Module-4

Infinite Series :- If u_n is a function of n, defined for all integral values of n, an expression of the form $u_1+u_2+u_3 --- u_n + ---$ containing infinite number of terms is called an infinite series and is usually denoted by $\sum_{n=1}^{\infty} u_n$ (or) $\sum u_n$ where u_n is the n^{th} term of series (or) the general term of the infinite series.

Suppose S_n is the sum of the 1^{st} n terms of a series and is denoted by $S_n = u_1+u_2+u_3+ --- +u_n$

We know that for a geometric series having the 1^{st} term a and the common ratio r. Then the S_n becomes

$$S_n = \frac{a(1-r^n)}{1-r} \quad r<1 \text{ and}$$

$$S_n = \frac{a(r^n-1)}{r-1} \quad r>1$$

Note

1. Suppose S_n be the sum of n number of terms of a series, then if $\lim_{n\to\infty} S_n = l$, where l is a finite value, then the given series is called the Convergent and if $\lim_{n\to\infty} S_n=\infty$. Then we say that the given series is a divergent

2.

If $\sum u_n$ is the sum of the positive terms (or) a series of positive terms. If $\lim_{n\to\infty} (u_n)^{1/n} = l$ is a infinite and the series is convergent if $l<1$ and divergent if $l>1$ and the test fails if $l=1$

3 a) $\lim_{n\to\infty} n^{1/n} = 1$

b) $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e = 2.7$

c) $\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x$

1 Test the Convergence of a Series $1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+----$

Soln :- $S_n = 1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}$ ---- is a geometric Series with common ratio $\frac{1}{3} < 1$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$S_n = \frac{1\left(1-\left(1/3\right)^n\right)}{1-1/3}$$

$$S_n = \frac{\left[1-\frac{1}{3^n}\right]}{2/3}$$

$$S_n = \frac{3}{2}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \lim_{n\to\infty} \frac{3}{2}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \frac{3}{2} \lim_{n\to\infty}\left[1-\frac{1}{3^n}\right]$$

$$\Rightarrow \frac{3}{2}\left[1-\frac{1}{\infty}\right]$$

$$\Rightarrow \frac{3}{2}(1-0)$$

$$\lim_{n\to\infty} S_n = \frac{3}{2} = \text{finite}$$

Does the Series is Convergent

2) Test the convergence of the series $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+---$

Soln Let $S=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+---$

Here the n^{th} term of a series is $u_n=\frac{1}{n(n+1)}=\frac{1}{n}-\frac{1}{n+1}$

$$S_n=\sum u_n$$

$$\Rightarrow S_n=\sum\left|\frac{1}{n}-\frac{1}{n+1}\right|$$

$$\Rightarrow S_n=\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+-------$$
$$+\left(\frac{1}{n-1}-\frac{1}{n}\right)+\left(\frac{1}{n}-\frac{1}{n+1}\right)$$

$$\Rightarrow S_n=1-\frac{1}{n+1}$$

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{n(n+1)}{2}$$

$$S_n=\frac{1}{2}\lim_{n\to\infty} n^2\left[1+\frac{1}{n}\right]=\infty$$

Does the series is divergent

3) Test the convergent of $1+2+2^2+2^3+---$

Soln Let $S=1+2+2^2+2^3+---$ is an infinite geometric ratio $r>1$

$$S_n=1+2+2^2+2^3+---+2^n$$

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$S_n=\frac{2^n-1}{2-1}$$

$$S_n=2^n-1$$

$$\lim_{n\to\infty} S_n=\lim_{S\to\infty} 2^n-1=\infty$$

$\therefore$ Does the series is divergent

4) Test The Convergent of $1+2+2^2+2^3+ - - -$

Soln Let $S = 1+2+2^2+2^3+ --- + 2^n$ is an infinite Geometric ratio $r > 1$

$$S_n = 1+2+2^2+2^3+ --- + 2^n$$

$$S_n = \frac{a(r^n - 1)}{r-1}$$

$$S_n = \frac{2^n - 1}{2-1}$$

$$S_n = 2^n - 1$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} 2^n - 1$$

$$= \infty$$

Does the Series is divergent

Cauchy's root test

5) find the nature of the Series $\sum_{n=1}^{\infty} a^{n^2} \cdot x^n$. $a < 1$

Soln :-

$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty} a^{n^2} \cdot x^n \quad a < 1$$

$$U_n = a^{n^2} x^n$$

$$\Rightarrow (U_n)^{1/n} = (a^{n^2} \cdot x^n)^{1/n} = (a^{n^2})^{1/2} \cdot (x^n)^{1/n}$$

$$\Rightarrow (U_n)^{1/n} = a^n \cdot x$$

$$\Rightarrow \lim_{n\to\infty} U_n^{1/n} = \lim_{n\to\infty} a^n \cdot x$$

$$\lim_{n\to\infty} U_n^{1/n} = x \lim_{n\to\infty} a^n = x(0) = 0$$

$$= \lim_{n\to\infty} U_n = 0 < 1$$ $\therefore$ Does the Series is Convergent

6] find the nature of the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$

Soln :-
$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^{n^2}$$

$$\Rightarrow U_n = \left(1+\frac{1}{n}\right)^{n^2}$$

$$\Rightarrow (U_n)^{1/n} = \left|\left(1+\frac{1}{n}\right)^{n^2}\right|^{1/n}$$

$$\Rightarrow (U_n)^{1/n} = \left[1+\frac{1}{n}\right]^{n}$$

$$\lim_{n\to\infty} (U_n)^{1/n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e = 2.7 > 1$$

Does the series is divergent

7] test for convergent $\sum_{n=1}^{\infty}\left(n+\frac{1}{n}\right)^{n^2}\cdot\left(\frac{1}{3^n}\right)$

Soln :-
$$\sum_{n=1}^{\infty} U_n = \sum_{n=1}^{\infty}\left|\frac{n+1}{n}\right|^{n^2}\cdot\frac{1}{3^n}$$

$$\Rightarrow U_n = \left(\frac{n+1}{n}\right)^{n^2}\cdot\frac{1}{3^n}$$

$$\Rightarrow U_n = \left(1+\frac{1}{n}\right)^{n^2}\cdot\frac{1}{3^n}$$

$$\Rightarrow (U_n)^{1/n} = \left[\left(1+\frac{1}{n}\right)^{n^2}\right]\cdot\frac{1}{3^n}\Bigg]^{1/n}$$

$$\Rightarrow U_n^{1/n} = \left[\left(1+\frac{1}{n}\right)^{n^2}\right]^{1/n}\cdot\left[\frac{1}{3^n}\right]^{1/n}$$

$$\Rightarrow U_n^{1/n} = \left(1+\frac{1}{n}\right)^{n}\cdot\frac{1}{3}$$

$$\Rightarrow U_n^{1/n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^{n}$$

$$\lim_{x\to\infty} U_n^{1/n} = \frac{1}{3}\left(1+\frac{1}{n}\right)^{n} = \frac{e}{3} = \frac{2.7}{3} = 0.9 < 1$$

Does the series is convergent

8] Discuss the Convergent $\sum_{n=1}^{\infty} \frac{(n+1)^n \cdot x^n}{n(n+1)}$

Soln:- $u_n = \frac{(n+1)^n \cdot x^n}{n(n+1)}$

$$\Rightarrow u_n = \frac{n^n\left(1+\frac{1}{n}\right)^n \cdot x^n}{n^n \cdot n^1}$$

$$\Rightarrow u_n = \frac{\left(1+\frac{1}{n}\right)^n \cdot x^n}{n}$$

$$\Rightarrow (u_n)^{1/n} = \left[\frac{\left(1+\frac{1}{n}\right)^n \cdot x^n}{n}\right]^{1/n}$$

$$\Rightarrow u_n^{1/n} = \frac{\left[\left(1+1/n\right)^n\right]^{1/n} \cdot (x^n)^{1/n}}{n^{1/n}}$$

$$\Rightarrow u_n^{1/n} = \frac{\left(1+\frac{1}{n}\right) \cdot x}{n^{1/n}}$$

$$\Rightarrow u_n^{1/n} = \frac{x\left(1+\frac{1}{n}\right)}{n^{1/n}}$$

$$\lim_{n\to\infty} u_n^{1/n} = x \cdot \lim_{n\to\infty} \frac{1+1/n}{n^{1/n}} = \frac{x(1+0)}{1} = x$$

$$\lim_{n\to\infty} u_n^{1/n} = x$$

$$\sum u_n = \begin{cases} \text{Convergent if } x<1 \\ \text{Divergent if } x>1 \end{cases}$$

D' alembeart's ratio test

Step 1 :- find the n^{th} term of Series say U_n

Step 2 :- find U_{n+1}

Step 3 :- find $\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = l$ (say)

Step 4 :- if $l<1$, then ΣU_n is convergent

if $l>1$, then ΣU_n is divergent

if $l=1$ Then test fails

q] Test for Convergence the Series $\frac{1^2}{2^1} + \frac{2^2}{2^2} + \frac{3^2}{2^3} + \frac{4^2}{2^4} + \cdots \; U_n = \frac{n^2}{2^n}$

Soln :-

$$U_{n+1} = \frac{(n+1)^2}{2^{n+1}}$$

$$\Rightarrow U_{n+1} = \frac{\frac{(n+1)^2}{2^{n+1}}}{\frac{n^2}{2^n}}$$

$$\Rightarrow U_{n+1} = \frac{(n+1)^2}{2^{n+1}} \times \frac{2^n}{n^2}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{n^2(1+1/n)^2}{2^n \cdot 2} \times \frac{2^n}{n^2}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\left(1+\frac{1}{n}\right)^2}{2} = \frac{1}{2}\left(1+\frac{1}{n}\right)^2$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{2} \lim_{n\to\infty} \left[1+\frac{1}{2}\right]^2 = \frac{1}{2}(1+0) = \frac{1}{2} < 1$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{1}{2} < 1$$

Does the Series is Convergent

10 Test for the convergent (or) divergent the series $\frac{3}{4+1} + \frac{3^2}{4^2+1} + \frac{3^3}{4^3+1} \cdots$

Soln :-

$$S = \frac{3}{4+1} + \frac{3^2}{4^2+1} + \frac{3^3}{4^3+1} + \cdots$$

The n^{th} Term S is $U_n = \frac{3^{n+1}}{4^{n+1}+1}$

$$U_{n+1} = \frac{3^{n+2}}{4^{n+2}+1}$$

$$\frac{U_{n+1}}{U_n} = \frac{3^{n+2}}{4^{n+2}+1} \times \frac{4^{n+1}+1}{3^{n+1}}$$

$$\frac{U_{n+1}}{U_n} = \frac{3 \cdot 3^{n+1}}{4 \cdot 4^{n+1}+1} \times \frac{4^{n+1}+1}{3^{n+1}}$$

$$= 3\left[\frac{4^{n+1}+1}{4 \cdot 4^{n+1}+1}\right]$$

$$= 3\left\{\frac{4^{n+1}\left[1+\frac{1}{4^{n+1}}\right]}{4^{n+1}\left[4+\frac{1}{4^{n+1}}\right]}\right\}$$

$$= 3.\left[\frac{1+\frac{1}{4^{n+1}}}{4+\frac{1}{4^{n+1}}}\right]$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 3. \lim_{n\to\infty} \frac{1+\frac{1}{4^{n+1}}}{4+\frac{1}{4^{n+1}}}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = 3\left(\frac{1+0}{4+0}\right)$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = \frac{3}{4} < 1 \quad \text{Convergent}$$

iii) Discuss the nature of the series $\sqrt{\frac{1}{2}}x + \sqrt{\frac{2}{3}}x^2 + \sqrt{\frac{3}{4}}x^3 + \cdots$

$\Rightarrow$

$$S = \sum_{n=1}^{\infty} \sqrt{\frac{n}{n+1}}\, x^n$$

$$\therefore\ U_n = \sqrt{\frac{n}{n+1}}\, x^n$$

$$U_{n+1} = \sqrt{\frac{n+1}{n+2}}\, x^{n+1}$$

$$\frac{U_{n+1}}{U_n} = \frac{\frac{\sqrt{n+1}}{\sqrt{n+2}} \times x^{n+1}}{\frac{\sqrt{n}}{\sqrt{n+1}}\, x^n}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\sqrt{n+1}}{\sqrt{n+2}} \cdot x^n \cdot x^1 \quad \frac{\sqrt{n+1}}{\sqrt{n}} \quad \frac{1}{x^n}$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{x^{n+1}}{\sqrt{n(n+2)}}$$

$$= \frac{x \cdot n\left(1+\frac{1}{n}\right)}{n\sqrt{1+\frac{2}{n}}}$$

$$\frac{U_{n+1}}{U_n} = \frac{x\left(1+\frac{1}{n}\right)}{\sqrt{1+\frac{2}{n}}}$$

$$\lim \frac{U_{n+1}}{U_n} = \lim_{n\to x} \frac{1+\frac{1}{n}}{\sqrt{1+\frac{2}{n}}} = x$$

$$\therefore\ \sum U_n = \begin{cases} \text{Convergence for } x<1 \\ \text{divergence for } x>1 \\ \text{fail at } x=1 \end{cases}$$

12 Discuss the nature of the series $\frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \cdots$

$\Rightarrow$ $S = \frac{1}{2} + \left(\frac{2}{3}\right)x + \left(\frac{3}{4}\right)x^2 + \left(\frac{4}{5}\right)x^3 + \cdots$

$$S = \sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right) x^{n-1}$$

$$U_n = \frac{n}{n+1} x^{n-1}$$

$$U_{n+1} = \frac{n+1}{n+2} x^{(n+1)-1}$$

$$U_{n+1} = \frac{n+1}{n+2} x^n$$

$$\Rightarrow \frac{U_{n+1}}{U_n} = \frac{\frac{n+1}{n+2} x^n}{\frac{n}{n+1} x^{n-1}}$$

$$\Rightarrow \frac{n+1}{n+2} x^n \cdot \frac{n+1}{n} \frac{1}{x^n \cdot x^{-1}}$$

$$\Rightarrow \frac{x(n+1)^2}{n(n+2)}$$

$$\Rightarrow \frac{x n^2 \left[1 + \frac{1}{n}\right]^2}{n^2 \left[1 + \frac{2}{n}\right]}$$

$$\lim_{n\to\infty} \frac{U_{n+1}}{U_n} = x \lim_{x\to\infty} \frac{\left[1 + \frac{1}{n}\right]^2}{\left[1 + \frac{2}{n}\right]}$$

$$\Rightarrow x \left[\frac{(1+0)}{(1+0)}\right] = x$$

$\sum U_n$ = Convergence when $x < 1$

Divergence when $x > 1$

13 Test for convergence (or) divergence of the series

$$\frac{1}{2\sqrt{1}}+\frac{x^2}{3\sqrt{2}}+\frac{x^4}{4\sqrt{3}}+\frac{x^6}{5\sqrt{4}}+----$$

$\Rightarrow$ $$S=\sum_{n=1}^{\infty}\frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$U_n=\frac{x^{2n-2}}{(n+1)\sqrt{n}}$$

$$U_{n+1}=\frac{x^{2(n+1)-2}}{(n+2)\sqrt{n+1}}$$

$$U_{n+1}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}$$

$$\therefore \frac{U_{n+1}}{U_n}=\frac{\dfrac{x^{2n}}{(n+2)\sqrt{n+1}}}{\dfrac{x^{2n-2}}{(n+1)\sqrt{n}}}$$

$$\Rightarrow \frac{U_{n+1}}{U_n}=\frac{x^{2n}}{(n+2)\sqrt{n+1}}\times\frac{(n+1)\sqrt{n}}{x^{2n}\,x^{-2}}$$

$$\Rightarrow \frac{x^2(n+1)\sqrt{n}}{(n+2)\sqrt{n+1}}$$

$$\Rightarrow \frac{x^2\sqrt{n+1}\,\cancel{\sqrt{n+1}}\,\sqrt{n}}{(n+2)\cancel{\sqrt{n+1}}}$$

$$\frac{U_{n+1}}{U_n}=\frac{x^2\sqrt{n(n+1)}}{n+2}$$

$$=\frac{x^2 n\left(\sqrt{1+\frac{1}{n}}\right)}{n\left(1+\frac{2}{n}\right)}$$

$$=\frac{x^2\sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}}$$

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$$\lim_{n\to\infty} = x^2 \lim_{n\to\infty} \frac{\sqrt{1+\frac{1}{n}}}{1+\frac{2}{n}} = x^2$$

$$\Sigma u_n = \begin{cases} \text{Convergence when } x^2<1 \\ \text{divergence when } x^2>1 \end{cases}$$

power Series Solution of Second order

consider a 2nd order differential Equation of a form
$P_0(x)\frac{d^2y}{dx^2} + P_1(x)\frac{dy}{dx} + P_2(x)y = 0 \longrightarrow (1)$ where $P_0(x), P_1(x). P_2(x)$ are the polynomial of the x and $P_0(x) \neq 0$. at $x=0$

Step 1 :- write the Soln of Eq^n (1) as a Series

$$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + ---- \rightarrow (2)$$

Step 2 :- find $y' = \sum_{r=1}^{\infty} a_r\, r x^{r-1}$, $y'' = \sum_{r=1}^{\infty} a_r.\, r x^{(n-1)(n-2)}$

Subetitude the y, y', y'' in Eqn (1) which results in an infinite Series

Step 3 :- In General when the co-Efficient of x^r is Equivated To 0, We Obtained recurrence relation, which helps us to determine the Constants a_2, a_3, a_4 ----- in Terms of a_0 & a_1

Step 4 :- thus we get the power Series Solution of the ODE, in The form of $y = a_0 F_1(x) + a_1 F_2(x)$

Note :- Some related Series

1) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ---$.

2) $e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + ---$

3) $\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + ----$

4) $\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + - - -$

5) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - - -$

6) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - - -$

14 Obtain Series Solution of $\frac{d^2y}{dx^2} + y = 0$

<u>Given</u> $\frac{d^2y}{dx^2} + y = 0 \rightarrow (1)$

$P_0(x) = 1 \neq 0$ for $x = 0$, $P_1(x) = 0$. $P_2(x) = 1$

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + - - - \rightarrow (2)$

$y' = \sum_{r=1}^{\infty} a_r r (x)^{n-1} \longrightarrow (3)$

$y'' = \sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r x^r = 0 \rightarrow (4)$

$\therefore$ the co-efficient x^r is $a_{n+2}(r+2)(r+1) + a_r$

$\Rightarrow a_{r+2}(r+1)(r+2) + a_r = 0$

$\Rightarrow (r+1)(r+2) a_{r+2} = -a_r$

$a_{r+2} = -\frac{a_r}{(r+1)(r+2)} \longrightarrow (5)$

when $r = 0$

$a_2 = -\frac{a_0}{2}$

when $r = 1 \Rightarrow a_3 = \frac{-a_1}{6}$

when $r = 2 \Rightarrow a_4 = \frac{-a_2}{12} = -\frac{(a_1/2)}{12} = \frac{a_0}{24}$

when $r = 3 \Rightarrow a_5 = -\frac{a_3}{20} = -\frac{a_1/6}{20} = -\frac{a_1}{120}$

when $r = 4 \Rightarrow a_6 = \frac{-a_4}{30} = -\frac{a_0/12}{30} = -\frac{a_2}{720}$

$$y = a_0 + a_1 x - \frac{a_0 x^2}{2} - \frac{a_1 x^3}{6} + \frac{a_0 x^4}{24} + \frac{a_1 x^5}{120} - \frac{a_0 x^6}{720} + \cdots$$

$$y = a_0\left[1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \cdots\right] + a_1\left[x - \frac{x^3}{6} + \frac{x^5}{120}\right]$$

15 Let us obtain the power series solution of the eqn $\frac{d^2y}{dx^2} - y = 0$

$\Rightarrow$ $$a_{r+2} = \frac{a_r}{(r+2)(r+1)} \quad (r \geq 0)$$

by putting $r = 0, 1, 2, 3, 4, \cdots$ we obtain

$$a_2 = \frac{a_0}{2},\ a_3 = \frac{a_1}{6},\ a_4 = \frac{a_0}{24},\ a_5 = \frac{a_1}{120},\ a_6 = \frac{a_0}{720}$$

Substituting these values in the expended from of $y = \sum_{r=0}^{\infty} a_r x^r$ we obtain

$$y = a_0\left[1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right] + a_1\left[x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right] \longrightarrow (1)$$

we have $(D^2 - 1)y = 0$ where $D = \frac{d}{dx}$

Auxillary Eq is $m^2 - 1 = 0$

$m = \pm 1$

$$y = c_1 e^x + c_2 e^{-x} \longrightarrow (2)$$

$$y = a_0\left[\frac{e^x + e^{-x}}{2}\right] + a_1\left[\frac{e^x - e^{-x}}{2}\right]$$

(or)

$$y = \left[\frac{a_0 - a_1}{2}\right] e^x + \left[\frac{a_0 - a_1}{2}\right] e^{-x}$$

$$y = c_1 e^x + c_2 e^{-x} \quad \text{where } c_1 = \frac{a_0 + a_1}{2}$$

$$c_2 = \frac{a_0 - a_1}{2}$$

16 Solve $\frac{d^2y}{dx^2}+xy=0$ by obtaining the solution in the form of series

$\Rightarrow$ $P_0(x)=1$, $P_1(x)=0$, $P_2(x)=x$

Let $\sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \text{-----}$ ②

$y' = \sum_{r=1}^{\infty} a_r \cdot r x^{r-1} \longrightarrow$ ③

$y'' = \sum_{r=2}^{\infty} a_r \cdot r(r-1) x^{r-2} \longrightarrow$ ④

① $\Rightarrow$ $\sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r x^r \cdot x$

$\sum_{r=2}^{\infty} a_r r(r-1) x^{r-2} + \sum_{r=0}^{\infty} a_r x^{r+1} \longrightarrow$ ⑤

The co-efficient x^r is $a_{r+2}(r+2)(r+1) + a_{r-1} \rightarrow$ ⑥

$a_{r+2}(r+2)(r+1) = -a_{r-1}$

$a_{r+2} = -\frac{a_{r-1}}{(r+2)(r+1)} \longrightarrow$ ⑦ $\quad x \geq 1,\ a_2 = 0$

when $r=1$, $a_3 = -\frac{a_0}{6}$

when $r=2$, $a_4 = -\frac{a_1}{12}$

when $r=3$, $a_5 = -\frac{a_2}{20} = 0$

when $r=4$, $a_6 = -\frac{a_3}{30} = -\frac{a_0/6}{30} = -\frac{a_0}{180}$

Frobenous method

Consider the Second order differential Eqn $p_0(x)\frac{d^2y}{dx^2} + p_1(x)\frac{dy}{dx} + p_2(x)y = 0$ for which $p_0(x) = 0$ when $x = 0$

Working procedure

1) Write the Solution of the Given D.E $\rightarrow$ ① as $y = \sum_{r=0}^{\infty} a_r x^{k+r} \rightarrow$ ②

$$y = x^k(a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots)$$

2) find $y' = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$, $y'' = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$ and Substitute the Same in the Equation ①

3) Equate the co-efficient of x^k [when $r=0$] at 0. we may get the polynomial of Second order in k & Gives two roots, Say k_1 & k_2

4) Also Equate the co-efficient of x^{k+1} [when $r=1$] to 0 and find the value of a_1

5) On both Side Equivate co-efficient of x^{k+r} and get the recurrence relation

6) find the Values a_2, a_3, a_3 ---- by The help of The recurence relation

7) Substitute all the Value A is the given Eqn ① and they $k=k_1$. we make get $y_1(x)$ act $k=k_1$ and Similarly $k=k_2$ we get $y_2(x)$

8) finally write the Soln of given D.E as $y = c_1y_1(x) + c_2y_2(x)$ where c_1 and c_2 are arbitary constants

17 Derivation of bessel's function using frobenous method

$\Rightarrow$ Consider a bessele's differential Eqn

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \longrightarrow ①$$

$y=0$

Here $P(x)=0$, when $x=0$

$\therefore$ the soln is $y=\sum_{r=0}^{\infty} a_r x^{k+r} \longrightarrow ②$

$$y'=\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1} \longrightarrow ③$$

$$y''=\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2} \longrightarrow ④$$

Substitude in Eq ①

$$① \Rightarrow x^2\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}+x\sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$$
$$+(x^2-n^2)\sum_{r=0}^{\infty} a_r x^{k+r}=0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r}+\sum_{r=0}^{\infty} a_r(k+r)x^{(k+r)}+\sum_{r=0}^{\infty} a_r x^{k+r+2}$$
$$-n^2\sum_{r=0}^{\infty} a_r x^{k+r}=0$$

$$\Rightarrow \sum_{r=0}^{\infty}\left[a_r(k+r)(k+r-1)+a_r(k+r)-n^2a_r\right]x^{k+r}+\sum_{r=0}^{\infty} a_r x^{k+r+2}=0$$

$$\Rightarrow \sum_{r=0}^{\infty} a_r((k+r)^2-n^2)x^{k+r}+\sum_{r=0}^{\infty} a_r x^{k+r+2}=0$$

$a_0(k^2-n^2)=0$

$a_0\neq 0$, $k^2-n^2=0$

$k=\pm n$

$k=-n$, $k=+n$

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$a_1[(k+1)^2 - n^2] = 0$

$\Rightarrow (k+1)^2 \neq n^2$

$a_1 = 0$

By making $a_r[(k+r)^2 - n^2] + a_{r-2} = 0$

$\Rightarrow a_r[(k+r)^2 - n^2] = -a_{r-2}$

$\Rightarrow a_r = \dfrac{-a_{r-2}}{(k+r)^2 - n^2}, \ \forall \ r \geq 2 \rightarrow (6)$

case (I):- when $k = n$

$(6) \Rightarrow a_r = \dfrac{-a_{r-2}}{(n+r)^2 - n^2}$

$a_r = -\dfrac{a_{r-2}}{n^2 + 2nr + r^2 - n^2}$

$a_r = -\dfrac{a_{r-2}}{2nr + r^2} \rightarrow (7) \quad \forall \ r \geq 2$

$a_2 = -\dfrac{a_0}{4n+4} = \dfrac{-a_0}{4(n+1)}$

$a_3 = \dfrac{-a_1}{6n+9} = 0$

$a_4 = \dfrac{-a_2}{8n+16}$

$= \dfrac{-1}{8n+16}\left[-\dfrac{a_0}{4(n+1)}\right]$

$= \dfrac{a_0}{8(n+2)(4(n+1))}$

$a_4 = \dfrac{a_0}{32(n+1)(n+2)}$

$a_5 = \dfrac{-a_3}{10r+25} = 0$

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$$a_6 = \frac{-a_4}{12(n+3)}$$

$$= \frac{-1}{12(n+3)}\left[\frac{a_0}{32(n+1)(n+2)}\right]$$

$$a_6 = \frac{-a_0}{384(n+1)(n+2)(n+3)}$$

$$(2) \Rightarrow y_1 = x^n\Big(a_0 + a_1x + \frac{-a_0}{4(n+1)}x^2 + 0\,x^3 + \frac{a_0x^4}{32(n+1)(n+2)}$$

$$+ 0 - \frac{a_0x^6}{384(n+1)(n+2)(n+3)} + \text{----}\Big]$$

$$y_1 = a_0x^n\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)}\right] \rightarrow (8)$$

case (2)

Illy for $k=-n$ we get

$$y_2 = a_0x^{-n}\left[1 - \frac{x^2}{4(-n+1)} + \frac{x^4}{32(-n+1)(-n+2)} - \frac{x^6}{384(-n+1)(-n+2)(-n+3)} + \text{----}\right] \rightarrow (9)$$

$\therefore$ the final soln of the given D.E

$$y = c_1y_1(x) + c_2y_2(x)$$

for the standarized of bessel of 1st kind let $a_0 = \frac{1}{2^n\sqrt{1+1}}$

$$y_1 = a_0 x^n \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots \right.$$

Let $y_0 = \dfrac{1}{2^n \Gamma(n+1)}$

$$① \Rightarrow \quad y_1(x) = J_n(x) = \frac{x^n}{2^n \Gamma(n+1)} \left| 1 - \frac{x^2}{4(n+1)} + \frac{x^4}{32(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots \right|$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{x^2}{4(n+1)(\Gamma(n+1))} + \frac{x^4}{32(n+1)\Gamma(n+1)(n+2)} - \frac{x^6}{384(n+1)(n+2)(n+3)} + \cdots \right]$$

$$= \left(\frac{x}{2}\right)^n \left[\frac{1}{\Gamma(n+1)} - \frac{x^2}{\Gamma(n+2)} + \frac{x^4}{32\,\Gamma(n+3)} - \frac{x^6}{384\,\Gamma(n+4)} + \cdots \right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \left[\frac{(-1)^0 \left(\frac{x}{2}\right)^{2(0)}}{\Gamma(n+0+1)\,0} + \frac{(-1)^1 (x/2)^{2(1)}}{\Gamma(n+1+1)\,(1)} + \frac{(-1)^2 (x/2)^{2(2)}}{(\Gamma(n+2+1))\,(2)} + \frac{(-1)^3 (x/2)^{2(3)}}{\Gamma(n+3+1)} + \cdots \right]$$

$$J_n(x) = \left(\frac{x}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r (x/2)^{2r}}{\Gamma(n+r+1)\, r}$$

$$\Rightarrow J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\Gamma(n+r+1)\, r!} \left(\frac{x}{2}\right)^{2r+n}$$

$$\text{Ily } J_{-n}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\overline{(1-n+r+1)}\, r!} \left(\frac{x}{2}\right)^{2r-n}$$

$$y = AJ_n(x) + BJ_{-n}(x)$$

18 Show that the Uezuall notation i) $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

ii) $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

$\Rightarrow$ w.k.t

$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(n+r+1)}\, r!} \left(\frac{x}{2}\right)^{2r+n} \longrightarrow ①$$

$$n = 1/2$$

$$① \Rightarrow J_{(1/2)}x = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(\tfrac{1}{2}+r+1)}\, r!} \left(\frac{x}{2}\right)^{2r+1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{3}{2})}\, r!} \left(\frac{x}{2}\right)^{2r} \left(\frac{x}{2}\right)^{1/2}$$

$$\Rightarrow J_{1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{3}{2})}\, r!} \left(\frac{x}{2}\right)^{2r}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\sqrt{\frac{3}{2}}(0!)} - \frac{1}{\sqrt{(\frac{5}{2})}(1!)} \left(\frac{x}{2}\right)^2 + \frac{1}{\sqrt{(-1/2)}(2!)} \left(\frac{x}{2}\right)^4 - \frac{1}{\sqrt{\frac{9}{2}}(3!)} \left(\frac{x}{2}\right)^6 + --- \right]$$

$$\Rightarrow J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{(\frac{1}{2})(\sqrt{1/2})} - \frac{x^2}{(\frac{3}{2})(1/2)(\sqrt{1/2}(4)} + \frac{x^4}{(\frac{5}{2})(\frac{3}{2})(1/2)\sqrt{1/2}(32)} - \frac{x^6}{(\frac{7}{2})(\frac{5}{2})(\frac{3}{2})(1/2)(\sqrt{\frac{1}{2}})384} + --- \right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}}\left[\frac{2}{\sqrt{\pi}} - \frac{x^2}{3\sqrt{\pi}} + \frac{x^4}{60\sqrt{\pi}} - \frac{x^6}{2520\sqrt{\pi}} + ---\right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \times \frac{2}{\sqrt{\pi}}\left[1 - \frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{5040} + ----\right]$$

$$J_{1/2}(x) = \frac{\sqrt{x}}{\sqrt{2}} \times \frac{\sqrt{2}\sqrt{2}}{\sqrt{\pi}} \times \frac{1}{\sqrt{x}\times\sqrt{x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ---\right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x$$

If $n = -1/2$

$$(1) \Rightarrow J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(-1/2)+r+1}\,r!}\left(\frac{x}{2}\right)^{2r-1/2}$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+1/2)}\,(r!)}\left(\frac{x}{2}\right)^{2r}\left(\frac{x}{2}\right)^{-1/2}$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\sum_{r=0}^{\infty} \frac{(-1)^r}{\sqrt{(r+\frac{1}{2})}\,(r!)}\left(\frac{x}{2}\right)^{2r}$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{x}\left[\frac{1}{\pi/2} - \frac{(x/2)^2}{(\sqrt{\frac{3}{2}})(1)} + \frac{(x/2)^4}{(\sqrt{\frac{15}{2}})(2)} - \frac{(x/2)^6}{\sqrt{(\frac{7}{2})}\,(6)} + --\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}}\left[\frac{1}{\pi/2} - \frac{x^2}{4(1/2)(\pi/2)} + \frac{x^4}{24\sqrt{\pi}} ---\right]$$

$$J_{-1/2}(x) = \frac{\sqrt{2}}{\sqrt{x}} \times \frac{1}{\sqrt{\pi}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ---\right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Orthogonal properties

Theoram

[19] **Statement** :- If α & β are the distinct root of the Eqn $J_n(x) = 0$, then $\int_0^1 x J_n(\alpha x) \cdot J_n(\beta x) = \begin{cases} 0. & \text{If } \alpha \neq \beta \\ \frac{1}{2}[J_n(\alpha)]^2 & \text{: } 2[\alpha = \beta) \end{cases}$

Proof :-

W.K.T $J_n(x)$ is the Soln of the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \longrightarrow (1)$$

lly $J_n(\lambda x)$ is a Soln for

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \lambda^2(x^2 - n^2)y = 0 \longrightarrow (2)$$

$J_n(\alpha x)$ is a Soln for the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\alpha^2 x^2 - n^2)y = 0 \longrightarrow (3)$$

& $J_n(\beta x)$ is a Soln for the Eqn

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\beta^2 x^2 - n^2)y = 0 \longrightarrow (4)$$

Let $u = J_n(\alpha x)$ and $v = J_n(\beta x)$, then

$$(3) \Rightarrow x^2 u'' + xu' + (\alpha^2 x^2 - n^2)u = 0 \longrightarrow (5)$$

$$(4) \Rightarrow x^2 v'' + xv' + (\beta^2 x^2 - n^2)v = 0 \longrightarrow (6)$$

$$(5)\ \frac{v}{x} \Rightarrow xu''v + vu' + \left(\alpha^2 x - \frac{n^2}{x}\right)uv = 0 \rightarrow (7)$$

$$(6)\ \frac{u}{x} \Rightarrow xuv' + uv' + \left(\beta^2 x - \frac{n^2}{x}\right)uv = 0 \rightarrow (8)$$

$⑦ - ⑧ \Rightarrow x(u''v - uv'') + (vu' - uv') + x(\alpha^2 - \beta^2) uv = 0$

$\Rightarrow x(u''v - uv'') + (vu' - uv') = x(\beta^2 - \alpha^2) uv$

$\Rightarrow \alpha[x(u'v - uv')] = x(\beta^2 - \alpha^2) uv$

$\Rightarrow \int_{x=0}^{1} \alpha(x(vu' - uv')] = (\beta^2 - \alpha^2) \int_0^1 x\, uv\, dx$

$\Rightarrow [x[vu_1 - uv_1]]_0^1 = (\beta^2 - \alpha^2) \int_0^1 x\, uv\, dx$

$\Rightarrow \int_0^1 x J_n(\alpha x) J_n'(\beta x)\, dx = \frac{1}{\beta^2 - \alpha^2} [J_n(\beta x) \alpha J_n(\alpha x) - \beta J_n(\beta x) J_n(\alpha x)]$

$\Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{1}{\beta^2 - \alpha^2} [\alpha J_n(\alpha J_n'(\beta) - \beta J_n(\alpha) J_n'(\beta)]$

If α & β are distinct we have $J_n(\alpha) = 0$, $J_n(\beta) = 0$

$⑨ \Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = 0$; If $\alpha \neq \beta$

If $\alpha = \beta$ we have $J_n(\alpha) = 0$

$⑨ \Rightarrow \int_0^1 x J_n(\alpha x) J_n(\beta x)\, dx = \frac{2 J_n'(x) J_n(\beta)}{\beta^2 - \alpha^2}$

$= \lim_{\beta \to \alpha} \alpha \frac{J_n(\alpha) - J_n(\beta)}{\beta^2 - \alpha^2} - \frac{0}{0}$

$= \lim_{\beta \to \alpha} \alpha \frac{J_n'(\alpha) J_n'(\beta)}{2\beta}$

$= \frac{J_n'(\alpha) J_n'(\beta)}{2\alpha}$

$= \frac{1}{2} [J_n(\alpha)]^2$ if $\alpha = \beta$

Legendre's differential Equation

Sol: Consider Legendre's linear differential eqn of 2nd order

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y=0 \longrightarrow (1)$$

Let $y = \sum_{r=0}^{\infty} a_r x^r \longrightarrow (2)$

$y' = \sum_{r=0}^{\infty} r a_r x^{r-1} \longrightarrow (3)$

$y'' = \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} \longrightarrow (4)$

$$(1) \Rightarrow (1-x^2)\sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - 2x\sum_{r=0}^{\infty} r a_r x^{r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^r =$$

$$\Rightarrow \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - \sum_{r=0}^{\infty} r(r-1) a_r x^r - \sum_{r=0}^{\infty} 2r a_r x^r + \sum_{r=0}^{\infty} n(n+1) a_r x^r = 0$$

$$\Rightarrow \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} - \left\{\sum_{r=0}^{\infty} \left| n(r-1) + 2r - n+1) \right.\right\} a_r\ x^r = 0$$

When $r=0$, the co-efficient of x^{-2} becomes $0(-1)$

$\therefore\ 0(-1)a_0 = 0$

$0 a_0 = 0$

$\Rightarrow a_0 \neq 0$

When $r=1$ the co-efficient of x^{-1} becomes

$\therefore\ 1\cdot(0)a_1 = 0$

$\Rightarrow a_1 \neq 0$

Compare the co-efficient of x^r on both sides

$$(5) \Rightarrow \left[(r+2)(r+1)a_{r+2}\right] - \left[r(r-1) + 2r - n(n+1)\right]a_r = 0$$

$$(r+1)(r+2)a_{r+2} - [r^2 - r + 2r - n(n+1)]a^r = 0$$

$$(r+1)(r+2)a_{r+2} - [r(r+1) - n(n+1)]a_r = 0$$

$$(r+1)(r+2)a_{r+2} = [r(r+1) - n(n+1)]a_r$$

$$\Rightarrow a_{r+2} = \frac{[r(r+1) - n(n+1)]a^r}{(r+1)(r+2)} \rightarrow (6)$$

when $r=0$

$$(6) \Rightarrow a_2 = \frac{[0 - n(n+1)]a_0}{2}$$

$$\Rightarrow a_2 = -\frac{n(n+1)}{2}a_0$$

lly when $r=1$

$$a_3 = \frac{[2 - n(n+1)]a_1}{6}$$

$$a_3 = \frac{[-n(n+1) - 2]a_1}{6}$$

$$a_3 = -\frac{[n^2 + n - 2]a_1}{6}$$

$$\Rightarrow a_3 = -\frac{[n^2 - n + 2n - 2]a_1}{6}$$

$$= -\frac{[n(n+1) + 2(n-1)]a_1}{6}$$

$$a_3 = -\frac{(n-1)(n+2)a_1}{6}$$

when $r=2$

$$(6) \Rightarrow a_4 = \frac{[-6 - n^2 - n]a_2}{12}$$

$$a_4 = -\frac{[n^2-2n+3n-6]a_2}{12}$$

$$= -\frac{[n(n-2)+3(n-2)]a_2}{12}$$

$$= -\frac{(n-2)(n+3)a_2}{12}$$

$$= -\left[\frac{(n-2)(n+3)}{12}\right]\left[-\frac{n(n+1)}{2}\right]a_0$$

$$a_4 = \frac{n(n+1)(n-2)(n+3)}{24}a_0$$

when $r=3$

$$a_5 = \frac{[12-n^2-n]a_3}{20}$$

$$a_5 = \frac{(-n^2+n-12)}{20}a_3$$

$$a_5 = -\frac{(n-3)(n+4)}{20}$$

$$= \left[\frac{(n-3)(n+4)}{20}\right]\left[-\frac{(n-1)(n+2)a_1}{6}\right]$$

$$a_5 = \frac{(n-1)(n-3)(n+2)(n+4)}{120}a_1$$

(2) $\Rightarrow$ $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots$

$\Rightarrow y = a_0 + a_1x - \frac{n(n+1)}{2}a_0x^2 - \frac{(n-1)(n-2)}{6}a_1x^3 + \frac{n(n+1)(n-2)(n-3)}{24}$

$a_0x^4 + \frac{(n-1)(n-3)(n+2)(n+4)}{120}a_1x^5 + \cdots$

$\Rightarrow y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n-2)(n+3)}{4!}x^4 + \cdots\right]$

$+ a_1\left[x - \frac{(n-1)(n+2)}{4!}x^3 + \frac{(n-1)(n-3)(n+2)(n+4)}{?}x^5\right]$

$y = a_0 y_1(x) + a_1 y_2(x)$

Legendre's differential Equation lead into polynomial

22. W.K.T the legendre's differential eq $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \rightarrow$ (1) By Taking the Soln $y = \sum_{r=0}^{\infty} a_r x^r$ we have the reccurance relation

$$a_{r+2} = \frac{n(n+1) - r(r+1)}{(r+1)(r+2)} a_r \rightarrow (2)$$

$y = a_0 y_1(x) + a_1 y_2(x)$ for a_0 It may be observed that the polynomial $y_1(x) \cdot y_2(x)$ Contained alternative power of x and general form The polynomial represents Either of them Increasing (or) decreasing in the powers of x and Can be represented in the form

$$y = f(x) = a_n x^n + a_{n-2} x^{n-2} + a_{n-4} x^{n-4} \text{ --- } x \rightarrow (3)$$

where $f(x) = a_0$ for n is Even

$a_1 x_1$ for n is add

When $r = n-2$

$$(2) \Rightarrow a_n = -\frac{n(n+1) - (n-2)(n-1)}{n(n-1)} a_{n-2}$$

$$a_n = -\frac{(n^2+n-n^2+3n-2)}{n(n-1)} a_{n-2}$$

$$a_n = -\frac{(4n-2)}{n(n-1)} a_{n-2}$$

$$a_n = -\frac{2(2n-1)}{n(n-1)} a_{n-2}$$

$$\Rightarrow a_{n-2} = -\frac{n(n-1)}{2(2n-1)} a_n$$

when $r=n-4$

$$② \Rightarrow a_{n-2} = -\left[\frac{n(n+1)-(n-4)(n-3)}{(n-3)(n-2)}\right] a_{n-4}$$

$$a_{n-2} = -\left[\frac{n^2+n-n^2+7n-12}{(n-2)(n-3)}\right] a_{n-4}$$

$$\Rightarrow a_{n-2} = \frac{[7n-12]}{(n-2)(n-3)} a_{n-4}$$

$$\Rightarrow a_{n-2} = \frac{-4(2n-3)}{(n-2)(n-3)} a_{n-4}$$

$$\Rightarrow a_{n-2} = -\frac{(n-2)(n-3)}{4(2n-3)} a_{n-2}$$

$$a_{n-4} = \left[\frac{-(n-2)(n-3)}{4(2n-3)}\right]\left[\frac{-n(n-1)}{2(2n-1)} a_n\right]$$

$$a_{n-4} = \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} a_n$$

$$③ \Rightarrow y = f(x) = a_n x^n - \frac{n(n-1)}{2(2n-1)} a_n x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} a_n x^{n-4} + - - -$$

$$y = f(x) = a_n \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)} x^{n-4} - - - - - G(x) \rightarrow ④\right.$$

where $g(x) = \begin{cases} a_0 (a_n & \text{for } n \text{ is Even} \\ a_1(x)\ a_n & \text{for } n \text{ is add} \end{cases}$

If the constant small a_n is choosen such that $y=f(x)$ becomes 1 when $x=1$ the polynomial so obtained are called Legender's polynomial denoted by $P_n(x)$

Let us choose $a_n = \frac{1.3.5.7 --- (2n-1)}{n!}$ the eq ① becomes

$$P_n(x) = \frac{1.3.5.7 --- (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{8(2n-1)(2n-3)}x^{n-4} + ---\right.$$

for $n=0,1,2,3$ ---- we can get the polynomials as

$P_0(x) = 1 \rightarrow$ (A)

$P_1(x) = x \rightarrow$ (B)

$P_2(x) = \frac{1}{2}(3x^2-1) \rightarrow$ (C)

$P_3(x) = \frac{1}{2}(5x^3 - 3x) \rightarrow$ (D)

$P_4(x) = \frac{1}{8}(35x^4 - 30x^2+3) \rightarrow$ (E)

in otherwards

$1 = P_0(x)$

$x = P_1(x)$

(C) $\Rightarrow 3x^2 - 1 = 2P_2(x) + 1$

$\Rightarrow 3x^2 = \frac{1}{3}[(2P_2(x)+1]$

$\Rightarrow x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$

(D) $\Rightarrow 5x^3 - 3x = 2P_3(x)$

$\Rightarrow 5x^3 - 2P_3(x) + 3x$

$\Rightarrow x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$

$\textcircled{E} \Rightarrow 35x^4 - 30x^2 + 3 = 8P_4(x)$

$\Rightarrow 35x^4 = 8P_4(x) + 30x^2 - 3$

$\Rightarrow x^4 = \frac{1}{35}\left[8P_4(x) + 30\frac{1}{3}\left(2P_2(x) + P_0(x)\right) - 3P_0(x)\right]$

$\Rightarrow x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$

Rodrigue's formula

22) for positive value of in the radrigue formula for legender polynomials can be defined as

$$P_1(n) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$P_0(x) = \frac{1}{1.1}(x^2-1) = 0$$

$$P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1) = \frac{2x}{2} = x$$

23) Show that $P_3\cos\theta = \frac{1}{8}(3\cos\theta + 5\cos3\theta)$

$\Rightarrow$ $P_3(x) = \frac{1}{2}[5x^3 - 3x] \rightarrow (1)$

$x = \cos\theta$

$(1) \Rightarrow P_3(\cos\theta) = \frac{1}{2}\left(5(\cos\theta)^3 - 3(\cos\theta)\right) \rightarrow (2)$

w.k.T $\cos3\theta = 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos^3\theta = \frac{1}{4}[\cos3\theta + 3\cos\theta]$

$(2) \Rightarrow P_3(\cos\theta) = \frac{1}{2}\left[\frac{5}{4}\cos3\theta + 3\cos\theta\right] - 3\cos\theta$

$\Rightarrow P_3(\cos\theta) = \frac{1}{2}\left[\frac{5\cos3\theta + 3\cos\theta}{4} - 12\cos\theta\right]$

$P_3\cos\theta = \frac{1}{8}[5\cos3\theta + 3\cos\theta]$

24 Use Rodrigue's formula S.T $P_4(\cos\theta) = \frac{1}{64}[35\cos\theta + 20\cos\theta + 9]$

W.K.T $\quad P_4(4) = \frac{1}{64}[35x^4 - 30x^2 + 3)$

$x = \cos\theta$

$$P_4(\cos\theta) = \frac{1}{8}\left[35\cos^4\theta - 30\cos^2\theta + 3\right]$$

$$= \frac{1}{8}\left[35(\cos^2\theta)^2 - 30\cos^2\theta + 3\right]$$

$$\cos^2\theta = \left(\frac{1+\cos 2\theta}{2}\right)$$

$$= \frac{1}{8}\left[35\left(\frac{1+\cos 2\theta}{2}\right)^2 - 30\left(\frac{1+\cos 2\theta}{2}\right) + 3\right]$$

$$= \frac{1}{8}\left[\frac{35}{4}(1 + 2\cos 2\theta + \cos^2 2\theta) - \frac{30}{2}(1+\cos 2\theta) + 3\right]$$

$$= \frac{1}{32}\left[35 + 70\cos 2\theta + 35\cos^2\theta - 60 - 60\cos 2\theta + 12\right]$$

$$= \frac{1}{32}\left[35\cos^2 2\theta + 10\cos 2\theta - 13\right]$$

$$= \frac{1}{32}\left[35\left(\frac{1+\cos 4\theta}{2}\right) + 10\cos 2\theta - 13\right]$$

$$= \frac{1}{32}\left[\frac{35 + 35\cos 4\theta + 20\cos 2\theta - 26}{2}\right]$$

$$= \frac{1}{64}\left[35\cos 4\theta + 20\cos 2\theta + 9\right]$$

25 Express $f(x) = x^3+2x^2-x-3$ in terms of lengthder's polynomial

$$\Rightarrow \quad f(x) = x^3+2x^2-x-3(1)$$

W.K.T

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$f(x) = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right] + \frac{2}{3}\left[2P_2(x) + P_0(x)\right] - P_1(x) - 3P_0(x)$$

$$= \frac{1}{15}\left[3\left[2P_2(x)+3P_1(x)\right] + 10\left[2P_2(x)+P_0(x)\right] - 15P_1(x) - 45P_0(x)\right]$$

$$\Rightarrow f(x) = \frac{1}{15}\left[6P_3(x)+9P_1(x)+20P_2(x)+10P_0(x)-15P_1(x)-45P_0(x)\right]$$

$$f(x) = \frac{1}{15}\left[6P_3(x)+20P_2(x)-6P_1(x)-35P_0(x)\right]$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{2}{3}P_1(x) - \frac{7}{3}P_0(x)$$

26 Express $f(x) = x^4+3x^3-x^2+5x-2$ in terms of legender's polynomial

$$\Rightarrow \quad f(x) = x^4+3x^3-x^2+5x-2(1) \longrightarrow (1)$$

W.K.T

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$x^4 = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right]$$

$$(1) \Rightarrow f(x) = \frac{1}{35}\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right] + \frac{3}{5}\left[2P_3(x) + 3P_1(x)\right]$$
$$- \frac{1}{3}\left[2P_2(x) + P_6(x)\right] + 5P_1(x) - 2P_0(x)$$

$$= \frac{1}{105}\Big\{3\left[8P_4(x) + 20P_2(x) + 7P_0(x)\right] + 63\left[2P_3(x) + 3P_1(x)\right]$$
$$- 35\left[2P_2(x) + P_0(x)\right] + 525P_1(x) - 210P_0(x)\Big\}$$

$$= \frac{1}{105}\left[24P_4(x) + 60P_2(x) + 20P_0(x) + 126P_3(x) + 184P_1(x)\right.$$
$$\left. - 70P_2(x) - 35P_0(x) + 525P_1(x) - 210P_0(x)\right]$$

$$\Rightarrow f(x) = \frac{1}{105}\left[24P_4(x) + 126P_3(x) - 10P_2(x) + 74P_1(x) - 224P_0(x)\right]$$

$$f(x) = \frac{8}{35}P_4(x) + \frac{6}{5}P_3(x) - \frac{2}{21}P_2(x) + \frac{34}{5}P_1(x) - \frac{32}{5}P_0(x)$$

27) If $x^3 + 2x^2 - x + 1 = aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$ find the values of a, b, c, d

Let $f(x) = x^3 + 2x^2 - x + 1$

$$1 = P_0(x)$$

$$x^2 = P_1(x)$$

$$x^2 = \left[\frac{1}{3}\left\{(2P_2(x) + P_0(x)\right\}\right]$$

$$x^3 = \frac{1}{5}\left[2P_2(x) + 3P_1(x)\right]$$

$$① \Rightarrow f(x) = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right] + \frac{2}{3}\left\{2P_2(x) + P_0(x)\right] - P_1(x) + P_0(x)\Big]$$

$$= \frac{1}{15}\Big\{3\left[2P_3(x) + 3P_1(x)\right] + 10\left[2P_2(x) + P_0(x)\right] - 15P_1(x) + 15P_0(x)\Big\}$$

$$= \frac{1}{15}\left[6P_3(x) + 9P_1(x) + 20P_2(x) + 10P_0(x) - 15P_1(x) + 15P_0(x)\right]$$

$$= \frac{1}{15}\left[25P_0(x) - 6P_1(x) + 20P_2(x) + 6P_3(x)\right]$$

$$aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x) = \frac{5}{3}P_0(x) + \left(\frac{-2}{5}\right)P_1(x) + \left(\frac{4}{3}\right)P_2(x) + \left(\frac{2}{5}\right)P_3(x)$$

$$a = \frac{5}{3},\ b = -\frac{2}{5},\ c = \frac{4}{3},\ d = \frac{2}{5}$$

[29] Express $f(x) = x^3 - 5x^2 + 14x + 5$ in terms of legendre's polynomial

$$f(x) = x^3 - 5x^2 + 14x + 5(1) \longrightarrow ①$$

$$1 = P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}\left[2P_2(x) + P_0(x)\right]$$

$$x^3 = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right]$$

$$① \Rightarrow f(x) = \frac{1}{5}\left[2P_3(x) + 3P_1(x)\right] - \frac{5}{3}\left[2P_2(x) + P_0(x)\right] + 14P_1(x) + 5P_0(x)$$

$$= \left[6P_3(x) + 9P_1(x) - 50P_2(x) - 25P_0(x) + 210P_1(x) + 75P_0(x)\right]$$

$$= \frac{1}{15}\left[6P_3(x) - 50P_2(x) + 219P_1(x) + 50P_0(x)\right]$$

$$\frac{2}{5}P_3(x) - \frac{10}{3}P_2(x) + \frac{73}{5}P_1(x) + \frac{10}{3}P_0(x)$$

25) Express $f(x) = 3x^3 - x^2 + 5x - 2(1)$ in terms of legender's polynomial

$$\Rightarrow f(x) = 3x^3 - x^2 + 5x - 2(1)$$

$$1 - P_0(x)$$

$$x = P_1(x)$$

$$x^2 = \frac{1}{3}[2P_2(x) + P_0(x)]$$

$$x^3 = \frac{1}{5}[2P_3(x) + 3P_1(x)]$$

$$f(x) = \frac{3}{5}[2P_3(x) + 3P_1(x)] - \frac{1}{3}[2P_2(x) + P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{3}{5}[2P_3(x) + 3P_1(x)] - \frac{1}{3}[2P_2(x) + P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{6}{5}P_3(x) + \frac{9}{5}P_1(x) - \frac{2}{3}P_2(x) - \frac{1}{3}P_0(x)] + 5P_1(x) - 2P_0(x)$$

$$= \frac{1}{15}[18P_3(x) + 27P_1(x) - 10P_2(x) - 5P_0(x) + 75P_1(x) - 30P_0(x)]$$

$$= \frac{1}{15}[-35P_0x + 102P_1(x) - 10P_2(x) + 18P_3(x)]$$

$$= \frac{-35}{15}P_0(x) + \frac{102}{15}P_1(x) - \frac{10}{15}P_2(x) + \frac{18}{15}P_3(x)]$$

$$= \frac{-7}{3}P_0(x) + \frac{34}{5}P_1(x) - \frac{2}{3}P_2(x) + \frac{6}{5}P_3(x)$$