6}{5} + 2 = -3B$

$\Rightarrow \frac{-6+10}{5} = -3B$

$\Rightarrow B = -\frac{4}{15}$

$\therefore$ ① $\Rightarrow \frac{F(z)}{z} = \frac{13}{5} \cdot \frac{1}{(5z-1)} - \frac{4}{15} \cdot \frac{1}{(5z+2)}$

$\Rightarrow F(z) = \frac{13}{15 \times 5} \cdot \frac{z}{(z - \frac{1}{5})} - \frac{4}{15 \times 5} \frac{z}{(z + \frac{2}{5})}$

$\Rightarrow F(z) = \frac{13}{75}\left[\frac{z}{z - 1/5}\right] - \frac{4}{75}\left[\frac{z}{z - (\frac{-2}{5})}\right]$

$\Rightarrow z^{-1}[F(z)] = \frac{13}{75} z^{-1}\left[\frac{z}{z - 1/5}\right] - \frac{4}{75} z^{-1}\left[\frac{z}{z - (\frac{-2}{5})}\right]$

$\Rightarrow f(n) = \frac{13}{75}\left(\frac{1}{5}\right)^n - \frac{4}{75}\left(\frac{-2}{5}\right)^n$

④ $\frac{18z^2}{(2z-1)(4z+1)}$

Let $F(z) = \frac{18z^2}{(2z-1)(4z+1)}$

$\Rightarrow \frac{F(z)}{z} = \frac{18z}{(2z-1)(4z+1)} \rightarrow$ ①

$\frac{18z}{(2z-1)(4z+1)} = \frac{A}{(2z-1)} + \frac{B}{(4z+1)}$

$\Rightarrow 18z = A(4z+1) + B(2z-1) \rightarrow$ ②

when $z = 1/2$

② $\Rightarrow 9 = 3A \Rightarrow A = 3$

when $z = -\frac{1}{4}$

$$② \Rightarrow 18\left(-\frac{1}{4}\right) = B\left[2\left(-\frac{1}{2}\right)-1\right]$$

$$\Rightarrow -\frac{9}{2} = -\frac{3}{2}B$$

$$\Rightarrow B = 3$$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{3}{(2z-1)} + \frac{3}{(4z+1)}$$

$$\Rightarrow F(z) = \frac{3z}{(2z-1)} + \frac{3z}{(4z+1)}$$

$$\Rightarrow F(z) = \frac{3}{2}\left[\frac{z}{z-1/2}\right] + \frac{3}{4}\left[\frac{z}{z+1/4}\right]$$

$$\therefore z^{-1}[F(z)] = \frac{3}{2} z^{-1}\left[\frac{z}{z-1/2}\right] + \frac{3}{4} z^{-1}\left[\frac{z}{z-(-1/4)}\right]$$

$$\Rightarrow f(n) = \frac{3}{2}\left(\frac{1}{2}\right)^n + \frac{3}{4}\left(-\frac{1}{4}\right)^n$$

⑤ $\dfrac{2z^2+3z}{(z+2)(z-4)}$

$$\Rightarrow F(z) = \frac{2z^2+3z}{(z+2)(z-4)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{2z+3}{(z+2)(z-4)} \rightarrow ①$$

$$\frac{2z+3}{(z+2)(z-4)} = \frac{A}{(z+2)} + \frac{B}{(z-4)}$$

$$2z+3 = A(z-4) + B(z+2) \rightarrow ②$$

when $z = 4$ | when $z = -2$

$② \Rightarrow 11 = B(6)$ | $-1 = A(-6)$

$B = \frac{11}{6}$ | $A = 1/6$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{1}{6}\frac{1}{(z+2)} + \frac{11}{6}\frac{1}{(z-4)}$$

$$\Rightarrow F(z) = \frac{1}{6}\frac{z}{(z+2)} + \frac{11}{6}\frac{z}{(z-4)}$$

$$\Rightarrow z^{-1}[F(z)] = \frac{1}{6}z^{-1}\left[\frac{z}{z+2}\right] + \frac{11}{6}z^{-1}\left[\frac{z}{z-4}\right]$$

$$\Rightarrow f(n) = \frac{1}{6}(-2)^n + \frac{11}{6}(4)^n$$

⑥ $\dfrac{8z^2}{(2z-1)(4z-1)}$

$$\Rightarrow F(z) = \frac{8z^2}{(2z-1)(4z-1)}$$

$$\Rightarrow \frac{F(z)}{z} = \frac{8z}{(2z-1)(4z-1)} \rightarrow ①$$

$$\frac{8z}{(2z-1)(4z-1)} = \frac{A}{(2z-1)} + \frac{B}{(4z-1)}$$

$$8z = A(4z-1) + B(2z-1) \rightarrow ②$$

when $z = \frac{1}{4}$

$② \Rightarrow 2 = B\left(2\left(\frac{1}{4}\right)-1\right)$

$\Rightarrow 2 = B\left(\frac{1}{2}-1\right)$

$\Rightarrow 2 = B\left(\frac{1-2}{2}\right)$

$\Rightarrow B = -4$

$z = \frac{1}{2}$

$② \Rightarrow 4 = A(2-1)$

$\Rightarrow 4 = A$

$\Rightarrow A = 4$

$$\therefore ① \Rightarrow \frac{F(z)}{z} = \frac{4}{(2z-1)} - \frac{4}{(4z-1)}$$

$$\Rightarrow F(z) = 4\frac{z}{(2z-1)} - 4\frac{z}{(4z-1)}$$

$$\Rightarrow z^{-1}[F(z)] = 4z^{-1}\left[\frac{z}{2z-1}\right] - 4z^{-1}\left[\frac{z}{4z-1}\right]$$

$$= \frac{4}{2}z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] - \frac{4}{4}z^{-1}\left[\frac{z}{z-1/4}\right]$$

$$\Rightarrow f(n) = 2\left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n$$

Difference Equations:

Step ①: Express the given difference equation in the notation of $y_n, y_{n+1}, y_{n+2}, \ldots$

Step ②: Apply z-transform on both sides and substitute

$$z[y_{n+2}] = z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right]$$

$$z[y_{n+1}] = z\left[\bar{y}(z) - y_0\right]$$

$$z[y_n] = \bar{y}(z)$$

Step ③: write $\bar{y}(z)$ has a function of z, hence apply the inverse z-transform and find $y(n)$.

① Solve the difference equation using z-transform $y_{n+2} - 4y_n = 0$, subject to the conditions $y_0 = 0$, $y_1 = 2$.

Given $y_{n+2} - 4y_n = 0 \quad y_0 = 0 \quad y_1 = 2$

$$\Rightarrow z[y_{n+2}] - 4z[y_n] = z[0]$$

$$\Rightarrow \left[z^2\bar{y}(z) - y_0 - \frac{y_1}{z}\right] - 4\bar{z}\,z = 0$$

$$\Rightarrow \left[z^2\bar{y}(z) - 0 - \frac{2}{z}\right] - 4\bar{y}(z) = 0$$

$$\Rightarrow z^2\bar{y}(z) - 2z - 4\bar{y}(z) = 0$$

$$\Rightarrow (z^2 - 4)\bar{y}(z) = 2z$$

$$\Rightarrow \bar{y}(z) = \frac{2z}{(z^2-4)}$$

$$\Rightarrow \bar{y}(z) = \frac{2z}{(z+2)(z-2)}$$

$$\Rightarrow \frac{\bar{y}(z)}{z} = \frac{2}{(z-2)(z+2)} \rightarrow ①$$

$$\frac{z}{(z-2)(z+2)} = \frac{A}{(z-2)} + \frac{B}{(z+2)}$$

$$\Rightarrow 2 = A(z+2) + B(z-2) \rightarrow ②$$

when $z = 2$ when $z = -2$

② $\Rightarrow 2 = 4A$ ② $\Rightarrow 2 = -4B$

$\Rightarrow A = \frac{1}{2}$ $\Rightarrow B = \frac{-1}{2}$

$\therefore$ ① $\Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{2} \cdot \frac{1}{z-2} - \frac{1}{2} \cdot \frac{1}{z+2}$

$\Rightarrow \bar{y}(z) = \frac{1}{2} \cdot \frac{z}{z-2} - \frac{1}{2} \cdot \frac{z}{z+2}$

$\Rightarrow z^{-1}[\bar{y}(z)] = \frac{1}{2} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{2} z^{-1}\left[\frac{z}{z+2}\right]$

$\Rightarrow y(n) = \frac{1}{2}(2)^n - \frac{1}{2}(-2)^n$

② Using z-transform solve the difference equation $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$, subject to the conditions $y_0 = 0, y_1 = 0$.

Sol: Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ $y_0 = 0, \ y_1 = 0$

$\Rightarrow z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$

$\Rightarrow z^2\left[\bar{y}(z) - y_0 - \frac{y_1}{z}\right] + 6z[\bar{y}(z) - y_0] + 9\bar{y}(z) = \frac{z}{z-2}$

$\Rightarrow z^2\bar{y}(z) + 6z\bar{y}(z) + 9\bar{y}(z) = \frac{z}{z-2}$

$\Rightarrow (z^2 + 6z + 9)\bar{y}(z) = \frac{z}{z-2}$

$\Rightarrow \bar{y}(z) = \frac{z}{(z-2)(z+3)^2}$

$\Rightarrow \frac{\bar{y}(z)}{z} = \frac{1}{(z-2)(z+3)^2} \rightarrow$ ①

$\frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \frac{C}{(z+3)^2}$

$1 = A(z+3)^2 + B(z-2)(z+3) + C(z-2) \rightarrow$ ②

when $z = 2$

② $\Rightarrow 1 = A(2+3)^2$

$\Rightarrow 1 = 25A \Rightarrow A = 1/25$

when $z=-3$

② $\Rightarrow$ $1 = C(-5)$

$\Rightarrow$ $C = -\frac{1}{5}$

$A + B = 0$

$\Rightarrow$ $B = -A$

$\Rightarrow$ $B = -\frac{1}{25}$

$\therefore$ ① $\Rightarrow$ $\frac{\bar{y}(z)}{z} = \frac{1}{25}\cdot\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{5}\cdot\frac{1}{(z+3)^2}$

$\Rightarrow \bar{y}(z) = \frac{1}{25}\cdot\frac{z}{z-2} - \frac{1}{25}\cdot\frac{z}{z+3} - \frac{1}{5}\cdot\frac{z}{(z+3)^2}$

$\Rightarrow \bar{y}(z) = \frac{1}{25}\cdot\frac{z}{z-2} - \frac{1}{25}\cdot\frac{z}{z-(-3)} + \frac{1}{5}\cdot\frac{(-3z)}{(z-(-3))^2}$

$\Rightarrow z^{-1}[\bar{y}(z)] = \frac{1}{25}\cdot z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25}\, z^{-1}\left[\frac{z}{z-(-3)}\right] + \frac{1}{15}\cdot\frac{(-3z)}{(z-(-3))^2}$

$\Rightarrow y(n) = \frac{1}{25}\cdot(2^n) - \frac{1}{25}(-3)^n + \frac{1}{15}(-3)^n.n.$

③ Solve $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$. Given $u_0 = 0$, $u_1 = 1$ by using z-transform.

Given $u_{n+2} - 3u_{n+1} + 2u_n = 2^n$. $u_0 = 0$ $u_1 = 1$

$\Rightarrow z[u_{n+2}] - 3z[u_{n+1}] + 2z[u_n] = z[2^n]$

$\Rightarrow z^2[\bar{u}(z) - u_0 - \frac{u_1}{z}] - 3z[\bar{u}(z) - u_0] + 2\bar{u}(z) = \frac{z}{z-2}$

$\Rightarrow z^2\bar{u}(z) - z - 3z\bar{u}(z) + 2\bar{u}(z) = \frac{z}{z-2}$

$\Rightarrow (z^2 - 3z + 2)\bar{u}(z) = \frac{z}{z-2} + z$

$\Rightarrow (z-1)(z-2)\bar{u}(z) = \frac{z + z(z-2)}{z-2}$

$\Rightarrow (z-1)(z-2)\,\bar{u}(z) = \frac{z^2 - z}{z-2}$

$\Rightarrow \bar{u}(z) = \frac{z^2 - z}{(z-1)(z-2)^2}$

$$\Rightarrow \bar{u}(z) = \frac{z\cancel{(z-1)}}{\cancel{(z-1)}(z-2)^2}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-2)^2}$$

$$\Rightarrow \bar{u}(z) = \frac{1}{2} \cdot \frac{2z}{(z-2)^2}$$

$$\Rightarrow z^{-1}[\bar{u}(z)] = \frac{1}{2} z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$\Rightarrow u(n) = \frac{1}{2} n 2^n$$

$$\Rightarrow u(n) = 2^{n-1} n.$$

④ Solve the difference equation $u_{n+2} + 2u_{n+1} + u_n = n$, subject to the conditions $u_0 = 0$, $u_1 = 0$.

Given $u_{n+2} + 2u_{n+1} + u_n = n$, $u_0 = 0$, $u_1 = 0$

$$\Rightarrow z[u_{n+2}] + 2z[u_{n+1}] + z[u_n] = z[n]$$

$$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 2z[\bar{u}(z)] + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow z^2\bar{u}(z) + 2z\bar{u}(z) + \bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow (z^2 + 2z + 1)\bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow (z+1)^2\bar{u}(z) = \frac{z}{(z-1)^2}$$

$$\Rightarrow \bar{u}(z) = \frac{z}{(z-1)^2(z+1)^2}$$

$$\Rightarrow \frac{\bar{u}(z)}{z} = \frac{1}{(z-1)^2(z+1)^2} \rightarrow ①$$

$$\therefore \frac{1}{(z-1)^2(z+1)^2} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{(z+1)} + \frac{D}{(z+1)^2}$$

$$\Rightarrow 1 = A(z-1)(z+1)^2 + B(z+1)^2 + C(z+1)(z-1)^2 + D(z-1)^2 \rightarrow ②$$

when $z=1$

② $\Rightarrow 1 = 4B$

$B = \frac{1}{4}$

when $z=-1$

② $\Rightarrow 1 = 4D$

$D = \frac{1}{4}$

$A+B+C+D = 1$

$\Rightarrow A + \frac{1}{4} + C + \frac{1}{4} = 1$

$\Rightarrow -A + C = \frac{1}{2} \rightarrow$ ③

when $A + C = 0 \rightarrow$ ④

③ + ④ $\Rightarrow 2C = \frac{1}{2}$

$C = \frac{1}{4}$

$A = -\frac{1}{4}$

$\therefore$ ① $\Rightarrow \frac{\bar{u}(z)}{z} = -\frac{1}{4}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{4}\cdot\frac{1}{(z+1)} + \frac{1}{4}\frac{1}{(z+1)^2}$

$\Rightarrow \bar{u}(z) = -\frac{1}{4}\frac{z}{z-1} + \frac{1}{4}\cdot\frac{z}{(z-1)^2} + \frac{1}{4}\cdot\frac{z}{(z+1)} + \frac{1}{4}\cdot\frac{z}{(z+1)^2}$

$\Rightarrow z^{-1}[\bar{u}(z)] = -\frac{1}{4} z^{-1}\left[\frac{z}{z-1}\right] + \frac{1}{4} z^{-1}\left[\frac{z}{(z-1)^2}\right] + \frac{1}{4} z^{-1}\left[\frac{z}{z-(-1)}\right]$
$-\frac{1}{4}\left[\frac{(-1)z}{(z-(-1))^2}\right]$

$\Rightarrow u(n) = -\frac{1}{4} + \frac{1}{4}n + \frac{1}{4}(-1)^n - \frac{1}{4}(-1)^n . n$

⑤ Solve the difference equation $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, subject to the conditions $u_0 = 0$, $u_1 = 1$.

Given $u_{n+2} + 4u_{n+1} + 3u_n = 3^n$, $u_0 = 0$, $u_1 = 1$

$\Rightarrow z[u_{n+2}] + 4z[u_{n+1}] + 3z[u_n] = z[3^n]$

$\Rightarrow z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] + 4z[\bar{u}(z) - u_0] + 3\bar{u}(z) = \frac{z}{z-3}$

$\Rightarrow z^2\bar{u}(z) - z + 4z\,\bar{u}(z) + 3\bar{u}(z) = \frac{z}{z-3}$

$\Rightarrow [z^2 + 4z + 3]\,\bar{u}(z) = \frac{z}{z-3} + z$

$\Rightarrow (z+3)(z+1)\,\bar{u}(z) = \frac{z + z(z-3)}{z-3}$

$\Rightarrow \bar{u}(z) = \frac{z + z^2 - 3z}{(z+1)(z-3)(z+3)}$

$$\Rightarrow \bar{u}(z) = \frac{z^2-2z}{(z+1)(z+3)(z-3)}$$

$$\Rightarrow \frac{\bar{u}(z)}{z} = \frac{z-2}{(z+1)(z+3)(z-3)} \rightarrow ①$$

$$\frac{z-2}{(z+1)(z+3)(z-3)} = \frac{A}{z+1} + \frac{B}{z+3} + \frac{C}{z-3}$$

$$\Rightarrow z-2 = A(z+3)(z-3) + B(z+1)(z-3) + C(z+1)(z+3) \rightarrow ②$$

when $z=-3$

② $\Rightarrow -5 = B(-2)(-6)$

$\Rightarrow -5 = 12B$

$\Rightarrow B = \frac{-5}{12}$

$z=3$

② $\Rightarrow 1 = C(4)(6)$

$\Rightarrow 1 = 24C$

$\Rightarrow C = \frac{1}{24}$

$z=-1$

② $\Rightarrow -3 = A(2)(-4)$

$\Rightarrow -3 = -8A$

$\Rightarrow A = \frac{3}{8}$

$$\therefore ① \Rightarrow \frac{\bar{u}(z)}{z} = \frac{3}{8}\frac{1}{z+1} - \frac{5}{12}\frac{1}{z+3} + \frac{1}{24}\cdot\frac{1}{z-3}$$

$$\Rightarrow \bar{u}(z) = \frac{3}{8}\frac{z}{z+1} - \frac{5}{12}\cdot\frac{z}{z+3} + \frac{1}{24}\cdot\frac{z}{z-3}$$

$$\Rightarrow z^{-1}[\bar{u}(z)] = \frac{3}{8} z^{-1}\left[\frac{z}{z+1}\right] - \frac{5}{12} z^{-1}\left[\frac{z}{z+3}\right] + \frac{1}{24} z^{-1}\left[\frac{z}{z-3}\right]$$

$$\Rightarrow u(n) = \frac{3}{8}(-1)^n - \frac{5}{12}(-3)^n + \frac{1}{24}(3)^n$$