Module - 5

NUMERICAL SOLUTION FOR SECOND ORDER DIFFERENTIAL EQUATION

Consider the differential equation of second order

$a_0\dfrac{d^2y}{dx^2} + a_1\dfrac{dy}{dx} + a_2 y = Q \rightarrow (1)$, where a_0, a_1, a_2 are the functions of 'x'.

Let $\dfrac{dy}{dx} = y' = z = f(x,y,z)$, then

$(1) \Rightarrow \dfrac{dz}{dx} = g(x,y,z)$ to the initial conditions $y(x_0) = y_0$,

$y'(x_0) = y'_0$

$\Rightarrow z(x_0) = z_0$.

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R-K METHOD:

$$y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$z(x_1) = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4], \text{ where}$$

$k_1 = hf(x_0, y_0, z_0)$, $l_1 = hg(x_0, y_0, z_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$, $l_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$, $l_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$, $l_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3)$

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① Solve $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1$, for $x = 0.1$, correct to 4 decimals using initial conditions $y_{(0)} = 1$, $y'_{(0)} = 0$, by using R-K method of 4^{th} order.

Sol:- Given: $\frac{d^2y}{dx^2} - x^2\frac{dy}{dx} - 2xy = 1 \rightarrow ①$

$$\text{Let } \frac{dy}{dx} = y' = z = f(x, y, z)$$

$$\therefore ① \Rightarrow \frac{dz}{dx} - x^2 z - 2xy = 1$$

$$\Rightarrow \frac{dz}{dx} = 1 + x^2 z + 2xy = g(x, y, z)$$

and given

$$y(0) = 1, \quad y'(0) = 0$$

$$\Rightarrow x_0 = 0, \ y_0 = 1, \quad y'_0 = z_0 = 0, \quad h = 0.1$$

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$K_1 = hf(x_0, y_0, z_0)$

$= (0.1)\, f(0, 1, 0)$

$= (0.1)(0)$

$K_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.1)\, g(0, 1, 0)$

$= (0.1)[1 + (0)(0) + 2(0)(1)]$

$l_1 = 0.1$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1)\, f(0.05, 1, 0.05)$

$= (0.1)\, f(0.05)$

$K_2 = 0.005$

$\therefore l_2 = (0.1)\, g(0.05, 1, 0.05)$

$= (0.1)(1.100125)$

$l_2 = 0.11$

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$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1) f(0.05, 1.0025, 0.055)$

$= (0.1)(0.055)$

$K_3 = 0.0055$

$\therefore l_3 = (0.1) g(0.05, 1.0025, 0.055)$

$= (0.1) g(1.1003875)$

$l_3 = 0.11$

$K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$= (0.1) f(0 + 0.1, 1 + 0.0055, 0 + 0.11)$

$= (0.1) f(0.1, 1.0055, 0.11)$

$= (0.1)(0.11)$

$K_4 = 0.011$

$\therefore y(x_1) = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 1 + \frac{1}{6}[0 + 2 \times 0.005 + 2 \times 0.0055 + 0.011]$

$y(0.1) \simeq \underline{1.00533}$

② Using RK method, solve $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2$. for $x = 0.2$. correct to 4 decimals using initial conditions $y_{(0)} = 1$, $y'_{(0)} = 0$.

Sol:- Given: $\frac{d^2y}{dx^2} = x\left(\frac{dy}{dx}\right)^2 - y^2 \longrightarrow ①$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = xz^2 - y^2 = g(x, y, z)$

and $y'_{(0)} = 0$, $y_{(0)} = 1$

$\Rightarrow x_0 = 0$, $y_0 = 1$ $y'_0 = z_0 = 0$, $h = 0.2$

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$k_1 = hf(x_0, y_0, z_0)$

$= (0.2)(0)$

$k_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.2)\, g(0,1,0)$

$= (0.2)(-1)$

$l_1 = -0.2$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.2)\, f(0.1, 1, -0.1)$

$= (0.2)(-0.1)$

$k_2 = -0.02$

$\therefore l_2 = (0.2)\, g(0.1, 1, -0.1)$

$= (0.2)(-0.999)$

$l_2 = -0.2$

$k_3 = (0.2)\, f(0.1, 0.99, -0.1)$

$= (0.2)(-0.1)$

$k_3 = -0.02$

$\therefore l_3 = (0.2)\, g(0.1, 0.99, -0.1)$

$= (0.2)(-0.989)$

$l_3 = -0.1978$

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$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$

$= (0.2)\, f(0.2, 0.98, -0.1978)$

$= (0.2)(-0.1978)$

$k_4 = -0.03956$

$\therefore y(x_1) = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$y(0.2) = 1 + \frac{1}{6}[0 - 0.04 - 0.04 - 0.03956]$

$\Rightarrow y(0.2) \simeq 0.98$

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(3) Find $y(0.1)$ using R-K method given that $y'' = xy' - y$, $y(0) = 3$, $y'(0) = 0$.

Sol: Given: $y'' = xy' - y \rightarrow (1)$

Let $\frac{dy}{dx} = y' = z = f(x,y,z)$

$\therefore (1) \Rightarrow \frac{dz}{dx} = xz - y = g(x,y,z)$

and $y(0) = 3$, $y'(0) = 0$

$\Rightarrow x_0 = 0$, $y_0 = 3$ $\quad y_0' = z_0 = 0$ $\quad h = 0.1$

$K_1 = hf(x_0, y_0, z_0)$

$= (0.1)\, f(0, 3, 0)$

$= (0.1)(0)$

$K_1 = 0$

$\therefore l_1 = hg(x_0, y_0, z_0)$

$= (0.1)\, g(0, 3, 0)$

$= (0.1)(-3)$

$l_1 = -0.3$

$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1)\, f\left(0 + \frac{0.1}{2}, 3 + \frac{0}{2}, 0 - \frac{0.3}{2}\right)$

$= (0.1)(-0.15)$

$K_2 = -0.015$

$\therefore l_2 = hg(0.05, 3, -0.15)$

$= (0.1)\, g(-3.015)$

$l_2 = -0.30075$

$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1)\, f\left(0.05, 3 - \frac{0.015}{2}, -\frac{0.3015}{2}\right)$

$= (0.1)\, f(0.05, 2.9925, -0.15075) = -0.015075$

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$\therefore l_3 = (0.1)\, g(0.05, 2.9925, -0.15075)$

$l_3 = -0.30000$

$K_4 = hf(x_0+h, y_0+K_3, z_0+l_3)$

$= (0.1) f(0.1, 2.9849, -0.30000)$

$K_4 = -0.03$

$\therefore y_{(0.1)} = 3 + \frac{1}{6}[0 - 0.03 - 0.03015 - 0.03]$

$\Rightarrow y_{(0.1)} \cong 2.9849$

④ Using R-K method to solve $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right)$, $y_{(0)} = 1$, $y'_{(0)} = 0.5$, find y at $x = 0.1$.

Sol: Given: $\frac{d^2y}{dx^2} = x^3\left(y + \frac{dy}{dx}\right) \rightarrow ①$

Let $\frac{dy}{dx} = y' = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = x^3(y+z) = g(x, y, z)$

and $y_{(0)} = 1$, $y'_{(0)} = 0.5$

$\Rightarrow x_0 = 0, y_0 = 1 \quad y'_0 = z_0 = 0.5$

$\Rightarrow K_1 = hf(x_0, y_0, z_0)$ $\qquad \Rightarrow l_1 = hg(x_0, y_0, z_0)$

$= 0.1\, f(0, 1, 0.5)$ $\qquad = 0.1\, g(0, 1, 0.5)$

$K_1 = 0.05$ $\qquad \Rightarrow l_1 = 0$

$\Rightarrow K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}, z_0 + \frac{l_1}{2}\right)$

$= (0.1) f(0.05, 1.025, 0.5)$

$\Rightarrow K_2 = 0.05$

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$\Rightarrow K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}, z_0 + \frac{l_2}{2}\right)$

$= (0.1)\, f(0.05, 1.025, 0.5)$

$\Rightarrow K_3 = 0.05$

$\Rightarrow l_3 = (0.1)\, g(0.05, 1.025, 0.5)$

$l_3 = 0$

$\Rightarrow K_4 = hf(x_0 + h, y_0 + K_3, z_0 + l_3)$

$= (0.1)\, f(0.1, 1.05, 0.5)$

$K_4 = 0.05$

$y_{(0.1)} = 1 + \frac{1}{6}[0.05 + 0.1 + 0.1 + 0.05]$

$y_{(0.1)} \cong 1.05$

① Apply Milne's method to compute $y_{(0.8)}$ given that $y'' = 1 - 2yy'$

x	0	0.2	0.4	0.6
y	0	0.02	0.0795	0.1762
$y' = z$	0	0.1996	0.3972	0.5689

Sol:- Given: $y'' = 1 - 2yy' \rightarrow ①$

Let $y' = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = 1 - 2yz = g(x, y, z)$

and given

$x_0 = 0$ $\quad y_0 = 0$ $\quad y'_0 = z_0 = 0$

$x_1 = 0.2$ $\quad y_1 = 0.02$ $\quad y'_1 = z_1 = 0.1996$

$x_2 = 0.4$ $\quad y_2 = 0.0795$ $\quad y'_2 = z_2 = 0.3972$

$x_3 = 0.6$ $\quad y_3 = 0.1762$ $\quad y'_3 = z_3 = 0.5689$

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$\therefore\ f_0 = f(x_0, y_0, z_0) = 0$

$f_1 = f(x_1, y_1, z_1) = 0.1996$

$f_2 = f(x_2, y_2, z_2) = 0.3972$

$f_3 = f(x_3, y_3, z_3) = 0.5689$

$g_0 = 1 - 2y_0 z_0 = 1$

$g_1 = 1 - 2y_1 z_1 = 0.9920$

$g_2 = 1 - 2y_2 z_2 = 0.93684$

$g_3 = 1 - 2y_3 z_3 = 0.7995$

$\therefore\ y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + f_3]$

$= \frac{4 \times 0.2}{3}[2 \times 0.1996 - 0.3972 + 2 \times 0.5689]$

$y_4^{(P)} = 0.3039$

$\Rightarrow z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$

$= \frac{4 \times (0.2)}{3}[1.984 - 0.93684 + 1.599]$

$z_4^{(P)} = 0.7056$

$\therefore\ f_4^{(P)} = 0.7056$

$y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$

$= 0.0795 + \frac{0.2}{3}[0.3972 + 4 \times 0.5689 + 0.7056]$

$y_4^{(C)} = 0.3047$

$y_{(0.8)} = 0.3047$

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② Apply Milne's predictor - corrector method to compute $y(0.4)$ given the differential equation $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx}$ and the following table of initial values.

x	0	0.1	0.2	0.3
y	1	1.1103	1.2427	1.3990
z	1	1.2103	1.4427	1.6990

Sol: Given: $\frac{d^2y}{dx^2} = 1 + \frac{dy}{dx} \rightarrow ①$

Let $y' = \frac{dy}{dx} = z = f(x,y,z)$

$\therefore$ ① $\Rightarrow \frac{dz}{dx} = 1 + z = g(x,y,z)$

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and given

$x_0 = 0$ $\quad y_0 = 1$ $\quad y_0' = z_0 = 1$

$x_1 = 0.1$ $\quad y_1 = 1.1103$ $\quad y_1' = z_1 = 1.2103$

$x_2 = 0.2$ $\quad y_2 = 1.2427$ $\quad y_2' = z_2 = 1.4427$

$x_3 = 0.3$ $\quad y_3 = 1.3990$ $\quad y_3' = z_3 = 1.6990$

$\therefore f_0 = f(x_0, y_0, z_0) = 1$ $\quad g_0 = 1 + z_0 = 2$

$f_1 = f(x_1, y_1, z_1) = 1.2103$ $\quad g_1 = 1 + z_1 = 2.2103$

$f_2 = f(x_2, y_2, z_2) = 1.4427$ $\quad g_2 = 1 + z_2 = 2.4427$

$f_3 = f(x_3, y_3, z_3) = 1.6990$ $\quad g_3 = 1 + z_3 = 2.6990$

$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 1 + \frac{4 \times 0.1}{3}[2 \times 1.2103 - 1.4427 + 2 \times 1.6990]$

$y_4^{(P)} = 1.5835$

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$$\Rightarrow z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2 \times 2.2103 - 2.4427 + 2 \times 2.6990]$$

$$\Rightarrow z_4^{(P)} = 1.9835$$

$$\therefore f_4^{(P)} = 1.9835$$

$$\Rightarrow y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.2427 + \frac{0.1}{3}[1.4427 + 4 \times 1.6990 + 1.9835]$$

$$y_4^{(C)} = 1.5835$$

$$y_{(0.4)} = 1.5835$$

③ Apply Milne's method to find $y_{(0.4)}$ for the given D.E $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0$.

x	0	0.1	0.2	0.3
y	1	1.03995	1.138036	1.29865
z	0.1	0.6995	1.2580	1.8730

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Sol: Given: $\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} - 6y = 0 \rightarrow ①$

Let $y' = \frac{dy}{dx} = z = f(x, y, z)$

$\therefore ① \Rightarrow \frac{dz}{dx} = -3xz + 6y$

$$\frac{dz}{dx} = 6y - 3xz = g(x, y, z)$$

and

$x_0 = 0$ $\quad y_0 = 1$ $\quad y'_0 = z_0 = 0.1$

$x_1 = 0.1$ $\quad y_1 = 1.03995$ $\quad y'_1 = z_1 = 0.6995$

$x_2 = 0.2$ $\quad y_2 = 1.138036$ $\quad y'_2 = z_2 = 1.2580$

$x_3 = 0.3$ $\quad y_3 = 1.29865$ $\quad y'_3 = z_3 = 1.8730$

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$\therefore f_0 = f(x_0, y_0, z_0) = 1$ $\qquad g_0 = 6y_0 - 3x_0 z_0 = 6$

$f_1 = f(x_1, y_1, z_1) = 0.6995$ $\qquad g_1 = 6y_1 - 3x_1 z_1 = 6.02985$

$f_2 = f(x_2, y_2, z_2) = 1.2580$ $\qquad g_2 = 6y_2 - 3x_2 z_2 = 6.073416$

$f_3 = f(x_3, y_3, z_3) = 1.8730$ $\qquad g_3 = 6y_3 - 3x_3 z_3 = 6.1062$

$$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$$

$$= 1 + \frac{4 \times 0.1}{3}[2 \times 0.6995 - 1.2580 + 2 \times 1.8730]$$

$$\Rightarrow y_4^{(P)} = 1.5183$$

$$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$$

$$= 0.1 + \frac{4 \times 0.1}{3}[2 \times 6.02985 - 6.02985 + 2 \times 6.073416]$$

$$z_4^{(P)} = 2.5236$$

$$\therefore f_4^{(P)} = 2.5236$$

$$\therefore y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$$

$$= 1.138036 + \frac{0.1}{3}[1.2580 + 4 \times 1.8730 + 2.5236]$$

$$y_4^{(C)} = 1.5138$$

(4) Use Milne's method, obtain an approximate solution at the point $x = 0.8$ of the problem $y'' = 2yy'$, given $y(0) = 0$, $y'(0) = 1$, $y(0.2) = 0.2027$ $y'(0.2) = 1.041$ $y(0.4) = 0.4228$, $y'(0.4) = 1.179$ $y(0.6) = 0.6841$ $y'(0.6) = 1.468$.

Sol: Given: $y'' = 2yy'$ $\rightarrow$ (1)

Let $y' = z = f(x, y, z)$

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$\therefore (1) \Rightarrow \frac{dz}{dx} = 2yz = g(x,y,z)$

and

$x_0 = 0 \qquad y_0 = 0 \qquad y_0' = z_0 = 1$

$x_1 = 0.2 \qquad y_1 = 0.2027 \qquad y_1' = z_1 = 1.041$

$x_2 = 0.4 \qquad y_2 = 0.4228 \qquad y_2' = z_2 = 1.179$

$x_3 = 0.6 \qquad y_3 = 0.6841 \qquad y_3' = z_3 = 1.468$

$\therefore f_0 = f(x_0, y_0, z_0) = 1 \qquad g_0 = 2y_0z_0 = 0$

$f_1 = f(x_1, y_1, z_1) = 1.041 \qquad g_1 = 2y_1z_1 = 0.4220$

$f_2 = f(x_2, y_2, z_2) = 1.179 \qquad g_2 = 2y_2z_2 = 0.9969$

$f_3 = f(x_3, y_3, z_3) = 1.468 \qquad g_3 = 2y_3z_3 = 2.0085$

$\therefore y_4^{(P)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= \frac{4\times 0.2}{3}[2\times 1.041 - 1.179 + 2\times 1.468]$

$y_4^{(P)} = 1.0237$

$\therefore z_4^{(P)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$

$= 1 + \frac{4\times 0.2}{3}[2\times 0.4220 - 0.9969 + 2\times 2.0085]$

$\Rightarrow z_4^{(P)} = 2.0304$

$\therefore f_4^{(P)} = 2.0304$

$\therefore y_4^{(C)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(P)}]$

$= 0.4228 + \frac{0.2}{3}[1.179 + 4\times 1.468 + 2.0304]$

$y_4^{(C)} = 1.02823$

$y_{(0.8)} = 1.02823$

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(5) Obtain the solution of the equation $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx}$, by computing the value of the dependent variable corresponding to the value 1.4 of the independent variable by applying Milne's method using the following data.

x	1	1.1	1.2	1.3
y	2	2.2156	2.4649	2.7514
z	2	2.3178	2.6725	2.0657

Sol: Given: $2\frac{d^2y}{dx^2} = 4x + \frac{dy}{dx} \rightarrow (1)$

Let $y' = \frac{dy}{dx} = z = f(x, y, z)$

$\therefore (1) \Rightarrow 2\frac{dz}{dx} = 4x + z$

$\frac{dz}{dx} = 2x + \frac{z}{2} = g(x, y, z) \rightarrow (2)$

and

$x_0 = 1 \quad y_0 = 2 \quad y'_0 = z_0 = 2$

$x_1 = 1.1 \quad y_1 = 2.2156 \quad y'_1 = z_1 = 2.3178$

$x_2 = 1.2 \quad y_2 = 2.4649 \quad y'_2 = z_2 = 2.6725$

$x_3 = 1.3 \quad y_3 = 2.7514 \quad y'_3 = z_3 = 2.0657$

$\therefore f_0 = f(x_0, y_0, z_0) = 2 \qquad g_0 = 2x_0 + \frac{z_0}{2} = 3$

$f_1 = f(x_1, y_1, z_1) = 2.3178 \qquad g_1 = 2x_1 + \frac{z_1}{2} = 3.3589$

$f_2 = f(x_2, y_2, z_2) = 2.6725 \qquad g_2 = 2x_2 + \frac{z_2}{2} = 3.73625$

$f_3 = f(x_3, y_3, z_3) = 2.0657 \qquad g_3 = 2x_3 + \frac{z_3}{2} = 3.63285$

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$\therefore y_4^{(p)} = y_0 + \frac{4h}{3}[2f_1 - f_2 + 2f_3]$

$= 2 + \frac{4\times 0.1}{3}[2\times 2.3178 - 2.6725 + 2\times 2.0657]$

$\Rightarrow y_4^{(p)} = 2.8126$

$\therefore z_4^{(p)} = z_0 + \frac{4h}{3}[2g_1 - g_2 + 2g_3]$

$= 2 + \frac{4\times 0.1}{3}[2\times 3 - 3.73625 + 2\times 3.63285]$

$z_4^{(p)} = 3.2706$

$\therefore f_4^{(p)} = 3.2706$

$\Rightarrow y_4^{(c)} = y_2 + \frac{h}{3}[f_2 + 4f_3 + f_4^{(p)}]$

$= 2.4649 + \frac{0.1}{3}[2.6725 + 4\times 3.0657 + 3.2706]$

$\Rightarrow y_4^{(c)} = 3.07176$

$y(1.4) = 3.07176$

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Calculus of Variations:-

Euler's Theorem:-

A necessary condition for the integral $I = \int_{x_1}^{x_2} f(x, y, y')\,dx$ where $y(x_1) = y_1$, $y(x_2) = y_2$, to be as extremum that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

Proof: Let the curve $y = y(x)$ passing through the points $P(x_1, y_1)$ and $Q(x_2, y_2)$ and make I as extremar. Also let $y = y(x) + h\alpha(x)$ be the neighbouring curve passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$ be an extremar.

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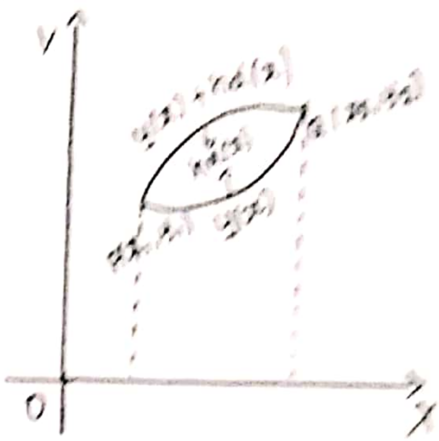
the curves both are coincident at P and Q.

$\Rightarrow \alpha(x_1)=0 , \alpha(x_2)=0$



Given : $I = \int_{x_1}^{x_2} f(x, y, y')\,dx \rightarrow (1)$

$$\Rightarrow I = \int_{x_1}^{x_2} f[x, \; y(x)+h\alpha(x), \; y'(x)+h\alpha'(x)]\,dx$$

$$\Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial h}\right] dx$$

$$= \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}(0) + \frac{\partial f}{\partial y}(0) + \frac{\partial f}{\partial y'}(\alpha'(x))\right] dx$$

$$\therefore \frac{dI}{dh} = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}(\alpha(x)) + \frac{\partial f}{\partial y'}(\alpha'(x))\right] dx$$

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$$\frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y}\,dx + \int_{x_1}^{x_2} \alpha'(x)\frac{\partial f}{\partial y'}\,dx$$

$$\Rightarrow \frac{dI}{dh} = \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial y'}\int_{x_1} \alpha'(x)\,dx - \int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\int \alpha(x)\,dx\right] dx$$

$$= \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial y'}\left[\alpha(x)\right]_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\alpha(x)\,dx$$

$$= \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y}\,dx + \frac{\partial f}{\partial y'}\left[\alpha(x_2)-\alpha(x_1)\right] - \int_{x_1}^{x_2}\left[\alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx\right]$$

$$= \int_{x_1}^{x_2} \alpha(x)\frac{\partial f}{\partial y}\,dx - \int_{x_1}^{x_2}\alpha(x)\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)dx$$

$$\frac{dI}{dh} = \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right]\alpha(x)\,dx$$

for the extremum of I, then $\frac{dI}{dh} = 0$

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① Find the extremal for the function $\int_0^{\pi/2}(y^2 - y'^2 - 2y\sin x)\,dx$, $y(0)=0$, $y(\frac{\pi}{2})=1$.

Sol: Given: $I = \int_{x_1}^{x_2} f(x,y,y')\,dx$

$$= \int_0^{\pi/2} [y^2 - y'^2 - 2y\sin x]\,dx$$

$\therefore f(x,y,y') = y^2 - y'^2 - 2y\sin x$

$\therefore$ WKT for the extremum of I

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow (2y - 2\sin x) - \frac{d}{dx}(-2y') = 0$$

$$\Rightarrow (y - \sin x) - \frac{d}{dx}\left(-\frac{dy}{dx}\right) = 0$$

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$$\Rightarrow y - \sin x + \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} + y = \sin x$$

$$\Rightarrow (D^2+1)y = \sin x$$

$\therefore$ Auxiliary equation is

$$m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1$$

$$\Rightarrow m = 0 \pm i$$

$\therefore y_c = c_1\cos x + c_2\sin x$

$\therefore y_p = \dfrac{\sin x}{D^2+1}$

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$$= \frac{-x}{2(1)} \cos x$$

$$y_p = -\frac{x}{2} \cos x$$

$\therefore y = y_c + y_p$

$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{x}{2} \cos x \rightarrow ①$

where $x = 0 \Rightarrow y = 0$

$① \Rightarrow 0 = c_1(1)$

$c_1 = 0$

where $x = \frac{\pi}{2} \Rightarrow y = 1$

$① \Rightarrow 1 = c_1(0) + c_2(1) - 0$

$c_2 = 1$

$① \Rightarrow y = \sin x - \frac{x}{2} \cos x$

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② On what curves can the functional $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx$, $y(0) = 0$, $y(\frac{\pi}{2}) = 0$ be extremised.

Sol: Given: $\int_0^{\pi/2} (y'^2 - y^2 + 2xy)\,dx = \int_{x_1}^{x_2} f(x, y, y')\,dx$

$\therefore f(x, y, y') = y'^2 - y^2 + 2xy$

$\therefore \frac{\partial f}{\partial y} = 0 - 2y + 2x = 2x - 2y$

$\frac{\partial f}{\partial y'} = 2y' - 0 + 0 = 2y'$

$\therefore$ WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow (2x - 2y) - \frac{d}{dx}(2y') = 0$

$\Rightarrow x - y - \frac{d}{dx}(y') = 0$

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$\Rightarrow x - y - y'' = 0$

$\Rightarrow y'' + y = x$

$\Rightarrow (D^2+1)y = x \qquad \because D = \frac{d}{dx}$

$\therefore$ The Auxillary equation is $m^2+1=0$

$m = 0 \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

$\Rightarrow y_p = \frac{x}{D^2+1}$

$y_p = (1+D^2)^{-1} x$

$y_p = (1 - D^2 + D^4 - D^6 + \dots)x$

$y_p = x$

$\therefore y = y_c + y_p$

$\Rightarrow y = c_1 \cos x + c_2 \sin x + x \rightarrow (1)$

when $x = 0 \Rightarrow y = 0$

$(1) \Rightarrow 0 = c_1(1) + 0 + 0$

$c_1 = 0$

when $x = \frac{\pi}{2} \Rightarrow y = 0$

$\therefore (1) \Rightarrow 0 = c_1(0) + c_2(1) + \frac{\pi}{2}$

$\Rightarrow c_2 = \frac{-\pi}{2}$

$$\boxed{y = \frac{-\pi}{2}\sin x + x}$$

(3) Find the extremal of the functional $\int_0^{\pi/2} (y'^2 - y^2 + 4y\cos x)\,dx$ $y(0) = 0 = y(\pi/2)$.

Given: Let $I = \int_{x_1}^{x_2} f(x,y,y')\,dx = \int_0^{\pi/2} (y'^2 - y^2 + 4y\cos x)\,dx$

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$\therefore f(x, y, y') = y'^2 - y^2 + 4y\cos x$

$\therefore \dfrac{\partial f}{\partial y} = 0 - 2y + 4\cos x = -2y + 4\cos x$

$\dfrac{\partial f}{\partial y'} = 2y'$

$\therefore$ WKT $\dfrac{\partial f}{\partial y} - \dfrac{d}{dx}\left(\dfrac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow -2y + 4\cos x - \dfrac{d}{dx}(2y') = 0$

$\Rightarrow -y + 2\cos x - y'' = 0$

$\Rightarrow y'' + y = 2\cos x$

$\Rightarrow (D^2 + 1)y = 2\cos x$

$\therefore$ The Auxillary equation is

$m^2 + 1 = 0$

$m^2 = -1$

$m = 0 \pm i$

$\therefore y_c = c_1\cos x + c_2\sin x$

$\therefore y_p = \dfrac{2\cos x}{D^2 + 1}$

$\dfrac{\cos ax}{D^2 + a^2} = \dfrac{x}{2a}\sin ax$

$= \dfrac{2x}{2(1)}\sin x$

$\dfrac{\sin ax}{D^2 + a^2} = -\dfrac{x}{2a}\cos ax$

$y_p = x\sin x.$

$\therefore y = y_c + y_p$

$y = c_1\cos x + c_2\sin x + x\sin x \rightarrow (1)$

where $x = 0 \Rightarrow y = 0$

$\therefore (1) \Rightarrow 0 = c_1(1) + 0 + 0$

$c_1 = 0$

when $x = \pi/2 \Rightarrow y = 0$

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$① \Rightarrow 0 = c_1(0) + c_2(1) + \frac{\pi}{2}(1)$

$c_2 = \frac{-\pi}{2}$

$\therefore y = \frac{-\pi}{2}\sin x + x \sin x$

④ Find the extremal of the function $\int_0^1 [y'^2 - y^2 - ye^{2x}]dx$ that passes through the point $(0,0)$ $(1, \frac{1}{e})$.

Sol: Let $I = \int_{x_1}^{x_2} f(x,y,y')dx = \int_0^1 [y'^2 - y^2 - ye^{2x}]dx$

$\therefore f(x,y,y') = y'^2 - y^2 - ye^{2x}$

$\Rightarrow \frac{\partial f}{\partial y} = -2y - e^{2x}$

$\Rightarrow \frac{\partial f}{\partial y'} = 2y'$

$\therefore$ WKT

$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$\Rightarrow -2y - e^{2x} - \frac{d}{dx}(2y') = 0$

$\Rightarrow -y - \frac{1}{2}e^{2x} - y'' = 0$

$\Rightarrow y'' + y = \frac{-1}{2}e^{2x}$

$\Rightarrow (D^2+1)y = \frac{-1}{2}e^{2x}$

$\therefore$ The Auxiliary equation is

$m^2 + 1 = 0$

$\Rightarrow m = 0 \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

$\therefore y_p = \frac{-1}{2} \frac{e^{2x}}{D^2+1}$

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$$= -\frac{1}{2}\,\frac{e^{2x}}{4+1}$$

$$y_p = -\frac{e^{2x}}{10}$$

$$\therefore y = y_c + y_p$$

$$\Rightarrow y = c_1\cos x + c_2\sin x - \frac{e^{2x}}{10} \rightarrow (1)$$

when $x=0 \Rightarrow y=0$

$$\therefore (1) \Rightarrow 0 = c_1(1) - \frac{1}{10}$$

$$c_1 = 0.1$$

when $x=1 \Rightarrow y = \frac{1}{e}$

$$\Rightarrow \frac{1}{e} = c_1\cos(1) + c_2\sin(1) - \frac{e^2}{10}$$

$$\Rightarrow 0.3679 = (0.1)(0.5403) + c_2(0.8457) - 0.7389$$

$$c_2 = 1.2511$$

$$y = (0.1)\cos x + (1.2511)\sin x - \frac{e^{2x}}{10}$$

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(5) Find the extremal of the function $\int_{x_1}^{x_2}(y'^2 - y^2 + 2y\sec x)\,dx$

Sol: Let $I = \int_{x_1}^{x_2} f(x,y,y')\,dx = \int_{x_1}^{x_2}(y'^2 - y^2 + 2y\sec x)\,dx$

$$\therefore f(x,y,y') = y'^2 - y^2 + 2y\sec x$$

$$\frac{\partial f}{\partial y'} = 2y'$$

$$\therefore \text{wkt} \quad \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$$

$$\Rightarrow 2y + 2\sec x - \frac{d}{dx}(2y') = 0$$

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$$\Rightarrow -y + \sec x - \frac{d}{dx}(y') = 0$$

$$\Rightarrow -y + \sec x - y'' = 0$$

$$\Rightarrow y'' + y = \sec x$$

$$\Rightarrow (D^2+1)y = \sec x$$

$\therefore$ The Auxillary equation is

$$m^2 + 1 = 0$$

$$m = 0 \pm i$$

$$\therefore y_c = c_1 \cos x + c_2 \sin x$$

$\therefore$ Let $y = Ay_1 + By_2$

when $y_1 = \cos x$, $y_2 = \sin x$

$$y_1' = -\sin x \quad , \quad y_2' = \cos x$$

$$w = y_1 y_2' - y_2 y_1'$$

$$= (\cos x)(\cos x) - (\sin x)(-\sin x)$$

$$w = 1$$

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$$\therefore A = -\int \frac{y_2 Q(x)}{w} dx + K_1$$

$$= -\int \frac{\sin x \cdot \sec x}{1} dx + K_1$$

$$= -\int \frac{\sin x}{\cos x} dx + K_1$$

$$= \int -\frac{\sin x}{\cos x} dx + K_1$$

$$A = \log(\cos x) + K_1$$

$$\therefore B = \int \frac{y_1 Q(x)}{w} dx + K_2$$

$$= \int \frac{\cos x \cdot \sec x}{1} dx + K_2$$

$$= \int 1\, dx + K_2$$

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$B = x + K_2$

$\therefore \; y = [\log(\cos x) + K_1 \int \log x + (x + K_2)\sin x]$

⑥ Prove that geodesic of a plane surface are straight lines.

Sol:- Let $S = \int_{x_1}^{x_2} \frac{ds}{dx}\, dx$

$$= \int_{x_1}^{x_2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

$$S = \int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$$

$$\therefore f(x, y, y') = \sqrt{1 + y'^2}$$

$$\therefore \frac{\partial f}{\partial y} = 0$$

$$\frac{\partial f}{\partial y'} = \frac{1}{2\sqrt{1 + y'^2}}\, 2y'$$

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{1 + y'^2}}$$

$\therefore$ WKT $\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0$

$$\Rightarrow 0 - \frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2}}\right] = 0$$

$$\Rightarrow \frac{d}{dx}\left[\frac{y'}{\sqrt{1 + y'^2}}\right] = 0$$

$$\Rightarrow \frac{\left(\sqrt{1 + y'^2}\right)y'' - y' \cdot \frac{1}{2\sqrt{1 + y'^2}} \cdot 2y'y''}{\left(\sqrt{1 + y'^2}\right)^2} = 0$$

$$\Rightarrow y''\sqrt{1 + y'^2} - \frac{y'^2 \cdot y''}{\sqrt{1 + y'^2}} = 0$$

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$$\Rightarrow y''(1+y'^2) - y'^2 \cdot y'' = 0$$

$$\Rightarrow y'' + y'^2 y'' - y'^2 y'' = 0$$

$$y'' = 0$$

$$D^2 y = 0 \,, \quad D = \frac{d}{dx}$$

$\therefore$ A.E. is $m^2 = 0$

$$m = 0, 0$$

$\therefore \boxed{y = c_1 + c_2 x}$ → Straight line equation.

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