Maharaja Education Trust (R), Mysuru

Maharaja Institute of Technology Mysore

Belawadi, Sriranga Pattana Taluk, Mandya – 571 477

Approved by AICTE, New Delhi,

Affiliated to VTU, Belagavi & Recognized by Government of Karnataka

Lecture Notes on

ADDITIONAL MATHEMATICS – I (18MATDIP31)

Prepared by

Department of Mathematics

Vision/ ಆಶಯ

"To be recognized as a premier technical and management institution promoting extensive education fostering research, innovation and entrepreneurial attitude"

ಸಂಶೋಧನೆ, ಆವಿಷ್ಕಾರ ಹಾಗೂ ಉದ್ಯಮಶೀಲತೆಯನ್ನು ಉತ್ತೇಜಿಸುವ ಅಗ್ರಮಾನ್ಯ ತಾಂತ್ರಿಕ ಮತ್ತು ಆಡಳಿತ ವಿಜ್ಞಾನ ಶಿಕ್ಷಣ ಕೇಂದ್ರವಾಗಿ ಗುರುತಿಸಿಕೊಳ್ಳುವುದು.

Mission/ ಧ್ಯೇಯ

- To empower students with indispensable knowledge through dedicated teaching and collaborative learning.

  ಸಮರ್ಪಣಾ ಮನೋಭಾವದ ಬೋಧನೆ ಹಾಗೂ ಸಹಭಾಗಿತ್ವದ ಕಲಿಕಾಕ್ರಮಗಳಿಂದ ವಿದ್ಯಾರ್ಥಿಗಳನ್ನು ಅತ್ಯುತ್ಕೃಷ್ಟ ಜ್ಞಾನಸಂಪನ್ನರಾಗಿಸುವುದು.

- To advance extensive research in science, engineering and management disciplines.

  ವೈಜ್ಞಾನಿಕ, ತಾಂತ್ರಿಕ ಹಾಗೂ ಆಡಳಿತ ವಿಜ್ಞಾನ ವಿಭಾಗಗಳಲ್ಲಿ ವಿಸ್ತೃತ ಸಂಶೋಧನೆಗಳೊಡನೆ ಬೆಳವಣಿಗೆ ಹೊಂದುವುದು.

- To facilitate entrepreneurial skills through effective institute - industry collaboration and interaction with alumni.

  ಉದ್ಯಮ ಕ್ಷೇತಗಳೊಡನೆ ಸಹಯೋಗ, ಸಂಸ್ಥೆಯ ಹಿರಿಯ ವಿದ್ಯಾರ್ಥಿಗಳೊಂದಿಗೆ ನಿರಂತರ ಸಂವಹನಗಳಿಂದ ವಿದ್ಯಾರ್ಥಿಗಳಿಗೆ ಉದ್ಯಮಶೀಲತೆಯ ಕೌಶಲ್ಯ ಪಡೆಯಲು ನೆರವಾಗುವುದು.

- To instill the need to uphold ethics in every aspect.

  ಜೀವನದಲ್ಲಿ ನೈತಿಕ ಮೌಲ್ಯಗಳನ್ನು ಅಳವಡಿಸಿಕೊಳ್ಳುವುದರ ಮಹತ್ವದ ಕುರಿತು ಅರಿವು ಮೂಡಿಸುವುದು.

- To mould holistic individuals capable of contributing to the advancement of the society.

  ಸಮಾಜದ ಬೆಳವಣಿಗೆಗೆ ಗಣನೀಯ ಕೊಡುಗೆ ನೀಡಬಲ್ಲ ಪರಿಪೂರ್ಣ ವ್ಯಕ್ತಿತ್ವವುಳ್ಳ ಸಮರ್ಥ ನಾಗರೀಕರನ್ನು ರೂಪಿಸುವುದು.

Maharaja Institute of Technology Mysore

Department of Mathematics

Vision

To promote a comprehensive, innovative and dynamic learning and research environment.

Mission

- **To encourage the students to develop reasoning ability, analytical skills and an awareness of logic and also to inculcate research culture in them.**

- **To provide an exemplary Mathematics program to prepare students not only for competence in their academics/professions but also for life long learning.**

Program Outcomes

1. **Engineering knowledge**: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.
2. **Problem analysis**: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.
3. **Design/development of solutions**: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
4. **Conduct investigations of complex problems**: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.
5. **Modern tool usage**: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.
6. **The engineer and society**: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.
7. **Environment and sustainability**: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for sustainable development.
8. **Ethics**: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.
9. **Individual and team work**: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.
10. **Communication**: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
11. **Project management and finance**: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one's own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.
12. **Life-long learning**: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.

Maharaja Institute of Technology Mysore

Department Of Mathematics

Subject: ADDITIONAL MATHEMATICS – I **Subject Code:** 18DIPMATAT31

Course Overview

To familiarize the important tools of complex trigonometry , Differential Calculus and Ordinary differential equations ODE's required to analyze the engineering problems and apply the knowledge of interpolation/extrapolation and Integral Calculus technique whenever analytical methods fail or very complicated, to offer solutions.

SEE Question paper pattern:

The SEE question paper will be set for 100 marks and the marks scored will be proportionately reduced to 60.

- The question paper will have ten full questions carrying equal marks.
- Each full question carries 20 marks.
- There will be two full questions (with a maximum of four sub questions) from each module.
  Each full question will have sub questions covering all the topics under a module.
- The students will have to answer five full questions, selecting one full question from each module.

IA Question paper pattern:

- The question paper will have four questions.
- Each full Question consisting of 15 marks.
- There will be 2 full questions (with a maximum of three sub questions) from each module.
- Each full question will have sub questions covering all the topics under a module.
  The students will have to answer any 2 full questions, selecting one full question from each module

Course Objectives

- To provide basic concepts of complex trigonometry, vector algebra, differential and integral calculus.
- To provide an insight into vector differentiation and first order ODE's.

Maharaja Institute of Technology Mysore

Department Of Mathematics

Subject: ADDITIONAL MATHEMATICS – I **Subject Code:** 18DIPMATAT31

Course Outcomes

CO's	DESCRIPTION OF THE OUTCOMES
18DIPMAT31.1	Apply concepts of complex numbers and vector algebra to analyze the problems arising in related area
18DIPMAT31.2	Use derivatives and partial derivatives to calculate rate of change of multivariate functions.
18DIPMAT31.3	Analyze position, velocity and acceleration in two and three dimensions of vector valued functions.
18DIPMAT31.4	Learn techniques of integration including the evaluation of double and triple integrals.
18DIPMAT31.5	Identify and solve first order ordinary differential equations.

CO/PO	PO 1	PO 2	PO 3	PO 4	PO 5	PO 6	PO 7	PO 8	PO 9	PO 10	PO 11	PO 12
18DIPMAT31.1	3	-										
18DIPMAT31.2	-	3										
18DIPMAT31.3	3	-										
18DIPMAT31.4		3										
18DIPMAT31.5	3	-										
Average of CO'S	3	3										

Faculty Signature				
Dr. A H Srinivasa	Ajay Kumar M	Indumathi R S	Seema S	Nataraj K
Sindhushree M V	Ajay C K	Purushothama S	Vinayak Bhandari	

Institute Level		
Criteria 8 Main Coordinator	NBA Convener	Principal

Maharaja Institute of Technology Mysore

Department of Mathematics

Subject: **ADDITIONAL MATHEMATICS – I** **Subject Code:** **18DIPMATAT31**

Syllabus

Module-1
Complex Trigonometry
Complex Numbers: Definitions and properties. Modulus and amplitude of a complex number, Argand's diagram, De-Moivre's theorem (without proof).

Vector Algebra:
Scalar and vectors. Addition and subtraction and multiplication of vectors- Dot and Cross products, problems.

Module-2
Differential Calculus:
Review of successive differentiation-illustrative examples. Maclaurin's series expansions-Illustrative examples. Partial Differentiation: Euler's theorem-problems on first order derivatives only. Total derivatives-differentiation of composite functions. Jacobians of order two-Problems.

Module-3
Vector Differentiation:
Differentiation of vector functions. Velocity and acceleration of a particle moving on a space curve. Scalar and vector point functions. Gradient, Divergence, Curl-simple problems. Solenoidal and irrotational vector fields-Problems.

Module-4
Integral Calculus
Review of elementary integral calculus. Reduction formulae for $\sin^n$, $\cos^n x$ (with proof) and $\sin^m x \cos^n x$ (without proof) and evaluation of these with standard limits-Examples. Double and triple integrals-Simple examples.

Module-5
Ordinary differential equations (ODE's)
Introduction-solutions of first order and first-degree differential equations: exact, linear differential equations. Equations reducible to exact and Bernoulli's equation.

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Syllabus

MODULE-1

Complex Trigonometry: Complex numbers: Definitions and properties. Modulus and amplitude of a complex number, Argand's diagram, De-Moivre's theorem(without proof).
Vector Algebra: Scalar and vectors. Addition , subtraction and multiplication of vectors -Dot and cross products –problems.

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COMPLEX NUMBERS

Defination:-

A number of the form $Z = x + iy$ where x and y are real numbers and $i = \sqrt{-1}$ is called a Complex number.

Here x is called the Real part (R) of Z and y is called the Imaginary part (I) of Z.

A pair of complex numbers $Z = x + iy$ and $\bar{Z} = x - iy$ are said to be conjugate of each other.

MODULUS AND AMPLITUDE OF A COMPLEX NUMBER

The number $r = +\sqrt{x^2 + y^2}$ is called Modulus of $Z = x + iy$ and is written as $\mathrm{mod}(x + iy)$ or $|Z| = |x + iy|$

The angle θ is called the Amplitude or Argument of $Z = x + iy$ and is written as $\mathrm{amp}(x + iy)$ or $\arg(x + iy)$

PROPERTIES OF COMPLEX NUMBERS

1. **If $x_1 + iy_1 = x_2 + iy_2$ then $x_1 - iy_1 = x_2 - iy_2$**
2. **Two complex numbers $x_1 + iy_1$ and $x_2 + iy_2$ are said to be equal when**

$$R(x_1 + iy_1) = R(x_2 + iy_2) \quad \text{i.e.} \quad x_1 = x_2$$

and

$$I(x_1 + iy_1) = I(x_2 + iy_2) \quad \text{i.e.} \quad y_1 = y_2$$

3. **Sum, difference, product and quotient of any two complex numbers is itself a complex number.**

If $x_1 + iy_1$ and $x_2 + iy_2$ be two given complex numbers, then

(i)their sum $= (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$

(ii)their difference $= (x_1 + iy_1) - (x_2 + iy_2) = (x_1 - x_2) + i(y_1 - y_2)$

(iii) their product $= (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 - x_2y_1)$

(iv) their product $= \frac{x_1+iy_1}{x_2+iy_2} = \frac{(x_1+iy_1)(x_2-iy_2)}{(x_2+iy_2)(x_2-iy_2)} = \left(\frac{x_1x_2+y_1y_2}{x_2{}^2+y_2{}^2}\right) + i\left(\frac{x_1y_1-x_1y_2}{x_2{}^2+y_2{}^2}\right)$

4. **Every complex number $x + iy$ can always be expressed in the form $r(\cos\theta + i\sin\theta)$**

Put $x = r\cos\theta$ and $y = r\sin\theta$

Squaring and adding we get, $x^2 + y^2 = r^2$

$$\text{i.e. } r = \sqrt{x^2 + y^2} \quad \textbf{(taking positive square root only)}$$

Dividing we get, $\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Thus $x + iy = r(\cos\theta + i\sin\theta)$ **where** $r = \sqrt{x^2 + y^2}$ **and** $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

5. **The conjugate of $Z = x + iy$ is $\bar{Z} = x - iy$ then,**

(i) $R(z) = \frac{Z+\bar{Z}}{2}$ **and** $I(z) = \frac{Z-\bar{Z}}{2}$

(ii) $|Z| = \sqrt{R^2(z) + I^2(z)} = |\bar{Z}|$

(iii) $Z\bar{Z} = |Z|^2$

(iv) $\overline{Z_1 + Z_2} = \overline{Z_1} + \overline{Z_2}$

(v) $\overline{Z_1 Z_2} = \overline{Z_1} \cdot \overline{Z_2}$

(vi) $\overline{\left(\frac{Z_1}{Z_2}\right)} = \frac{\overline{Z_1}}{\overline{Z_2}}$ **where** $\overline{Z_2} \neq 0$

PROBLEMS:-

1. Find the modulus and amplitude of $\frac{(3+\sqrt{2}i)^2}{1+2i}$

   Solution:- $\frac{(3+\sqrt{2}i)^2}{1+2i} = \frac{9-2-6\sqrt{2}i}{(1+2i)(1-2i)} = \frac{7-12\sqrt{2}-i(6\sqrt{2}+14)}{5}$

   $$\text{Modulus} = \frac{\sqrt{(7-12\sqrt{2})^2+(6\sqrt{2}+14)^2}}{5} = \frac{11\sqrt{5}}{5}$$

   $$\text{Amplitude} = \tan^{-1}\left(\frac{6\sqrt{2}+14}{12\sqrt{2}-7}\right)$$

2. Reduce $1 - \cos\alpha + i\sin\alpha$ to the modulus amplitude form

   Solution:- Put $1 - \cos\alpha = r\cos\theta$ and $\sin\alpha = r\sin\theta$

   $$r = (1-\cos\alpha)^2 + \sin^2\alpha = 2 - 2\cos\alpha = 4\sin^2\left(\frac{\alpha}{2}\right)$$

   $$\text{i.e. } r = \sin\left(\frac{\alpha}{2}\right)$$

   $$\text{and } \tan\theta = \frac{\sin\alpha}{1-\cos\alpha} = \frac{2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\sin^2\left(\frac{\alpha}{2}\right)} = \cot\left(\frac{\alpha}{2}\right) = \tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)$$

   $$\therefore \theta = \left(\frac{\pi-\alpha}{2}\right)$$

   Thus $1 - \cos\alpha + i\sin\alpha = 2\sin\left(\frac{\alpha}{2}\right)\left[\cos\left(\frac{\pi-\alpha}{2}\right) + i\sin\left(\frac{\pi-\alpha}{2}\right)\right]$

3. Find the real values of x, y so that $-3 + ix^2y$ and $x^2 + y + 4i$ may represent complex conjugate numbers.

   Solution:- If $Z = -3 + ix^2y$ then $\bar{Z} = x^2 + y + 4i$ so that

   $$Z = (x^2 + y) - 4i$$

   $$-3 + ix^2y = x^2 + y - 4i$$

   Equating Real and Imaginary parts from both sides we get,

   $$-3 = x^2 + y \text{ and } x^2y = -4$$

   Eliminating $x, (y+3)y = -4$

   Or $y^2 + 3y - 4 = 0$ i.e. $y = 1$ or $y = -4$

   When $y = 1$, $x^2 = -3 - 1$ or $x = +2i$ which is not feasible.

   When $y = -4$, $x^2 = 1$ or $x = \pm 1$

Hence $x = 1, y - 4$ or $x = -1, y = -4$

4. Express $1 + \sin\alpha + i\cos\alpha$ in the modulus-amplitude form.
5. Express $\frac{2-\sqrt{3}i}{1+i}$ in the form $Z = x + iy$

GEOMETRIC REPRESENTATION OF COMPLEX NUMBERS

If (r, θ) be the polar co-ordinates then r is the modulus of Z and θ is its amplitude. The points whose Cartesian coordinates are (x, y)uniquely represents the complex number $Z = x + iy$ on the complex plane Z. The diagram in which this representation is carried out is called Argand diagram.

PROBLEMS:-

1. Find the square root and cube root of $-\sqrt{3} + i$

Solution:- Let $-\sqrt{3} + i = r\,\mathrm{cis}\,\theta = Z$

Then $r = \sqrt{x^2 + y^2} = \sqrt{3 + 1} = 2$

Since $x < 0$ and $y > 0$ we have

$$\theta = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

$$\therefore \theta = \frac{5\pi}{6}$$

$$\therefore Z = -\sqrt{3} + i = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$$

Replacing $\left(\frac{5\pi}{6}\right)$ by $2\pi k + \left(\frac{5\pi}{6}\right)$ where $k \in z$ we get,

$$Z = -\sqrt{3} + i = 2\left[\cos\left(2\pi k + \frac{5\pi}{6}\right) + i\sin\left(2\pi k + \frac{5\pi}{6}\right)\right]$$

$$Z = -\sqrt{3} + i = 2\mathrm{cis}\left(2\pi k + \frac{5\pi}{6}\right)$$

$$\therefore Z_K = -\sqrt{3} + i = 2\mathrm{cis}\left(\frac{12\pi k + 5\pi}{6}\right) \quad \text{-----------}(1)$$

By taking the square root in (1) we have,

$$Z_K = \left(-\sqrt{3}+i\right)^{1/2} = 2^{1/2}\left[\text{cis}\left(\frac{12\pi k+5\pi}{6}\right)\right]^{1/2}$$

$$Z_K = \left(-\sqrt{3}+i\right)^{1/2} = 2^{1/2}\left[\text{cis}\left(\frac{12\pi k+5\pi}{12}\right)\right] \text{ where } k = 0, 1$$

When $k = 0$, we get $Z_0 = 2^{1/2}\left[\text{cis}\left(\frac{5\pi}{12}\right)\right]$

When $k = 1$, we get $Z_1 = 2^{1/2}\left[\text{cis}\left(\frac{17\pi}{12}\right)\right]$

$\therefore$ The square roots of $-\sqrt{3}+i$ are $2^{1/2}\left[\text{cis}\left(\frac{5\pi}{12}\right)\right]$ and $2^{1/2}\left[\text{cis}\left(\frac{17\pi}{12}\right)\right]$

By taking the cube root in (1) we have,

$$Z_K = \left(-\sqrt{3}+i\right)^{1/3} = 2^{1/3}\left[\text{cis}\left(\frac{12\pi k+5\pi}{6}\right)\right]^{1/3}$$

$$Z_K = \left(-\sqrt{3}+i\right)^{1/3} = 2^{1/3}\left[\text{cis}\left(\frac{12\pi k+5\pi}{18}\right)\right] \text{ where } k = 0, 1, 2$$

When $k = 0$, we get $Z_0 = 2^{1/3}\left[\text{cis}\left(\frac{5\pi}{18}\right)\right]$

When $k = 1$, we get $Z_1 = 2^{1/3}\left[\text{cis}\left(\frac{17\pi}{18}\right)\right]$

When $k = 2$, we get $Z_2 = 2^{1/3}\left[\text{cis}\left(\frac{29\pi}{18}\right)\right]$

$\therefore$ The cube roots of $-\sqrt{3}+i$ are $2^{1/3}\left[\text{cis}\left(\frac{5\pi}{18}\right)\right]$, $2^{1/3}\left[\text{cis}\left(\frac{17\pi}{18}\right)\right]$ & $2^{1/3}\left[\text{cis}\left(\frac{29\pi}{18}\right)\right]$

2. Find the cube root of $1 + i\sqrt{3}$

Solution:- Let $-\sqrt{3} + i = r\,\text{cis}\ \theta = Z$

Then $r = \sqrt{x^2+y^2} = \sqrt{1+\left(\sqrt{3}\right)^2} = \sqrt{1+3} = 2$

$$\cos\theta = \frac{x}{r} = \frac{1}{2} \text{ and } \sin\theta = \frac{y}{r} = \frac{\sqrt{3}}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

$$\therefore Z = 1 + i\sqrt{3} = 2\left[\cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)\right]$$

Replacing $\left(\frac{\pi}{3}\right)$ by $2\pi k + \left(\frac{\pi}{3}\right)$ where $k \in z$ we get,

$$Z = 1 + i\sqrt{3} = 2\left[\cos\left(2\pi k + \frac{\pi}{3}\right) + i\sin\left(2\pi k + \frac{\pi}{3}\right)\right]$$

$$Z = 1 + i\sqrt{3} = 2\text{cis}\left(2\pi k + \frac{\pi}{3}\right)$$

$$\therefore Z_K = 1 + i\sqrt{3} = 2\text{cis}\left(\frac{6\pi k + \pi}{3}\right) \text{------------(1)}$$

By taking the cube root in (1) we have,

$$Z_K = \left(1 + i\sqrt{3}\right)^{1/3} = 2^{1/3}\left[\text{cis}\left(\frac{6\pi k + \pi}{9}\right)\right]^{1/3}$$

$$Z_K = \left(1 + i\sqrt{3}\right)^{1/3} = 2^{1/3}\left[\text{cis}\left(\frac{6\pi k + \pi}{9}\right)\right] \text{ where } k = 0, 1, 2$$

When $k = 0$, we get $Z_0 = 2^{1/3}\left[\text{cis}\left(\frac{\pi}{9}\right)\right]$

When $k = 1$, we get $Z_1 = 2^{1/3}\left[\text{cis}\left(\frac{7\pi}{9}\right)\right]$

When $k = 2$, we get $Z_2 = 2^{1/3}\left[\text{cis}\left(\frac{13\pi}{9}\right)\right]$

$\therefore$ **The cube roots of $-\sqrt{3} + i$ are** $2^{1/3}\left[\text{cis}\left(\frac{\pi}{9}\right)\right]$, $2^{1/3}\left[\text{cis}\left(\frac{7\pi}{9}\right)\right]$ & $2^{1/3}\left[\text{cis}\left(\frac{13\pi}{9}\right)\right]$

DE MOIVRE'S THEOREM

If $'n'$ is any integer which is positive or negative then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

NOTE:-

(i) $\text{cis}\theta_1 \cdot \text{cis}\theta_2 \cdots\cdots \text{cis}\theta_n = \text{cis}(\theta_1 + \theta_2 + \cdots + \theta_n)$

(ii) $(\cos\theta - i\sin\theta)^n = \cos n\theta - i\sin n\theta = (\cos\theta - i\sin\theta)^{-n}$

(iii) $(\text{cis } m\theta)^n = \text{cis } mn\theta = (\text{cis } n\theta)^m$

PROBLEMS:-

1. Simplify $\frac{(\cos 3\theta+i\sin 3\theta)^4(\cos 4\theta-i\sin 4\theta)^5}{(\cos 4\theta+i\sin 4\theta)^3(\cos 5\theta+i\sin 5\theta)^{-4}}$

Solution:- We have, $\frac{(\cos 3\theta+i\sin 3\theta)^4(\cos 4\theta-i\sin 4\theta)^5}{(\cos 4\theta+i\sin 4\theta)^3(\cos 5\theta+i\sin 5\theta)^{-4}}$

$$=\frac{(\cos\theta+i\sin\theta)^{12}(\cos\theta+i\sin\theta)^{-20}}{(\cos\theta+i\sin\theta)^{12}(\cos\theta+i\sin\theta)^{-20}}=1$$

$$\therefore \frac{(\cos 3\theta+i\sin 3\theta)^4(\cos 4\theta-i\sin 4\theta)^5}{(\cos 4\theta+i\sin 4\theta)^3(\cos 5\theta+i\sin 5\theta)^{-4}}=1$$

2. P.T. $(1+\cos\theta+i\sin\theta)^n+(1+\cos\theta-i\sin\theta)^n=2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{n\theta}{2}\right)$

Solution:- Put $1+\cos\theta=r\cos\alpha$ and $\sin\theta=r\sin\alpha$

$$\therefore r^2=(1+\cos\theta)^2+\sin^2\theta=2+2\cos\theta=4\cos^2\left(\frac{\theta}{2}\right)$$

$$\therefore r=2\cos\left(\frac{\theta}{2}\right)$$

And $\tan\ \alpha=\frac{\sin\theta}{1+\cos\theta}=\frac{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{2\cos^2\left(\frac{\theta}{2}\right)}=\tan\left(\frac{\theta}{2}\right)$

$$\therefore \alpha=\left(\frac{\theta}{2}\right)$$

Consider LHS

$$=[r(1+\cos\alpha+i\sin\alpha)]^n+[r(1+\cos\alpha-i\sin\alpha)]^n$$

$$=r^n[(\cos\alpha+i\sin\alpha)^n+(\cos\alpha-i\sin\alpha)^n]$$

$$=r^n[\cos n\alpha+i\sin n\alpha+\cos n\alpha-i\sin n\alpha]$$
$$=r^n\ 2\cos n\alpha$$
$$=2^{n+1}\cos^n\left(\frac{\theta}{2}\right)\cdot\cos\left(\frac{n\theta}{2}\right)$$

3. P.T. $\frac{(\cos 5\theta-i\sin 5\theta)^2(\cos 7\theta+i\sin 7\theta)^{-3}}{(\cos 4\theta-i\sin 4\theta)^9(\cos\theta+i\sin\theta)^5}=1$

4. P.T. $\left(\frac{\cos\theta+i\sin\theta}{\sin\theta+i\cos\theta}\right)^4=\cos 8\theta+i\sin 8\theta$

5. P.T. $(1+\sin\theta+i\cos\theta)^n+(1+\sin\theta-i\cos\theta)^n=2^{n+1}\cos^n\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cdot\cos\left(\frac{n\pi}{4}-\frac{n\theta}{2}\right)$

VECTOR ALGEBRA

Definition: A quantity which is completely specified by its magnitude and direction is called a vector.

A vector is represented by a line segment. Thus $\overrightarrow{AB}$represents a vector whose magnitude is the length AB and the direction from A to B.

Definition: A quantity which has only magnitude and no direction is called scalar.

Addition, Subtraction and Multiplication of Vectors.

Vectors are added according to the triangle law of addition. Let $\overrightarrow{OP}$ and $\overrightarrow{PQ}$ represent a two vectors then $\overrightarrow{OQ}$ is called sum of the vectors. i.e, $\overrightarrow{OP} + \overrightarrow{PQ} = \overrightarrow{OQ}$

The subtraction of a vector $\overrightarrow{PQ}$ $from$ $\overrightarrow{OP}$ is taken to be the addition of -$\overrightarrow{PQ}$ to$\overrightarrow{OP}$. i.e, $\overrightarrow{OP} + \left(-\overrightarrow{PQ}\right)$.

Let $\overrightarrow{OP} = A. we\ have\ A + A = 2A\ and - A + (-A) = -2A$

Where both 2A and $-2A$ denotes the vector of magnitude twice that of A.

Direction cosines: Let $\overrightarrow{OP}$ $makes\ an\ angle\ \alpha, \beta, \gamma\ with\ OX, OY, OZ$ Respectively, then $cos\alpha, cos\beta, cos\gamma$ are called direction cosines of the line whish are denoted by l, m, n. Also $cos^2\alpha + cos^2\beta + cos^2\gamma =$ 1.

Distance between two points.

If

$P(x_1, y_1 z_1) and\ Q(x_2, y_2, z_2) is \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

Problems:

1. Show that the points A(-4 , 9 ,6) , B(-1 , 6,6) and C(0 , 7 , 10) from a right angled isosceles triangle. Find the direction cosines of AB.
   Solution: we have,
   $AB = \sqrt{(-1+4)^2 + (6-9)^2 + (6-6)^2} = 3\sqrt{2}$
   $BC = \sqrt{(0+1)^2 + (7-6)^2 + (10-6)^2} = 3\sqrt{2}$
   $CA = \sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2} = 6$
   Since $AB^2 + BC^2 = CA^2$ $and\ AB = BC, it\ follows\ that\ \Delta ABC$
   is a right angled isosceles triangle.
   The direction ratios of $\overrightarrow{AB}\ are - 1 + 4, 6 - 9, 6 - 6$.
   Its direction cosines are $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, 0 .

2. S.T the points (0 ,4 , 1) (2 , 3 ,-1) (4, 5 ,0) and (2 , 6 , 2) are vertices of a square.

3. S.T the position vectors of the vertices of a triangle $A = 3\sqrt{3}i - 3j$, $B = 6j, C = 3\sqrt{3}i + 3j\ from\ a\ isosceles\ triangle.$

4. If the lines makes an angles α, β, γ with the axes, prove that $cos2\alpha + cos2\beta + cos2\gamma = -1.$

SCALAR OR DOT PRODCUT

Definition: The dot product of two vectors A and B is defined as the scalar $abcos\theta, where\ \theta$ is the angle between A and B.
Thus , A.B = $abcos\theta$.

VECTOR OR CROSS PRODUCT

Definition: The cross product of vectors A and B is defined as a vector such that its magnitude is $absin\theta, \theta$being the angle between A and B, its direction is perpendicular to the plane A and B, and it forms with A and B a right handed system.
Thus $A \times B = ab\ sin\theta\ N$.

Also, $A \times B = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Problems:

1. Find the sides and angles of the triangle whose vertices are i-2j+2k, 2i+j-k and 3i-j+2k.

Solution: let $\overrightarrow{OA} = i - 2j - 2k, \overrightarrow{OB} = 2i + j = k,\ \overrightarrow{OC}\ 3i - j + 2k\ then\ \overrightarrow{BC} = i - 2j + 3k\ \overrightarrow{CA} = -2i - j\ \overrightarrow{AB} = i + 3j - 3k$
$\therefore BC = \sqrt{14}\ CA = \sqrt{5}\ AB = \sqrt{19}$
Now the direction cosines of AB and AC are,
$1/\sqrt{19}$, $3/\sqrt{19}$, $-3/\sqrt{19}$ and $2/\sqrt{5}$, $1/\sqrt{5}$, 0 .
$cosA = \frac{1}{\sqrt{19}}.\frac{2}{\sqrt{5}} + \frac{3}{\sqrt{19}}.\frac{1}{\sqrt{5}} + \frac{3}{\sqrt{19}}.0 = \sqrt{\frac{5}{19}} \quad A = \cos^{-1}\sqrt{5/19}.$
the direction cosines of BC and BA are,

$1/\sqrt{14}$, $-2/\sqrt{14}$, $3/\sqrt{14}$ and $-1/\sqrt{19}$, $-3/\sqrt{19}$, $3/\sqrt{19}$.
$cosB = \frac{1}{\sqrt{14}}.\frac{-1}{\sqrt{19}} + \frac{-2}{\sqrt{14}}.\frac{-3}{\sqrt{19}} + \frac{3}{\sqrt{14}}.\frac{3}{\sqrt{19}} = \sqrt{\frac{14}{19}} \quad B = \cos^{-1}\sqrt{14/19}.$
the direction cosines of CA and CB are,

$-2/\sqrt{5}$, $-1/\sqrt{5}$, 0 and $-1/\sqrt{14}$, $2/\sqrt{14}$, $-3/\sqrt{19}$.
$cosC = \frac{-2}{\sqrt{5}}.\frac{-1}{\sqrt{14}} + \frac{-1}{\sqrt{5}}.\frac{2}{\sqrt{14}} + 0.\frac{-3}{\sqrt{14}} = 0 \quad C = 90^{\circ}.$

2. If $A = 4i + 3j + k$, $B = 2i - j + 2k$, find a unit vector N perpendicular to vector A and B such that A,B,N form a right handed system. Also find the angle between the vectors A and B .

   Solution : $A \times B = \begin{vmatrix} i & j & k \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix} = 7i - 6j - 10k.$

   $|A \times B| = \sqrt{185}.$

   Unit vector N $\perp$ to A and B $= \frac{A\times B}{|A\times B|} = \frac{7i-6j-10k}{\sqrt{185}}.$

   Also $a = \sqrt{4^2 + 3^2 + 1^2} = \sqrt{26}$ $and\ b = 3$

   w.k.t. $|A \times B| = ab\ sin\theta.$ $sin\theta = \frac{\sqrt{185}}{3\sqrt{26}}.$

3. A line makes an angle $\alpha, \beta, \gamma, \delta$ with the diagonals of the cube, prove that $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = 4/3.$
4. P.T the area of the triangle whose vertices are A,B,C is $\frac{1}{2}[B \times C + C \times A + A \times B].$
5. Calculate the area of the triangle whose vertices are A(1,0,-1) B(2,1,5) and C(0,1,2).
6. P.T $\sin(A + B) = sinA\ cosB + cosA\ sinB.$
7. P.T $\cos(A + B) = \cos AcosB - sinA\ sinB.$
8. In any triangle ABC , prove that

   (i) $\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC}.$

   (ii) $a = bcosC + ccosB.$

   (iii) $a^2 = b^2 + c^2 - 2bccosA.$

Maharaja Institute of Technology Mysore

Department of Mathematics

Subject: ADDITIONAL MATHEMATICS – I **Subject Code:** 18DIPMATAT31

Syllabus

Module-2

Differential Calculus:

Review of successive differentiation-illustrative examples. Maclaurin's series expansions-Illustrative examples. Partial Differentiation: Euler's theorem-problems on first order derivatives only. Total derivatives-differentiation of composite functions. Jacobians of order two-Problems.

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Taylor's and Maclaurin's Series expansion:

Consider a function $y = f(x)$ then the series expansion of a function $y = f(x)$ at a point x=a is given by,

$$\boldsymbol{y(x) = y(a) + \frac{(x-a)}{1!} y_1(a) + \frac{(x-a)^2}{2!} y_2(a) + \frac{(x-a)^3}{3!} y_3(a) + \ldots + \frac{(x-a)^n}{n!} y_n(a)}$$

This expansion is known as **Taylor's series expansion.**

Put a=0 in the above equation we get

$$\boldsymbol{y(x) = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots \infty = \sum_{n=0}^{\infty} \frac{x^n}{n!} y_n(0)}$$

This expansion is known as **Maclaurin's series expansion.**

PROBLEMS:

Using Maclaurin's series, expand the following functions

1. e^x

Sol: Let $y(x) = e^x$

Consider the Maclaurin's series expansion,

$$y(x) = y(0) + \frac{x}{1!} y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \ldots$$

$y(x) = e^x \Rightarrow y(0) = e^0 = 1$, $y_1(x) = e^x \Rightarrow y_1(0) = e^0 = 1$, $y_2(x) = e^x \Rightarrow y_2(0) = e^0 = 1$

$y_3(x) = e^x \Rightarrow y_3(0) = e^0 = 1$

Therefore, the Maclaurin's series expansion becomes,

$$\boldsymbol{y(x)} = e^x = \boldsymbol{1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}} + \ldots$$

2. $\sin x$ and $\cos x$

Sol: Let , $y(x) = \sin x \Rightarrow y(0) = \sin 0 = 0$

$\boldsymbol{y_1(x) = cos x \Rightarrow y_1(0) = cos 0 = 1}$

$\boldsymbol{y_2(x) = -sin x \Rightarrow y_2(0) = -sin 0 = 0}$

$\boldsymbol{y_3(x) = -cosx \Rightarrow y_3(0) = -cos0 = -1}$, $\boldsymbol{y_4(x) = sinx \Rightarrow y_4(0) = sin0 = 0}$

$\boldsymbol{y_5(x) = cosx \Rightarrow y_5(0) = cos0 = 1}$

Therefore, the Maclaurin's series expansion becomes,

$\boldsymbol{y}(\boldsymbol{x}) = sinx = \frac{\boldsymbol{x}}{\boldsymbol{1}!} - \frac{\boldsymbol{x^3}}{\boldsymbol{3}!} + \frac{\boldsymbol{x^5}}{\boldsymbol{5}!} + \ldots$

Again for $y(x) = cos\,x \Rightarrow y(0) = cos0 = 1$

$y_1(x) = sinx \Rightarrow y_1(0) = sin0 = 0,\ y_2(x) = cosx \Rightarrow y_2(0) = cos0 = 1,$

$y_3(x) = -sinx \Rightarrow y_3(0) = -sin0 = 0, y_4(x) = -cosx \Rightarrow y_4(0) = -cos0 = -1$

$y_5(x) = sinx \Rightarrow y_5(0) = sin0 = 0$

Therefore, the Maclaurin's series expansion becomes,

$\boldsymbol{y}(\boldsymbol{x}) = cosx = \boldsymbol{1} + \frac{\boldsymbol{x^2}}{\boldsymbol{2}!} - \frac{\boldsymbol{x^4}}{\boldsymbol{4}!} + \ldots$

3. $\boldsymbol{log(1+x)\ upto\ x^4}$

Sol: Let, $y(x) = log(1+x) \Rightarrow y(0) = log1 = 0$

$$y_1(x) = \frac{1}{1+x} \Rightarrow y_1(0) = 1$$

$$y_2(x) = -\frac{1}{(1+x)^2} \Rightarrow y_2(0) = -1$$

$y_3(x) = \frac{2}{(1+x)^3} \Rightarrow y_3(0) = 2, y_4(x) = -\frac{6}{(1+x)^4} \Rightarrow y_4(0) = -6$

Therefore, the Maclaurin's series expansion becomes,

$\boldsymbol{y}(\boldsymbol{x}) = log(1+x) = \boldsymbol{x} - \frac{\boldsymbol{x^2}}{\boldsymbol{2}!} + \frac{\boldsymbol{2x^3}}{\boldsymbol{3}!} - \frac{\boldsymbol{6x^4}}{\boldsymbol{4}!} + \ldots$

4.Find the first four non-zero terms in the expansion $y(x) = \frac{x}{e^{x-1}}$ using Maclaurin's series

Sol: Consider $y(x) = \frac{x}{e^{x-1}} \Rightarrow y(0) = 0$

$$y = \frac{x}{e^x \cdot e^{-1}} = e(e^x x)$$

$$y_1 = e[e^{-x} + x \cdot (-e^{-x})] \ \Rightarrow y_1(0) = e$$

$y_2 = e\{-e^{-x} - [x(-e^{-x}) + e^{x}]\} = e[-2e^{-x} + xe^{-x}]$

since, $y = e(e^{-x}x) \Rightarrow \frac{y}{e} = e^{-x}\, x$

Therefore, $y_2 = e[-2e^{-x} + \frac{y}{e}] \Rightarrow y_2(0) = -2e,\ y_3 = e[2e^{-x} + \frac{y_1}{e}] \Rightarrow y_3(0) = 3e$

$y_4 = e[-2e^{-x} + \frac{y_2}{e}] \ \Rightarrow y_4(0) = -4e$

Therefore, the Maclaurin's series expansion becomes,

$y(x) = \frac{x}{e^{x-1}} = \frac{\boldsymbol{x}}{\mathbf{1}!}(\boldsymbol{e}) + \frac{\boldsymbol{x^2}}{\mathbf{2}!}(-\mathbf{2}\boldsymbol{e}) + \frac{\boldsymbol{x^3}}{\mathbf{3}!}(\mathbf{3}\boldsymbol{e})\text{-}\frac{\boldsymbol{x^4}}{\mathbf{4}!}(-\mathbf{4}\boldsymbol{e})$

Expand 5. $sin(e^{x} - 1)$ upto x^4

Sol: Let, $y(x) = sin(e^{x} - 1) \Rightarrow y(0) = 0$

$y_1 = e^{x}\, cos(e^{x} - 1) \Rightarrow y_1(0) = 1\ ,\ y_2 = y_1 - e^{2x}y \Rightarrow y_2(x) = 1$

$y_3 = y_2 - [e^{2x}y_1 + 2e^{2x}y]\ \Rightarrow y_3(x) = 0$

$y_4 = y_3 - [e^{2x}y_2 + 2e^{2x}y_1] - 2[e^{2x}y_1 + 2e^{2x}y]\ \Rightarrow y_4(x) = -3 - 2(1 + 0) = -5$

Therefore, the Maclaurin's series is,

$\boldsymbol{sin}(e^{x} - \mathbf{1}) = \frac{\boldsymbol{x}}{\mathbf{1}!} + \frac{\boldsymbol{x^2}}{\mathbf{2}!}\text{-}5\frac{\boldsymbol{x^4}}{\mathbf{4}!} + \ldots$

EX:1. Expand $\sqrt{1 + sin2x}\ \ upto\ \ x^4$

2. Expand $log(sec\, x)$ in ascending powers of x upto the first three non-vanishing terms.

3. Expand $log(cos\, x)\ \ \ upto\ \ the\ term\ containing\ x^6$.

4. Expand $log(1 + cos\, x)\ \ \ upto\ \ the\ term\ containing\ x^4$.

Euler's Theorem:-

Statement of Euler's theorem:- If u is a homogenous function of x and y with degree n, then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

Worked Examples

Verify Euler's theorem for

i). $u=\left(\frac{xy}{x+y}\right)$ ii) $u=ax^2+2hxy+by^2$

Solution:-

(i) Consider $u=\frac{xy}{x+y}=\frac{x^2(y/x)}{x(1+y/x)}=x\left(\frac{y/x}{1+y/x}\right)=x\phi\left(y/x\right)=x^1\phi\left(y/x\right)$

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu \text{ ....}(*)$$

This shows that $u=\frac{xy}{x+y}$ is a homogenous function of degree1 (1=n). Hence, Euler's theorem when applied to u becomes

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=(1)u \text{ ....}(1)\ (n=1)$$

we verify the equation (*) by showing actually LHS=RHS

now as $u=\left(\frac{xy}{x+y}\right), \frac{\partial u}{\partial x}=\frac{y^2}{(x+y)^2}$ & $\frac{\partial u}{\partial y}=\frac{x^2}{(x+y)^2}$

$\therefore$ LHS of (*) $=x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

$$=x\left(\frac{y^2}{(x+y)^2}\right)+y\left(\frac{x^2}{(x+y)^2}\right)=\frac{xy(x+y)}{(x+y)^2}=\frac{xy}{(x+y)}=\text{RHS}$$

This verifies the result (1)

(ii) Consider $u=ax^2+2hxy+by^2=x^2\left(a/x^2+2h\left(y/x\right)+b\,y^2/x^2\right)$

$=x^2\phi\left(y/x\right)$,which means u is a homogenous function of degree 2. Hence, on applying Euler's theorem to u, we get

$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u$ …………(1)

Now, as $u=ax^2+2hxy+by^2$, we see that $\frac{\partial u}{\partial x}=2(ax+hy)$ & $\frac{\partial u}{\partial y}=2(ay+hx)$

Consider, LHS of (1) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}$

$=x(2(ax+hy))+y(2(ay+hx))$

$=2ax^2+2hxy+2hxy+2ay^2=2(ax^2+2hxy+by^2)$

$=2u$ = RHS

This verifies the result (1)

(2) If $\sin u=\left(\frac{x^2y^2}{x+y}\right)$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\tan u$

Solution:- Consider $\sin u=\left(\frac{x^2y^2}{x+y}\right)=f$,say We note that

$f=\left(\frac{x^2y^2}{x+y}\right)=\frac{x^4\left(y/x\right)^2}{x\left(1+y/x\right)}=x^3\phi\left(y/x\right)$.Thus, `f` is a homogenous function of degree 3.

Applying Euler's theorem to f, we get

$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=3f$ As $f=\sin u$,we see that

$x\frac{\partial}{\partial x}(\sin u)+y\frac{\partial}{\partial y}(\sin u)=3(\sin u)$

i.e. $x\left\{(\cos u)\frac{\partial u}{\partial x}\right\}+y\left\{(\cos u)\frac{\partial u}{\partial y}\right\}=3\sin u$

or we get $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=3\tan u$ as desired

If $u=e^{\left(\frac{x^3+y^3}{3x+4y}\right)}$, show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=2u\log u$

Solution: - Consider $u = e^{\left(\frac{x^3+y^3}{3x+4y}\right)}$ $\quad \therefore \quad \log u = \left(\dfrac{x^3+y^3}{3x+4y}\right) = f$,say

$\therefore\ f = \dfrac{x^3+y^3}{3x+4y} = \dfrac{x^3\left(1+{y^3}/{x^3}\right)}{x\left(3+4\left({y}/{x}\right)\right)} = x^2\phi\left({y}/{x}\right)$.Thus, f is a homogenous function of degree 2.

Applying Euler's theorem to f, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = 2f$$

i.e $\quad x\dfrac{\partial}{\partial x}(\log u) + y\dfrac{\partial}{\partial y}(\log u) = 2(\log u)$

or $x\left(\dfrac{1}{u}\dfrac{\partial u}{\partial x}\right) + y\left(\dfrac{1}{u}\dfrac{\partial u}{\partial y}\right) = 2\log u$

or $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 2u\log u$

4. If $\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$ Prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$

Solution:- Let $\cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}} = \dfrac{x\left(1+{y}/{x}\right)}{\sqrt{x}\left(1+\sqrt{{y}/{x}}\right)} = x^{\frac{1}{2}}\phi\left({y}/{x}\right) = f$,

Say Hence, f is a homogenous function of degree ½

On applying Euler's theorem to f, we get

$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = \dfrac{1}{2}f$ Since $f = \cos u$

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

or $x\left\{\sin u\dfrac{\partial u}{\partial x}\right\} + y\left\{-\sin u\dfrac{\partial u}{\partial y}\right\} = \dfrac{1}{2}\cos u$ or $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\dfrac{\cos u}{\sin u}$

or $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = -\dfrac{1}{2}\cot u$

5. If $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$, show that $xu_x + yu_y = \sin 2u$ .Hence to evaluate $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$

Solution: - Consider $u = \tan^{-1}\left(\dfrac{x^3+y^3}{x+y}\right)$

$\therefore \quad \tan u = \dfrac{x^3+y^3}{x+y} = f$, say

Now, $\therefore \quad f = \dfrac{x^3+y^3}{x+y} = \dfrac{x^3\left(1+\dfrac{y^3}{x^3}\right)}{x\left(1+\dfrac{y}{x}\right)} = x^2\phi\left(\dfrac{y}{x}\right)$.So, f is a homogenous function of degree2. Applying Euler's theorem to f we get

$x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = 2f$ As $f = \tan u$, we see that

$$x\frac{\partial}{\partial x}(\tan u) + y\frac{\partial}{\partial y}(\tan u) = 2\tan u$$

$$x\left\{\sec^2 u\frac{\partial u}{\partial x}\right\} + y\left\{\sec^2 u\frac{\partial u}{\partial y}\right\} = 2\tan u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = 2\sin u\cos u$$

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = \sin 2u$ ....(*),as required.

To get the value of $x^2u_{xx} + 2xyu_{xy} + y^2u_{yy}$ we proceed as follows Differentiating (*) partially w.r.t.x & y we get, respectively

$x\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} + y\dfrac{\partial^2 u}{\partial x\partial y} = (2\cos 2u)\dfrac{\partial u}{\partial x}$ ....(1)

$x\dfrac{\partial^2 u}{\partial y\partial x} + y\dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial u}{\partial y} = (2\cos 2u)\dfrac{\partial u}{\partial y}$ ....(2)

Multiplying eq (1) by x, and eq(2) by y and adding thereafter we get

$$x^2\frac{\partial^2 u}{\partial x}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=(2\cos 2u-1)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=(2\cos 2u-1)(\sin 2u)$$

$$x^2u_{xx}+2xyu_{xy}+y^2u_{yy}=\sin 4u-\sin 2u$$

6. If $u=x^2\tan^{-1}\left(y/x\right)-y^2\tan^{-1}\left(x/y\right)$, show that

$$x^2\frac{\partial^2 u}{\partial x}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2u$$

Solution:- We note that $u=v-w$ where $v=x^2\tan^{-1}\left(y/x\right)$ & $w=y^2\tan^{-1}\left(x/y\right)$

So that v and w are homogenous functions of degree 2. Applying the corollary to the Euler's theorem to v and w, we obtain

$$x^2\frac{\partial^2 v}{\partial x}+2xy\frac{\partial^2 v}{\partial x\partial y}+y^2\frac{\partial^2 v}{\partial y^2}=2(2-1)v=2v$$

$$y^2\frac{\partial^2 w}{\partial x}+2xy\frac{\partial^2 w}{\partial x\partial y}+y^2\frac{\partial^2 w}{\partial y^2}=2(2-1)w=2w$$

Taking the difference of these two expressions, we get

$$x^2\frac{\partial^2 u}{\partial x}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=2u$$

Problem set for practice

If $u=\sin^{-1}\left(\frac{x^2y^2}{x+y}\right)$,show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\tan u$

If $z=\log\left(\frac{x^2y^2}{x+y}\right)$,show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=1$

If $u=\cot^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$,prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{4}\sin 2u$

If $z=\tan^{-1}\left(\frac{x^3y^3}{x^3+y^3}\right)$,prove that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{3}{2}\sin 2u$

TOTAL DERIVATIVES:

If u=f(x,y) where x & y are functions of t i.e x=x(t) & y=y(t) then u is a composite function of a single variable t.

$\frac{du}{dt} = \frac{\delta u}{\partial x} \cdot \frac{dx}{dt} + \frac{\delta u}{\delta y} . \frac{dy}{dt}$ is called total derivative of u w.r.t t.

$Examples$: 1. $u = x^2y + xy^2$ $where\ x = at$, $y = 2at$ then find $\frac{du}{dt}$

Sol: We know that, $\frac{du}{dt} = \frac{\delta u}{\partial x} \cdot \frac{dx}{dt} + \frac{\delta u}{\delta y} . \frac{dy}{dt}$

$\frac{\delta u}{\delta x} = 2xy + y^2$ & $\frac{\delta u}{\delta y} = x^2 + 2xy$

$$\frac{dx}{dt} = a \quad \& \quad \frac{dy}{dt} = 2a$$

$$\frac{du}{dt} = (2xy + y^2)a + (x^2 + 2xy)2a$$

Put $x = at$, $y = 2at$

$$\frac{du}{dt} = 18a^3t^2$$

2.If $u = sin\left(\frac{x}{y}\right)$ $where\ x = e^t$ & $y = t^2$

Sol: We know that, $\frac{du}{dt} = \frac{\delta u}{\partial x} \cdot \frac{dx}{dt} + \frac{\delta u}{\delta y} . \frac{dy}{dt}$

$\frac{\delta u}{\delta x} = \left[cos\left(\frac{x}{y}\right)\right]\left(\frac{1}{y}\right)$ $\quad$ $\frac{\delta u}{\delta y} = \left[cos\left(\frac{x}{y}\right)\right]\left(\frac{-x}{y^2}\right)$

$$\frac{dx}{dt} = e^t \quad \& \quad \frac{dy}{dt} = 2t$$

Therefore, $\frac{du}{dt} = \left[cos\left(\frac{x}{y}\right)\right]\left(\frac{1}{y}\right)\ e^t + \left[cos\left(\frac{x}{y}\right)\right]\left(\frac{-x}{y^2}\right)\ 2t$

Put $x = e^t$ & $y = t^2$ we get,

$$\frac{du}{dt} = \left[cos\left(\frac{e^t}{t^2}\right)\right] \cdot \frac{e^t}{t^2}\left[1 - \frac{2}{t}\right]$$

If $u = x^2 + y^2$ $where\ x = e^t\ cos\, t$ & $y = e^t\ sin\, t$

Sol: $\frac{du}{dt} = \frac{\delta u}{\partial x} \cdot \frac{dx}{dt} + \frac{\delta u}{\delta y} . \frac{dy}{dt}$

$$\frac{\delta u}{\delta x} = 2x \ \& \ \frac{\delta u}{\delta y} = 2y$$

$$\frac{dx}{dt} = e^t(-sin\,t) + e^t\,cos\,t$$

$$\frac{dy}{dt} = e^t\,cos\,t + e^t\,sin\,t$$

$$\frac{du}{dt} = 2x[e^t(\cos t - \sin t)] + 2y[e^t(\cos t + \sin t)]$$

Put $x = e^t\,cos\,t \ \& \ y = e^t\,sin\,t$

$$\frac{du}{dt} = 2e^{2t}$$

3.If $u = e^x\,sin(yz)\ where\ x = t^2, y = t - 1, z = \frac{1}{t}\ at\ t = 1$

Sol: We know that $\frac{du}{dt} = \frac{\delta u}{\partial x} \cdot \frac{dx}{dt} + \frac{\delta u}{\delta y} . \frac{dy}{dt} + \frac{\delta u}{\delta z} . \frac{dz}{dt}$

$\frac{\delta u}{\delta x} = e^x\,sin(yz) \qquad \frac{dx}{dt} = 2t$

$\frac{\delta u}{\delta y} = e^x[cos(yz)]z \qquad \frac{dy}{dt} = 1$

$\frac{\delta u}{\delta z} = e^x[cos(yz)]y \qquad \frac{dz}{dt} = -\frac{1}{t^2}$

$\frac{du}{dt}$ =$[e^x\,sin(yz)\ \ (2t)]$ +$e^x[cos(yz)]z + [e^x[cos(yz)]y] - \frac{1}{t^2}$

$$\frac{du}{dt} = e\ at\ t = 1$$

EX: If $u = x^2 + y^2 + z^2\ \ where\ x = e^t, y = e^t\,cos\,t\ \ \& \ z = e^t\,sin\,t$

Composite function containing two variables

If z=f(x,y) is a function of 2 variables x & y and x & y are functions of another variable u & v then z is a composite function of u & v.

The partial derivative of z w.r.t u & v.

$$\frac{\delta z}{\delta u}=\frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta u}+\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta u}$$

$$\frac{\delta z}{\delta v}=\frac{\delta z}{\delta x}\cdot\frac{\delta x}{\delta v}+\frac{\delta z}{\delta y}\cdot\frac{\delta y}{\delta v}$$

PROBLEMS:

1. **If $\boldsymbol{z=f(x,y)}$ where $\boldsymbol{x=e^{u}+e^{-v}}$ & $\boldsymbol{y=e^{-u}-e^{v}}$**

then $\frac{\delta z}{\delta u}-\frac{\delta z}{\delta v}=x\cdot\frac{\delta z}{\delta x}-y\cdot\frac{\delta z}{\delta y}$

sol: $\frac{\delta x}{\delta u}=e^{u}$ & $\frac{\delta x}{\delta v}=-e^{-v}$

$$\frac{\delta y}{\delta u}=-e^{-u} \;\&\; \frac{\delta y}{\delta v}=-e^{v}$$

Therefore,

$$\frac{\delta z}{\delta u}=\frac{\delta z}{\delta x}\cdot(e^{u})+\frac{\delta z}{\delta y}\cdot(-e^{-u})$$

$$\frac{\delta z}{\delta v}=\frac{\delta z}{\delta x}\cdot(-e^{-v})+\frac{\delta z}{\delta y}\cdot(-e^{v})$$

subtract the above equations we get,

$$\frac{\delta z}{\delta u}-\frac{\delta z}{\delta v}=(e^{u}+e^{-v})\left(\frac{\delta z}{\delta x}\right)-(e^{-u}-e^{v})\left(\frac{\delta z}{\delta y}\right)$$

$$\frac{\delta z}{\delta u}-\frac{\delta z}{\delta v}=x.\frac{\delta z}{\delta u}-y.\frac{\delta z}{\delta v}$$

2. **EX: If Z=f(x,y) where $\boldsymbol{x=e^{u}\cos v}$ & $\boldsymbol{y=e^{u}\sin v}$ then P.T**

$$\left(\frac{\delta z}{\delta u}\right)^{2}+\left(\frac{\partial z}{\delta v}\right)^{2}=e^{2u}\left[\left(\frac{\delta z}{\delta x}\right)^{2}+\left(\frac{\delta z}{\delta y}\right)^{2}\right]$$

If $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right)$ $then\ P.T\ x\frac{\delta u}{\delta x} + y\frac{vu}{\delta_y} + z\frac{\delta u}{\delta z} = 0$

Sol: Here, u is a composite function of x,y & z.

Given, $u = f\left(\frac{x}{y}, \frac{y}{z}, \frac{z}{x}\right) \Rightarrow f(p, q, r)$

Where $P = \frac{x}{y}$, $q = \frac{y}{z}$, $r = \frac{z}{x}$

$$\frac{\delta p}{\delta x} = \frac{1}{y} \ , \frac{\delta q}{\delta x} = 0 \ , \qquad \frac{\delta r}{\delta x} = \frac{-z}{x^2}$$

$$\frac{\delta p}{\delta y} = -\frac{x}{y^2}, \frac{\delta q}{\delta y} = \frac{1}{z} \ , \frac{\delta r}{\delta y} = 0$$

$$\frac{\delta p}{\delta z} = 0, \frac{\delta q}{\delta z} = \frac{-y}{z^2} \ , \frac{\delta r}{\delta z} = \frac{1}{x}$$

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta p}\frac{\delta p}{\delta x} + \frac{\delta u}{\delta q}\frac{\delta q}{\delta x} + \frac{\delta u}{\delta r}\frac{\partial r}{\delta x} \Rightarrow \frac{\delta u}{\delta x} = \frac{\delta u}{\delta p}.\frac{1}{y} + 0 + \frac{\delta u}{\delta r}\frac{-z}{x^2}$$

$$\frac{\delta u}{\delta y} = \frac{\delta u}{\delta p}\frac{\delta p}{\delta y} + \frac{\delta u}{\delta q}\frac{\delta q}{\delta y} + \frac{\delta u}{\delta r}\frac{\partial r}{\delta y} \Rightarrow \frac{\delta u}{\delta x} = \frac{\delta u}{\delta p}.\frac{-x}{y^2} + \frac{\delta u}{\delta q}.\frac{1}{z} + 0$$

$$\frac{\delta u}{\delta z} = \frac{\delta u}{\delta p}\frac{\delta p}{\delta z} + \frac{\delta u}{\delta q}\frac{\delta q}{\delta z} + \frac{\delta u}{\delta r}\frac{\partial r}{\delta z} \Rightarrow \frac{\delta u}{\delta x} = 0 + \frac{\delta u}{\delta q}.\frac{-y}{z^2} + \frac{\delta u}{\delta r}\frac{1}{x}$$

Therefore, $x\frac{\delta u}{\delta x} + y\frac{vu}{\delta_y} + z\frac{\delta u}{\delta z} = 0$

3. **If** $u = f(2x - 3y, 3y - 4z, 4z - 2x)$ then **P.T** $\frac{1}{2}\boldsymbol{u_x} + \frac{1}{3}\boldsymbol{u_y} + \frac{1}{4}\boldsymbol{u_z} = \mathbf{0}$

Sol: Given, $u = f(2x - 3y, 3y - 4z, 4z - 2x)$

$u = f(p, q, r)$ where

$p = 2x - 3y \quad q = 3y - 4z \quad r = 4z - 2x$

$$\frac{\delta p}{\delta x} = 2 \qquad \frac{\delta q}{\delta x} = 0 \qquad \frac{\delta r}{\delta x} = -2$$

$$\frac{\delta p}{\delta y} = -3 \qquad \frac{\delta q}{\delta y} = 3 \qquad \frac{\delta r}{\delta y} = 0$$

$\frac{\delta p}{\delta z} = 0 \qquad \frac{\delta q}{\delta z} = -4 \qquad \frac{\delta r}{\delta z} = 4$

Here, u is a composite function of x, y & z.

$$\frac{\delta u}{\delta x} = \frac{\delta u}{\delta p}\frac{\delta p}{\delta x} + \frac{\delta u}{\delta q}\frac{\delta q}{\delta x} + \frac{\delta u}{\delta r}\frac{\partial r}{\delta x} \Rightarrow \frac{\delta u}{\delta x} = \frac{\delta u}{\delta p}.2 + 0 + \frac{\delta u}{\delta r}(-2)$$

$$\frac{\delta u}{\delta y} = \frac{\delta u}{\delta p}\frac{\delta p}{\delta y} + \frac{\delta u}{\delta q}\frac{\delta q}{\delta y} + \frac{\delta u}{\delta r}\frac{\partial r}{\delta y} \Rightarrow \frac{\delta u}{\delta x} = \frac{\delta u}{\delta p}.(-3) + \frac{\delta u}{\delta q}.3 + 0$$

$$\frac{\delta u}{\delta z} = \frac{\delta u}{\delta p}\frac{\delta p}{\delta z} + \frac{\delta u}{\delta q}\frac{\delta q}{\delta z} + \frac{\delta u}{\delta r}\frac{\partial r}{\delta z} \Rightarrow \frac{\delta u}{\delta x} = 0 + \frac{\delta u}{\delta q}.(-4) + \frac{\delta u}{\delta r}4$$

Therefore, $\frac{1}{2}u_x + \frac{1}{3}u_y + \frac{1}{4}u_z = 0$

EX:1. If $u = f\left(xz, \frac{y}{z}\right)$ then S.T $xu_x - yu_y - zu_z = 0$

If $u = f(x^2 + 2yz, y^2 + 2zx)$ then P.T

$(y^2 - zx)u_x + (x^2 - yz)u_y + (z^2 - xy)u_z = 0$

JACOBIANS:

If u & v are functions of two independent variables x & y then the Jacobian of u & v w.r.t x & y is denoted by J.

i.e $J = \frac{\delta(u,v)}{\delta(x,y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

Similarly, $J = \frac{\delta(u,v,\omega)}{\delta(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

Problems:

1. **If $u = x + y + z,\ v = y + z, w = z$ then find its Jacobian.**

Sol: $We\ know\ that, J = \frac{\delta(u,v,\omega)}{\delta(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

$J = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} \Rightarrow J = 1$

If $ux = yz\ \ , vy = xz\ ,\ wz = xy\ \ then\ find\ J.$

Sol: $We\ know\ that, J = \frac{\delta(u,v,\omega)}{\delta(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$

J=$\begin{vmatrix} \frac{-yz}{x^2} & \frac{z}{x} & \frac{y}{x} \\ \frac{z}{y} & \frac{-xz}{y^2} & \frac{x}{y} \\ \frac{y}{z} & \frac{x}{z} & \frac{-xy}{z^2} \end{vmatrix}$

On expanding we get, J=4

2. If $x = r\,sin\,\theta\,cos\,\phi, y = r\,sin\,\theta\,sin\,\phi, z = r\,cos\,\theta$ then find J.

Sol: J=$\begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix}$

=$\begin{vmatrix} sin\,\theta\,cos\,\phi & r\,cos\theta\,cos\,\phi & -r\,sin\,\theta\,sin\,\phi \\ sin\,\theta\,sin\,\phi & r\,cos\,\theta\,sin\,\phi & r\,sin\,\theta\,cos\,\phi \\ cos\,\theta & -r\,sin\,\theta & 0 \end{vmatrix}$

On expanding we get, J= $r^2\,sin\,\theta$

3. If $x = u^2 - v^2, y = v^2 - w^2, z = w^2 - u^2$ then find J

Sol: J=$\begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_\omega \end{vmatrix}$=$\begin{vmatrix} 2u & -2v & 0 \\ 0 & 2v & -2w \\ -2u & 0 & 2w \end{vmatrix}$

On expanding we get, J=0

EX: If $u = xy^2, v = yz^2\ \&\ \omega = zx^2$ then find its Jacobian.

Maharaja Institute of Technology Mysore

Department of Mathematics

Syllabus

Subject:- Additional Mathematics - I **Subject Code:-** 18MATDIP31

Module-3

Vector Differentiation:

Differentiation of vector functions. Velocity and acceleration of a particle moving on a space curve. Scalar and vector point functions. Gradient, Divergence, Curl-simple problems. Solenoidal and irrotational vector fields-Problems.

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Vector Differentiaton

Vector is a quantity having both magnitude and direction

Derivative of a Vector Valued function

Let the Position vector of a point in p(x,y,z) space be $\vec{\gamma} = xi + yj + zk$.
If x,y,z are all functions of a single parameter r then $\vec{\gamma}$ is said to be a vector function of t.
NOTE: i. $\hat{\imath}.\hat{\imath} = 1 = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k}$
ii. $\hat{\imath} \cdot \hat{\jmath} = 0 = j \cdot \hat{k} = \hat{k} \cdot \hat{\jmath}$
iii. $\hat{\imath} \times \hat{\jmath} = 0 = j \times \hat{k} = \hat{k} \times \hat{\jmath}$

Velocity and Acceleration: If $\vec{\gamma} = xi + yj + zk$ represents the position vector of a point moving along a curve then i. Velocity$(\vec{v}) = \frac{d\vec{\gamma}}{dt}$ is the velocity of the particle at any time t.
ii. Acceleration$(\vec{a}) = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ represents the rate of change of velocity$(\vec{v})$ and is the acceleration of the particle at any time t.
Angle between two vectors is $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}$
Unit normal vector$(\hat{n})$ is given by $(\hat{n}) = \frac{\vec{n}}{|\vec{n}|}$
Unit Tangent vector $(\hat{T}) = \frac{\vec{T}}{|\vec{T}|}$ $where\ \vec{T} = \frac{d\gamma}{dt}$
If $\vec{a} = a_1 i + a_2 j + a_3 k\ then\ |\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

PROBLEMS:

1. If $x = t^2 + 1, y = 4t - 3, z = 2t^2 - 6t$ represents parametric equation of the curve then determine i. the unit tangent vector at any point on t
   ii. The angle between the tangents at t=1 and t=2
   sol: Let $\vec{\gamma} = xi + yj + zk$ be the position vector at any time t
   then $\vec{\gamma} = (t^2 + 1)i + (4t - 3)j + (2t^2 - 6t)k$
   $\vec{T} = \frac{d\vec{\gamma}}{dt} = 2ti + 4j + (4t - 6)k$ is the tangent vector
   Unit Tangent vector $(\hat{T}) = \frac{\vec{T}}{|\vec{T}|} = \frac{(2t)i + 4j + (4t-6)k}{\sqrt{(2t)^2 + 4^2 + (4t-6)^2}}$
   $\hat{T} = \frac{2[ti + 2j + (2t-3)k]}{\sqrt{20t^2 - 48t + 52}} = \frac{[ti + 2j + (2t-3)k]}{\sqrt{5t^2 - 12t + 13}}$

ii. $\vec{T} = 2ti + 4j + (4t - 6)k$ is the tangent vector

When t=1, $(\vec{T})_{t=1} = 2i + 4j - 2k = \vec{A}$

When t=2, $(\vec{T})_{t=2} = 4i + 4j + 2k = \vec{B}$

Let θ be the angle between the tangents then we have $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}$

$$\cos\theta = \frac{(2i+4j-2k)\cdot(4i+4j+2k)}{\sqrt{2^2+4^2+(-2)^2}\sqrt{4^2+4^2+2^2}} = \frac{(2)(4)+(4)(4)+(-2)(2)}{\sqrt{24}\sqrt{36}} = \frac{5}{3\sqrt{6}}$$

Therefore $\theta = \cos^{-1}\left(\frac{5}{3\sqrt{6}}\right)$ is the required angle.

2. Find the angle between the tangents to the curve
   $\vec{\gamma} = \left(t - \frac{t^3}{3}\right)i + t^2 j + \left(t + {t^3}/{3}\right)k$ at t=± 3
   Sol: We know that $\vec{T} = \frac{d\vec{\gamma}}{d-L} = (1-t^2)i + 2tj + (1+t^2)k$ is the tangent vector
   When $t = 3 \Rightarrow (\vec{T})_{t=3} = -8i + 6j + 10k = \vec{A}$
   When $t = -3 \Rightarrow (\vec{T})_{t=-3} = -8i - 6j + 10k = \vec{B}$

Let θ be the angle between the tangents then we have $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\cdot|\vec{B}|}$

$$= \frac{(-8)(-8)+(6)(-6)+(10)(10)}{\sqrt{(-8)^2+6^2+(10)^2(-8)^2+(-6)^2+(10)^2}} = \frac{16}{25}$$

Therefore $\theta = \cos^{-1}\left(\frac{16}{25}\right)$ is the required angle.

3. Find the unit tangent to the curve

$$\vec{\gamma} = (\cos 2t)i + (\sin 2t)j + tk \; at \; x = \frac{1}{\sqrt{2}}$$

$$\vec{T} = \frac{d\vec{\gamma}}{dt} = (-2\sin 2t)i + (2\cos 2t)j + K$$

$$|\vec{T}| = \overline{(-z\sin 2t)^2 + \overline{(a\cos 2t)^2 + 1}} = \sqrt{5}$$

We know that, $\hat{T} = \frac{\vec{T}}{|\vec{T}|} i = \frac{(-2\sin 2t)i + (2\cos 2t)j + k}{\sqrt{5}}$

Given $x = \frac{1}{\sqrt{2}} \Rightarrow t = \frac{\Pi}{8}$

Therefore $\hat{T} = \frac{(-\sqrt{2})i + (\sqrt{2})j + k}{\sqrt{5}}$ is the unit tangent vector.

EXERCISE:

1. A particle moves along a curve whose parametric equations are $x = e^t, y = 2\cos 3t, z = 2\sin 3t$ where t is the time. Find the velocity and acceleration at any time t and also their magnitude at t=0.
2. Find the angle between the tangents to the curve $\vec{r} = t^2 i + 2tj - t^3 k \; at \; t = \pm 1$
3. Find the unit tangent vector at any time t $\vec{r} = (3\cos t)i + (3\sin t)j + 4tk$

<u>Gradient, Divergence and Curl</u>

Scalar and Vector fields

Definition:

If every point p(x,y,z) of a region R in space there corresponds to a scalar ϕ(x,y,z) then ϕ is called scalar point function.

Eg: $i. \phi = x^2 + y^2 + z^2 \quad ii. \phi = xy^2z^3$

If every point p(x,y,z) of a region R in space there corresponds to a vector $\vec{A}$(x,y,z) then $\vec{A}$ is called vector point function.

Eg: $i. \vec{A} = x^2i + y^2j + z^2k \;\; ii. \vec{A} = (xyz)i + (yz)j + zk$

NOTE: The Vector Differential operator ∇ is defined by

$$\nabla = \frac{\delta}{\delta x}i + \frac{\delta}{\delta y}j + \frac{\delta}{\delta z}k = \Sigma\frac{\delta}{\delta x}i$$

Gradient of a scalar field:

If $\phi(x, y, z)$ is a continuously differentiable scalar function then the gradient of ϕ is denoted by grad ϕ and grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x}i + \frac{\delta\phi}{\delta y}j + \frac{\delta\phi}{\delta z}k = \Sigma\frac{\delta\phi}{\delta x}i$

Clearly, $\nabla\phi$ is a vector quantity.

Divergence of a vector field:

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function then divergence of $\vec{A}$ is denoted by $\nabla.\vec{A}$ or div$\vec{A}$

If $\vec{A} = a_1i + a_2j + a_3k$ where a_1, a_2, a_3 are all functions of x,y,z then

div$\vec{A} = \nabla.\vec{A}$=$\left(\frac{\delta}{\delta x}i + \frac{\delta}{\delta y}j + \frac{\delta}{\delta z}k\right)(a_1i + a_2j + a_3k)$

$\nabla.\vec{A} = \frac{\delta a_1}{\delta x} + \frac{\delta a_2}{\delta y} + \frac{\delta a_3}{\delta z}$

Clearly, $\nabla.\vec{A}$ is a scalar quantity

CURL of a vector field:

If $\vec{A}(x, y, z)$ is a continuously differentiable vector function then curl of $\vec{A}$ is denoted by $\nabla \times \vec{A}$ or curl$\vec{A}$

curl$\vec{A}$ = $\nabla \times \vec{A}$= $\begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ a_1 & a_2 & a_3 \end{vmatrix}$

$\nabla \times \vec{A}$=$i\left(\frac{\delta a_3}{\delta y}-\frac{\partial a_2}{\delta z}\right)-\left(\frac{\delta a_3}{\delta x}-\frac{\delta a_1}{\partial z}\right)+k\left(\frac{\delta a_2}{\delta x}-\frac{\delta a_1}{\delta y}\right)$

Clearly, $\nabla \times \vec{A}$ is a vector quantity.

Directional Derivative:

If φ(x, y, z) is a scalar function and $\vec{d}$ is a given direction then $\nabla\phi \cdot \hat{n}$ *where* $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$ is called as the Directional Derivative of φ along $\hat{n}$.

NOTE: i. The angle between the two surfaces is defined to be equal to angle between their normal

If $\phi_1(x,y,z) = C_1$ and $\phi_2(x,y,z) = C_2$ be the equation of two surfaces then $\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$ where θ is the angle between the normal.

ii. If $\theta = \frac{\Pi}{2}$ then $\cos\frac{\pi}{2} = O$ *implies* $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ then the surfaces are said to intersect each other orthogonally or at right angles.

PROBLEMS:

Find the unit vector normal to the following surfaces

i.$x^2y + 2xz = 4$ *at* $(2,-2,3)$

Sol: Let $\varphi = x^2y + 2xz - 4$

Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x}i + \frac{\delta\phi}{\delta y}j + \frac{\delta\phi}{\delta z}k$

$$\nabla\phi = (2xy + 2z)i + x^2j + 2xk$$

$$(\nabla\phi)_{(2,-2,3)} = -2i + 4j + 4k$$

Therefore, the required unit vector normal is $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$\hat{n} = \frac{2(-i+2j+2k)}{\sqrt{2^2(1+4+4)}}$=$\frac{-i+2j+2k}{3}$

ii. $xy^3z^2 = 4$ *at* $(-1,-1,2)$

Sol: $\varphi = xy^3z^2 - 4$

Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k$

$$\nabla\phi = (y^3z^2)i + (3xy^2z^2)j + (2xy^3z)k$$

$(\nabla\phi) = -4i - 12j + 4k$ at (-1,-1,2)

Therefore the required unit vector normal is $\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$

$$\hat{n} = \frac{4(-i - 3j + k)}{\sqrt{4^2(1 + 9 + 1)}} = \frac{-i - 3j + k}{\sqrt{11}}$$

EX: $x^2y - 2xz + 2y^2z^4 = 10 \ \ at \ \ (2,1,-1)$

<u>Find the Directional Derivative of the following:</u>

i. $\phi = x^2yz + 4xz^2$ at (1,-2,-1) along 2i-j-2k

<u>Sol</u>: Given $\phi = x^2yz + 4xz^2$

Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k$

$$\nabla\phi = (2xyz + 4z^2)i + x^2zj + (x^2y + 8xz)k$$

$\nabla\phi = 8i - j - 10k$ at (1,-2-1)

The unit vector normal along the direction 2i-j-2k is $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$

$$\hat{n} = \frac{2i - j - 2k}{\sqrt{4 + 1 + 4}} = \frac{2i - j - 2k}{3}$$

Therefore the required directional derivative is, $\nabla\phi \cdot \hat{n}$=$(8i - j - 10k)\left(\frac{2i-j-2k}{3}\right)$

$$\nabla\phi \cdot \hat{n} = \frac{1}{3}(16 + 1 + 20) = \frac{37}{3}$$

ii. $\phi = 4xz^3 - 3x^2y^2z$ at (2,-1,2) along 2i-3j+6k

Sol: Given $\phi = 4xz^3 - 3x^2y^2z$

Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k$

$$\nabla\phi = (4z^3 - 6xy^2z)i - 6x^2yz + (12xz^2 - 3x^2y^2)k$$

$$\nabla\phi = 8i + 48j + 84k \;\; at\ (2,-1,2)$$

i. The unit vector normal along the direction 2i-3j+6k is $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$

$$\hat{n} = \frac{2i - 3j + 6k}{\sqrt{4+9+36}} = \frac{2i - 3j + 6k}{7}$$

Therefore, the required directional derivative is,

$$\nabla\phi \cdot \hat{n} = (8i + 48j + 84k)\left(\frac{2i-3j+6k}{7}\right) = \frac{1}{7}(16 - 144 + 504)$$

$$\nabla\phi \cdot \hat{n} = \frac{376}{7}$$

ii. $\phi = xy + yz + zx$ at (1,2,3) along 3i+4j+5k

Sol: Given $\phi = xy + yz + zx$

Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x} i + \frac{\delta\phi}{\delta y} j + \frac{\delta\phi}{\delta z} k$

$\nabla\phi = (y+z)i + (x+z)j + (y+x)k$

$\nabla\phi = 5i + 4j + 3k\ at$ (1,2,3)

The unit vector normal along the direction 3i+4j+5k is $\hat{n} = \frac{\vec{d}}{|\vec{d}|}$

$$\hat{n} = \frac{3i + 4j + 5k}{\sqrt{9+16+25}} = \frac{3i + 4j + 5k}{5\sqrt{2}}$$

Therefore, the required directional derivative is, $\nabla\phi \cdot \hat{n} = (5i + 4j + 3k)\left(\frac{3i+4j+5k}{5\sqrt{2}}\right)$

$$= \frac{1}{5\sqrt{2}}(15 + 16 + 15)$$

$$\nabla\phi \cdot \hat{n} = \frac{46}{5\sqrt{2}}$$

EX: $\phi = xy^2 + yz^3$ at (2,-1,1) along i+2j+2k

2. Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$ and $z = x^2 + y^2 - 3$ at (2,-1,2)

Sol: Angle between the surfaces is the angle between their normal

$\nabla\phi$ is the vector normal to the surface and if θ is the angle between two normals then $\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$

We know that Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x}i + \frac{\delta\phi}{\delta y}j + \frac{\delta\phi}{\delta z}k$

Let $\phi_1 = x^2 + y^2 + z^2 - 9$ and $\phi_2 = x^2 + y^2 - z - 3$

$\nabla\phi_1 = 2xi + 2yj + 2zk$ and $\nabla\phi_2 = 2xi + 2yj - k$

At the points (2,-1,2) $\nabla\phi_1 = 4i - 2j + 4k$ and $\nabla\phi_2 = 4i - 2j - k$

$$cos\,\theta = \frac{(4i-2j+4k)(4i-2j-k)}{\sqrt{4^2+(-2)^2+4^2}\sqrt{4^2+(-2)^2+(-1)^2}} = \frac{16}{6\sqrt{21}}$$

$Therefore, \theta = \cos -1\left(\frac{8}{3\sqrt{21}}\right)$ is the angle between the two surfaces.

3.Find the angle between the normal to the surface $xy = z^2$ at (4,1,2) & (3,3,-3)

Sol: $Let\ \ \phi = xy - z^2$

We know that Grad $\phi = \nabla\phi = \frac{\delta\phi}{\delta x}i + \frac{\delta\phi}{\delta y}j + \frac{\delta\phi}{\delta z}k$

$\nabla\phi$=yi+xj-2zk and $\nabla\phi = i + 4j - 4k = \vec{A}\ at\ (4,1,2)$ & $\nabla\phi = 3i + 3j + 6k = \vec{\boldsymbol{B}}\ at\ (3,3,-3)$

If θ is the angle between the vectors $\vec{A}$ & $\vec{\boldsymbol{B}}$ then we have $cos\,\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$

$$cos\,\theta = \frac{(i+4j-4k)(3i+3j+6k)}{\sqrt{1^2+4^2+(-4)^2}\sqrt{3^2+3^2+6^2}}$$

$$\cos\theta = \frac{-1}{\sqrt{22}} \Rightarrow \theta = \cos^{-1}\left(-\frac{1}{\sqrt{22}}\right)$$

$$\theta = \Pi \pm cos^{-1}\left(\frac{1}{\sqrt{22}}\right)$$

EX: Show that the surfaces $4x^2 + z^3 = 4\ and\ 5x^2 - 2yz - 9x = 0$ intersect each other orthogonally at the point (1,-1,2)

PROBLEMS ON GRADIENT, DIVERGENCE AND CURL

1. Given $\vec{A} = x^2yzi + y^2zxj + z^2xyk$ then find div $\vec{A}$,curl $\vec{A}$ and $\nabla^2\vec{A}$

   Sol: Given $\vec{A} = x^2yzi + y^2zxj + z^2xyk$= $a_1i + a_2j + a_3k$

   i. div$\vec{A} = \nabla.\vec{A} = \frac{\delta a_1}{\delta x} + \frac{\delta a_2}{\delta y} + \frac{\delta a_3}{\delta z}$ =2xyz+2xyz+2xyz=6xyz

   ii. curl$\vec{A}$ = $\nabla \times$ $\vec{A}$= $\begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x^2yz & y^2zx & z^2xy \end{vmatrix}$

$=i\left\{\frac{\delta}{\delta y}(z^2xy)-\frac{\delta}{\partial z}(y^2zx)\right\}$-j$\left\{\frac{\delta}{\delta x}(z^2xy)-\frac{\delta}{\delta z}(x^2yz)\right\}+k\left\{\frac{\delta}{\delta x}(y^2zx)-\frac{\delta}{\delta y}(x^2yz)\right\}$

$=i(z^2x-y^2x)-j(z^2y-x^2y)+k(y^2z-x^2z)$

curl$\vec{A}$= $x(z^2-y^2)i-y(z^2-x^2)j+z(y^2-x^2)k$

iii. $\nabla^2\vec{A}=\frac{\delta^2\vec{A}}{\delta x^2}+\frac{\sigma^2\vec{A}}{\delta y^2}+\frac{\delta^2\vec{A}}{\delta z^2}$

$$=\frac{\delta^2(x^2yzi+y^2zxj+z^2xyk)}{\delta x^2}+\frac{\sigma^2\vec{A}}{\delta y^2}+\frac{\delta^2\vec{A}}{\delta z^2}$$

$=\frac{\delta}{\delta x}(2xyzi+y^2z\,j+z^2yk)$+$\frac{\delta}{\delta y}(x^2zi+2xyzj+z^2xk)+\frac{\delta}{\delta z}(x^2yi+y^2xj+2xyzk)$

=2yzi+2xzj+2xyk=2(yz+xz+xy)

2. Find the $div\,\overrightarrow{F}\,and\;\;curl\,\vec{F}\;\;where\,\overrightarrow{F}=\nabla(x^3+y^3+z^3-3xyz)$

Sol: Let $\phi=x^3+y^3+z^3-3xyz$

$$\vec{F}=\nabla\phi=\frac{\delta\phi}{\delta I}i+\frac{\delta\dot{\phi}}{\delta y}j+\frac{\delta\phi}{\sigma z}k$$

$$\vec{F}=(3x^2-3yz)i+(3y^2-3xz)j+(3z^2-3xy)k$$

Implies $\vec{F}=f_1i+f_2j+f_3k$

$div\,\vec{F}=\nabla\cdot\vec{F}\cdot=\frac{\delta f_1}{\delta x}+\frac{\delta f_2}{\delta y}+\frac{\delta f_3}{\delta z}$

=6x+6y+6z=6(x+y+z)

curl$\vec{F}$ = $\nabla\times\;\vec{F}$= $\begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ (3x^2-3yz) & (3y^2-3xz) & (3z^2-3xy) \end{vmatrix}$

$$=i\left\{\frac{\delta}{\delta^y}(3z^2-3xy)-\frac{\delta}{\delta z}(3y^2-3xz)\right\}-j\left\{\frac{\delta}{\delta^y}(3z^2-3xy)-\frac{\delta}{\delta z}(3x^2-3yz)\right\}$$
$$+k\left\{\frac{\delta}{\delta^x}(3y^2-3xz)-\frac{\delta}{\delta y}(3x^2-3yz)\right\}$$
$$=i\{-3x-(-3x)\}-j\{-3y-(-3y)\}+k\{-3z-(-3z)\}$$

curl$\vec{F}=0$

3. If $\vec{F}=\nabla(xy^3z^2)$ then find div$\vec{F}$ and curl$\vec{F}$ at the point (1,-1,1)

Sol:$Let\;\phi=xy^3z^2$ and

$\vec{F}=\nabla\phi=\frac{\delta\phi}{\delta I}i+\frac{\delta\dot{\phi}}{\delta y}j+\frac{\delta\phi}{\sigma z}k$

$$\vec{F}=(y^3z^2)i+(3xy^2z^2)j+(2xy^3z)K=f_1i+f_2j+f_3k$$

$div\,\vec{F} = \nabla \cdot \vec{F} \cdot= \frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} + \frac{\delta f_3}{\delta z}$

$$div\,\vec{F} = 0 + 6xyz^2 + 2xy^3 = 2xy(3z^2 + y^2)$$

$$div\,\vec{F} = -8\ at\ (1,-1,1)$$

curl$\vec{F}$ = $\nabla\times\ \vec{F}$= $\begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix}$

=$i(6xy^2z - 6xy^2z) - j(2y^3z - 2y^3z) + k(3y^2z^2 - 3y^2z^2)$

curl$\vec{F} = 0$

4. If $\vec{F} = (3x^2y - z)i + (xz^3 + y^4)j - 2x^3z^2k$ then find grad(div$\vec{F}$) at (2,-1,0)

   Sol: $div\,\vec{F} = \nabla \cdot \vec{F} \cdot= \frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} + \frac{\delta f_3}{\delta z}$

   $div\,\vec{F} = 6xy + 4y^3 - 4x^3z = \phi$

   grad(div$\vec{F}$) $= \nabla\phi = \frac{\delta\phi}{\delta x}i + \frac{\delta\dot{\phi}}{\delta y}j + \frac{\delta\phi}{\sigma z}k$

   grad(div$\vec{F}$) $= \langle 6y - 12x^2z\rangle i + (6x + 12y^2) - 4x^3k$

   At the point (2,-1,0) we have grad(div$\vec{F}$) $= -6i + 24j - 32k$

5. If $\vec{F} = (x + y + 1)i + j - (x + y)k$ then show that $\vec{F}.curl\vec{F} = 0$

   Sol:Curl$\vec{F} = \nabla\times\ \vec{F}$= $\begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$=i(-1-0)-j(1-0)+k(0-1)

   Curl$\vec{F} = -i + j - k$

   $\vec{F}$.Curl$\vec{F} = (x + y + 1)i + j - (x + y)k$. (-i+j-k)=$-x - y - 1 + 1 + x + y = 0$

EX:1. Find curl(curl$\vec{A}$) given that $\vec{A} = xyi + y^2zj + z^2yk$

2. Find grad(div$\vec{A}$) and div(curl$\vec{A}$) for Find $\vec{A}$=$x^2i + 3yj + x^3k$

3. If $\vec{A} = xy^2i + 2x^2yzj - 3y^2zk$ then find div$\vec{A}$, curl$\vec{A}$ and div(curl$\vec{A}$) at (2,1,1)

SOLENOIDAL AND IRROTATIONAL VECTOR FIELDS

A vector field $\vec{F}$ is said to be Solenoidal if div$\vec{F}$=0 and Irrotational if curl$\vec{F} = 0$

Irrotational field is also called as **conservative field or Potential field**

When $\vec{F}$ is Irrotational there always exists a scalar point function ϕ *such that* $\nabla\phi = \vec{F}$

ϕ is called a scalar potential of $\vec{F}$.

PROBLEMS:

1. Show that $\vec{F} = \frac{xi+yj}{x^2+y^2}$ is both Solenoidal & Irrotational.

Sol: Given $\vec{F} = \frac{x}{x^2+y^2} i + \frac{y}{x^2+y^2} j$

$$\vec{F} = f_1 i + f_2 j$$

$div\,\vec{F} = \frac{\delta f_1}{\delta x} + \frac{\delta f_2}{\delta y} = \frac{\delta}{\delta x}\left(\frac{x}{x^2+y^2}\right) + \frac{\delta}{\delta y}\left(\frac{y}{x^2+y^2}\right)$

$= \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} = 0$

$Therefore\ \ div\,\vec{F} = 0 \Rightarrow \vec{F}$ is Solenoidal

Curl$\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ \frac{x}{x^2+y^2} & \frac{y}{x^2+y^2} & 0 \end{vmatrix} = i(0-0) - j(0-0) + k\left\{\frac{\sigma}{\delta x}\left(\frac{y}{x^2+y^2}\right)\right\} -$

$\frac{\sigma}{\delta y}\left(\frac{x}{x^2+y^2}\right)$

Curl$\vec{F} = \left\{\frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2}\right\}k = 0$

Therefore Curl$\vec{F} = 0$ implies $\vec{F}$ is Irrotational.

2. Show that $\vec{F} = (y+z)i + (z+x)j + (x+y)k$ is Irrotational. Also find a scalar point function $\phi\ such\ that\ \ \nabla\phi = \vec{F}$.

   Sol: First we have to show that Curl$\vec{F} = 0$

   Curl$\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ y+z & z+x & x+y \end{vmatrix}$

   = $i(1-1) - j(1-1) + k(1-1) = 0$ implies $\vec{F}$ is Irrotational

   Consider, $\nabla\phi = \vec{F}$

$$\frac{\delta\phi}{\delta x} i + \frac{\delta\dot{\phi}}{\delta y} j + \frac{\delta\phi}{\delta z} k = (y+z)i + (z+x)j + (x+y)k$$

$\frac{\delta\phi}{\delta x} = y+z \qquad \frac{\delta\dot{\phi}}{\delta y} = z+x \qquad \frac{\delta\phi}{\delta z} = x+y$

Integrate w.r.t x integrate w.r.t y integrate w.r.t z

$\phi = \int (y+z)\,dx + f_1(y,z) \quad \phi = \int (z+x)\,dy + f_2(x,z) \quad \phi = \int (x+y)\,dz + f_3(x,y)$

$\phi = xy + xz + f_1(y,z)$

$\phi = yz + xy + f_2(x,z) \qquad f_1(y,z) = yz \quad f_2(x,z) = xz \quad f_3(x,y) = xy$

$\phi = xz + yz + f_3(x,y)$, Therefore $\phi = xy + xz + yz$

EX:SHOW THAT $\vec{F} = (2xy^2 + yz)i + (2x^2y + xz + 2yz^2)j + (2y^2z + xy)k$ is a conservative force field and also find its scalar potential.

3. Find the value of the constant 'a' such that $\vec{F} = (axy - z^3)i + (a-2)x^2 j + (1-a)xz^2 k$ is Irrotational and hence find a scalar function.

   Sol: We need to find a such that Curl$\vec{F} = 0$

$$\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = 0$$

$=i(0-0) - \dot{j}\{(1-a)z^2 - (-3z^2)\} + k\{2x(a-2) - ax\} = 0$

$= \quad -j(z^2 - az^2 + 3z^2) + k(2ax - 4x - ax) = 0$

$= \quad (a-4)z^2 j + (a-4)xk = 0$

Implies a=4

$$\vec{F} = (4xy - z^3)i + 2x^2 j - 3xz^2 k$$

Consider, $\nabla\phi = \vec{F}$

$$\frac{\delta\phi}{\delta x}i + \frac{\delta\dot{\phi}}{\delta y}j + \frac{\delta\phi}{\delta z}k = (4xy - z^3)i + 2x^2 j - 3xz^2 k$$

$\frac{\delta\phi}{\delta x} = (4xy - z^3)$ $\quad \frac{\delta\dot{\phi}}{\delta y} = 2x^2$ $\quad \frac{\delta\phi}{\delta z} = -3xz^2$ Integrate w.r.t x

integrate w.r.t y integrate w.r.t z

$\phi = \int (4xy - z^3)\,dx + f_1(y,z)$ $\quad \phi = \int 2x^2\,dy + f_2(x,z)$ $\quad \phi = \int -3xz^2\,dz + f_3(x,y)$

$\phi = 2x^2y - xz^3 + f_1(y,z)$

$\phi = 2x^2y + f_2(x,z)$ $\quad f_1(y,z) = 0$ $\quad f_2(x,z) = -xz^3$ $\quad f_3(x,y) = 2x^2y$

$\phi = -xz^3 + f_3(x,y)$

Therefore $\phi = 2x^2y - xz^3$

4. Find the constants 'a' and 'b' such that $\vec{F} = (axy + z^3)i + (3x^2 - z)j + (bxz^2 - y)k$ is irrotational and also find its scalar function.

   Sol: We need to find 'a' and 'b' such that Curl$\vec{F} = 0$

$$\text{Curl}\vec{F} = \nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\delta}{\delta x} & \frac{\delta}{\delta y} & \frac{\delta}{\delta z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix} = 0$$

$$i(-1+1) - j(bz^2 - 3z^2) + k(6x - ax) = 0$$

$$Implies\ b = 3 \ \ and\ \ a = 6$$

$$\vec{F} = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$$

Consider, $\nabla\phi = \vec{F}$

$$\frac{\delta\phi}{\delta x}i + \frac{\delta\dot{\phi}}{\delta y}j + \frac{\delta\phi}{\delta z}k = (6xy + z^3)i + (3x^2 - z)j + (3xz^2 - y)k$$

$\frac{\delta\phi}{\delta x} = (6xy + z^3)$ $\quad \frac{\delta\dot{\phi}}{\delta y} = (3x^2 - z)$ $\quad \frac{\delta\phi}{\delta z} = (3xz^2 - y)$

Integrate w.r.t x integrate w.r.t y integrate w.r.t z

$\phi = \int (6xy + z^3)\, dx + f_1(y, z)$ $\phi = \int (3x^2 - z)\, dy + f_2(x, z)$ $\phi = (3xz^2 - y)\, dz + f_3(x, y)$

$\phi = 3x^2y + xz^3 + f_1(y, z)$
$\phi = 3x^2y - yz + f_2(x, z)$ $f_1(y, z) = -yz$ $f_2(x, z) = xz^3$ $f_3(x, y) = 2x^2y$

$\phi = xz^3 - yz + f_3(x, y)$

Therefore $\phi = 3x^2y + xz^3 - yz$

EX:1. Find the constants a,b and c such that $\vec{F} = (x + y + az)i + (bx + 2y - z)j + (x + cy + 2z)k$ is irrotational and also find its scalar potential

2. Find the value of constant 'a' such that $\vec{F} = y(ax^2 + z)i + x(y^2 - z^2)j + 2xy(z - xy)k$ is solenoidal.

Maharaja Institute of Technology Mysore

Department of Mathematics

Subject: **ADDITIONAL MATHEMATICS – I** **Subject Code:** **18DIPMATAT31**

Syllabus

Module-4

Integral Calculus

Review of elementary integral calculus. Reduction formulae for $\sin^n$, $\cos^n x$ (with proof) and $\sin^m x \cos^n x$ (without proof) and evaluation of these with standard limits-Examples. Double and triple integrals-Simple examples.

Index

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Reduction Formulae.

Obtain the reduction formula for $\int sin^n\, x\, dx$ and hence deduce $\int_0^{\pi/2} sin^n x\, dx$.

Solution : $\int sin^n x\, dx = \int sin^{n-1}x\, sinx\, dx$

$$= sin^{n-1}x(-cosx) - \int (n-1)sin^{n-2}x\, cosx\, (-cosx)dx.$$

$$= -sin^{n-1}x\, cosx + (n-1)\int sin^{n-2}x\, dx - (n-1)\int sin^n x\, dx.$$

$$\int sin^n x\, dx = -\frac{sin^{n-1}x\, cosx}{n} + \frac{n-1}{n}\int sin^{n-2}\, dx\text{.......(1)}$$

Let

$$I_n = \int_0^{\pi/2} sin^n x\, dx = \left[\frac{sin^{n-1}x\ cosx}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} sin^{n-2}x\, dx$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

Case (1) when n is odd

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}\ ,\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}\ , \dots \dots . I_5 = \frac{4}{5}\, I_3\ \ , I_3 = \frac{2}{3}\, I_1$$

And $I_1 = \int_0^{\pi/2} sinx\, dx = 1$

Case (2) when n is even

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4} I_{n-4} = \frac{n-5}{n-4} I_{n-6}\ , \dots \dots . I_4 = \frac{3}{4}\, I_2\ \ , I_2 = \frac{1}{2}\, I_0,$$

And $I_0 = \int_0^{\pi/2} sinx\,^0 dx = \pi/2$

$$therefore\ we\ have\ \ I_n = \frac{(n-1)(n-3)(n-5)\dots 3.1}{n(n-2)(n-4)\dots .4.2} \cdot \frac{\pi}{2}$$

Obtain the reduction formula for $\int cos^n\, x\, dx$ and hence deduce $\int_0^{\pi/2} cos^n x\, dx$.

Solution : $\int cos^n x\, dx = \int cos^{n-1} x\, cosx\, dx$

$$= cos^{n-1}x(sinx) - \int (n-1)cos^{n-2}x(-\,sinx)\,(sinx)dx.$$

$$= cos^{n-1}x\, sinx + (n-1)\int cos^{n-2}x\, dx - (n-1)\int cos^n x\, dx.$$

$$\int cos^n x\, dx = -\frac{cos^{n-1}x\, sinx}{n} + \frac{n-1}{n}\int cos^{n-2}\, dx \ldots\ldots(1)$$

Let

$$I_n = \int_0^{\pi/2} cos^n x\, dx = \left[\frac{cos^{n-1}x\ \ sinx}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} cos^{n-2}x\, dx$$

$$I_n = \frac{n-1}{n} I_{n-2}$$

Case (1) when n is odd

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4}\,,\ I_{n-4} = \frac{n-5}{n-4} I_{n-6}\,, \ldots \ldots . I_5 = \frac{4}{5}\, I_3\ \ , I_3 = \frac{2}{3}\, I_1$$

And $I_1 = \int_0^{\pi/2} cosx\, dx = 1$

Case (2) when n is even

$$I_{n-2} = \frac{n-3}{n-2} I_{n-4} I_{n-4} = \frac{n-5}{n-4} I_{n-6}\,, \ldots \ldots . I_4 = \frac{3}{4}\, I_2\ \ , I_2 = \frac{1}{2}\, I_0,$$

And $I_0 = \int_0^{\pi/2} cosx^0 dx = \pi/2$

$$therefore\ we\ have\ \ I_n = \frac{(n-1)(n-3)(n-5)\ldots 3.1}{n(n-2)(n-4)\ldots .4.2} \cdot \frac{\pi}{2}$$

Note:

$$\int_0^{\pi/2} \sin^m x \cos^n x\ \ dx = \frac{(m-1)(m-3)(m-5)\ldots .\times(n-1)(n-3)(n-5)\ldots .}{(m+n)(m+n-2)(m+n-4)\ldots ..} \times k$$

$$\text{where}\ \ k = \begin{cases} \pi/2\ \text{,if both m and n are even} \\ 1\ \text{,other wise.} \end{cases}$$

Problems: Evaluate the following:

1. $\int_0^{\pi/2} cos^6x\, dx$

Solution : Given n =6 even

$$\int_0^{\pi/2} cos^6x\, dx = \frac{5.3.1.\pi}{6.4.2.2} = \frac{5\pi}{16}$$

2. $\int_0^a \frac{x^7}{\sqrt{a^2-x^2}}\, dx$

Solution:$put\ x = a\ sin\theta\ , dx = a\cos\theta\, d\theta\ \ if\ x \to 0 to\ a\ , \theta \to 0\ to\ \pi/2$

$$\int_0^a \frac{x^7}{\sqrt{a^2-x^2}}\, dx = \int_0^{\pi/2} \frac{a^7 sin^7\theta}{a\ cos\theta}\ a\ cos\theta\ d\theta$$

$$= \int_0^{\pi/2} a^7 sin^7 d\theta = a^7\ \frac{6.4.2}{7.5.3.1} = \frac{16}{35}a^7$$

3. $\int_0^{\pi/6} cos^4 3\theta\ sin^3 6\theta\ d\theta$

Solution : $put\ 3\theta = x\ \ , dx = 3d\theta\ \ and\ if\ \theta \to 0\ to \frac{\pi}{6}\ \ , x \to 0\ to\ \pi/2$

$$\int_0^{\pi/6} cos^4 3\theta\ sin^3 6\theta\ d\theta = \int_0^{\pi/6} cos^4 3\theta(\ 2sin3\theta\ cos3\theta)^3\ d\theta$$

$$= 8\int_0^{\pi/6} sin^3 3\theta cos^7 3\theta\ d\theta$$

$$= \frac{8}{3}\ \int_0^{\pi/2} sin^3 x cos^7 x\ dx$$

$$= \frac{8}{3}\frac{2.6.4.2}{10.8.6.4.2} = \frac{1}{15}$$

ASSIGNMENT PROBLEMS

Evaluate the following:

1. $\int_0^{\pi/2} cos^9 x \ dx.$
2. $\int_0^{\pi/6} sin^5 3\theta \ d\theta.$
3. $\int_0^{\pi/2} sin^{15} x \, cos^3 x \, dx.$
4. $\int_0^1 \frac{x^9}{\sqrt{1-x^2}} \, dx$
5. $\int_0^4 x^3 \sqrt{4x - x^2} \, dx$

DOUBLE AND TRIPLE INTERGALS

Double integrals: $\iint_R f(x,y)\,dA = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)\,dy\,dx.$

Triple integrals: $\iiint_R f(x,y,z)dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)dz\,dy\,dx.$

Problems

Evaluate the following:

1. $\int_0^1\int_x^{\sqrt{x}} x\,y\;dy\,dx. = \int_0^1 x\left[\frac{y^2}{2}\right]_x^{\sqrt{x}} dx = \frac{1}{2}\int_0^1 (x^2 - x^3)\,dx$

$$= \frac{1}{2}\left[\left(\frac{x^3}{3}\right) - \left(\frac{x^4}{4}\right)\right]_0^1 = \frac{1}{24}.$$

2. $\int_0^{\pi/2}\int_0^{\pi} \sin(x+y)\;dy\,dx. = \int_0^{\pi/2}[-\cos(x+y)]_0^{\pi}\,dy$

$$= 2\int_0^{\pi/2} cosy\,dy = 2[siny]_0^{\pi/2} = 2.$$

Assignment Problems:

1. $\int_0^2\int_0^{2-y} x\,y\,dx\,dy.$
2. $\int_0^4\int_0^{x^2} e^{\frac{y}{x}}\,dy\,dx.$
3. $\int_0^{\pi}\int_0^{siny} dx\,dy.$
4. $\int_0^{\pi}\int_{2sin\theta}^{4sin\theta} r^3 dr\,d\theta.$
5. $\int_{b/2}^{b}\int_0^{\pi/2} r\,d\theta\,dr.$

Evaluate the following:

1. $\int_0^2 \int_1^3 \int_1^2 xy^2z \; dz\, dy\, dx$

$$=\int_0^2 x\, dx \times \int_1^3 y^2 dy \times \int_1^2 z\, dz = 26.$$

2. $\int_{-c}^{c} \int_{-b}^{b} \int_{-a}^{a} (x^2 + y^2 + z^2)\, dz\, dy\, dx.$

$$= \int_{-c}^{c} \int_{-b}^{b} x^2 [z]_{-a}^{a} + y^2 [z]_{-a}^{a} + \left[\frac{z^3}{3}\right]_{-a}^{a} \; dy\, dx.$$

$$= \int_{-c}^{c} \int_{-b}^{b} 2a\, x^2 + 2ay^2 + 2\frac{a^3}{3} \; dy\, dx.$$

$$= \int_{-c}^{c} 4abx^2 + \frac{4ab^3}{3} + \frac{4a^3b}{3} \; dx$$

$$= \frac{8abc}{3}[a^2 + b^2 + c^2]$$

3. $\int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z}\, dz\, dy\, dx.$

$$= \int_0^{\log 2} \int_0^{x} [e^{x+y+z}]_0^{x+\log y}\, dz\, dy.$$

$$= \int_0^{\log 2} \int_0^{x} e^{2x+y} \; y - e^{x+y} \; dz\, dy$$

$$= \int_0^{\log 2} [ye^{2x+y} - e^{2x+y}]_0^x - [e^{x+y}]_0^x \; dx.$$

$$= \int_0^{log2} (x-1)e^{2x} + e^x\, dx$$

$$= \frac{8}{3} log2 - \frac{19}{9}.$$

4. $\int_0^{\log 2} \int_0^{x} \int_0^{x+\log y} e^{x+y+z} \; dz\, dy\, dx.$

$$= \int_0^{\log 2} \int_0^{x} [e^{x+y+z}]_0^{x+\log y}\, dz\, dy.$$

$$= \int_0^{\log 2} \int_0^{x} e^{2x+y} \; y - e^{x+y} \; dz\, dy$$

$$= \int_0^{\log 2} [ye^{2x+y} - e^{2x+y}]_0^x - [e^{x+y}]_0^x \; dx.$$

$$= \int_0^{log2} (x-1)e^{2x} + e^x\, dx$$

$$= \frac{8}{3} log2 - \frac{19}{9}.$$

5. $\int_0^1 \int_0^{1-x} \int_0^{1-x-y} \frac{dz\, dy\, dx}{(1+x+y+z)^3}$

$$= \int_0^1 \int_0^{1-x} \left[\frac{-1}{2(1+x+y+z)^2}\right]_0^{1-x-y} dy\, dx$$

$$= \int_0^1 \int_0^{1-x} \frac{-1}{2}\left[\frac{1}{4} - \frac{1}{(1+x+y)^2}\right] dy\, dx$$

$$= \frac{-1}{2}\int_0^1 \frac{1}{4}[y]_0^{1-x} - \left[\frac{-1}{(1+x+y)}\right]_0^{1-x} dx$$

$$= \frac{-1}{2} \int_0^1 \frac{3}{4} - \frac{x}{4} - \frac{1}{1+x} \ dx$$

$$= -\frac{1}{2}\left[\frac{3}{4}[x]_0^1 - \frac{1}{4}\left[\frac{x^2}{2}\right]_0^1 - [\log(1+x)]_0^1\right]$$

$$= -\frac{1}{2}[\frac{5}{8} - log2]$$

6. $\int_0^1 \int_{y^2}^1 \int_0^{1-x} x\ dz\, dx\, dy$

$$= \int_0^1 \int_{y^2}^1 x\, [z]_0^{1-x}\, dx\, dy$$

$$= \int_0^1 \int_{y^2}^1 x - x^2\, dx\, dy$$

$$= \int_0^1 \left[\frac{x^2}{2}\right]_{y^2}^1 - \left[\frac{x^3}{3}\right]_{y^2}^1 dy \ = \int_0^1 \frac{1}{6} - \frac{y^4}{2} + \frac{y^6}{3}\, dy = \frac{4}{35}$$

Assignment Problems:

1. $\int_{-1}^1 \int_0^z \int_{x-z}^{x+z} (x+y+z) dy\, dx\, dz$
2. $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z} dz\ dy\, dx$
3. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} xyz\, dz\, dy\, dx$
4. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^0 \frac{dz\, dy\, dx}{\sqrt{x^2+y^2+z^2}}$
5. $\int_0^\infty \int_0^\infty \int_0^\infty e^{-(x+y+z)}\ dx\, dy\, dz$

Maharaja Institute of Technology Mysore

Department of Mathematics

Syllabus

Subject:- Additional Mathematics - I **Subject Code:-** 18MATDIP31

Module-5

ORDINARY DIFFERENTIAL EQUATIONS (ODE'S)

Introduction-solutions of first order and first degree differential equations: exact, linear differential equations. Equations reducible to exact and Bernoulli's equation.

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Introduction

Many problems in all branches of science and engineering when analysed for putting in a mathematical form assumes the form of a differential equation. An engineer or an applied mathematician will be mostly interested in obtaing a solution for the associated equation without bothering much on the rigorous aspects like the proof, validilty , condition , region of existence etc. Accordingly the study of differential equations at various levels is focussed on the methods of solving the equations.

EXACT DIFFERENTIAL EQUATIONS

Statement: The necessary and the sufficient condition for the DE $M(x,y)dx + N(x,y)dy = 0$ to be an exact equation is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Further the solution of the exact solution is given by

$$\int M dx + \int N(y)dy = c$$

Where in the first term we integrate M wrt x keeping y fixed and $N(y)$ indicate the terms in N without x

Example1 : Solve $(2x + y + 1)dx + (x + 2y + 1)dy = 0$

Solution: Let $M = 2x + y + 1$ and $N = x + 2y + 1$

$$\frac{\partial M}{\partial y} = 1 \text{ and } \quad \frac{\partial N}{\partial x} = 1$$

Since

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact

The solution is

$$\int M dx + \int N(y)dy = c$$

$$\int 2x + y + 1\, dx + \int (2y + 1)dy = c$$

$x^2 + xy + x + y^2 + y = c$ is the required solution

Example 2: Solve $(y^3 - 3x^2y)dx - (x^3 - 3xy^2)dy = 0$

Solution: Let $M = y^3 - 3x^2y$ and $N = x^3 - 3xy^2$

$\frac{\partial M}{\partial y} = 3y^2 - 3x^2$ and $\frac{\partial N}{\partial x} = -3x^2 + 3y^2$

Since

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, the given equation is exact

The solution is

$$\int M dx + \int N(y) dy = c$$

$$\int (y^3 - 3x^2y)\, dx + \int 0\, dy = c$$

$xy^3 - x^3y = c$ is the required solutiom

(3) Solve $\boldsymbol{(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)\, dy = 0}$

(4) **Solve**$\boldsymbol{\frac{dy}{dx} = \frac{ycosx+siny+y}{sinx+xcosy+x}}$

Equation Reducible to the Exact Form

Sometimes the given DE which is not an exact equation can be transformed into an exact equation by multiplying with some function known as Integrating Factor (IF)

Worked Problems

1. Solve $\boldsymbol{(4xy + 3y^2 - x)dx + x(x + 2y)dy = 0}$

Solution: $M = 4xy + 3y^2 - x$ and $N = x(x + 2y)$

$\frac{\partial M}{\partial y} = 4x + 6y$ and $\frac{\partial N}{\partial x} = 2x + 2y$ The equation is not exact

Consider $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2x + 4y = 2(x + 2y)$ $close\ to\ N$

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{x}$

Hence, $e^{\int f(x)dx}$ is an integrating factor

$$e^{\int f(x)dx} = e^{\int \frac{2}{x} dx} = x^2$$

Multiplying the given equation by x^2 we now have,

$M = 4xy + 3y^2 - x$ and $N = x^4 + 2x^3y$

$\frac{\partial M}{\partial y} = 4x^3 + 6x^2 y$ and $\frac{\partial N}{\partial x} = 4x^3 + 6x^2 y$

Solution of the exact equation is

$$\int M dx + \int N(y) dy = c$$

$$\int (4x^3 y + 3x^2 y^2 - x^3)\, dx + \int 0\, dy = c$$

$x^4 y + x^3 y^2 - \frac{x^4}{4} = c$ is the required solution

(2) Solve $\boldsymbol{y(2x - y + 1)dx + x(3x - 4y + 3)dy = 0}$

Solution: Let $M = y(2x - y + 1)$ and $N = x(3x - 4y + 3)$

$\frac{\partial M}{\partial y} = 2xy - 2y + 1$ and $\frac{\partial N}{\partial x} = 6x - 4y + 3$ The equation is not exact

Consider $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -4x + 2y - 2 = -2(2x - y + 1)$ $close\ to\ M$

Now, $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{y} = g(y)$

Hence, $e^{-\int g(y)dy}$ is an integrating factor

$$e^{-\int g(y)dy} = e^{\int \frac{2}{y} dy} = y^2$$

Multiplying the given equation by x^2 we now have,

$M = 2xy^3 - y^4 + 3y^3$ and $N = 3x^2 y^2 - 4xy^3 + 3xy^2$

Solution of the exact equation is

$$\int M dx + \int N(y) dy = c$$

$$\int (2xy^3 - y^4 + y^3)\, dx + \int 0\, dy = c$$

Thus, $x^2 y^3 - xy^4 + xy^3 = c$ is the required solution

(3) Solve $\boldsymbol{(8xy - 9y^2)dx + 2(x^2 - 3xy)dy = 0}$

Solution: Let $M = 8xy - 9y^2$ and $N = 2(x^2 - 3xy)$

$\frac{\partial M}{\partial y} = 8x - 18y$ and $\frac{\partial N}{\partial x} = 4x - 6y$ The equation is not exact

Consider $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 4(x - 3y)\ close\ to\ N$

Now, $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{2}{x} = f(x)$

Hence, $e^{\int f(x)dx}$ is an integrating factor

$$e^{\int f(x)dx} = e^{\int \frac{2}{x}dx} = x^2$$

Multiplying the given equation by x^2 we now have,

$M = 8x^3y - 9x^2y^2$ and $N = 2x^4 - 6x^3y$

$\frac{\partial M}{\partial y} = 8x^3 - 18x^2y$ and $\frac{\partial N}{\partial x} = 8x^3 - 18x^2y$

Solution of the exact equation is

$$\int Mdx + \int N(y)dy = c$$

$$\int 8x^3y - 9x^2y^2\ dx + \int 0\ dy = c$$

$2x^4y - 3x^3y^2 = c$ is the required solution

(4) **Solve** $[\boldsymbol{y^4 + 2y}]\boldsymbol{dx} + [\boldsymbol{xy^3 + 2y^4 - 4x}]\boldsymbol{dy} = \boldsymbol{0}$

(5) **Solve** $(\boldsymbol{x^2 + y^2 + x})\boldsymbol{dx} + \boldsymbol{xydy} = \boldsymbol{0}$

<u>Bernoulli's Differential Equation</u>

The DE of the form $\frac{dy}{dx} + Py = Qy^n$ where P and Q are functions of x is called as Bernoulli's Differential Equation in y.

Worked Problems

(1) **Solve** $\boldsymbol{\frac{dy}{dx} + \frac{y}{x} = y^2x}$

Solution: This is Bernoulli's equation. Dividing the given equation by y^2 we have

$$\frac{1}{y^2}\frac{dy}{dx}+\frac{1}{yx}=x----(1)$$

Put $\frac{1}{y}=t \quad \frac{1}{y^2}\frac{dy}{dx}=\frac{dt}{dx}$

Hence, (1) becomes $\frac{-dt}{dx}+\frac{1}{x}=x$ or $\frac{dt}{dx}-\frac{t}{x}=-x$

This equation is a linear equation of the form $\frac{dt}{dx}+Pt=Q$, where

$P=-\frac{1}{x}$ and $Q=-x$

$$e^{\int Pdx}=e^{-\int\frac{1}{x}dx}=\frac{1}{x}$$

The solution is $te^{\int Pdx}=\int Qe^{\int Pdx}dx+c$

$$t\frac{1}{x}=\int -x.\frac{1}{x}\ dx+c$$

Thus $\frac{1}{xy}=-x+c$ is the required solution

(2) Solve $\frac{dy}{dx}-\frac{1}{2}\left(1+\frac{1}{x}\right)y+\frac{3y^3}{x}=0$

Solution: This is Bernoulli's equation. Dividing the given equation by y^3 we have

$$\frac{1}{y^3}\frac{dy}{dx}-\frac{1}{2}\left(1+\frac{1}{x}\right)\frac{1}{y^2}=\frac{-3}{x}----(1)$$

Put $\frac{1}{y^2}=t \quad \frac{-2}{y^3}\frac{dy}{dx}=\frac{dt}{dx}$ or $\frac{1}{y^3}\frac{dy}{dx}=-\frac{1}{2}\frac{dt}{dx}$

Hence, (1) becomes $\frac{-1}{2}\frac{dt}{dx}-\frac{t}{x}\left(1+\frac{1}{x}\right)=\frac{-3}{x}$

$$\frac{dt}{dx}+\left(1+\frac{1}{x}\right)t=\frac{6}{x}$$

This equation is a linear equation of the form$\frac{dt}{dx}+Pt=Q$, where

$P=1+\frac{1}{x}$ and $Q=\frac{6}{x}$

$$e^{\int Pdx}=e^{\int 1+\frac{1}{x}dx}=e^x x$$

The solution is $te^{\int Pdx}=\int Qe^{\int Pdx}dx+c$

$$te^x x=\int\frac{6}{x}e^x x\ dx+c$$

Thus $\frac{e^x x}{y^2} = 6e^x + c$ is the required solution

(3) **Solve** $\boldsymbol{r sin\theta - cos\theta \frac{dr}{d\theta} = r^2}$

Solution: we have $cos\theta \frac{dr}{d\theta} - rsin\theta = -r^2$

Dividing by r^2 we get $\frac{cos\theta}{r^2}\frac{dr}{d\theta} - \frac{1}{r}sin\theta = -1 ------ (1)$

Put $\frac{1}{r} = y$ and differentiate w.r.t θ

$\frac{-1}{r^2}\frac{dr}{d\theta} = \frac{dy}{d\theta}$ and hence (1) becomes

$-cos\theta \frac{dy}{d\theta} - ysin\theta$=-1 or $\frac{dy}{d\theta} + ytan\theta = sec\theta$

This equation is a linear equation of the form$\frac{dt}{dx} + Pt = Q$, where
$P = tan\theta$ and $Q = sec\theta$

$$e^{\int Pd\theta} = e^{\int tan\theta\, d\theta} = sec\theta$$

The solution is $ye^{\int Pd\theta} = \int Qe^{\int Pd\theta} d\theta + c$

$$ysec\theta = \int sec^2\theta \; d\theta + c$$

$$ysec\theta = tan\theta + c$$

$\frac{sec\theta}{r} = tan\theta + c$ is the required solution

(4) **Solve** $\boldsymbol{\frac{dy}{dx} + ytanx = y^2 secx}$

(5) **Solve** $\boldsymbol{y(2xy + e^x)dx - e^x dy = 0}$

(6) **Solve** $\boldsymbol{x^3 \frac{dy}{dx} - x^2 y = -y^4 cosx}$