||Jai Sri Gurudev||

S. J. C. INSTITUTE OF TECHNOLOGY, CHICKBALLAPUR

Department of Mathematics

LECTURE NOTES

CALCULUS & LINEAR ALGEBRA (18MAT11)

Prepared By:

Purushotham.P
Assistant Professor
SJC Institute of Technology
Email id: ppurushothamp48@gmail.com
Phone: 8197481658

CALCULUS AND LINEAR ALGEBRA

MODULE-4

DIFFERENTIAL EQUATION

If an equation contain one dependant variable and then derivative with respect to one or more independent Variable is called Differential equation.

There are are two types of D. E

1. Ordinary differential equation
2. partial differential equation

=> Ordinary differential equation :- If an equation contain one dependant variable and its derivative with respect to one independant variable.

ex :- $x^2 \frac{d^2y}{dx^2} - 3x \frac{dy}{dx} = 0$

=> Partial differential equation :- If an equation Contain one depeandant variable and derivatives with respect to two or more variable

ex :- $x \frac{dz}{dx} + y \frac{dz}{dy} = 2z$

⇒ Order and degree of differential equation

The highest derivative in a given D.E is called order it's power is called degree

ex :- $x^2\left(\frac{d^3y}{dx^3}\right)^4 - 2x\left(\frac{d^2y}{dx^2}\right)^5 + \left(\frac{dy}{dx}\right)^5 + y = 0$

The highest derivative is $\frac{d^3y}{dx^3}$ and power 4

order of D.E = 3

degree of D.E = 4

⇒ Solution of 1st order and 1st degree D.E

Generally 1st order and 1st degree D.E is in the form $\frac{dy}{dx} = f(x,y)$

⇒ Exact Differential equation.

Step 1 :- write the given D.E is in the form $M(x,y)dx + N(x,y)dy = 0$

Step 2 :- Identify the M and N find $\frac{\partial M}{\partial y}$, $\frac{\partial N}{\partial y}$

Step 3 :- if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ we say that given D.E is exact D.E

Step 4 :- Write the Solution for E.D.E

$\int M(x,y)\, dx + \int$ (The terms which don't contain x in N) $dy = c$

Problems :-

1. Solve $\frac{dy}{dx} + \frac{2x+3y-1}{3x+4y-2} = 0$

$(2x+3y-1)\, dx + (3x+4y+2)dy = 0$

$M = (2x+3y-1)$ $\quad N = (3x+4y+2)$

$$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$$

The given D.E is E.DE

$\int (xy)\,dx + \int$ (the term which don't contain x) $dy = c$

$$\Rightarrow \int (2x+3y-1)\,dx + \int (4y+2)\,dy = c$$

$$\Rightarrow \frac{2x^2}{2} + 3yx - x + \frac{4y^2}{2} + 2y = c$$

$$\Rightarrow x^2 + 3yx - x + 2y^2 + 2y = c$$

$$\Rightarrow x^2 + 2y^2 - x + 2y + 3yx = c$$

02. Solve $(2x+y+1)\,dx + (x+2y+1)\,dy = 0$

$$(2x+y+1)\,dx + (x+2y+1)\,dy = 0$$

$$M = (2x+y+1) \qquad N = (x+2y+1)$$

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = 1$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1$$

$\therefore$ The DE is E.D.E

$\Rightarrow \int M(x,y)\,dx + \int$ (The term which don't contain x in N) $dy = c$

$$\Rightarrow \int (2x+y+1)\,dx + \int (2y+1)\,dy = c$$

$$\Rightarrow x^2 + yx + x + y^2 + y = c$$

3. Solve $(y^3 - 3x^2y)\,dx - (x^3 - 3xy^2)\,dy = 0$

$$(y^3 - 3x^2y)\,dx + (-x^3 + 3xy^2)\,dy = 0$$

$$M = y^3 - 3x^2y \qquad N = 3xy^2 - x^3$$

$$\frac{\partial M}{\partial y} = 3y^2 - 3x^2 \qquad \frac{\partial N}{\partial x} = 3y^2 - 3x^2$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3y^2 - 3x^2$$

$\therefore$ The DE is E DE

$$\Rightarrow \int (y^3 - 3x^2y)\,dx + \int 0 = c \Rightarrow xy^3 - x^3y = c$$

$$xy[x^2 - y^2] = c$$

4. Solve $[5x^4 + 3x^2y^2 - 2xy^3]dx + [2x^3y - 3x^2y^2 - 5y^5]dy$

$[5x^4 + 3x^2y^2 - 2xy^3]\,dx + [2x^3y - 3x^2y^2 - 5y^4]\,dy$

$M = 5x^4 + 3x^2y^2 - 2xy^3 \qquad N = 2x^3y - 3x^2y^2 - 5y^4$

$\frac{\partial M}{\partial y} = 6x^2y - 6xy^2 \qquad \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 6x^2y - 6xy^2$$

$\therefore$ The given DE is EDE

$\Rightarrow \int (5x^4 + 3x^2y^2 - 2xy^3)\,dx + \int -5y^4\,dy = c$

$\Rightarrow 5\frac{x^5}{5} + \frac{3x^3}{3}y^2 - \frac{2x^2}{2}y^3 - \frac{5y^5}{5} = c$

$\Rightarrow x^2 - y^5 - x^3y^2 - x^2y^3 = c$

5. Solve $\frac{dy}{dx} + \frac{(x+3y-4)}{(3x+9y-2)} = 0$

$\frac{dy}{dx} + \frac{x+3y-4}{(3x+9y-2)} = 0$

$(x + 3y - 4)\,dx + (3x + 9y - 2)\,dy = 0$

$M = x + 3y - 4, \quad N = 3x + 9y - 2$

$\frac{\partial M}{\partial y} = 3 \qquad \frac{\partial N}{\partial x} = 3$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 3$$

$\therefore$ The given D.E is EDE

$\Rightarrow \int (x + 3y - 4)\,dx + \int (9y - 2)\,dy = c$

$\Rightarrow \frac{x^2}{2} + 3yx - 4x + \frac{9y^2}{2} - 2y = c$

$\Rightarrow$ Reductable exact Differential equation.

Step ① :- consider the differential equation

$M(x,y)\,dx + N(x,y)\,dy = 0$

Step ② :- find $\frac{\partial M}{\partial y}$ an $\frac{\partial N}{\partial x}$ and check equality

Step ③ :- If it is not equal then find $\left|\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right|$

case ① :- if $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$

$$I.F = e^{\int f(x)dx}$$

Multiply the Integration factor to the given DE and follow same procedure

Case ②:- if $\frac{1}{M}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = g(y)$

$$IF = e^{\int g(y)\,dy}$$

And proceed the same.

1. Solve $(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0$

$(4xy + 3y^2 - x)\,dx + (x^2 + 2yx)dy = 0 \rightarrow ①$

$M = 4yx + 3y^2 - x$ $\qquad N = x^2 + 2yx$

$\frac{\partial M}{\partial y} = 4x + 6y$ $\qquad \frac{\partial N}{\partial x} = 2x + 2y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\Rightarrow \frac{1}{N}\left[\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}\right] = \frac{(4x + 6y - 2x - 2y)}{x^2 + 2yx} = \frac{2x + 4y}{x^2 + 2yx} = \frac{2(x + 2y)}{x^2 + 2yx}$

$\Rightarrow \frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{2}{x} = f(x)$

$\Rightarrow IF = e^{\int f(x)\,dx} = e^{\int 2/x\,dx} = e^{2\log x} = e^{\log x^2} = x^2$

$I.F \times ① = x^2(4xy + 3y^2 - x)\,dx + x^2(x^2 + 2yx)dy = 0$

$(4x^3y + 3x^2y^2 - x^3)\,dx + (x^4 + 2yx^3)dy = 0$

$M' = 4x^3y + 3x^2y^2 - x^3$ $\qquad N' = x^4 + 2yx^3$

$\frac{\partial M'}{\partial y} = 4x^3 + 6x^2y$ $\qquad \frac{\partial N'}{\partial x} = 4x^3 + 6yx^2$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 4x^3 + 6x^2 y$$

The given solution is E DE

$$\Rightarrow \int (4x^3 y + 3x^2 y^2 - x^2)\, dx - \int 0\, dy = c$$

$$\Rightarrow \frac{4x^4}{4} y + \frac{3x^3}{3} y^2 - \frac{x^4}{4} = c$$

$$\Rightarrow x^4 y + x^3 y^2 - \frac{x^4}{4} = c$$

2. Solve $y(2x - y + 1) + x(3x - 4y + 3) dy = 0$

$$(2xy - y^2 + y)\, dx + (3x^2 - 4yx + 3x)\, dy = 0 \rightarrow (1)$$

$M = 2xy - y^2 + y \qquad N = 3x^2 - 4yx + 3x$

$\frac{\partial M}{\partial y} = 2x - 2y + 1 \qquad \frac{\partial N}{\partial x} = 6x - 4y + 3$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{6x - 4y + 3 - 2x + 2y - 1}{2xy - y^2 + y} = \frac{4x - 2y + 2}{y(2x - y + 1)}$$

$$= \frac{2(2x - y + 1)}{y(2x - y + 1)}$$

$$\Rightarrow \frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{2}{y} = g(y)$$

$$\Rightarrow I.F = e^{\int g(y)\, dy} = e^{2\int 1/y\, dy} = e^{\log y^2} = y^2$$

$$\Rightarrow I.F \times (1) :- \quad y^2(2xy - y^2 + y)\, dx + y^2(3x^2 - 4yx + 3x)\, dy = 0$$

$$(2xy^3 + y^4 + y^3)\, dx + (3x^2 y^2 - 4xy^3 + 3xy^2)\, dy = 0$$

$M' = (2xy^3 - y^4 + y^3) \qquad N' = (3x^2 y^2 - 4xy^3 + 3xy^2)$

$\frac{\partial M'}{\partial y} = 6xy^2 - 4y^3 + \qquad \frac{\partial N'}{\partial x} = 6x^2 y - 4y^3 + 3y^2$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 6xy^2 - 4y^3 + 3y^2$$

$$\Rightarrow \int (2xy^3 - y^4 + y^3)\, dx + \int 0\, dy = c \quad \Rightarrow x^2 y^3 - xy^4 + xy^3 = c$$

3. $(x^2+y^2+x)\,dx + xy\,dy = 0$

$(x^2+y^2+x)dx + xy\,dy = 0 \rightarrow (1)$

$M = x^2+y^2+x \qquad N = xy$

$\frac{\partial M}{\partial y} = 2y \qquad \frac{\partial N}{\partial y} = y$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{2y-y}{xy} = \frac{y}{xy} = \frac{1}{x} = f(x)$

$\Rightarrow$ I.F $= e^{\int f(x)dx} = e^{\int 1/x\,dx} = e^{\log x} = x$

$\Rightarrow$ I.F $\times (1) = (x^3+xy^2+x^2)dx + x^2y\,dy = 0$

$M' = x^3+xy^2+x^2 \qquad N' = x^2y$

$\frac{\partial M'}{\partial y} = 2xy \qquad \frac{\partial N'}{\partial x} = 2xy$

$$\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x} = 2xy$$

$\Rightarrow \int (x^3 + y^2x + x^2)dx + \int 0\,dy = c$

$\Rightarrow \frac{x^4}{4} + \frac{y^2x^2}{2} + \frac{x^3}{3} = c$

4. Solve $y(2xy+1)\,dx - x\,dy = 0$

$(2xy^2+y)\,dx - x\,dy = 0$

$M = 2xy^2+y \qquad N = -x$

$\frac{\partial M}{\partial y} = 4xy+1 \qquad \frac{\partial N}{\partial x} = -1$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{-1+4xy-1}{y(2xy+1)} = \frac{-2(x2y+1)}{y(2xy+1)} = -\frac{2}{y} = g(y)$

$\Rightarrow$ I.F $= e^{-2\int 1/y\,dy} = e^{-2\log y} = e^{\log 1/y^2} = 1/y^2$

$\Rightarrow$ IF $\times (1) = \left(\frac{2xy^2}{y^2} + \frac{y}{y^2}\right)dx - \left(\frac{x}{y^2}\right)dy = 0$

$$(2x + 1/y)\,dx - x/y^2\,dy = 0$$

$M' = 2x + 1/y$ $\quad\quad$ $N' = -1/y^2$

$\frac{\partial M'}{\partial y} = \frac{-1}{y^2}$ $\quad\quad$ $\frac{\partial N'}{\partial x} = -1/y^2$

$\frac{\partial M'}{\partial y} = -1/y^2$ $\quad\quad$ $\frac{\partial N'}{\partial x} = -1/y^2$

$$\Rightarrow \int (2x + 1/y)\,dx = c$$

$$\Rightarrow \frac{2x^2}{2} + \frac{x}{y} = c$$

$$\Rightarrow x^2 + \frac{x}{y} = c$$

5. Solve $y(x+y)\,dx + (x+2y-1)\,dy = 0$

$$(xy + y^2)\,dx + (x + 2y - 1)\,dy = 0 \rightarrow (1)$$

$M = xy + y^2$ $\quad\quad$ $N = x + 2y - 1$

$\frac{\partial M}{\partial y} = x + 2y$ $\quad\quad$ $\frac{\partial N}{\partial x} = 1$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$$\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{x+2y-1}{x+2y-1} = 1 = f(x)$$

$$I.F = e^{\int f(x)dx} = e^{\int dx} = e^x$$

$$\Rightarrow I.F \times (1) \Rightarrow e^x(xy + y^2)\,dx + e^x(x + 2y - 1)\,dy = 0$$

$$(xye^x + e^x y^2)\,dx + (e^x x + 2ye^x - e^x)\,dy = 0$$

$M' = xye^x + e^x y^2$ $\quad\quad$ $N' = e^x x - 2ye^x - e^x$

$\frac{\partial M'}{\partial y} = xe^x + e^x y 2$ $\quad\quad$ $\frac{\partial N'}{\partial x} = xe^x + 2ye^x$

$\therefore$ The given solution is EDE

$$\Rightarrow \int (xye^x + e^x y^2)\,dx = c$$

$$\Rightarrow y(xe^x - e^x) + y^2(e^x) = c$$

$$\Rightarrow e^x[(xy - y) + y^2] = c$$

6. Solve $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

$(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$

$M = 3x^2y^4 + 2xy$ $\quad N = 2x^3y^3 - x^2$

$\frac{\partial M}{\partial y} = 12x^2y^3 + 2x$ $\quad \frac{\partial N}{\partial x} = 6x^2y^3 - 2x$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{6x^2y^3 - 2x - 12x^2y^3 - 2x}{(3x^2y^4 + 2xy)} = \frac{2(x^2y^3 3 + 2x)}{y(3x^2y^3 + 2x)}$

$= \frac{2}{y}$ $(g)(y)$

$\Rightarrow IF = e^{\int g(y)dy} = e^{\int 2/y\, dy} = e^{\log y^2} = y^2$

① $\times I.F = y^2(3x^2y^4 + 2xy)dx + y^2(2x^3y^3 - x^2)dy = 0$

$(3x^2y^6 + 2xy^3)dx + 2x^3y^5 - x^2y^2\, dy = 0$

$M' = 3x^2y^6 + 2xy^3$ $\quad N' = 2x^3y^5 - x^2y^2$

$\frac{\partial M'}{\partial y} = 18x^2y^5 + 6xy^2$ $\quad \frac{\partial N'}{\partial x} = 6x^2y^5 - 2xy^2$

$\therefore$ The Reductable D.E is

$\Rightarrow \int(3x^2y^6 + 2xy^3)dx = c$

$\Rightarrow x^3y^6 + x^2y^3 = c$

7. Solve $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0 \rightarrow$ ①

$M = y^4 + 2y$ $\quad N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$ $\quad \frac{\partial N}{\partial x} = y^3 - 4$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{(y^3 - 4 - 4y^3 - 2)}{y^4 + 2y} = \frac{3(y^3 + 2)}{y(y^4 + 2)}$ $(\div \text{ by } y)$

$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{3[y^4 + 2y]}{y(y^4 + 2y)} = \frac{3}{y} = g(y)$

$\Rightarrow I.F = e^{\int g(y)dy} = e^{3\int 1/y\, dy} = e^{\log y^3} = y^3$

$I.F \times (1) = y^3(y^4+ay)dx + y^3[xy^3+ay^4-4x]dy = 0$

The given Reductable D.E is

$$\Rightarrow \int (y^7+ay^4)dx + \int ay^7 dy = c$$

$$\Rightarrow xy^7 + axy^4 + \frac{ay^8}{8} = c$$

$$\Rightarrow xy^7 + axy^4 + \frac{y^8}{4} = c$$

Linear differential equation of 1^{st} degree and 1^{st} degree

The general Linear differential equation of 1^{st} degree and 1^{st} order

$$\frac{dy}{dx} + P(x)y = Q(x)$$

identify P.Q and also find $I.F = e^{\int P(x)dx}$

Write the Solution $y \times I.F = \int Q(x)\, IF\, dx + c$

in other form.

The Solution of $\frac{dx}{dy} + P(y)x = Q(y)$

$$\therefore IF \times x = \int Q(y)\, IF dy + c$$

$$I.F = e^{\int P(y)dy}$$

1. Solve $\frac{dy}{dx} - \frac{y}{x} = ax^2$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = ax^2$$

$$P = -1/x \qquad Q = ax^2$$

$$I.F = \left[e^{\int P(x)dx}\right] = e^{-\int 1/x\, dx} = e^{\log(1/x)} = \frac{1}{x}$$

$$IF \times (y) = \int Q(x)\, IF\, dx + c$$

$$\Rightarrow \frac{y}{x} = \int ax^2 \frac{1}{x} dx + c$$

$$\Rightarrow \frac{y}{x} = \int ax^2 \frac{1}{x} dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x\,dx + c$$

$$\Rightarrow \frac{y}{x} = \int 2x\,dx + C$$

$$\Rightarrow \frac{y}{x} = x^2 + C$$

$$\Rightarrow y = x^2 + Cx$$

2. Solve $\frac{dy}{dx} + y\cot x = \cos x$

$$\frac{dy}{dx} + y\cot x = \cos x$$

$P = \cot x \qquad Q = \cos x$

$$\Rightarrow I.F = e^{\int P(x)\,dx} = e^{\int \cot x\,dx} = e^{\log(\sin x)} = \sin x$$

$$\Rightarrow I.F \times y = \int Q(x)\, I.F\, dx + C$$

$$\Rightarrow y\sin x = \int \cos x \sin x\, dx + c$$

$$\Rightarrow y\sin x = \frac{1}{2}\int (\sin 2x)\,dx + c$$

$$\Rightarrow y\sin x = \frac{-\cos 2x}{4} + c$$

Bernoulis differential equation

The general Solution of Bernoulis differential equation

$$\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n \rightarrow (1)$$

divide eq^n (1) by y^n

$$eq^n (1) \Rightarrow \frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{1}{y^{n-1}} = Q(x) \rightarrow (2)$$

let $\frac{1}{y^{n-1}} = u$

And differentiating w r t x we get

$$\Rightarrow (-n+1)\, y^{-n}\frac{dy}{dx} = \frac{du}{dx}$$

$$\frac{1}{y^n}\frac{dy}{dx} = \frac{1}{(1-n)}\frac{du}{dx}$$

then eq^n (2) $\Rightarrow \frac{1}{1-n}\frac{du}{dx} + P(x)u = Q(x)$

$$\frac{du}{dx} + (1-n)P(x)u = (1-n)Q(x)$$

$$\frac{du}{dx} + P'(x)u = Q'(x)$$

1. Solve $\frac{dy}{dx} + \frac{y}{x} = y^2 x$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{y}{x}\right) = xy^2$$

$$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x \longrightarrow (1)$$

let $u = 1/y \longrightarrow (2)$

$D \rightarrow x$

$$\Rightarrow \frac{du}{dx} = \frac{-1}{y^2}\left(\frac{dy}{dx}\right)$$

$$\Rightarrow -\frac{du}{dx} = \frac{1}{y^2}\left(\frac{dy}{dx}\right) \longrightarrow (a)$$

Apply in eq^n (a) in eq^n (1)

$$\Rightarrow -\frac{du}{dx} + \frac{u}{x} = x$$

$$\Rightarrow \frac{du}{dx} + \left(-\frac{1}{x}\right)u = -x$$

$$P = -1/x \quad , \quad Q = -x$$

$$\Rightarrow I.F = e^{\int P(x)dx} = e^{-\int 1/x \, dx} = e^{\log(1/x)} = 1/x$$

$$u \times IF = \int Q(x)\, I.F \, dx + c$$

$$\Rightarrow \frac{u}{x} = \int -\frac{x}{x}\, dx + c$$

$$\Rightarrow \frac{u}{x} = -\int dx + c$$

$$\Rightarrow \frac{u}{x} = -x + c$$

$$\Rightarrow u = -x^2 + cx$$

$$\Rightarrow \frac{1}{y} \mp x^2 = cx$$

2. Solve $\frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$

$$\Rightarrow \frac{dy}{dx} - y\tan x = \frac{\sin x \cos^2 x}{y^2}$$

$$\Rightarrow y^2\frac{dy}{dx} + (-\tan x)y^3 = \sin x\cos^2 x \rightarrow ①$$

let $u = y^3$

$$\frac{du}{dx} = 3y^2\frac{dy}{dx}$$

$$\frac{1}{3}\frac{du}{dx} = y^2\frac{dy}{dx} \rightarrow ⓐ$$

apply eq^n ⓐ in ①

$$\frac{1}{3}\frac{du}{dx} + (-\tan x)u = \sin x\cos^2 x$$

$$\frac{du}{dx} + (-3\tan x)u = 3\sin x\cos^2 x$$

let $P = -3\tan x$ $\quad Q = 3\sin x\cos^2 x$

$$I.F = e^{\int P(x)\,dx} = e^{-\int \tan x\, 3\,dx} = e^{-3\log(\sec x)} = e^{\log(1/\sec^3 x)}$$

$$= \frac{1}{\sec^3 x} = \cos^3 x$$

$$u \times I.F = \int Q(x)\, I.F\, dx + c$$

$$u \times \cos^3 x = \int 3\sin x\cos^2 x\cos^3 x\, dx + c$$

$$u\cos^3 x = 3\int(\sin x\cos^5 x)\,dx + c$$

let $t = \cos x$

$$-dt = +\sin x\, dx$$

$$ut^3 = -3\int t^5\,dt + c$$

$$ut^3 = -\frac{3t^6}{6} + c$$

$$y^3\cos^3 x = \frac{-\cos^6 x}{2} + c$$

3. Solve $xy(1+xy^2)\frac{dy}{dx} = 1$

$$xy(1+xy^2)\frac{dy}{dx} = 1$$

$$\frac{dx}{dy} = xy + x^2y^3$$

$$\frac{dx}{dy} - xy = x^2y^3$$

$$\frac{1}{x^2}\frac{dx}{dy} + \left(-\frac{1}{x}\right)y = y^3 \longrightarrow ①$$

Let $U = \frac{-1}{x}$

D -y

$$\frac{du}{dy} = \frac{1}{x^2}\frac{dx}{dy}$$

eqn ① $\Rightarrow \frac{du}{dy} + uy = y^3$

$Q = y^3$

$P = y$

$$I.F = e^{\int P(y)\,dy} = e^{\int y\,dy} = e^{y^2/2}$$

$$U \times IF = \int Q(y)\, IF\, dy + c$$

$$Ue^{y^2/2} = \int y^3 e^{y^2/2} dy + c$$

$$\frac{-e^{y^2/2}}{x} = \int (y^2 e^{y^2/2})(y\,dy) + c$$

$$\frac{-e^t}{x} = \int (2t\, e^t)\,dt + c$$

$$= 2\int (t-1)\,dt + c$$

$$\frac{-e^t}{x} = 2(t-1)\,e^t + c$$

$$\frac{-e^{y^2/2}}{x} = 2\left(y^2/_2 - 1\right) e^{y^2/2} + c$$

$$\frac{-e^{y^2/2}}{x} = y^2 e^{y^2/2} - 2e^{y^2/2} + c$$

4. Solve $dy + [x \sin 2y - x^3\cos^2 y]dx = 0$

$$dy + [x\sin 2y - x^3\cos^2 y]dx = 0$$

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

$$\frac{1}{\cos^2 y}\frac{dy}{dx} + \frac{x2\sin y\cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + \tan y (2x) = x^3 \rightarrow ①$$

Let $u = 2\tan y$

$D \rightarrow x$

$$\frac{du}{dx} = 2\sec^2 y \frac{dy}{dx}$$

$$\frac{1}{2}\frac{du}{dx} = \sec^2 y \frac{dy}{dx} \rightarrow ⓐ$$

Apply eq^n ⓐ in ①

$$\frac{1}{2}\frac{du}{dx} + ux = x^3$$

$P = 2x \qquad Q = 2x^3$

$$I.F = e^{\int P(x)dx} = e^{2\int x\,dx} = e^{x^2}$$

$$u \times I.F = \int Q(x)\, I.F\, dx + C$$

$$ue^{x^2} = \int e^{x^2} 2x^3\, dx + C$$

$$e^{x^2} \times 2\tan y = \int e^{x^2} x^2\, 2(x\,dx) + C$$

$$e^t \times 2\tan y = \int e^t (t)\, dt + C$$

$$= (t-1)e^t + C$$

$$2\tan y\, e^{x^2} = (x^2-1)e^{x^2} + C$$

$$2\tan y = (x^2-1) + \frac{C}{e^{x^2}}$$

5. Solve $(r\sin\theta - r^2)d\theta - \cos\theta\, dr = 0$

$$(r\sin\theta - r^2)d\theta - \cos\theta\, dr = 0$$

$$\cos\theta\, dr = (r\sin\theta - r^2)d\theta$$

$$\frac{dr}{d\theta} = \frac{r\sin\theta}{\cos\theta} - \frac{r^2}{\cos\theta}$$

$$\frac{dr}{d\theta} - r\tan\theta = \frac{-r^2}{\cos\theta}$$

$$\frac{1}{r^2}\frac{dr}{d\theta} + \left(\frac{-1}{r}\right)\tan\theta = \frac{-1}{\cos\theta} \rightarrow ①$$

let $U = -\frac{1}{r}$

$D \to \theta$

$$\frac{du}{d\theta} = \frac{1}{r^2}\frac{dr}{d\theta} \to (a)$$

$$eq^n \ (1) \Rightarrow \frac{du}{d\theta} + U\tan\theta = \frac{-1}{\cos\theta}$$

$$P = \tan\theta \qquad Q = -\frac{1}{\cos\theta}$$

$$I.F = e^{\int P(\theta)\,d\theta} = e^{\int \tan\theta\, d\theta} = e^{\log \sec\theta} = \sec\theta$$

The solution is

$$U \times I F = \int Q(\theta)\ I.F\, d\theta + c$$

$$U\sec\theta = \int -\frac{1}{\cos\theta}\sec\theta\, d\theta + c$$

$$U\sec\theta = -\int \sec^2\theta\, d\theta + c$$

$$-\frac{\sec\theta}{r} = -\tan\theta + c$$

$$\tan\theta - \frac{\sec\theta}{r} = c$$

6. Solve $\frac{dx}{dy} = \frac{x + \sqrt{xy}}{y}$ or $\frac{dy}{dx} = \frac{y}{\sqrt{xy} + x}$

$$\frac{dx}{dy} = \frac{x}{y} + \frac{\sqrt{xy}}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{xy}}{y}$$

$$\frac{dx}{dy} - \frac{x}{y} = \frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{1}{\sqrt{x}}\frac{dx}{dy} + (-\sqrt{x})\frac{1}{y} = \frac{1}{\sqrt{y}} \to (1)$$

$U = (-\sqrt{x})$

$D \to y$

$$\frac{du}{dy} = \frac{-1}{2\sqrt{x}}\frac{dx}{dy}$$

$$eq^n \text{ (1)} \Rightarrow -2\frac{du}{dy} + \frac{u}{y} = \frac{1}{\sqrt{y}}$$

$$\frac{du}{dy} + \left(\frac{-1}{2y}\right)u = \frac{-1}{2\sqrt{y}}$$

$$P = \frac{-1}{2y} \qquad Q = \frac{-1}{2\sqrt{y}}$$

$$I.F = e^{\int P(x)\,dy} = e^{-\int 1/2y\,dy} = \frac{1}{\sqrt{y}}$$

$$P = \frac{-1}{2y} \qquad Q = \frac{-1}{2\sqrt{y}}$$

$$I.F = e^{\int P(y)\,dy} = e^{-\int 1/2y\,dy} = \frac{1}{\sqrt{y}}$$

$$u.I.F = \int Q.I.F\,dy + c$$

$$\frac{-\sqrt{x}}{\sqrt{y}} = \frac{1}{2}\int \frac{1}{y}\,dy + c$$

$$\frac{\sqrt{x}}{\sqrt{y}} = \frac{1}{2}\log y + c$$

$$\frac{\sqrt{x}}{\sqrt{y}} - \frac{1}{2}\log y = c$$

7. $x^3\frac{dy}{dx} - x^2y = -y^4\cos x$

$$x^3\frac{dy}{dx} - x^2y = -y^4\cos x$$

$$\frac{dy}{dx} + \left(-\frac{1}{x}\right)y = \frac{-y^4\cos x}{x^3}$$

$$\frac{1}{y^4}\frac{dy}{dx} + \left(-\frac{1}{x}\right)\frac{1}{y^3} = -\frac{\cos x}{x^3} \rightarrow \text{(1)}$$

Put $u = \frac{1}{y^3}$

$D \rightarrow x$

$$\frac{du}{dx} = \frac{-3}{y^4}\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{du}{dx} = \frac{1}{y^4}\frac{dy}{dx}$$

$$-\frac{1}{3}\frac{du}{dx} - \left(\frac{u}{x}\right) = \frac{\cos x}{x^3}$$

$$\frac{dy}{dx} + \frac{3y}{x} = \frac{3\cos x}{x^3}$$

Let $P = 3/x$, $Q = \frac{3\cos x}{x^3}$

$$I.F = e^{\int P dx} = e^{\int 3/x \, dx} = e^{\log x^3} = x^3$$

The solution is given by

$$I.F \times U = \int Q \, IF \, dx + c$$

$$\frac{x^3}{y^3} = \int \frac{3\cos x}{x^3} x^3 dx + c$$

$$\frac{x^3}{y^3} = 3\int \cos x \, dx + c$$

$$\frac{x^3}{y^3} = 3\sin x + c$$

Orthogonal trajectory :-

Orthagonal trajectory is a curve which is $\perp$'lar to the given curve

Working procedure :-

Carterian form :

Step ① :- Consider the given $F(x, y, c) = 0$

where is the parameter

Step ② :- Construct D E which is free from the paramet -er

$f(x, y, dy/dx) = 0$

Replace $\frac{dy}{dx}$ as $-\frac{dx}{dy}$ in the above we get

$g(x, y, -dx/dy) = 0$

Solve the above and we get

$G(x, y, c') = 0$

which is O. T of Given Curve

Polar form :

consider the curve $f(r, \theta, c) = \theta$

where c is parameter

consider D.E which is free from parameter

$$f(r, \theta, dr/d\theta) = 0$$

Replace $\frac{dr}{d\theta}$ as $-\frac{r^2 d\theta}{dr}$ in the above we get

$$g\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0$$

Solve the above and get

$$G(r, \theta, c') = \theta$$

which is require O.T of a given curve

1. Find orthogonal trajectory of parameter $y^2 = 4ax$

Given $y^2 = 4ax \rightarrow (1)$

$D \rightarrow x$

$$2y \frac{dy}{dx} = 4a$$

in eq^n (2) $y^2 = 2y\left(\frac{dy}{dx}\right)x$

$$y = 2x\left(\frac{dy}{dx}\right) \rightarrow (2)$$

$$\frac{dy}{dx} = -\frac{dx}{dy}$$

$$y = 2x \frac{dx}{dy}$$

$$y\,dy = -2x\,dx$$

Integration OBS

$$\int y\,dy = -2\int x\,dx$$

$$\frac{y^2}{2} = -\frac{2x^2}{2} + c$$

$$y^2 = -2x^2 + 2c$$

$$y^2 + 2x^2 = 2c$$

2. Find the orthogonal trajectory of $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$ where λ is parameter.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1 \rightarrow (1)$$

$D \rightarrow x$

$$\frac{2x}{a^2}+\frac{2yy_1}{b^2+\lambda}=0$$

$$\frac{x}{a^2}+\frac{2yy_1}{b^2+\lambda}=0$$

$$\frac{yy_1}{b^2+\lambda}=\frac{-x}{a^2}$$

$$\frac{1}{b^2+\lambda}=\frac{-x}{a^2yy_1}$$

$$b^2-\lambda=a^2yy_1 \Rightarrow (2)$$

eqn (1) $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{\frac{-a^2yy_1}{x}}=1$

$$x^2-\frac{xy}{y_1}=a^2$$

Put $y_1=\frac{-1}{y_1}$

$$x^2-\frac{xy}{(-1/y_1)}=a^2$$

$$x^2+xyy_1=a^2$$

$$xyy_1=a^2-x^2$$

$$yy_1=\frac{a^2}{x}-x$$

$$y\frac{dy}{dx}=\frac{a^2}{x}-x$$

Integration OBS

$$\int y\,dy=\int\frac{a^2}{x}dx-\int x\,dx$$

$$\frac{y^2}{2}=a^2\log x-\frac{x^2}{2}+c$$

$$\int y\,dy = \int \frac{a^2}{x}\,dx - \int x\,dx$$

$$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c$$

$$y^2 + x^2 = 2a^2 \log x + 2c$$

3. Show that $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ is a self orthogonal where λ is a parameter.

$$\frac{x^2}{a^2+\lambda^2} + \frac{y^2}{b^2+\lambda} = 1 \rightarrow (1)$$

$$\frac{2x}{a^2+\lambda} + \frac{2yy_1}{b^2+\lambda} = 0$$

$$\frac{x}{a^2+\lambda} + \frac{yy_1}{b^2+\lambda} = 0$$

$$\Rightarrow x(b^2+\lambda) + (a^2+\lambda)(yy_1) = 0$$

$$\Rightarrow b^2x + \lambda x + a^2yy_1 + \lambda yy_1 = 0$$

$$\Rightarrow (x+yy_1)\lambda = -b^2x - a^2yy_1$$

$$\lambda = \frac{-b^2x - a^2yy_1}{x+yy_1}$$

$$a^2+\lambda = a^2 - \frac{b^2x - a^2yy_1}{x+yy_1} \quad ; \quad b^2+\lambda = b^2 - \frac{b^2x - a^2yy_1}{x+yy_1}$$

$$\Rightarrow a^2+\lambda = \frac{a^2(x+yy_1) - b^2x - a^2yy_1}{x+yy_1} \quad ; \quad \Rightarrow b^2+\lambda = \frac{b^2(x+yy_1) - b^2x + a^2yy_1}{x+yy_1}$$

$$\Rightarrow a^2+\lambda = \frac{a^2x + a^2yy_1 - b^2x - a^2yy_1}{x+yy_1} \quad ;$$

$$\Rightarrow b^2+\lambda = \frac{-(a^2-b^2)yy_1}{x+yy_1}$$

$$\Rightarrow a^2+\lambda = \frac{(a^2+b^2)x}{x+yy_1}$$

Then eq^n (1) $\Rightarrow$ $$\frac{x^2}{\frac{(a^2-b^2)x}{x+xy_1}} + \frac{y^2}{\frac{-(a^2-b^2)yy_1}{x+yy_1}} = 1$$

$$x(x+yy_1) - y(x+yy_1)/y_1 = a^2-b^2$$

$$(x - yy_1)\left(x - \frac{y}{y_1}\right) = a^2 - b^2 \rightarrow ②$$

Put $y_1 = \frac{-1}{y_1}$

$$(x - y/y_1)(x + yy_1) = a^2 - b^2 \rightarrow ③$$

The eqⁿ ② and ③ are same

∴ The eqⁿ is self orthogonal

4. Show that $y^2 = 4a(x+a)$ is self orthogonal a is a parameter.

Consider $y^2 = 4a(x+a) \rightarrow ①$

D → x

$$2y\frac{dy}{dx} = 4a$$

$$a = \frac{2yy_1}{4}$$

$$a = \frac{yy_1}{2}$$

in eqⁿ ① $y^2 = 2yy_1(x + yy_1/2)$

$$y^2 = y(2xy_1 + yy_1^2)$$

$$y = 2xy_1 + yy_1^2 \rightarrow ②$$

This is D E of given family

Replacing $y_1 = \frac{-1}{y_1}$

$$y = 2x\left(-\frac{1}{y_1}\right) + y\left(-\frac{1}{y_1}\right)^2$$

$$y = -\frac{2x}{y_1} + \frac{y}{y_1^2}$$

$$y = \frac{-2xy_1 + y}{y_1^2}$$

$$yy_1^2 + 2xy_1 + y = 0 \rightarrow ③$$

This is the D.E of orthogonal family which is same as ② being D.E of the given family

Thus the family of parabola $y^2 = 4a(x+a)$ is orthogonal

5. find the orthogonal trajectory of $r^n = a^n \cos n\theta$ where a is a parameter.

$$r^n = a^n \cos n\theta$$

$$D \to \theta$$

$$n r^{n-1} \frac{dr}{d\theta} = -a^n \sin n\theta \,(n)$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = -\frac{a^n \sin n\theta}{r^n}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{-a^n \sin n\theta}{a^n \cos n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r}\left[r^2 \frac{d\theta}{dr}\right] = \tan n\theta$$

$$-\frac{1}{r} dr = \frac{1}{\tan n\theta} d\theta$$

Integration OBS

$$\int \frac{1}{r} dr = \int \frac{1}{\tan n\theta} d\theta$$

$$\log r = \int \cot n\theta \, d\theta$$

$$\log r = \int \cot n\theta \, d\theta$$

$$\log r = \frac{\log(\sin n\theta)}{n} + \log k$$

$$n \log r^n = \log \sin n\theta + n \log k$$

$$\log r^n = \log(\sin n\theta) + \log k^n$$

$$\log r^n = \log[\sin n\theta \, k^n]$$

$$r^n = k^n \sin n\theta$$

6. Find the orthogonal trajectory of $r^n = a^n \sin n\theta$ where a is parameter.

$$r^n = a^n \sin n\theta$$

$$D \rightarrow \theta$$

$$\frac{dr}{d\theta} n r^{n-1} = a^n (\cos n\theta) n$$

$$\frac{r^n}{r} \frac{dr}{d\theta} = a^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{r^n}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{a^n \cos n\theta}{a^n \sin n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = \cot n\theta$$

Replace $\left(\frac{dr}{d\theta}\right) = -\frac{r^2 d\theta}{dr}$

$$\frac{1}{r}\left(-r^2 \frac{d\theta}{dr}\right) = \cot n\theta$$

$$\frac{1}{r}(dr) = \frac{1}{\cot n\theta} d\theta$$

Integration on B S

$$-\int \frac{1}{r} dr = -\int \tan n\theta \, d\theta$$

$$\log k - \log r = \frac{+\log(\sec \theta n)}{n}$$

$$n \log k - n \log r = \log \sec n\theta$$

$$\log k^n - \log r^n = \log \sec n\theta$$

$$\log(r^n) + \log k^n = \log(\sec n\theta)$$

$$\log\left(\frac{k^n}{r^n}\right) = \log(\sec n\theta)$$

$$r^n = k^n \frac{1}{\sec n\theta}$$

$$\Rightarrow r^n = k^n \cos n\theta$$

7. Find the orthogonal trajectory of $r^n \sin n\theta = a^n$ where a is parameter.

$$r^n \sin n\theta = a^n$$

$D \rightarrow \theta$

$$n r^{n-1} \frac{dr}{d\theta} \sin n\theta + r^n \cos n\theta (n) = 0$$

$$\frac{r^n}{r} \frac{dr}{d\theta} \sin n\theta = -r^n \cos n\theta$$

$$\frac{1}{r} \frac{dr}{d\theta} = \frac{-\cos n\theta}{\sin n\theta}$$

$$\frac{1}{r} \frac{dr}{d\theta} = -\cot n\theta$$

Replace $\frac{dr}{d\theta} = -r^2 \frac{d\theta}{dr}$

$$\frac{1}{r} \cdot \frac{-r^2 d\theta}{dr} = -\cot n\theta$$

$$r \frac{d\theta}{dr} = \cot n\theta$$

$$\frac{1}{r} dr = \frac{1}{\cot n\theta} d\theta$$

Integration o B S

$$\int \frac{1}{r} dr = \int \tan n\theta \, d\theta$$

$$\log r = \frac{\log (\sin n\theta)}{n} + \log k$$

$$n \log r = -\log (\cos n\theta) + n \log k$$

$$\log r^n + \log (\cos n\theta) = \log k^n$$

$$\log (r^n \cos n\theta) = \log k^n$$

$$r^n (\cos n\theta) = k^n$$

8. find the orthogonal trajectory of family of curve $r = 2a \cos\theta$ where 'a' is parameter.

$$r = 2a\cos\theta \rightarrow (1)$$

$D \rightarrow \theta$

$$\frac{dr}{d\theta} = -2a\sin\theta \rightarrow (2)$$

divide eq (2) and (1)

$$r\frac{d\theta}{dr} = -\cot n\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replace $\left(\frac{dr}{d\theta}\right) = -r^2\frac{d\theta}{dr}$

$$-\frac{1}{r}\, r^2\frac{d\theta}{dr} = -\tan n\theta$$

$$r\frac{d\theta}{dr} = \tan n\theta$$

$$\frac{1}{r}\,dr = \frac{1}{\tan n\theta}\,d\theta$$

Integrate OBS

$$\int\frac{1}{r}\,dr = \int\frac{1}{\tan n\theta}\,d\theta$$

$$\log r = \int \cot n\theta\, d\theta$$

$$\log r = \log \sin n\theta + \log k$$

$$\log r = \log k \sin n\theta$$

$$r = k\sin\theta$$

9. Find the orthogonal trajectory of curve $r = a(1-\cos\theta)$ where 'a' is parameter.

$$r = a(1-\cos\theta) \rightarrow (1)$$

$D \rightarrow \theta$

$$\frac{dr}{d\theta} = a\sin\theta \rightarrow (2)$$

eqn (2) by (1)

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin\theta}{1-\cos\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{2\sin\theta/2\cos\theta/2}{2\sin^2\theta/2}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan\theta/2$$

Replace $\frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}$

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \tan\theta/2$$

$$r\frac{d\theta}{dr} = -\tan\theta/2$$

$$\frac{1}{r}dr = -\frac{1}{\tan(\theta/2)}d\theta$$

Integrate OBS

$$\int\frac{1}{r}dr = -\int\cot(\theta/2)\,d\theta$$

$$\log r = -\frac{\log(\sec(\theta/2))}{(1/2)} + \log k$$

$$\log r = -2\log\sec(\theta/2) + \log k$$

$$\log r = \log\cos^2(\theta/2) + \log k$$

$$\log r = \log(k\cos^2\theta/2)$$

$$r = k\cos^2\theta/2$$

$$r = k\left(\frac{1+\cos\theta}{2}\right)$$

$$r = \frac{k}{2}[1+\cos\theta]$$

$$r = b[1+\cos\theta]$$

10. Find the orthogonal trajectory of $r^n\cos n\theta = a^n$ where 'a' is parameter.

$$r^n\cos n\theta = a^n$$

$$D \to \theta$$

$$nr^{n-1}\frac{dr}{d\theta}(\cos n\theta - r^n\sin n\theta(n)) = 0$$

$$n\frac{r^n}{r}\frac{dr}{d\theta}\cos n\theta = r^n\sin n\theta\,(n)$$

$$\frac{1}{r}\frac{dr}{d\theta}\cos n\theta = \sin n\theta$$

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\sin n\theta}{\cos n\theta}$$

$$\frac{1}{r}\frac{dr}{d\theta} = \tan n\theta$$

Replace $\frac{dr}{d\theta} = -\frac{d\theta}{dr} r^2$

$$\frac{1}{r}\left[-r^2\frac{d\theta}{dr}\right] = \tan n\theta$$

$$\frac{1}{\tan n\theta} d\theta = -\frac{1}{r} dr$$

Integration DBS

$$\int \cot n\theta \, d\theta = -\int \frac{1}{r} dr$$

$$\log(\sin n\theta)\frac{1}{n} = -\log r + \log k$$

$$\log(\sin n\theta) + n\log r = n\log k$$

$$\log(\sin n\theta \, r^n) = \log k^n$$

$$r^n \sin n\theta = k^n$$

<u>Newton's law of cooling</u>:-

let 'T' be the temperature of a body, T_0 be the temp, of surrounding medium at any time 't', by the property of the Newton's law of cooling the rate of temperture of body is directly porpostional to difference Between temperature of the and it's surrounding medium.

$$\frac{dT}{dt} \propto (T - T_0)$$

$$\frac{dT}{dt} = -k[T - T_0]$$

$$dT = -k[T - T_0]dt$$

Integration DBs

$$\int dT = -\int k[T - T_0]dt$$

$$\int \frac{1}{[T - T_0]} dT = -k\int dt$$

$$\log[T - T_0] = -kt + c$$

$$T - T_0 = e^{-Kt+c}$$

$$T = T_0 + e^{-Kt+c}$$

$$T = T_0 + e^{-Kt} e^{c}$$

$$\boxed{T = T_0 + \lambda e^{-Kt}}$$

1. a copper ball originally at 180° cool down 80°c in 20 min. 40°c what will be the ball after 40 min from the origin.

$T_A = 40°c$

$t = 0$, $T = 80°c$

$t = 20$ $T = 60°c$

Suppose T be a temperture of copper ball and T_A is the temperture of the air at any time (t)

w.k.T

$$T = T_0 + \lambda e^{-Kt} \rightarrow (1)$$

given temp/. of air $T_0 = 40°c$

$$eq^n \; (1) \Rightarrow T = 40 + \lambda e^{-Kt} \rightarrow (2)$$

Given $T = 80°c$ when $t = 0$

$$eq^n \; (2) \Rightarrow 80 = 40 + \lambda e^{-K(0)}$$

$$\Rightarrow \lambda = 40$$

$$eq \; (2) \Rightarrow T = 40 + 40 e^{-Kt} \rightarrow (3)$$

$T = 60°c$ $t = 20$

$$(3) \Rightarrow 40 + 40 e^{-Kt} = 60$$

$$20 = 40 e^{-20K}$$

$$e^{-K20} = 0.5$$

$$-20K = \log_e(0.5)$$

$$K = 0.03465$$

③ $\Rightarrow T = 40 + 40e^{(-0.0346)t} \rightarrow$ ④

when $t = 40$ min

eq ④ $\Rightarrow T = 40 + 40e^{(-0.0346)40}$

$T = 40 + 40e^{-1.384}$

$T = 50°c$

2. The temperature of air is 30°C a metal ball cools from 100°c to 70°c in 15min find How many long will it taken for metal to reach the temperature of 40°c

$\Rightarrow$ Suppose T be the temperature of metal ball T_0 is temp/. of air at time 't'

W.K.T $T = T_0 + \lambda e^{-kt} \rightarrow$ ①

given that $T_0 = 30°c$

$T = 30 + \lambda e^{-kt} \rightarrow$ ②

Given $T = 100°c$ at $t = 0$

② $\Rightarrow 100 = 30 + \lambda e^{-kt}$

$\lambda = 70$

② $\Rightarrow T = 30 + 70e^{-kt} \rightarrow$ ③

$T = 70°c$, $t = 15$ min

③ $\Rightarrow 40 = 30 + 70e^{-(0.0373t)}$

$\frac{10}{70} = e^{-0.0373t}$

$-0.0373t = \ln(0.1428)$

$-0.0373t = -1.9463$

$t = 52$ mic 18 ec

3. A body is of heated 110°c and placed in air at 10°c after 1 hour, it's temp. becomes 60°c how much additional time is required to cool.

$\Rightarrow$ Suppose T be the temp/. of the body, T_0 is temp/. of air

$$T = T_0 + \lambda e^{-Kt} \rightarrow (1)$$

$$T_0 = 10^\circ$$

$$(1) \Rightarrow T = 10 + \lambda e^{-Kt} \rightarrow (2)$$

$T = 110^\circ C$ at $t = 0$

$$(2) \Rightarrow 110 = 10 + \lambda e^{-0(K)}$$

$$\lambda = 100$$

$$T = 10 + 110e^{-60K} \rightarrow (3)$$

Given, $T = 60^\circ C$, $t = 60^\circ$

$$(3) \Rightarrow 60 = 10 + 110e^{-60K}$$

$$0.5 = e^{-60K}$$

$$-60K = -0.69314$$

$$K = 0.01155$$

$$(3) \Rightarrow T = 10 + 100e^{-(0.01155t)} \rightarrow (4)$$

also given $T = 30$ $t = ?$

$$30 = 10 + 100e^{-(0.01155)t}$$

$$\ln(0.2) = -0.01155t$$

$$t = 139.345$$

t = 2 hours 19 min 34 Sec

4. A bottle of mineral water at a room temp. 72F is kept in refrigitor where the temp. is 44F after half of an hour, what are cooled 61F, what is the temp. of mineral water another half an hour.

$\Rightarrow$ From newton's law

$$T = T_0 + \lambda e^{-Kt} \rightarrow (1)$$

$$T_0 = 44F$$

$$(1) \Rightarrow T = 44 + \lambda e^{-Kt} \rightarrow (2)$$

$T = 72F$, when $t = 0$

$$72 = 44 + \lambda e^{-0K}$$

$$\lambda = 28$$

$$② \Rightarrow T = 44 + 28 e^{-Kt} \rightarrow ③$$

$$T = 61F \quad , t = 30$$

$$③ \Rightarrow 61 = 44 + 28 e^{-30K}$$

$$K = 0.01663$$

$$③ \Rightarrow T = 44 + 28 e^{-0.01663(60)}$$

$$T = 44 + 10.323$$

$$T = 54.32^{\circ}F$$

5. A body in at $25^{\circ}C$ whose from $100^{\circ}C$ to $75^{\circ}C$ in one minute. find the temperture of a body at 3 minutes.

$\Rightarrow$ From Newton's law

$$T = T_0 + \lambda e^{-Kt} \rightarrow ①$$

Given that air temperture $T_0 = 25$

$$① \Rightarrow T = 25 + \lambda e^{-Kt} \rightarrow ②$$

They given that $T = 100^{\circ}C$ at $t = 0$

$$② \Rightarrow 100 = 25 + \lambda e^{-K(0)}$$

$$\lambda = 75$$

$$② \Rightarrow T = 25 + 75 e^{-Kt} \rightarrow ③$$

$T = 75F$ at $t = 1$

$$③ \Rightarrow 75 e^{-K} + 25 = 75$$

$$e^{-K} = 0.666$$

$$K = 0.406$$

$$③ \Rightarrow T = 25 + 75 e^{-0.406t} \rightarrow ④$$

given $T = ?$ $t = 3$

$$T = 25 + 75 e^{(-0.406 \times 3)}$$

$$T = 25 + 22.186$$

$$T = 47.186 C^{\circ}$$

Flow of electricity :- [L-R circuits]

The electrical circuit may have 3 passive elements they are resistance (R), inductance (L), capacitance (C) and active element be voltage source with a emf source with (E) to a current (I) at any time 't'.

①. A series circuit with resistance (R), inductance (L) and emf (E) governed by D.E. $L\frac{di}{dt} + Ri = E$, where L, R constant and intial, the current is zero. find the current at any time (t).

⇒ Given that resistance b/w current, resistance, and inductance, E is $L\frac{di}{dt} + Ri = E$, R, L constant

$$L\frac{di}{dt} + Ri = E \rightarrow ①$$

$$\frac{di}{dt} + \left(\frac{R}{L}\right)i = \frac{E}{L} \rightarrow ②$$

$$P = R/L \qquad Q = E/L$$

$$I.F = e^{\int P dt} = e^{\int R/L\, dt} = e^{(R/L)t}$$

The Solution is given by $i \times I.F = \int Q\, IF\, dt + c$

$$i \times e^{(R/L)t} = \int \frac{E}{L} e^{(R/L)t} dt + c$$

$$i \times e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{R/L} + K$$

$$i \times e^{(R/L)t} = E \frac{e^{(R/L)t}}{R} + K \rightarrow ③$$

$$i(t) = \frac{E}{R} + Ke^{-R/L\, t} \rightarrow ④$$

Given $i = 0$, $t = 0$

$$0 = \frac{E}{R} + Ke^{-0}$$

$$K = E/R \rightarrow ⓐ$$

Substitute eq^n ③ in ④

$$i(t) = \frac{E}{R} - \frac{E}{R}e^{-(R/L)t}$$

$$i(t) = \frac{E}{R}\left[1 - e^{-(R/L)t}\right]$$

2. An Inductance 2H and resistance 20Ω are Connected to a Series emf 'E' holds if the current is intial zero, when t = 0 find the current at end 0.01 sec if E = 100 volts ?

W.K.T the D.E of a L-R Circuit is

$$L\frac{di}{dt} + Ri = E \rightarrow ①$$

Gives the Current at any time 't' is

$$i(t) = \frac{E}{R}\left[1 - e^{-(R/L)t}\right] \longrightarrow ②$$

Given E = 100V, L = 2H, R = 20Ω to find Current t = 0.01 sec

$$i_{(0.01)} = \frac{100}{20}\left[1 - e^{-(20/2)(0.01)}\right]$$

$$= 5\left[1 - e^{-0.1}\right]$$

$$= 5\left[1 - 0.9048\right]$$

$$i_{(0.01)} = 5\left[0.09516\right]$$

$$i_{(0.01)} = 0.4758 \text{ Amp,}$$

3. L-R Series Circuit D.E acted on by an emf 'E' Sinωt, Satisfy if there is no current in the circuit intial, after the value of current any time (t).

⇒ The given D.E of L-R Circuit

$$L\frac{di}{dt} + Ri = E\sin\omega t$$

$$\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}\sin\omega t$$

$$P = R/L \quad , \quad Q = E/L \sin\omega t$$

$$I.F = e^{\int P dt} = e^{\int R/L\, dt} = e^{(R/L)t}$$

$$i \times IF = \int Q\, I.F\, dt + K$$

$$i e^{(R/L)t} = \int \frac{E}{L} \sin\omega t \; e^{(R/L)t} dt + K$$

$$i e^{(R/L)t} = \frac{E}{L} \int e^{(R/L)t} \sin\omega t \, dt + K$$

$$i e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{\sqrt{\frac{R^2}{L^2} + \omega^2}} \sin[\omega t - \tan^{-1}(\omega/R_L)] + K$$

$$i e^{(R/L)t} = \frac{E}{L} \frac{e^{(R/L)t}}{\sqrt{R^2 + \omega^2 + L^2}} \sin[\omega t - \tan^{-1}(\omega L/R)] + K$$

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin[\omega t - \tan^{-1}(\omega L/R)] + K e^{(R/L)t} \rightarrow ①$$

w.k.T , $t = 0$, $i = 0$

$$0 = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(\omega(0) - \tan^{-1}(\omega L/R)] + K e^{(R/L)0}$$

$$\frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin(-\phi) + K = 0 \qquad \therefore -\phi = -\tan^{-1}(\omega L/R)$$

$$K = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} (\sin\phi) \rightarrow ⓐ$$

Substitute eqn ⓐ in ①

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin[\omega t - \phi] + \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin\phi \; e^{(R/L)t}$$

4. Solve the D.E $L\frac{di}{dt} + Ri = 200(\sin(300t))$ when $L = 0.05$ and $R = 100\,\Omega$. find the value of current (I) at any time 't', intial no current in circuit what values does apporach after long time.

$\Rightarrow$ W.K.T

for D.E $L\frac{di}{dt} + Ri = 200 \sin(300)t$

The value of current at any time (t) in initial current is zero.

$$i(t) = \frac{E}{\sqrt{R^2+\omega^2L^2}} \sin(\omega t - \phi) + e^{(R/L)t} \sin\phi \rightarrow (1)$$

by comparing given eq^n with general

$R = 100$, $L = 0.05$, $E = 200$, $\omega = 300$

$$\sqrt{R^2+\omega^2L^2} = \sqrt{(100)^2 + (300^2)(0.05)^2} = 101.1187$$

$$\phi = \tan^{-1}(\omega L/R) = \tan^{-1}\left(\frac{300 \times 0.05}{100}\right) = 8.5307$$

$$(1) \Rightarrow i(t) = \frac{200}{101.1187} \sin(300t - 8.5307) + e^{(100/0.05)t} \sin(8.5307)$$

$$\Rightarrow i(t) = 1.9778 \sin(300t - 8.5307) + e^{2000t}(0.14833)$$

Non Linear differential equation :-

For $P = \frac{dy}{dx}$ the polynomial

$$A_n p^n + A_1 p^{n-1} + A_2 p^{n-2} + A_3 p^{n-3} + \dots + A_n p^{n-n} = 0 \rightarrow (1)$$

is called Linear D.E

For which $A_0, A_1, A_2, A_3 \dots A_n$ are the f^n of x and y then the solution of eq^n can be done by the method of Solvable P, as fallow.

$(1) \Rightarrow [P - f_1(x,y)][P - f_2(x,y)], [P - f_3(x,y)] \dots [P - f_n(x,y)] = 0$

$\Rightarrow P - f_1(x,y) = 0 \dots P - f_2(x,y) = 0 \dots [P - f_n(x,y)] = 0$

$\Rightarrow P = f_1(x,y) \dots P = f_2(x,y) \dots P = f_n(x,y)$

$\Rightarrow f_1(x,y,c) = 0, \dots F_2(x,y,c_2) = 0 \dots F_n(x,y,C_n) = 0$

$\therefore F_1(x, y, c_1), F_2(x, y, c_2) \ldots \ldots F_n(x, y, c_n) = 0$

1. $\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0$

$$\left(\frac{dy}{dx}\right)^2 - 7\left(\frac{dy}{dx}\right) + 12 = 0$$

Let $P = \frac{dy}{dx}$

$$P^2 - 7P + 12 = 0$$

$$P(P-4) - 3(P-4) = 0$$

$$P - 4 = 0 \quad , \quad P - 3 = 0$$

$$P = 4 \quad , \quad P = 3$$

$$\frac{dy}{dx} = 4 \quad , \quad \frac{dy}{dx} = 3$$

$$dy = 4dx \qquad dy = 3dx$$

Integrate OBS

$$\int dy = \int 4dx \qquad \int dy = \int 3dx$$

$$y = 4x + c_1 \qquad y = 3x + c_2$$

$$y = 4x - c_1 = 0 \qquad y = 3x - c_2 = 0$$

$\therefore$ The Solution is

$$(y - 4x - c_1)(y - 3x - c_2) = 0$$

2. Solve $y\left(\frac{dy}{dx}\right)^2 + (x-y)\left(\frac{dy}{dx}\right) - x = 0$

$$y\left(\frac{dy}{dx}\right)^2 + (x-y)\left(\frac{dy}{dx}\right) - x = 0 \rightarrow (1)$$

Let $P = \frac{dy}{dx}$

in eq^n (1) $yP^2 + (x-y)P - x = 0$

$$yp^2 + xp - yp - x = 0$$

$$yp^2 - yp + xp - x = 0$$

$$yp(p-1) + x(p-1) = 0$$

Case ① :-

$p - 1 = 0$

$\Rightarrow p = 1$

$\Rightarrow \frac{dy}{dx} = 1$

$\Rightarrow \frac{dy}{dx} = 1$

$dy = dx$

$\Rightarrow \int dy = \int dx$

$\Rightarrow y - x + c_1 = 0$

Case ② :-

$yp + x = 0$

$\Rightarrow yp = -x$

$\Rightarrow y\frac{dy}{dx} = -x$

$\Rightarrow \int y\, dy = -\int x\, dx$

$\Rightarrow \frac{y^2}{2} = \frac{-x^2}{2} + C_2$

$\Rightarrow \frac{1}{2}[y^2 + x^2] = C_2$

$y^2 + x^2 - 2C_2 = 0$

$\therefore$ The Solution is given by

$$(y - x - c_1)(y^2 + x^2 - 2c_2) = 0$$

3. Solve $xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\frac{dy}{dx} + xy = 0$

$$xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2)\left(\frac{dy}{dx}\right) + xy = 0 \rightarrow ①$$

Let $p = \frac{dy}{dx}$

eqⁿ ① $\Rightarrow xy(p^2) - (x^2 + y^2)(p) + xy = 0$

$$xp[yp - x] - y[yp - x] = 0$$

$$(xp - y)(yp - x) = 0$$

Case ① :-

$px - y = 0$

$px = y$

$x\frac{dx}{dy} = y$

$\int \frac{1}{x}dx = \int \frac{1}{y}dy$

$\log x = \log y + \log c_1$

$\log x = \log(y_1 c_1)$

$x = yc_1$

Case ② :-

$yp - x = 0$

$yp = x$

$y\frac{dy}{dx} = x$

$\int y\, dy = \int x\, dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c_2$

$y^2 - x^2 - 2c_2 = 0$

$\therefore$ The Solution is $(x - yc_1)(y^2 - x^2 - 2c_2) = 0$

4. Solve $\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x}$

$$\frac{dy}{dx} - \frac{dx}{dy} = \frac{x}{y} - \frac{y}{x} \rightarrow (1)$$

Let $P = \frac{dy}{dx}$ $\quad \frac{dx}{dy} = \frac{1}{P}$

$(1) \Rightarrow$ $P - \frac{1}{P} = \frac{x}{y} - \frac{y}{x}$

$$\frac{P^2-1}{P} = \frac{x^2-y^2}{xy}$$

$$(P^2-1)xy = P(x^2-y^2)$$

$$yxp^2 - xy = px^2 - py^2$$

$$xy[p^2-1] = p[x^2-y^2] = 0$$

$$xyp^2 - xy - x^2p + y^2p = 0$$

$$y[yp-x] + xp[yp-x] = 0$$

$$(xp+y)(yp-x) = 0$$

$xp+y = 0$

$xp = -y$

$p = -\frac{y}{x}$

$\frac{dy}{dx} = -\frac{y}{x}$

$\int \frac{1}{y}\,dy = -\int \frac{1}{x}\,dx$

$\log y + \log x = \log c$

$\log\left(\frac{y}{x}\right) = \log c_1$

$\frac{y}{x} = c_1$

$y = xc_1$

$y - c_1x = 0$

$yp - x = 0$

$yp = x$

$p = \frac{x}{y}$

$\frac{dy}{dx} = \frac{x}{y}$

$\int y\,dy = \int x\,dx$

$\frac{y^2}{2} = \frac{x^2}{2} + c_2$

$y^2 - x^2 = 2c_2$

$y^2 - x^2 - 2c_2 = 0$

$\therefore$ The Solution is $(y - c_1x)(y^2 - x^2 - 2c_2) = 0$

5. Solve $p^2 + 2py\cot x = y^2$

$$p^2 + 2py\cot x - y^2 = 0 \rightarrow (1)$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4(-y^2)}}{2(1)}$$

$$p = \frac{-2y\cot x \pm 2y\sqrt{\cot^2 x + 1}}{2}$$

$$p = -y\cot x \pm y\,\mathrm{cosec}\,x$$

Case (1) :- $p = -y\cot x + y\,\mathrm{cosec}\,x$

$$\frac{dy}{dx} = -y[\mathrm{cosec}\,x - \cot x]dx$$

$$\int \frac{1}{y}dy = \int \mathrm{cosec}\,x - \cot x\,dx$$

$$\log y = \log\tan(x/2) - \log\sin x + \log c_1$$

$$= \log\left[c_1\frac{\tan(x/2)}{\sin x}\right]$$

$$= \log\left[c_1\frac{\sin(x/2)/\cos(x/2)}{2\sin(x/2)\cos(x/2)}\right]$$

$$= \log\left[\frac{c_1}{2\cos^2(x/2)}\right]$$

$$= \log\left[\frac{c_1}{1+\cos x}\right]$$

$$y = \frac{c_1}{(1+\cos x)}$$

$$y(1+\cos x) - c_1 = 0$$

Case (2) :- $p = -y\cot x - y$

$$p = -y\cot x - y\,\mathrm{cosec}\,x$$

$$p = -y[\cot x + \mathrm{cosec}\,x]$$

$$\frac{dy}{dx} = -y[\cot x + \mathrm{cosec}\,x]$$

$$-\int \frac{1}{y}dy = \int(\cot x + \mathrm{cosec}\,x)dx$$

$$-\int \frac{1}{y}\,dy = \int (\cot x + \operatorname{cosec} x)\,dx$$

$$-\log y = \log(\sin x) + \log \tan(x/2) + \log C_2$$

$$\log(1/y) = \log[\sin x \cdot \tan x/2 \; C_2]$$

$$\Rightarrow y(1-\cos x) - C_2 = 0$$

$\therefore$ The Solution is $(y(1+\cos x)-C_1)(y(1-\cos x)-C_2 = 0$

$\Rightarrow$ Clairaut's equation :-

The D.E of the form $y = P(x) + f(x)$ is said to be Clairaut's form and it's Solution is obtained by replacing $P \to C$ the Solution becomes, $y = f(x) + f(x)$ is Called general Solution of Clairaut's and D.E differenti-ate partially w r t c, we get function interms of c and again replace or Substitute in given solution it is called Singular Solution.

1. Solve the clairaut's equation $y = px + \frac{a}{p}$

given $y = px + \frac{a}{p} \to (1)$

which is in clairaut's form $y = P(x) + F(x)$

$\therefore$ The Solution is $y = Cx + \frac{a}{c} \to (2)$

$$D \to c$$

$$x - \frac{a}{c^2} = 0$$

$$x = \frac{a}{c^2}$$

$$c^2 = \frac{a}{x}$$

$$c = \sqrt{\frac{a}{x}}$$

$\therefore$ The Sigular Solution is

$$(2) \Rightarrow y = x\sqrt{\frac{a}{x}} + \frac{a}{\frac{\sqrt{a}}{\sqrt{x}}}$$

$\Rightarrow y = \sqrt{x}\sqrt{a} + \sqrt{x}\sqrt{a} \Rightarrow y = 2\sqrt{ax} \Rightarrow y^2 = 4ax$

2. Show that equation $xp^2 + px - py - y + 1 = 0$ where use Clairaut's equation.

$$xp^2 + px - py + 1 - y = 0$$

$$xp(p+1) - y(p+1) = -1$$

$$(p+1)(xp - y) = -1$$

$$px - y = \frac{-1}{(p+1)}$$

$$-y = -px - \frac{1}{p+1}$$

$$y = px + \frac{1}{p+1} \rightarrow (1)$$

eqn (1) is Clairaut's equation

$$y = cx + \frac{1}{c+1}$$

$$D \rightarrow x$$

$$0 = x - \frac{1}{(c+1)^2}$$

$$\frac{1}{(c+1)^2} = x$$

$$c+1 = \frac{1}{\sqrt{x}}$$

$\therefore$ The solution of the following

$$y = \left(\frac{1}{\sqrt{x}} - 1\right) x + \frac{1}{\frac{1}{\sqrt{x}} - 1 + 1}$$

$$y = \sqrt{x} - x + \sqrt{x}$$

$$y = 2\sqrt{x} - x$$

3. Show that the equation $xp^3 - yp^2 + 1 = 0$

$$xp^3 - yp^2 + 1 = 0$$

$$p^2 [xp - y] = -1$$

$$px - y = \frac{-1}{p^2}$$

$$-y = \frac{-1}{p^2} + px \rightarrow (1)$$

(1) is a clairaut's equation

$$y = cx + \frac{1}{c^2}$$

Partial D → c

$$0 = x - \frac{2}{c^3}$$

$$\frac{2}{c^3} = x$$

$$c^3 = \frac{2}{x}$$

$$c = \sqrt[3]{\frac{2}{x}}$$

The Solution is

$$y = \sqrt[3]{2/x} + (x/2)^{2/3}$$

4. $(px - y)(py + x) = 2p$ is reduced to clairaut's form taking $X = x^2 \quad Y = y^2$

⇒ Given that $(px - y)(py + x) = 2p \rightarrow$ (1)

$$X = x^2 \qquad Y = y^2$$

$$\frac{dX}{dx} = 2x \qquad \frac{dY}{dy} = 2y$$

w.k.T

$$p = \frac{dy}{dx}$$

$$p = \frac{dy}{dY} \frac{dY}{dX} \frac{dX}{dx}$$

$$p = \frac{1}{2y} P 2x$$

$$p = \frac{x}{y} P$$

$$p = \frac{\sqrt{X}}{\sqrt{Y}} P$$

(1) ⇒ $\left[\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X} - \sqrt{Y}\right]\left[\frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{Y} + \sqrt{X}\right] = 2\frac{\sqrt{X}}{\sqrt{Y}} P$

$$\left[\frac{px-y}{\sqrt{y}}\right]\left[(p+1)\right]\sqrt{x} = \frac{2\sqrt{x}p}{\sqrt{y}}$$

$$(px-y)(p+1) = 2p$$

$$px-y = \frac{2p}{p+1}$$

$$y = px + \left(\frac{-2p}{p+1}\right)$$

The reduced eq^n is

$$y = cx + \left(\frac{-2c}{c+1}\right)$$

5. Solve $e^{4x}(p-1) + e^{2y}p^2 = 0$ by using Subsititution $u = 2e^{2x}$ and $v = e^{2y}$

$$\Rightarrow e^{4x}(p-1) + e^{2y}p^2 = 0$$

$$u = e^{2x}, \quad v = e^{2y}$$

$$\frac{du}{dx} = 2e^{2x} \qquad \frac{dv}{dy} = 2e^{2y}$$

w.k.T

$$p = \frac{dy}{dx}$$

$$p = \frac{dy}{dy}\ \frac{dv}{du}\ \frac{du}{dx}$$

$$p = \frac{1}{2e^{2y}}\ P\ 2e^{2x}$$

$$p = \frac{e^{2x}}{e^{2y}}\ P$$

$$p = \frac{u}{v}\ P$$

$$① \Rightarrow e^{4x}\left(\frac{u}{v}P - 1\right) + e^{2y}\left(\frac{u^2P^2}{v^2}\right) = 0$$

$$u^2\left(\frac{Pv - v}{v}\right) + v\left(\frac{u^2}{v^2}\right)P^2 = 0$$

$$Pu - v = -P^2$$

$$v = P^2 + Pu$$

The resultant equation is

$$V = UC + C^2$$

$$e^{2y} = Ce^{2x} + c^2$$

6. Find the general and Singular Solution $x^2[y-px] = p^2y$ by taking into clairaut's form using Substitution $X = x^2$, $Y = y^2$

$$\Rightarrow x^2[y - px] = p^2y \rightarrow ①$$

$$X = x^2 \qquad Y = y^2$$

$$\frac{dX}{dx} = 2x \qquad \frac{dY}{dy} = 2y$$

$$p = \frac{dy}{dx}$$

$$p = \frac{dy}{dY}\ \frac{dY}{dX}\ \frac{dX}{dx}$$

$$p = \frac{1}{2y}\ P\ 2x$$

$$p = \frac{\sqrt{X}}{\sqrt{Y}}\ P$$

$$① \Rightarrow X\left[\sqrt{Y} - \frac{\sqrt{X}}{\sqrt{Y}} P\sqrt{X}\right] = \frac{X}{Y} P^2\sqrt{Y}$$

$$X\left[\frac{Y - XP}{\sqrt{Y}}\right] = \frac{XP^2}{\sqrt{Y}}$$

$$Y - PX = P^2$$

∴ The Solution is in Clairat's Solution

$$Y^2 = CX^2 + C^2 \qquad Y = CX + C^2 \rightarrow ②$$

$$D \rightarrow C$$

$$0 = X^2 + 2C$$

$$2C = x^2$$

$$C = x^2/2$$

∴ The Singular Solution is

$$y^2 = \left(-\frac{x^2}{2}\right) x^2 + \left(\frac{x^2}{2}\right)^2$$

$$y^2 = \frac{-2x^4 + x^4}{4}$$

$$4y^2 = -x^4$$

$$4y^2 + x^4 = 0$$

7. Solve the $y^2(y - px) = x^4p^2$ by reducing it is in Clariat's form, taking the substitution $X = \frac{1}{x}$, $Y = \frac{1}{y}$

$\Rightarrow$ Given,

$$y^2(y - px) = x^4p^2 \rightarrow (1)$$

$$X = \frac{1}{x}, \quad Y = \frac{1}{y}$$

$$\Rightarrow \frac{dX}{dx} = -\frac{1}{x^2}, \quad \frac{dY}{dy} = -\frac{1}{y^2}$$

$$\frac{dy}{dx} = \frac{dy}{dx} \frac{dY}{dx} \frac{dX}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{dy}{\left(\frac{dY}{dy}\right)} \left(\frac{dY}{dX}\right)\left(\frac{dX}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{(-1/y^2)} P \cdot \left(-\frac{1}{x^2}\right)$$

$$p = \frac{Py^2}{x^2}$$

$$\Rightarrow p = \frac{P\, 1/Y^2}{1/X^2}$$

$$\Rightarrow p = \frac{X^2}{Y^2} P$$

$$(1) \Rightarrow \frac{1}{Y^2}\left[\frac{1}{Y} - \frac{X^2}{Y^2} P \frac{1}{X}\right] = \frac{1}{X^4} \frac{X^4}{Y^4} P^2$$

$$\Rightarrow \frac{1}{y^2}\left[\frac{1}{x} - \frac{px}{y^2}\right] = \frac{p^2}{y^4}$$

$$\Rightarrow \frac{[y-px]}{y^4} = \frac{p^2}{y^4}$$

$$\Rightarrow y - px = p^2$$

$$\Rightarrow px + p^2 = y \longrightarrow (2)$$

$\therefore$ (2) is clariat's equation

$\therefore$ The Solution is,

$$y = cx + c^2$$

$$\frac{1}{y} = \frac{c}{x} + c^2 \,//$$