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Department of Mathematics

Lecture Notes

Transform Calculus, Fourier Series & Numerical Techniques (18MAT31)

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Module - 1

LAPLACE TRANSFORM

Definition: Changing the function from one domain to another domain without changing its actual value is called Transformation.

Mathematically, suppose $F(t)$ be a real valid function defined on $[-\infty, +\infty]$ for any kernal or a function $K(s,t)$ such that $T[F(t)] = \int_{-\infty}^{\infty} K(s,t) \cdot F(t)\,dt = f(s)$ is called the Transformation of $F(t)$. where T is called the Transformation operator s is real (or) complex.

If the kernal $K(s,t)$ is convergent, we have

$$K(s,t) = \begin{cases} e^{-st}, & t > 0 \\ 0, & t < 0 \end{cases}$$

then $T[F(t)] = \int_0^{\infty} e^{-st} F(t)\,dt = L[F(t)] = f(s)$ is called the Laplace transform of $F(t)$ in the positive time t, where

L is called the Laplace transform operator.

If $K(s,t) = e^{ist}$, then

① => $T[F(t)] = \int_{-\infty}^{\infty} e^{ist} F(t)\,dt = F[F(t)] = f(s)$ is called the Fourier transform of $F(t)$.

where as F is the fourier transformation operator.

__Notations:-__ Suppose the given functions are in the capitals of alphabetics $F(t), G(t), H(t)$ ---- then their Laplace transforms are $f(s), g(s), h(s)$...

If the functions are small letters like $f(t), g(t), h(t)$ --- then their Laplace transforms to be in the form $\bar{f}(s), \bar{g}(s), \bar{h}(s)$...

__Properties of Laplace transforms:-__

1. Suppose $L[F(t)] = f(s)$, $L[G(t)] = g(s)$ and then for any c_1 and c_2 such that $L[c_1 F(t) + c_2 G(t)] = c_1 L[F(t)] + c_2 L[G(t)]$.

2. If $L[F(t)] = f(s)$, then
   a) $L[e^{at} F(t)] = f(s-a)$
   b) $L[e^{-at} F(t)] = f(s+a)$

3. If $L[F(t)] = f(s)$, then $L[t^n F(t)] = (-1)^n \dfrac{d^n f(s)}{ds^n}$

4. If $L[F(t)] = f(s)$, then $L\left[\dfrac{F(t)}{t}\right] = \int_s^{\infty} f(s)\,ds$.

Standard Results of Laplace transform:-

1) Let $F(t) = 1$

wkt $L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt$

$$L[1] = \int_0^\infty e^{-st}\, 1\, dt$$

$$= \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \frac{-1}{s}\left[e^{-st}\right]_0^\infty$$

$$= \frac{-1}{s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{-1}{s}[0-1]$$

$$\boxed{L[1] = \frac{1}{s}}$$

2) Let $F(t) = e^{at}$

$$L[F(t)] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$L[F(t)] = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left(e^{-\infty} - e^{0}\right)$$

$$= \frac{-1}{s-a}(0-1)$$

$$\boxed{L[e^{at}] = \frac{1}{s-a}}$$

3) Let $F(t) = e^{-at}$

$$\therefore L[e^{-at}] = \int_0^\infty e^{-st} e^{-at} dt$$

$$= \int_0^\infty e^{-(s+a)t} dt$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-(s+a)t}\right]_0^\infty$$

$$= \frac{-1}{s+a}\left[e^{-\infty} - e^{0}\right]$$

$$L[e^{-at}] = \frac{-1}{s+a}[0-1]$$

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

4) $L[t^n] = \frac{n!}{s^{n+1}}, \forall n \in Z^+$

$$= \frac{\Gamma(n+1)}{s^{n+1}} \forall n \in Q$$

5) Let $F(t) = \cosh at$

$$F(t) = \frac{e^{at} + e^{-at}}{2}$$

$$L[F(t)] = \frac{1}{2} L[e^{at} + e^{-at}]$$

$$= \frac{1}{2}\{L(e^{at}) + L(e^{-at})\}$$

$$= \frac{1}{2}\left[\frac{1}{s-a} + \frac{1}{s+a}\right]$$

$$= \frac{1}{2}\left[\frac{s+a+s-a}{s^2-a^2}\right]$$

$$= \frac{1}{\cancel{2}} \frac{\cancel{2}s}{s^2-a^2}$$

$L[F(t)] = L[\cosh at] = \dfrac{s}{s^2-a^2}$

$|||^{ly}$ $F(t) = \sinh at$

$F(t) = \dfrac{e^{at}-e^{-at}}{2}$

$L[F(t)] = \frac{1}{2} L[e^{at} - e^{-at}]$

$= \frac{1}{2}\left[\frac{1}{s-a} - \frac{1}{s+a}\right]$

$= \frac{1}{2}\left[\frac{s+a-s+a}{s^2-a^2}\right]$

$= \frac{1}{\cancel{2}} \frac{\cancel{2}a}{s^2-a^2}$

$L[F(t)] = L[\sinh at] = \dfrac{a}{s^2-a^2}$

WKT $e^{i\theta} = \cos\theta + i\sin\theta$

$\Rightarrow e^{iat} = \cos at + i\sin at$

$\Rightarrow L[e^{ait}] = L[\cos at] + i\,L[\sin at] \rightarrow ①$

$\therefore L[e^{iat}] = L[e^{(ai)t}]$

$= \dfrac{1}{s-ai} = \dfrac{s+ai}{(s-ai)(s+ai)}$

$= \dfrac{s+ai}{s^2-(ai)^2}$

$L[e^{iat}] = \dfrac{s+ai}{s^2+a^2}$

$L[\cos at] + i\,L[\sin at] = \dfrac{s}{s^2+a^2} + i\dfrac{a}{s^2+a^2}$

$\therefore L[\cos at] = \dfrac{s}{s^2+a^2}$

$L[\sin at] = \dfrac{a}{s^2+a^2}$

Find the laplace transform for the following functions:

(1) $\sin 5t \cdot \cos 2t$

(2) $\sin t \cdot \sin 2t \cdot \sin 3t$

(3) $\cos t \cdot \cos 2t \cdot \cos 3t$

(4) $(3t+4)^3 + 5^t$

(5) $t^{-5/2} + t^{5/2}$

(6) $e^{-2t} \sinh(4t)$

(1) Let $F(t) = \sin 5t \cdot \cos 2t$

$$F(t) = \frac{1}{2}[2\sin 5t \cdot \cos 2t]$$

$$= \frac{1}{2}[\sin(5t+2t) + \sin(5t-2t)]$$

$$= \frac{1}{2}[\sin(7t) + \sin(3t)]$$

$$\Rightarrow L[F(t)] = \frac{1}{2} L[\sin 7t + \sin 3t]$$

$$\Rightarrow f(s) = \frac{1}{2}\left[\frac{7}{s^2+49} + \frac{3}{s^2+9}\right]$$

(2) Let $F(t) = \sin t \cdot \sin 2t \cdot \sin 3t$

$$= \frac{1}{2}[2\sin 2t \cdot \sin t] \sin 3t$$

$$= \frac{1}{2}[\cos(2t-t) - \cos(2t+t)] \sin 3t$$

$$= \frac{1}{2}[\cos t - \cos 3t] \sin 3t$$

$$= \frac{1}{2}[\sin 3t \cdot \cos t - \sin 3t \cdot \cos 3t]$$

$$= \frac{1}{4}[2\sin 3t \cdot \cos t - 2\sin 3t \cdot \cos 3t]$$

$$= \frac{1}{4}[\sin(4t) + \sin 2t - \sin 6t]$$

$$= \frac{1}{4} L[\sin 2t + \sin 4t - \sin 6t]$$

$$= \frac{1}{4}[L(\sin 2t) + L(\sin 4t) - L(\sin 6t)]$$

$$= \frac{1}{4}\left[\frac{2}{s^2+4} + \frac{4}{s^2+16} - \frac{6}{s^2-36}\right]$$

(3) Let $F(t) = \cos t \cdot \cos 2t \cdot \cos 3t$

$$= \frac{1}{2}[2\cos 2t \quad \cos t]\cos 3t$$

$$= \frac{1}{2}[\cos 3t + \cos t]\cos 3t$$

$$= \frac{1}{2}[\cos^2 3t + \cos 3t \cdot \cos t]$$

$$= \frac{1}{2}\left[\frac{1+\cos 2(3t)}{2} + \frac{2\cos 3t \cdot \cos t}{2}\right]$$

$$\Rightarrow F(t) = \frac{1}{4}[1 + \cos 6t + \cos 4t + \cos 2t]$$

$$\Rightarrow L[F(t)] = \frac{1}{4}L[1 + \cos 6t + \cos 4t + \cos 2t]$$

$$\Rightarrow F(t) = f(s) = \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{s}{s^2+4}\right]$$

(4) Let $F(t) = (3t+4)^3 + 5^t$

$$= (3t)^3 + 4^3 + 3(3t)^2 + 3(3t)(4)^2 + e^{\log_e(5^t)}$$

$$\Rightarrow F(t) = 27t^3 + 64 + 108t^2 + 144t + e^{t(\log 5)}$$

$$\Rightarrow L[F(t)] = 27L[t^3] + 64L[1] + 108L[t^2] + 144L[t] + L[e^{(\log 5)t}]$$

$$\Rightarrow f(s) = 27 \cdot \frac{3!}{s^4} + 64 \cdot \frac{1}{s} + 108 \cdot \frac{2!}{s^3} + 144\frac{1!}{s^2} + \frac{1}{s\log 5}$$

$$\Rightarrow f(s) = \frac{162}{s^4} + \frac{64}{s} + \frac{216}{s^3} + \frac{144}{s^2} + \frac{1}{s-\log 5}$$

(5) $F(t) = e^{-2t}\sinh 4t$

$$= e^{-2t}\left[\frac{e^{4t}-e^{-4t}}{2}\right]$$

$$F(t) = \frac{1}{2}[e^{2t} - e^{-6t}]$$

$$\Rightarrow L[F(t)] = \frac{1}{2} L[e^{2t} - e^{-6t}]$$

$$= \frac{1}{2}\left[\frac{1}{s-2} - \frac{1}{s+6}\right]$$

⑥ Let $F(t) = t^{-5/2} + t^{5/2}$

$$L[F(t)] = L[t^{-5/2}] + L[t^{5/2}]$$

$$f(s) = \frac{\Gamma(-5/2+1)}{s^{-5/2+1}} + \frac{\Gamma(5/2+1)}{s^{5/2+1}}$$

$$= \frac{\Gamma(-3/2)}{s^{-3/2}} + \frac{\Gamma(7/2)}{s^{7/2}}$$

$$= \frac{(-3/2)\Gamma\left(\frac{-3}{2}\right)}{\left(\frac{-3}{2}\right)s^{-3/2}} + \frac{\left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right)}{s^{7/2}}$$

$$= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{3}{2}\right)s^{-3/2}} + \frac{(5/2)\Gamma\left(\frac{5}{2}\right)}{s^{7/2}}$$

$$= \frac{\Gamma(-1/2)}{\left(\frac{-3}{2}\right)s^{-3/2}} + \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\left(-\frac{1}{2}\right)\Gamma(-1/2)}{\left(\frac{-1}{2}\right)\left(\frac{-3}{2}\right)s^{-3/2}} + \frac{\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\Gamma\left(\frac{-1}{2}+1\right)}{\frac{3}{4}s^{-3/2}} + \frac{15}{8}\frac{\Gamma\left(\frac{1}{2}\right)}{s^{7/2}}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)}{\frac{3}{4}s^{-3/2}} + \frac{15}{8}\frac{\Gamma(1/2)}{s^{7/2}}$$

$$\Rightarrow f(s) = \frac{4}{3}\frac{\sqrt{\pi}}{s^{-3/2}} + \frac{15}{8}\frac{\sqrt{\pi}}{s^{7/2}}$$

Find the laplace transform of the following functions:

① $e^{-2t}(2\cos 5t - \sin 5t)$

② $e^{-t}\cos^2 3t$

③ $e^{3t}\sin 5t . \sin 3t$

④ $e^{-4t}t^{-5/2}$

① Let $F(t) = 2\cos 5t - \sin 5t$

$$L[F(t)] = 2L[\cos 5t] - L[\sin 5t]$$

$$f(s) = 2\left[\frac{s}{s^2+25}\right] - \left[\frac{5}{s^2+25}\right]$$

$$f(s) = \frac{2s-5}{s^2+25}$$

$\therefore$ WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{-2t}F(t)] = f(s+2)$$

$$\Rightarrow L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2(s+2)-5}{(s+2)^2+25}$$

$$= \frac{2s+4-5}{s^2+4s+4+25}$$

$$\Rightarrow L[e^{-2t}(2\cos 5t - \sin 5t)] = \frac{2s-1}{s^2+4s+29}$$

② Let $F(t) = \cos^2 3t$

$$F(t) = \frac{1+\cos 6t}{2}$$

$$L[F(t)] = \frac{1}{2}L[1+\cos 6t]$$

$$= \frac{1}{2}\{L(1) + L(\cos 6t)\}$$

$$f(s) = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+36}\right]$$

$\therefore$ WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{-t}F(t)] = f(s+1)$$

$$L[e^{-t}\cos^2 3t] = \frac{1}{2}\left[\frac{1}{s+1} + \frac{s+1}{(s+1)^2+36}\right]$$

4) Let $e^{-4t}\, t^{-\frac{5}{2}}$

$$F(t) = t^{-5/2}$$

$$L[F(t)] = L[t^{-\frac{5}{2}}]$$

$$f(s) = \frac{\Gamma\left(-\frac{5}{2}+1\right)}{s^{-\frac{5}{2}+1}}$$

$$= \frac{\Gamma\left(-\frac{3}{2}\right)}{s^{-3/2}}$$

$$= \frac{\left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right)}{\left(-\frac{3}{2}\right)s^{-3/2}}$$

$$= \frac{\Gamma\left(-\frac{3}{2}+1\right)}{\left(-\frac{3}{2}\right)s^{-3/2}}$$

$$= \frac{\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{3}{2}\right)s^{-\frac{3}{2}}}$$

$$= \frac{\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)s^{-3/2}}$$

$$= \frac{\left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right)}{\frac{3}{4}\, s^{-3/2}}$$

$$= \frac{\Gamma\left(-\frac{1}{2}+1\right)}{\frac{3}{4}\, s^{-3/2}}$$

$$f(s) = \frac{\sqrt{\pi}}{\frac{3}{4}\, s^{-3/2}}$$

$$L[e^{at}F(t)] = f(s-a)$$

$$L[e^{-4t}F(t)] = f(s+4)$$

$$L[e^{-4t}\, t^{-5/2}] = \frac{\sqrt{\pi}}{\frac{3}{4}(s+4)^{-3/2}} = \frac{4\sqrt{\pi}}{3(s+4)^{-3/2}}$$

3) Let $F(t) = \sin 5t \cdot \sin 3t$

$$F(t) = \frac{1}{2}(2\sin 5t \cdot \sin 3t)$$

$$F(t) = \frac{1}{2}[\cos 2t - \cos 8t]$$

$$L[F(t)] = \frac{1}{2} L[\cos 2t - \cos 8t]$$

$$f(s) = \frac{1}{2}\left[\frac{s}{s^2+4} - \frac{s}{s^2+64}\right]$$

WKT $L[e^{at}F(t)] = f(s-a)$

$$L[e^{3t}F(t)] = f(s-3)$$

$$L[e^{3t}\sin 5t \cdot \sin 3t] = \frac{1}{2}\left[\frac{(s-3)}{(s-3)^2+4} - \frac{(s-3)}{(s-3)^2+64}\right]$$

Find the laplace transform of the following functions:-

(6)

1. $t\cos at$
2. $t^2 \sin at$
3. $t^3 \cosh t$
4. $t^3 e^{4t} \cosh 3t$
5. $t e^{-2t} \sin 4t$

(1) Let $F(t) = \cos at$

$$L[F(t)] = L[\cos at]$$

$$f(s) = \frac{s}{s^2+a^2}$$

wkt $L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$$L[t\, F(t)] = (-1)^1 \frac{d}{ds} f(s)$$

$$L[t\cos at] = -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$$

$$= -\left[\frac{(s^2+a^2)(1) - s(2s)}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$$

$$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$$

$$L[t\cos at] = \frac{s^2-a^2}{(s^2+a^2)^2}$$

(2) Let $F(t) = \sin at$

$$L[F(t)] = L[\sin at]$$

$$f(s) = \frac{a}{s^2+a^2}$$

wkt $L[t^n F(t)] = (-1)^n \frac{d^n}{ds^n} f(s)$

$$L[t^2 F(t)] = (-1)^2 \frac{d^2}{ds^2} f(s)$$

$$L[t^2 F(t)] = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{a}{s^2+a^2}\right)\right]$$

$$L[t^2 \sin at] = \frac{d}{ds}\left[\frac{-2as}{(s^2+a^2)^2}\right]$$

$$= -2a \frac{d}{ds}\left[\frac{s}{(s^2+a^2)^2}\right]$$

$$= -2a \left[\frac{(s^2+a^2)^2(1) - s.2(s^2+a^2)(2s)}{(s^2+a^2)^4}\right]$$

$$= -2a \left[\frac{s^2+a^2-4s^2}{(s^2+a^2)^3}\right]$$

$$L[t^2 \sin at] = -2a \left[\frac{a^2-3s^2}{(s^2+a^2)^3}\right]$$

⑤ $t e^{-2t} \sin 4t$

Let $F(t) = t \sin 4t$

$$L[F(t)] = -\frac{d}{ds}\left(\frac{4}{s^2+16}\right)$$

$$f(s) = \left[\frac{(s^2+16)0 - 4(2s)}{(s^2+16)^2}\right]$$

$$= \frac{8s}{(s^2+16)^2}$$

$$L[e^{-2t} F(t)] = [f(s)]_{s \to s+2}$$

$$= \frac{8(s+2)}{[(s+2)^2+16]^2}$$

Find the Laplace transform of the following:-

① $\frac{\cos at - \cos bt}{t}$

$F(t) = \cos at - \cos bt$

$L[F(t)] = L[\cos at - \cos bt]$

$f(s) = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$L\left[\frac{F(t)}{t}\right] = \int_0^{\infty} f(s)\,ds$

$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)$ $\quad \therefore \int \frac{f'(x)}{f(x)}dx = \log[f(x)]$

$= \frac{1}{2}[\log(s^2+a^2) - \log(s^2+b^2)]_s^{\infty}$

$= \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^{\infty}$

$= \frac{1}{2}\left[\lim_{s\to\infty} \log \frac{s^2(1+a^2/s^2)}{s^2(1+b^2/s^2)} - \log\frac{s^2+a^2}{s^2+b^2}\right]$

$= \frac{1}{2}\left[0 - \log\frac{s^2+a^2}{s^2+b^2}\right]$

$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$

$L\left[\frac{\cos at - \cos bt}{t}\right] = \log\sqrt{\frac{s^2+b^2}{s^2+a^2}}$

② $\frac{2\sin t \sin 5t}{t}$

$F(t) = 2\sin t \sin 5t$

$= 2 \times \frac{1}{2}[\cos(4t) - \cos(6t)]$

$F(t) = \cos(4t) - \cos 6t$

$L\left[\frac{F(t)}{t}\right] = \log\sqrt{\frac{s^2+36}{s^2+16}}$

③ $2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$

Let $F(t) = 2^t + \frac{\cos 2t - \cos 3t}{t} + t\sin t$

Let $L[F(t)] = L\left[e^{\log 2 \cdot t}\right] + L\left[\frac{\cos 2t - \cos 3t}{t}\right] + L[t\sin t]$

$= \frac{1}{s-\log 2} + \log\sqrt{\frac{s^2+9}{s^2+4}} - \frac{d}{ds}\left(\frac{1}{s^2+1}\right)$

$L[F(t)] = \frac{1}{s-\log 2} + \log\sqrt{\frac{s^2+9}{s^2+4}} + \frac{2s}{(s^2+1)^2}$

④ $t^2 e^{-3t}\sin 2t$

Let $[F(t)] = L[t^2 e^{-3t}\sin 2t]$

$L[\sin 2t] = \frac{2}{s^2+4}$

$L[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2}\left[\frac{2}{s^2+4}\right]$

$L[t^2 \sin 2t] = \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{2}{s^2+4}\right)\right]$

$= \frac{d}{ds}\left[\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2}\right]$

$= \frac{d}{ds}\left[\frac{-4s}{(s^2+4)^2}\right]$

$= -4\frac{d}{ds}\left[\frac{s}{(s^2+4)^2}\right]$

$= -4\left[\frac{(s^2+4)(1) - s \cdot 2(s^2+4)(2s)}{(s^2+4)^4}\right]$

$= -4\left[\frac{(s^2+4) - 4s^2}{(s^2+4)^3}\right]$

$= -4\left[\frac{4 - 3s^2}{(s^2+4)^3}\right]$

$$L[e^{-3t}t^2\sin 2t] = 4\left[\frac{3s^2-4}{(s^2+4)^3}\right]_{s\to s+3}$$

$$= 4\left[\frac{3(s+3)^2-4}{[(s+3)^2+4]^3}\right]$$

⑤ t^3+4t^2-3t+5

$$L[t^3+4t^2-3t+5] = L[t^3]+4L[t^2]-3L[t]+5L[1]$$

$$= \frac{3!}{s^4} + 4\cdot\frac{2!}{s^3} - 3\cdot\frac{1!}{s^2} + 5\cdot\frac{1}{s}$$

$$= \frac{6}{s^4} + \frac{8}{s^3} - \frac{3}{s^2} + \frac{5}{s}$$

⑥ $\int_0^\infty t^3 e^{-t}\sin t\, dt = 0$

Let $\int_0^\infty e^{-st}F(t)dt = L[F(t)] \rightarrow ①$

Let $F(t) = t^3\sin t$

$① = \int_0^\infty e^{-st}t^3\sin t\,dt = L[t^3\sin t] \rightarrow ②$

WKT $L[\sin t] = \frac{1}{s^2+1}$

$$L[t^3\sin t] = (-1)^3\frac{d^3}{ds^3}\left(\frac{1}{s^2+1}\right)$$

$$= -\frac{d^2}{ds^2}\left[\frac{d}{ds}\left(\frac{1}{s^2+1}\right)\right]$$

$$= -\frac{d^2}{ds^2}\left[\frac{-2s}{(s^2+1)^2}\right]$$

$$= 2\frac{d^2}{ds^2}\left[\frac{s}{(s^2+1)^2}\right]$$

$$= 2\frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{(s^2+1)^2}\right)\right]$$

$$= 2\frac{d}{ds}\left[\frac{(s^2+1)^2(1) - s.2(s^2+1).(2s)}{(s^2+4)^4}\right]$$

$$= 2\frac{d}{ds}\left[\frac{s^2+1-4s^2}{(s^2+1)^3}\right]$$

$$= 2\frac{d}{ds}\left[\frac{1-3s^2}{(s^2+1)^3}\right]$$

$$= 2\left[\frac{(s^2+1)^3(-6s)-(1-3s^2)3(s^2+1)^2(2s)}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{(s^2+1)(-6s)+(-6s)(1-3s^2)}{(s^2+1)^4}\right]$$

$$= -12s\left[\frac{s^2+1+1-3s^2}{(s^2+1)^4}\right]$$

$$= -12s\left[\frac{-2s^2+2}{(s^2+1)^4}\right]$$

$$L[t^3\sin t] = 24s\left[\frac{s^2-1}{(s^2+1)^4}\right]$$

$$① \Rightarrow \int_0^\infty e^{-st}t^3\sin t\,dt = 24s\left[\frac{s^2-1}{(s^2+1)^4}\right] \rightarrow ③$$

if $s=1$

$$③ \Rightarrow \int_0^\infty e^{-t}t^3\sin t\,dt = 0$$

$$\therefore \int_0^\infty t^3e^{-t}\sin t\,dt = 0$$

⑦ Evaluate $\int_0^\infty t\,e^{-3t}\cos 2t\,dt$

Let $\int_0^\infty e^{-3t}F(t)\,dt = L[F(t)] \rightarrow ①$

Let $F(t) = t\cos 2t$

$$① \Rightarrow \int_0^\infty t\,e^{-3t}\cos 2t\,dt = L[t\cos 2t] \rightarrow ②$$

WKT $L[\cos 2t] = \frac{s}{s^2+4}$

$$\Rightarrow L[t\cos 2t] = (-1)\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$$

$$= -\left[\frac{1.(s^2+4)-s(2s)}{(s^2+4)^2}\right]$$

$$= -\left[\frac{(s^2+4)-2s^2}{(s^2+4)^2}\right]$$

$$= -\left[\frac{-s^2+4}{(s^2+4)^2}\right]$$

$$L[t\cos 2t] = \left[\frac{s^2-4}{(s^2+4)^2}\right]$$

① $\Rightarrow \int_0^\infty t\, e^{-3t}\cos 2t\, dt = \left[\frac{s^2-4}{(s^2+4)^2}\right]$

if $s=3$

③ $\Rightarrow \int_0^\infty t\, e^{-3t}\cos 2t = \left[\frac{9-4}{(9+4)^2}\right]$

$$\therefore \int_0^\infty t\, e^{-3t}\cos 2t = \frac{5}{169}$$

Laplace transform for periodic functions :-

Suppose for any $T \geq 0$, the function $f(t)$ is said to be a periodic function for $f(t+T) = f(t)$.

The Laplace transform of the periodic function $f(t)$ for the period T is defined as

$$L[f(t)] = \bar{f}(s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\, dt$$

① If $f(t) = \begin{cases} E, & \text{if } 0 \leq t \leq a/2 \\ -E, & \text{if } a/2 < t \leq a \end{cases}$ where $f(t+a) = f(t)$,

then show that $L[f(t)] = \frac{E}{s}\tanh\left[\frac{as}{4}\right]$.

Given $f(t) = \begin{cases} E, & \text{if } 0 \leq t \leq a/2 \\ -E, & \text{if } a/2 < t \leq a \end{cases}$

$$f(t+a) = f(t) \Rightarrow T = a$$

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^{\infty} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_{a/2}^{a} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} E\, e^{-st}\,dt + \int_{a/2}^{a} (-E) e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left[\int_0^{a/2} e^{-st}\,dt - \int_{a/2}^{a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-as}}\left\{\left[\frac{e^{-st}}{-s}\right]_0^{a/2} - \left[\frac{e^{-st}}{-s}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{1-e^{-as}}\left\{-\frac{1}{s}\left[e^{-st}\right]_0^{a/2} + \frac{1}{s}\left[e^{-st}\right]_{a/2}^{a}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-(e^{-as/2} - e^{0}) + [e^{-as} - e^{-as/2}]\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left\{-e^{-as/2} + 1 + e^{-as} - e^{-as/2}\right\}$$

$$= \frac{E}{s(1-e^{-as})}\left[1 - 2e^{-as/2} + e^{-as}\right]$$

$$= \frac{E}{s(1-e^{-as})}\left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2\right]$$

$$= \frac{E\,(1-e^{-as/2})^2}{s[(1)^2 - (e^{-as/2})^2]}$$

$$= \frac{E\,(1-e^{-as/2})^2}{s(1-e^{-as/2})(1+e^{-as/2})}$$

$$= \frac{E(1-e^{-as/2})}{s\,(1+e^{-as/2})}$$

$$= \frac{E}{s}\;\frac{(1-e^{-as/2})\,e^{as/4}}{(1+e^{-as/2})\,e^{as/4}}$$

$$= \frac{E}{s}\;\frac{e^{as/4} - e^{-as/4}}{e^{as/4} + e^{-as/4}}$$

$\therefore L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{4}\right)$

② If $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases}$ where $f(t+2a) = f(t)$ then

show that $L[f(t)] = \frac{1}{s^2} \tanh\left(\frac{as}{2}\right)$.

Given $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases}$

$f(t+2a) = f(t) \Rightarrow T = 2a$

$\Rightarrow L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\, dt$

$= \frac{1}{1-e^{-2as}} \left[\int_0^a e^{-st} f(t)\, dt + \int_a^{2a} e^{-st} f(t)\, dt \right]$

$\Rightarrow L[f(t)] = \frac{1}{1-e^{-2as}} \left[\int_0^a t e^{-st}\, dt + \int_a^{2a} (2a-t) e^{-st}\, dt \right]$

$\therefore \int_0^a t e^{-st}\, dt = t \int_0^a e^{-st}\, dt - \int_0^a \left[1 \int e^{-st}\, dt \right] dt$

$= -\frac{1}{s} \left[t e^{-st} \right]_0^a + \frac{1}{s} \int_0^a e^{-st}\, dt$

$= -\frac{1}{s} \left[t e^{-st} \right]_0^a - \frac{1}{s^2} \left[e^{-st} \right]_0^a$

$= -\frac{1}{s} \left[a e^{-as} - 0 \right] - \frac{1}{s^2} \left[e^{-as} - e^0 \right]$

$\int_0^a t e^{-st}\, dt = -\frac{a}{s} e^{-as} - \frac{1}{s^2} e^{-as} + \frac{1}{s^2}$

$\Rightarrow \int_0^{2a} (2a-t) e^{-st}\, dt = (2a-t) \int_0^{2a} e^{-st}\, dt - \int_0^{2a} \left[(-1) \int e^{-st}\, dt \right] dt$

$= -\frac{1}{s} \left[(2a-t) e^{-st} \right]_a^{2a} - \frac{1}{s} \int_0^{2a} e^{-st}\, dt$

$$= \frac{-1}{s}\left[(2a-t)e^{-st}\right]_a^{2a} + \frac{1}{s^2}\left[e^{-st}\right]_a^{2a}$$

$$= -\frac{1}{s}\left[0 - ae^{-as}\right] + \frac{1}{s^2}\left[e^{-2as} - e^{-as}\right]$$

$$\int_0^{2a}(2a-t)e^{-st}\,dt = \frac{a}{s}e^{-as} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}$$

$$L[f(t)] = \frac{1}{1-e^{-2as}}\left[-\frac{a}{s}e^{-as} - \frac{1}{s^2}e^{-as} + \frac{1}{s^2} + \frac{a}{s}e^{-as} + \frac{1}{s^2}e^{-2as} - \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\frac{1}{s^2} - \frac{2}{s^2}e^{-as} + \frac{1}{s^2}e^{-2as}\right]$$

$$= \frac{1}{s^2(1-e^{-2as})}\left[1 - 2(1)e^{-as} + (e^{-as})^2\right]$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$= \frac{1}{s^2}\,\frac{1-e^{-as}}{1+e^{-as}} \times \frac{e^{as/2}}{e^{as/2}}$$

$$= \frac{1}{s^2}\,\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+e^{-as/2}}$$

$$\therefore L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{2}\right)$$

③ If $f(t) = \begin{cases} E\sin(\omega t), & 0 \le t \le \frac{\pi}{\omega} \\ 0, & \frac{\pi}{\omega} < t \le \frac{2\pi}{\omega} \end{cases}$ is a periodic function of the period $\frac{2\pi}{\omega}$, then ST

$$L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi/\omega})}$$ for ω and E constants.

Given $f(t) = \begin{cases} E\sin(\omega t), & 0 \le t \le \pi/\omega \\ 0, & \pi/\omega < t \le \frac{2\pi}{\omega} \end{cases}$ and $T = \frac{2\pi}{\omega}$

$$\therefore L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{2\pi/\omega} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\pi/\omega} e^{-st} f(t)\,dt + \int_{\pi/\omega}^{2\pi/\omega} e^{-st} f(t)\,dt \right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} E \sin\omega t\,dt$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\pi/\omega} e^{-st} \sin\omega t\,dt$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}} \left\{ \frac{e^{-st}}{(-s)^2+\omega^2} \left[-s\sin\omega t - \omega\cos\omega t \right] \right\}_0^{\pi/\omega}$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}} \left\{ \left[\frac{e^{-\frac{\pi s}{\omega}}}{s^2+\omega^2} (0+\omega) \right] - \left[\frac{1}{s^2+\omega^2} (0-\omega) \right] \right\}$$

$$= \frac{E}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{\omega e^{-\pi s/\omega}}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2} \right]$$

$$= \frac{E\omega \left(1+e^{-\frac{\pi s}{\omega}}\right)}{(s^2+\omega^2)(1-e^{-\pi s/\omega})(1+e^{-\pi s/\omega})}$$

$$\Rightarrow L[f(t)] = \frac{E\omega}{(s^2+\omega^2)(1-e^{-\pi s/\omega})}$$

(4) If $f(t) = \begin{cases} E, & \text{if } 0 \le t \le a \\ -E, & \text{if } a < t \le 2a \end{cases}$ where $f(t+2a) = f(t)$, then show that $L[f(t)] = \frac{E}{s} \tanh\left(\frac{as}{2}\right)$.

Given $f(t) = \begin{cases} E, & \text{if } 0 \le t \le a \\ -E, & \text{if } a < t \le 2a \end{cases}$

$$f(t+2a) = f(t) \quad \therefore T = 2a$$

WKT

$$L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\,dt$$

$$L[f(t)] = \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st} f(t)\,dt + \int_a^{2a} e^{-st} f(t)\,dt\right\}$$

$$= \frac{1}{1-e^{-2as}}\left\{\int_0^a e^{-st} E\,dt + \int_a^{2a} e^{-st} (-E)\,dt\right\}$$

$$= \frac{E}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt + (-1)\int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\int_0^a e^{-st}\,dt - \int_a^{2a} e^{-st}\,dt\right]$$

$$= \frac{E}{1-e^{-2as}}\left[\left[\frac{e^{-st}}{-s}\right]_0^a - \left[\frac{e^{-st}}{-s}\right]_a^{2a}\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left\{-\left[e^{-st}\right]_0^a + \left[e^{-st}\right]_a^{2a}\right\}$$

$$= \frac{E}{s(1-e^{-2as})}\left[-e^{-as} + e^0 + e^{-2as} - e^{-as}\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[-2e^{-as} + 1 + e^{-2as}\right]$$

$$= \frac{E}{s(1-e^{-2as})}\left[(1)^2 + (e^{-as})^2 - 2(1)(e^{-as})\right]$$

$$= \frac{E(1-e^{-as})^2}{s(1-e^{-2as})}$$

$$= \frac{E(1-e^{-as})^2}{s\left[(1)^2 - (e^{-as})^2\right]}$$

$$= \frac{E(1-e^{-as})^{\cancel{2}}}{s\left[1+e^{-as}\right]\cancel{\left[1-e^{-as}\right]}}$$

$$= \frac{E(1-e^{-as})}{s(1+e^{-as})}$$

$$= \frac{E(1-e^{-as})e^{as/2}}{s(1+e^{-as})e^{as/2}}$$

$$= \frac{E}{s}\frac{(e^{as/2} - e^{-as/2})}{(e^{as/2} + e^{-as/2})}$$

$$L[f(t)] = \frac{E}{s}\tanh\left[\frac{as}{2}\right]$$

⑤ If $f(t) = t, \; 0 \le t \le a/2$

$a-t, \; a/2 < t \le a$ where $f(t+a) = f(t)$ then

Show that $L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{4}\right)$

Given $f(t) = \begin{cases} t, & 0 \le t \le a/2 \\ a-t, & a/2 < t \le a \end{cases}$

$f(t+a) = f(t) \Rightarrow T = a$

$$L[f(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} e^{-st} f(t)\,dt + \int_a^{a/2} e^{-st} f(t)\,dt\right]$$

$$= \frac{1}{1-e^{-as}}\left[\int_0^{a/2} t e^{-st}\,dt + \int_{a/2}^{a} e^{-st}(a-t)\,dt\right]$$

$$\int_0^{a/2} t e^{-st}\,dt = t\int_0^{a/2} e^{-st}\,dt - \int_0^{a/2}\left[1\int e^{-st}\,dt\right]$$

$$= -\frac{1}{s}\left[t e^{-st}\right]_0^{a/2} + \int_0^{a/2}\left[\frac{e^{-st}}{-s}\right]$$

$$= -\frac{1}{s}\left[\frac{a}{2}e^{-as/2} - 0\right] - \frac{1}{s^2}\left[e^{-st}\right]_0^{a/2}$$

$$= -\frac{1}{s}\left[\frac{a}{2}e^{-as/2}\right] - \frac{1}{s^2}\left[e^{-as/2} - e^0\right]$$

$$= -\frac{a}{2s}e^{-as/2} - \frac{1}{s^2}e^{-as/2} + \frac{1}{s^2}$$

$$\therefore \int_{a/2}^{a}(a-t)e^{-st}\,dt = (a-t)\int_{a/2}^{a} e^{-st}\,dt - \int_{a/2}^{a}\left[(-1)\int e^{-st}\,dt\right]dt$$

$$= -\frac{1}{s}\left[(a-t)e^{-st}\right]_{a/2}^{a} - \frac{1}{s}\int_{a/2}^{a} e^{-st}\,dt$$

$$= -\frac{1}{s}\left[(a-t)e^{-st}\right]_{a/2}^{a} + \frac{1}{s^2}\left[e^{-st}\right]_{a/2}^{a}$$

$$= -\frac{1}{s}\left[0 - (a - a/2)e^{-\frac{as}{2}}\right] + \frac{1}{s^2}\left[e^{-as} - e^{-as/2}\right]$$

$$\int_{a/2}^{a}(a-t)e^{-st}\,dt = \frac{a}{2s}e^{-as/2} + \frac{1}{s^2}e^{-as} - \frac{e^{-as/2}}{s^2}$$

$$\therefore L[f(t)] = \frac{1}{1-e^{-as}}\left[\frac{-a}{2s}e^{-as/2} - \frac{1}{s^2}e^{-as/2} + \frac{1}{s^2} + \frac{a}{2s}e^{-as/2} + \frac{1}{s^2}e^{-as} - \frac{e^{-as/2}}{s^2}\right]$$

$$= \frac{1}{1-e^{-as}}\left[-\frac{2}{s^2}e^{-as/2} + \frac{1}{s^2} + \frac{1}{s^2}e^{-as}\right]$$

$$= \frac{1}{s^2(1-e^{-as})}\left[(1)^2 - 2(1)(e^{-as/2}) + (e^{-as/2})^2\right]$$

$$= \frac{1}{s^2(1-e^{-as})}\left[1 - e^{-as/2}\right]^2$$

$$= \frac{1}{s^2}\frac{(1-e^{-as/2})^2}{(1)^2-(e^{-as/2})^2}$$

$$= \frac{1}{s^2}\frac{(1-e^{-as/2})^2}{(1-e^{-as/2})(1+e^{-as/2})}$$

$$= \frac{1}{s^2}\frac{(1-e^{-as/2})}{(1+e^{-as/2})} \times \frac{e^{as/4}}{e^{as/4}}$$

$$= \frac{1}{s^2}\frac{(e^{as/4} - e^{-as/4})}{(e^{as/4} + e^{-as/4})}$$

$$L[f(t)] = \frac{1}{s^2}\tanh\left(\frac{as}{4}\right)$$

Unit step (or) Heaviside function :- For any $a \geq 0$ the unit step (or) Heaviside function can be defined as $u(t-a)$ (or) $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a. \end{cases}$

The Laplace transform of the unit step function is $L[u(t-a)] = \frac{e^{-as}}{s}$.

1. $f(t)$ be a function, then $L[f(t-a)\,u(t-a)] = e^{-as}\bar{f}(s)$, where $\bar{f}(s) = L[f(t)]$.

2. If $f(t) = \begin{cases} f_1(t), & \text{for } t < a \\ f_2(t), & \text{for } t \geq a \end{cases}$, then the Heaviside form of $f(t)$ can be written as $f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a)$ and its Laplace transform is

$$L[f(t)] = L[f_1(t)] + L\{[f_2(t) - f_1(t)]u(t-a)\}$$

3. If $f(t) = \begin{cases} f_1(t), & \text{for } 0 \le t < a \\ f_2(t), & \text{for } a \le t < b \\ f_3(t), & \text{for } t \ge b \end{cases}$ then, the Heaviside form

of $f(t)$ can be written as

$$f(t) = f_1(t) + [f_2(t) - f_1(t)]u(t-a) + [f_3(t) - f_2(t)]u(t-b)$$

$$L f(t) = L[f_1(t)] + L\{[f_2(t) - f_1(t)]u(t-a)\} + L\{[f_3(t) - f_2(t)]u(t-b)\}.$$

① Find the Laplace transform for the following Heaviside function:-

① $(t^2-1)u(t-1)$

② $\sin t \cdot u(t-\pi)$

③ $(2t^2-t+1)u(t-1)$.

① Let $f(t-1) = t^2-1$

$$f(t) = (t+1)^2-1$$

$$f(t) = t^2+2t+1-1$$

$$f(t) = t^2+2t$$

$$L[f(t)] = L[t^2] + 2L[t]$$

$$\bar{f}(s) = \frac{2}{s^3} + \frac{2}{s^2}$$

WKT

$$\Rightarrow L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$$

$$\Rightarrow L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$$

$$\Rightarrow L[f(t^2-1)u(t-1)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2}\right] //$$

② $\sin t \cdot u(t-\pi)$

Let $f(t-\pi) = \sin t$

$$f(t) = \sin(t+\pi)$$

$$f(t) = -\sin t$$

$$L[f(t)] = -L[\sin t]$$

$$\Rightarrow \bar{f}(s) = -\frac{1}{s^2+1}.$$

WKT $L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$$\Rightarrow L[f(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{f}(s)$$

$$\Rightarrow L[\sin t \cdot u(t-\pi)] = \frac{-e^{-\pi s}}{s^2+1}$$

③ $(2t^2-t+1)u(t-1)$

$f(t-1) = 2t^2-t+1$

$f(t) = 2(t+1)^2-(t+1)+1$

$= 2(t^2+1+2t)-t$

$f(t) = 2t^2+3t+2$

$L[f(t)] = 2L[t^2]+3L[t]+2L[1]$

$= 2\cdot\frac{2!}{s^3}+3\frac{1!}{s^2}+\frac{2}{s}$

$L[f(t)] = \frac{4}{s^3}+\frac{3}{s^2}+\frac{2}{s}$

$\Rightarrow L[f(t-a)u(t-a)] = e^{-as}\bar{f}(s)$

$\Rightarrow L[f(t-1)u(t-1)] = e^{-s}\bar{f}(s)$

$\Rightarrow L[(2t^2-t+1)u(t-1)] = e^{-s}\left(\frac{4}{s^3}+\frac{3}{s^2}+\frac{2}{s}\right)$

① Express $f(t) = \begin{cases} \sin t, & \text{for } t<\pi \\ \pi-t, & \text{for } t\geq\pi \end{cases}$ to unit step function, hence find its Laplace transform.

Given $f(t) = \begin{cases} \sin t, & \text{for } t<\pi \\ \pi-t, & \text{for } t\geq\pi \end{cases}$

$\therefore f(t) = \sin t + [(\pi-t)-\sin t]u(t-\pi)$

$\Rightarrow L[f(t)] = L[\sin t] + L\{[(\pi-t)-\sin t]u(t-\pi)\} \rightarrow ①$

$\Rightarrow g(t-\pi) = (\pi-t)-\sin t$

$\Rightarrow g(t) = \pi-(t+\pi)-\sin(t+\pi)$

$\Rightarrow g(t) = \sin t - t$

$$L[g(t)] = L[\sin t] - L[t]$$

$$\Rightarrow \bar{g}(s) = \frac{1}{s^2+1} - \frac{1}{s^2}$$

$\therefore$ WKT $L[g(t-a)u(t-a)] = e^{-as}\bar{g}(s)$

$$\Rightarrow L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2}\right)$$

$$\Rightarrow \bar{f}(s) = \frac{1}{s^2+1} + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2}\right)$$

② Express $f(t) = \begin{cases} \sin t, & \text{for } 0 \le t < \pi/2 \\ \cos t, & \text{for } t \ge \pi/2 \end{cases}$ to unit step function, hence find its Laplace transform.

Given $f(t) = \begin{cases} \sin t, & \text{for } 0 \le t < \pi/2 \\ \cos t, & \text{for } t \ge \pi/2 \end{cases}$

$$\Rightarrow f(t) = \sin t + [\cos t - \sin t]u(t - \pi/2) \rightarrow ①$$

$$\Rightarrow L[f(t)] = L[\sin t] + L\{[\cos t - \sin t]u(t-\pi/2)\} \rightarrow ②$$

$$\Rightarrow g(t - \pi/2) = \cos t - \sin t$$

$$\Rightarrow g(t) = \cos(t + \pi/2) - \sin(t + \pi/2)$$

$$\Rightarrow g(t) = \sin t - \cos t$$

$$\Rightarrow L[g(t)] = -L[\sin t] - L[\cos t]$$

$$\Rightarrow \bar{g}(s) = \frac{-1}{s^2+1} - \frac{s}{s^2+1}$$

$$\Rightarrow \bar{g}(s) = -\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

WKT

$$\Rightarrow L[g(t-a)u(t-a)] = e^{-as}\bar{g}(s)$$

$$\Rightarrow L[g(t-\pi/2)u(t-\pi/2)] = e^{-\pi s/2}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

$$② \Rightarrow \frac{1}{s^2+1} - e^{-\pi/2}\left[\frac{1}{s^2+1} + \frac{s}{s^2+1}\right]$$

③ Express $f(t) = \begin{cases} \sin t, & 0 \le t < \pi \\ \sin 2t, & \pi \le t < 2\pi \\ \sin 3t, & t \ge 2\pi \end{cases}$ to unit step function, hence find its Laplace transform.

$\Rightarrow f(t) = \sin t + (\sin 2t - \sin t)u(t-\pi) + (\sin 3t - \sin 2t)u(t-2\pi)$

$\Rightarrow L[f(t)] = L[\sin t] + L[(\sin 2t - \sin t)u(t-\pi)] + L[(\sin 3t - \sin 2t)u(t-2\pi)] \rightarrow ①$

$\Rightarrow g(t-\pi) = \sin 2t - \sin t$

$\Rightarrow g(t) = \sin(2t + 2\pi) - \sin(t + \pi)$

$\Rightarrow g(t) = \sin 2t + \sin t$

$\Rightarrow L[g(t)] = L[\sin 2t] + L[\sin t]$

$\Rightarrow \bar{g}(s) = \dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}$

$\therefore L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{g}(s)$

$\Rightarrow L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\left[\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right]$

$\Rightarrow h(t-2\pi) = \sin 3t - \sin 2t$

$\Rightarrow h(t) = \sin(3t + 6\pi) - \sin(2t + 4\pi)$

$\Rightarrow h(t) = \sin 3t - \sin 2t$

$\Rightarrow L[h(t)] = L[\sin 3t] - L[\sin 2t] = \dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}$

$\therefore L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{h}(s)$

$\Rightarrow L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi}\left[\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right]$

$① \Rightarrow \bar{f}(s) = \dfrac{1}{s^2+1} + e^{-\pi s}\left[\dfrac{2}{s^2+4} + \dfrac{1}{s^2+1}\right] + e^{-2\pi s}\left[\dfrac{3}{s^2+9} - \dfrac{2}{s^2+4}\right]$ //

④ Express $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$ to unit step function, hence find its Laplace transform.

Given $f(t) = \begin{cases} 1, & 0 \le t < 1 \\ t, & 1 \le t < 2 \\ t^2, & t \ge 2 \end{cases}$

$\Rightarrow f(t) = 1 + (t-1)u(t-1) + (t^2-t)u(t-2)$

$\Rightarrow L[f(t)] = L[1] + L[(t-1)u(t-1)] + L[(t^2-t)u(t-2)] \rightarrow (1)$

$\Rightarrow$ Let $g(t-1) = t-1$

$\Rightarrow g(t) = t$

$\Rightarrow L[g(t)] = L[t]$

$\Rightarrow \bar{g}(s) = \frac{1}{s^2}$

$\therefore L[g(t-1)u(t-1)] = e^{-s}\bar{g}(s) = \frac{e^{-s}}{s^2}$

$\Rightarrow$ Let $h(t-2) = t^2 - t$

$\Rightarrow h(t) = (t+2)^2 - (t+2)$

$\Rightarrow h(t) = t^2 + 4t + 4 - t - 2$

$\Rightarrow h(t) = t^2 + 3t + 2$

$\Rightarrow L[h(t)] = L[t^2] + 3L[t] + 2L[1]$

$\Rightarrow \bar{h}(s) = \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}$

$\therefore L[h(t-2)u(t-2)] = e^{-2s}\bar{h}(s) = e^{-2s}\left[\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right]$

$(1) \Rightarrow \therefore \bar{f}(s) = \frac{1}{s} + \frac{e^{-s}}{s^2} + e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\right)$

(5) $f(t) = \begin{cases} t^2, & 0 \le t < 2 \\ 4t, & 2 \le t < 4 \\ 8, & t \ge 4 \end{cases}$ to unit step function hence find its Laplace transform.

$\Rightarrow f(t) = t^2 + (4t - t^2)u(t-2) + (8-4t)u(t-4)$

$L[f(t)] = L[t^2] + L[(4t-t^2)u(t-2)] + 2[(8-4t)u(t-4)]$

Let $g(t-2) = 4t - t^2$

$\Rightarrow g(t) = 4(t+2) - (t+2)^2$

$= 4t + 8 - (t^2 + 4t + 4)$

$= 4t + 8 - t^2 - 4t - 4$

$g(t) = 4 - t^2$

$L[g(t)] = L\,4[1] - L[t^2]$

$\bar{g}(s) = \frac{4}{s} - \frac{2}{s^3}$

$\therefore L[g(t-2)u(t-2)] = e^{-2s}\bar{g}(s)$

$= e^{-2s}\left[\frac{2}{s^3} - \frac{4}{s}\right]$

$\Rightarrow h(t-4) = 8 - 4t$

$\Rightarrow h(t) = 8 - 4(t+4)$

$= 8 - 4t + 16$

$\Rightarrow h(t) = -8 - 4t$

$\Rightarrow L[h(t)] = -L\,8(1) - 4L[t]$

$\Rightarrow L[h(t)] = -\frac{8}{s} - \frac{4}{s^2}$

$\Rightarrow \bar{h}(s) = -\left(\frac{4}{s^2} + \frac{8}{s}\right)$

$\therefore L[h(t-4)u(t-4)] = e^{-4s}\bar{h}(s)$

$= e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s}\right)$

$\therefore (1) \Rightarrow \bar{f}(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} - \frac{4}{s}\right) - e^{-4s}\left(\frac{4}{s^2} + \frac{8}{s}\right)$

(6) Express $f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ \cos 2t, & \pi \le t < 2\pi \\ \cos 3t, & t \ge 2\pi \end{cases}$ to unit step function, hence find its Laplace transform.

$\Rightarrow$ Given $f(t) = \begin{cases} \cos t, & 0 \le t < \pi \\ \cos 2t, & \pi \le t < 2\pi \\ \cos 3t, & t \ge 2\pi \end{cases}$

$\Rightarrow f(t) = \cos t + [(\cos 2t - \cos t)u(t-\pi)] + [(\cos 3t - \cos 2t)u(t-2\pi)] \rightarrow (1)$

$\Rightarrow L[f(t)] = L[\cos t] + L[(\cos 2t - \cos t)u(t-\pi)] + L[(\cos 3t - \cos 2t)u(t-2\pi)]$

$\Rightarrow g(t-\pi) = \cos 2t - \cos t$

$\Rightarrow g(t) = \cos 2(t+\pi) - \cos(t+\pi)$

$\Rightarrow g(t) = \cos(2t + 2\pi) - \cos(t+\pi)$

$\Rightarrow g(t) = \cos 2t + \cos t$

$\Rightarrow L[g(t)] = L[\cos(2t)] + L[\cos t]$

$\Rightarrow \bar{g}(s) = \frac{s}{s^2+4} + \frac{s}{s^2+1}$

$\therefore L[g(t-\pi)u(t-\pi)] = e^{-\pi s}\bar{g}(s)$

$= e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right]$

$\Rightarrow$ Let $h(t-2\pi) = \cos 3t - \cos 2t$

$\Rightarrow h(t) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$

$\Rightarrow h(t) = \cos 3t - \cos 2t$

$\Rightarrow L[h(t)] = L[\cos 3t] - L[\cos 2t]$

$\Rightarrow \bar{h}(s) = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$\Rightarrow L[h(t-2\pi)u(t-2\pi)] = e^{-2\pi s}\bar{h}(s)$

$= e^{-2\pi s}\left[\frac{s}{s^2+9} - \frac{s}{s^2+4}\right]$

(1) $\Rightarrow \bar{f}(s) = \frac{s}{s^2+1} + e^{-\pi s}\left[\frac{s}{s^2+4} + \frac{s}{s^2+1}\right] + e^{-2\pi s}\left[\frac{s}{s^2+9} + \frac{s}{s^2+4}\right]$

Inverse Laplace Transform:- Suppose $f(s)$ be the Laplace transform of $F(t)$, then the inverse Laplace transform of $f(s)$ can be defined as $L^{-1}[f(s)] = F(t)$, where L^{-1} is called the inverse Laplace Transform.

Some Important Results:-

1. $L^{-1}\left[\frac{1}{s}\right] = 1$
2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
3. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
4. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
5. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$
6. $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$
7. $L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{1}{a}\sinh at$
8. $L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!} \quad \forall\ n = 0, 1, 2, \ldots$
9. $L^{-1}[f(s+a)] = e^{-at}L^{-1}[f(s)] = e^{-at}F(t)$
10. $L^{-1}[f(s-a)] = e^{at}L^{-1}[f(s)] = e^{at}F(t)$
11. $L^{-1}[f^n(s)] = (-1)^n t^n F(t)$

Find the Laplace transform for the following:-

1. $\frac{1}{s(s+1)(s+2)}$
2. $\frac{4s+5}{(s+1)^2(s+2)}$
3. $\frac{s+2}{s^2-4s+3}$
4. $\frac{4s+5}{(s-1)^2(s+2)}$
5. $\frac{1}{(s-1)(s^2+1)}$
6. $\frac{3s+2}{s^2-s-2}$

① $\frac{1}{s(s+1)(s+2)}$

Let $f(s) = \frac{1}{s(s+1)(s+2)}$

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} \longrightarrow ①$$

$$1 = A(s+1)(s+2) + B(s)(s+2) + C(s)(s+1) \longrightarrow ②$$

when $s = 0$

② $\Rightarrow 1 = A(1)(2)$

$1 = 2A$

$A = \frac{1}{2}$

when $s = -1$

② $\Rightarrow 1 = B(-1)(-1+2)$

$1 = -B$

$B = -1$

when $s=-2$

$$② \Rightarrow 1 = C(-2)(-2+1)$$

$$1 = -2C(-1)$$

$$1 = 2C$$

$$C = 1/2$$

$$\therefore ① \Rightarrow f(s) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{s+1} + \frac{1}{2}\,\frac{1}{s+2}$$

$$\Rightarrow L^{-1}[f(s)] = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s+2}\right]$$

$$\Rightarrow F(t) = \frac{1}{2}(1) - e^{-t} + \frac{1}{2}e^{-2t}$$

$$\Rightarrow F(t) = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$$

② $$\frac{4s+5}{(s+1)^2(s+2)}$$

Let $$f(s) = \frac{1}{(s+1)^2(s+2)}$$

$$\frac{4s+5}{(s+1)^2(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+2)} \rightarrow ①$$

when $s=-1$

$$② \Rightarrow 1 = B(-1+2)$$

$$B = 1$$

when $s=-2$

$$② \Rightarrow -3 = C(-2+1)^2$$

$$C = -3$$

By comparing s^2 coefficient on both sides

we have $A+C=0$

$$A = -C$$

$$\boxed{A=3}$$

$$\therefore ① \Rightarrow f(s) = \frac{3}{s+1} + \frac{1}{(s+1)^2} - \frac{3}{s+2}$$

$$L^{-1}[f(s)] = 3L^{-1}\left[\frac{1}{s+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2}\right] - 3L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 3e^{-t}L^{-1}\left[\frac{1}{s}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2}\right] - 3e^{-2t}L^{-1}\left[\frac{1}{s}\right]$$

$F(t) = 3e^{-t} + te^{-t} - 3e^{-2t}$. (8)

③ $\frac{1}{(s-1)(s^2+1)}$

Let $f(s) = \frac{1}{(s-1)(s^2+1)}$

$\Rightarrow \frac{1}{(s-1)(s^2+1)} = \frac{A}{(s-1)} + \frac{Bs+C}{s^2+1} \rightarrow ①$

$\Rightarrow 1 = A(s^2+1) + (Bs+C)(s-1) \rightarrow ②$

$\Rightarrow 1 = As^2 + A + Bs^2 - Bs + Cs - C$

$\Rightarrow 1 = (A+B)s^2 + (C-B)s + (A-C) \rightarrow ③$

when $s = 1$

② $\Rightarrow 1 = A(1+1)$

$1 = 2A$

$A = \frac{1}{2}$

By comparing s^2 coefficient on both sides we have

$A + B = 0$

$B = -A$

$B = -\frac{1}{2}$

By comparing coefficient of s

$C - B = 0$

$C = B$

$C = \frac{-1}{2}$

$\frac{1}{(s-1)(s^2+1)} = \frac{1}{2(s-1)} + \frac{(-\frac{1}{2})s}{(s^2+1)} - \frac{1}{2}\frac{1}{(s^2+1)}$

$= \frac{1}{2(s-1)} - \frac{s}{2(s^2+1)} - \frac{1}{2}\frac{1}{(s^2+1)}$

$= \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{2}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{s^2+1}\right]$

$= \frac{1}{2}e^{t} - \frac{1}{2}\cos t - \frac{1}{2}e^{-t}\ t$ //

④ $\frac{s+2}{s^2-4s+13}$

Let $f(s) = \frac{s+2}{s^2 - 2(s)(2) + 2^2 - 2^2 + 13}$

$f(s) = \frac{s+2}{(s-2)^2 - 4 + 13}$

$f(s) = \frac{s+2}{(s-2)^2 + 9}$

$f(s) = \frac{(s-2)+2+2}{(s-2)^2+9}$

$f(s) = \frac{(s-2)+4}{(s-2)^2+9}$

$L^{-1}f(s) = L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+9}\right]$

$= e^{2t}L^{-1}\left[\frac{s+4}{s^2+9}\right]$

$= e^{2t}L^{-1}\left[\frac{s}{s^2+9} + \frac{4}{s^2+9}\right]$

$= e^{2t}L^{-1}\left[\frac{s}{s^2+9} + \frac{4}{s^2+9}\right]$

$= e^{2t}\left\{L^{-1}\left[\frac{s}{s^2+9}\right] + 4L^{-1}\left[\frac{1}{s^2+9}\right]\right\}$

$\Rightarrow L^{-1}f(s) = e^{2t}\left[\cos 3t + \frac{4}{3}\sin 3t\right]$

⑤ $\frac{4s+5}{(s-1)^2(s+2)}$

$f(s) = \frac{1}{(s-1)^2(s+2)}$

$\frac{4s+5}{(s-1)^2(s+2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s+2)} \rightarrow ①$

$4s+5 = A(s-1)(s+2) + B(s+2) + C(s-1)^2$

when $s = 1$

$9 = 3B \Rightarrow B = 9/3 \Rightarrow B = 3$

when $s = -2$

$4(-2)+5 = C(-3)^2$

$-3 = 9C$

$C = -\frac{1}{3}$

By comparing s^2 coefficient on both side we have

$A + C = 0$

$A = -C$

$A = -1/3$

$\therefore$ ① $\Rightarrow f(s) = \frac{1}{3(s-1)} + \frac{3}{(s-1)^2} - \frac{1}{3(s+2)}$

$\Rightarrow L^{-1}[f(s)] = \frac{1}{3} L^{-1}\left[\frac{1}{(s-1)}\right] + 3L^{-1}\left[\frac{1}{(s-1)^2}\right] - \frac{1}{3} L^{-1}\left[\frac{1}{(s+2)}\right]$

$= \frac{1}{3} e^{t} L^{-1}\left[\frac{1}{s}\right] + 3e^{t} L^{-1}\left[\frac{1}{s^2}\right] - \frac{1}{3} e^{-2t} L^{-1}\left[\frac{1}{s}\right]$

$\Rightarrow F(t) = \frac{1}{3} e^{t} + 3e^{t} t - \frac{1}{3} e^{-2t}$

⑥ $\frac{3s+2}{s^2-s-2}$

Let $f(s) = \frac{3s+2}{s^2-s-2}$

$= \frac{3s+2}{s^2+s-2s-2}$

$= \frac{3s+2}{s(s+1)-2(s+1)}$

$f(s) = \frac{3s+2}{(s-2)(s+1)}$

$\therefore \frac{3s+2}{(s-2)(s+1)} = \frac{A}{(s-2)} + \frac{B}{(s+1)} \rightarrow$ ①

$3s+2 = A(s+1) + B(s-2) \rightarrow$ ②

when $s = 2$

② $\Rightarrow 8 = A(3)$

$A = \frac{8}{3}$

when $s = -1$

② $\Rightarrow -1 = B(-3)$

$B = 1/3$

① $\Rightarrow f(s) = \frac{8}{3} \frac{1}{s-2} + \frac{1}{3} \frac{1}{s+1}$

$$L^{-1}[f(s)] = \frac{8}{3}L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{3}L^{-1}\left[\frac{1}{s+1}\right]$$

$$F(t) = \frac{8}{3}e^{2t} + \frac{1}{3}e^{-t}$$

⑦ Find the inverse laplace transform of $\frac{2s^2+5s-4}{s^3+s^2-2s}$

$$\Rightarrow f(s) = \frac{2s^2+5s-4}{s(s^2+s-2)}$$

$$= \frac{2s^2+5s-4}{s(s^2-s+2s-2)}$$

$$= \frac{2s^2+5s-4}{s[s(s-1)+2(s-1)]}$$

$$\Rightarrow f(s) = \frac{2s^2+5s-4}{s(s-1)(s+2)}$$

$$\therefore \frac{2s^2+5s-4}{s(s-1)(s+2)} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s+2)} \rightarrow ①$$

$$\Rightarrow 2s^2+5s-4 = A(s-1)(s+2) + B(s)(s+2) + C(s)(s-1) \rightarrow ②$$

when $s=1$

② $\Rightarrow 2+5-4 = B(3)$

$3 = B(3)$

$B = 1$

when $s=-2$

② $\Rightarrow 8-10-4 = C(-2)(-2-1)$

$-6 = C(6)$

$C = -1$

② $\Rightarrow 2 = A+B+C$

$2-A = B+C$

$-A = B+C-2$

$-A = 1-1-2$

$A = 2$

$$① \Rightarrow f(s) = 2\cdot\frac{1}{s} + \frac{1}{(s-1)} - \frac{1}{(s+2)}$$

$$L^{-1}[f(s)] = 2L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{1}{s+2}\right]$$

$$F(t) = 2 + e^{t} - e^{-2t}$$

Fi Find the inverse laplace Transform for the following:- (20)

① $\log\left|\frac{s^2+1}{s(s+1)}\right|$ ② $\log\left|\frac{s(s+5)}{(s^2+25)(s-7)}\right|$

③ $\log\left|\frac{s+a}{s+b}\right|$ ④ $\tan^{-1}\left(\frac{2}{s^2}\right)$

① $\log\left|\frac{s+a}{s+b}\right|$

Let $f(s) = \log\left|\frac{s+a}{s+b}\right|$

$f(s) = \log(s+a) - \log(s+b) \rightarrow ①$

diff. w.r.t s

$① \Rightarrow f'(s) = \frac{1}{s+a} - \frac{1}{s+b}$

$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left(\frac{1}{s+a}\right) - L^{-1}\left(\frac{1}{s+b}\right)$

$\Rightarrow (-1)^1 t^1 F(t) = e^{-at} - e^{-bt}$

$\Rightarrow -t\, F(t) = e^{-at} - e^{-bt}$

$\Rightarrow F(t) = \frac{e^{-at} - e^{-bt}}{-t}$

$\Rightarrow F(t) = \frac{e^{-bt} - e^{-at}}{t}$

② $\log\left|\frac{s^2+1}{s(s+1)}\right|$

Let $f(s) = \log\left|\frac{s^2+1}{s(s+1)}\right|$

$\Rightarrow f(s) = \log|s^2+1| - \log|s(s+1)|$

$\Rightarrow f(s) = \log(s^2+1) - \log s - \log(s+1) \rightarrow ①$

Diff. ① wrt 's'

$① \Rightarrow f'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$

$\Rightarrow L^{-1}[f(s)] = 2\left[L^{-1}\left(\frac{s}{s^2+1}\right) - L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)\right]$

$\Rightarrow (-1)^1 t^1 F(t) = 2\cos t - 1 - e^{-t}$

$$\Rightarrow F(t) = \frac{1+e^{-t}-2\cos t}{t}$$

③ $\log\left|\frac{s(s+5)}{(s^2+25)(s-7)}\right|$

Let $f(s) = \log\left|\frac{s(s+5)}{(s^2+25)(s-7)}\right|$

$$\Rightarrow f(s) = \log|s(s+5)| - \log|(s^2+25)(s-7)|$$

$$\Rightarrow f(s) = \log s + \log(s+5) - \log(s^2+25) - \log(s-7) \rightarrow ①$$

diff. ① wrt 's'

$$① \Rightarrow f'(s) = \frac{1}{s} + \frac{1}{s+5} - \frac{2s}{s^2+25} - \frac{1}{s-7}$$

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{s}\right] + L^{-1}\left[\frac{1}{s+5}\right] - 2L^{-1}\left[\frac{s}{s^2+25}\right] - L^{-1}\left[\frac{1}{s-7}\right]$$

$$\Rightarrow -t\,F(t) = 1 + e^{-5t} - 2\cos 5t - e^{7t}$$

$$\Rightarrow F(t) = \frac{2\cos 5t + e^{7t} - e^{-5t} - 1}{t}$$

④ $\tan^{-1}\left(\frac{2}{s^2}\right)$

Let $f(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

Diff. wrt 's'

$$\Rightarrow f'(s) = \frac{1}{1+\left(\frac{2}{s^2}\right)^2}\frac{d}{ds}\left(\frac{2}{s^2}\right)$$

$$\Rightarrow f'(s) = \frac{1}{1+\frac{4}{s^4}}\left(\frac{-4}{s^3}\right)$$

$$\Rightarrow f'(s) = \frac{s^4}{s^4+4}\left(\frac{-4}{s^3}\right)$$

$$\Rightarrow f'(s) = \frac{-4s}{s^4+4}$$

$$\Rightarrow f'(s) = \frac{-4s}{(s^2)^2+(2)^2}$$

$$f'(s) = \frac{-4s}{(s^2+2)^2 - 2(s^2)(2)}$$

$$= \frac{-4s}{(s^2+2)^2 - 4s^2}$$

$$= \frac{-4s}{(s^2+2)^2 - (2s)^2}$$

$$= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{-4s}{(s^2+2s+2)(s^2-2s+2)}$$

$$= \frac{(s^2-2s+2) - (s^2+2s+2)}{(s^2+2s+2)(s^2-2s+2)}$$

$$\Rightarrow f'(s) = \frac{1}{s^2+2s+2} - \frac{1}{s^2-2s+2}$$

$$= \frac{1}{s^2+2(s)(1)+1+1} - \frac{1}{s^2-2(s)(1)+1+1}$$

$$\Rightarrow f'(s) = \frac{1}{(s+1)^2+1} - \frac{1}{(s-1)^2+1}$$

$$\Rightarrow L^{-1}[f'(s)] = L^{-1}\left[\frac{1}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s-1)^2+1}\right]$$

$$\Rightarrow -tF(t) = e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right] - e^{t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$\Rightarrow -tF(t) = e^{-t}\sin t - e^{t}\sin t$$

$$\Rightarrow F(t) = \sin t\left[\frac{e^{t}-e^{-t}}{t}\right]$$

Laplace transform for convolution of two functions :-

Suppose $f(t)$ and $g(t)$ be the two functions and let $\bar{f}(s)$ and $\bar{g}(s)$ be the Laplace transforms of $f(t)$ and $g(t)$. Then the convolution of $f(t)$ and $g(t)$ can be defined as $f(t) * g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} f(t-u)g(u)\,du$

$$\therefore L[f(t)*g(t)] = L\left[\int_{u=0}^{t} f(u)g(t-u)\,du\right] = L\left[\int_{u=0}^{t} f(t-u)g(u)\,du\right] = \bar{f}(s).\bar{g}(s)$$

$$\therefore L^{-1}[\bar{f}(s).\bar{g}(s)] = f(t)*g(t) = \int_{u=0}^{t} f(u)g(t-u)\,du = \int_{u=0}^{t} f(t-u)g(u)\,du$$

Find the following by using laplace transform with the convolution theorem :-

1. $\frac{s}{(s-1)(s^2+4)}$
2. $\frac{1}{(s-1)(s^2+1)}$
3. $\frac{1}{s(s^2+a^2)}$
4. $\frac{s}{(s^2+a^2)^2}$
5. $\frac{s^2}{(s^2+a^2)^2}$
6. $\frac{1}{(s^2+a^2)(s^2+b^2)}$
7. $\frac{s}{(s^2+a^2)(s^2+b^2)}$
8. $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$

① $\frac{s}{(s-1)(s^2+4)}$

let $\bar{f}(s).\bar{g}(s) = \frac{s}{(s-1)(s^2+4)} = \frac{1}{(s-1)} \cdot \frac{s}{s^2+4}$

$\therefore \bar{f}(s) = \frac{1}{s-1}$. $\bar{g}(s) = \frac{s}{s^2+4}$

$\Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s-1}\right] = e^t = f(t)$

$\Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t = g(t)$

WKT $L^{-1}[\bar{f}(s).\bar{g}(s)] = f(t)*g(t)$

$$= \int_{u=0}^{t} f(t-u)g(u)\,du$$

$$= \int_{u=0}^{t} e^{t-u} \cos 2u \, du$$

$$= \int_{u=0}^{t} e^{t} . e^{-u} \cos 2u \, du$$

$$= e^{t} \int_{u=0}^{t} e^{-u} \cos 2u \, du$$

$$= e^{t} \left\{ \frac{e^{u}}{(-1)^2+2^2} \left[-\cos 2u + 2 \sin 2u \right] \right\}_{0}^{t}$$

$$= e^{t} \left\{ \frac{e^{-u}}{5} \left[2 \sin 2u - \cos 2u \right] \right\}_{0}^{t}$$

$$= e^{t} \left\{ \frac{e^{-t}}{5} \left[2 \sin 2t - \cos 2t \right] - \frac{1}{5} (0-1) \right\}$$

$$= e^{t} \left\{ \frac{e^{-t}}{5} \left[2 \sin 2t - \cos 2t \right] + \frac{1}{5} \right\}$$

2) $\dfrac{1}{s(s^2+a^2)}$

Let $\bar{f}(s) \cdot \bar{g}(s) = \dfrac{1}{s(s^2+a^2)} = \dfrac{1}{s} \cdot \dfrac{1}{(s^2+a^2)}$

$\therefore \bar{f}(s) = \dfrac{1}{s} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s}\right] = 1 = f(t)$

$\bar{g}(s) = \dfrac{1}{s^2+a^2} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{\sin at}{a} = g(t)$

$\therefore L^{-1}[\bar{f}(s) \cdot \bar{g}(s)] = f(t) * g(t) = \displaystyle\int_{u=0}^{t} f(t-u) \, g(u) \, du$

$$= \int_{u=0}^{t} 1 \cdot \frac{1}{a} \sin au \, du$$

$$= \frac{1}{a} \int_{u=0}^{t} \sin au \, du$$

$$= \frac{1}{a} \left[\frac{-\cos au}{a} \right]_{0}^{t}$$

$$= \frac{-1}{a^2} \left[\cos au \right]_{0}^{t}$$

$$= \frac{-1}{a^2} \left[\cos at - 1 \right]$$

$$= \frac{1 - \cos at}{a^2}$$

③ $\dfrac{s}{(s^2+a^2)^2}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{s}{(s^2+a^2)^2} = \dfrac{1}{(s^2+a^2)}\cdot\dfrac{s}{(s^2+a^2)}$

$\therefore \bar{f}(s) = \dfrac{1}{(s^2+a^2)} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \dfrac{1}{a}\sin at = f(t)$

$\bar{g}(s) = \dfrac{s}{(s^2+a^2)} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t) = \displaystyle\int_{u=0}^{t} f(t-u)g(u)\,du$

$= \dfrac{1}{a}\displaystyle\int_{u=0}^{t}\sin(at-au)\cos au\,du$

$= \dfrac{1}{2a}\displaystyle\int_{u=0}^{t} 2\sin(at-au)\cos au\,du$

$= \dfrac{1}{2a}\displaystyle\int_{u=0}^{t}\left[\sin(at-\cancel{au}+\cancel{au}) + \sin(at-au-au)\right]du$

$= \dfrac{1}{2a}\displaystyle\int_{u=0}^{t}\left[\sin at + \sin(at-2au)\right]du$

$= \dfrac{1}{2a}\left\{\sin at\displaystyle\int_{u=0}^{t} 1\cdot du + \int_{u=0}^{t}\sin(at-2au)\,du\right\}$

$= \dfrac{1}{2a}\left[u\sin at - \dfrac{\cos(at-2au)}{-2a}\right]_{u=0}^{t}$

$= \dfrac{1}{2a}\left[u\sin at + \dfrac{1}{2a}\cos(at-2au)\right]_{u=0}^{t}$

$= \dfrac{1}{2a}\left\{\left[t\sin at + \dfrac{1}{2a}\cos at\right] - \left[0 + \dfrac{1}{2a}\cos at\right]\right\}$

$= \dfrac{1}{2a}\left[t\sin at + \dfrac{1}{2a}\cos at - \dfrac{1}{2a}\cos at\right]$

$= \dfrac{1}{2a}t\sin at$

(4) $\dfrac{s^2}{(s^2+a^2)(s^2+b^2)}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{s^2}{(s^2+a^2)(s^2+b^2)} = \dfrac{s}{(s^2+a^2)}\cdot\dfrac{s}{(s^2+b^2)}$

$\therefore \bar{f}(s) = \dfrac{s}{s^2+a^2} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = f(t)$

$\bar{g}(s) = \dfrac{s}{s^2+b^2} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{s}{s^2+b^2}\right] = \cos bt = g(t)$

$$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t)*g(t) = \int_{u=0}^{t} f(t-u)\,g(u)\,du$$

$$= \int_{u=0}^{t}\cos(at-au)\cos bu\,du$$

$$= \frac{1}{2}\int_{u=0}^{t} 2\cdot\cos(at-au)\cos bu\,du$$

$$= \frac{1}{2}\int_{u=0}^{t}[\cos(at-au+bu)+\cos(at-au-bu)]\,du$$

$$= \frac{1}{2}\int_{u=0}^{t}\{\cos[at+(b-a)u]+\cos[at-(b+a)u]\}\,du$$

$$= \frac{1}{2}\left\{\sin\frac{[at+(b-a)u]}{b-a}+\cos\frac{[at-(b+a)u]}{b+a}\right\}_{u=0}^{t}$$

$$= \frac{1}{2}\left\{\left[\frac{\sin bt}{b-a}+\frac{\sin bt}{b-a}\right]-\left[\frac{\sin at}{b-a}-\frac{\sin at}{b+a}\right]\right\}$$

$$= \frac{1}{2}\left\{\left[\frac{1}{b-a}+\frac{1}{b+a}\right]\sin bt-\left[\frac{1}{b-a}-\frac{1}{b+a}\right]\sin at\right\}$$

$$= \frac{1}{2}\left\{\frac{2b}{b^2-a^2}\sin bt-\frac{2a}{b^2-a^2}\sin at\right\}$$

$$= \frac{1}{b^2-a^2}[b\sin bt - a\sin at]$$

⑤ $\dfrac{1}{(s-1)(s^2+1)}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{1}{(s-1)(s^2+1)} = \dfrac{1}{(s-1)}\cdot\dfrac{1}{(s^2+1)}$

$\therefore \bar{f}(s) = \dfrac{1}{(s-1)} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s-1}\right] = e^t = f(t)$

$\bar{g}(s) = \dfrac{1}{(s^2+1)} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{1}{s^2+1}\right] = \sin t = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(t-u)g(u)\,du$

$= \int_{u=0}^{t} e^{(t-u)}\sin u\,du$

$= \int_{u=0}^{t} e^{t}\cdot e^{-u}\sin u\,du$

$= e^{t}\int_{u=0}^{t} e^{-u}\sin u\,du$

$= e^{t}\left\{\dfrac{e^{-u}}{(-1)^2+1^2}[-\sin u - \cos u]\right\}_0^t$

$= e^{t}\left\{\dfrac{e^{-t}}{2}[-\sin t - \cos t] - \dfrac{1}{2}(-1)\right\}$

$= e^{t}\left\{\dfrac{e^{-t}}{2}[-\sin t - \cos t] + \dfrac{1}{2}\right\}$

⑥ $\dfrac{s^2}{(s^2+a^2)^2}$

Let $\bar{f}(s)\cdot\bar{g}(s) = \dfrac{s}{(s^2+a^2)}\cdot\dfrac{s}{(s^2+a^2)}$

$\bar{f}(s) = \dfrac{s}{s^2+a^2} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = f(t)$

$\bar{g}(s) = \dfrac{s}{s^2+a^2} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{s}{s^2+a^2}\right] = \cos at = g(t)$

$\therefore L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t) = \int_{u=0}^{t} f(t-u)g(u)\,du$

$= \int_{u=0}^{t} \cos(at-au)\cos au\,du$

$$= \frac{1}{2}\int_{u=0}^{t} 2\cos(at-au)\cos au \; du$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos(at-au+au) + \cos(at-au-au)]du$$

$$= \frac{1}{2}\int_{u=0}^{t} [\cos at + \cos(at-2au)]du$$

$$= \frac{1}{2}\left\{\cos at \int_{u=0}^{t} 1.du + \int_{u=0}^{t}\cos(at-2au)du\right\}$$

$$= \frac{1}{2}\left[u\cos at + \frac{\sin(at-2au)}{-2a}\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left[u\cos at - \frac{1}{2a}\sin(at-2au)\right]_{u=0}^{t}$$

$$= \frac{1}{2}\left\{\left[t\cos at - \frac{1}{2a}\sin(at-2at)\right] - \frac{1}{2a}\sin(0)\right\}$$

$$= \frac{1}{2}\left\{t\cos at + \frac{1}{2a}\sin at\right\}$$

$$= \frac{t\cos at}{2} + \frac{1}{4a}\sin at$$

(7) $\dfrac{1}{(s^2+a^2)(s^2+b^2)}$

Let $\bar{f}(s).\bar{g}(s) = \dfrac{1}{(s^2+a^2)} \cdot \dfrac{1}{(s^2+b^2)}$

$\bar{f}(s) = \dfrac{1}{(s^2+a^2)} \Rightarrow L^{-1}[\bar{f}(s)] = L^{-1}\left[\dfrac{1}{s^2+a^2}\right] = \sin at$

$\bar{g}(s) = \dfrac{1}{(s^2+b^2)} \Rightarrow L^{-1}[\bar{g}(s)] = L^{-1}\left[\dfrac{1}{s^2+b^2}\right] = \sin bt$

$\therefore L^{-1}[\bar{f}(s).\bar{g}(s)] = f(t) * g(t) = \displaystyle\int_{u=0}^{t} f(t-u)g(u)\,du$

$$= \int_{u=0}^{t} \sin(at-au).\sin bu\,du$$

$$= \frac{1}{2ab}\int_{u=0}^{t} 2\sin(at-au).\sin bu \; du$$

$$= \frac{1}{2ab}\int_{u=0}^{t}[\cos(at-au-bu) - \cos(at-au+bu)]du$$

$$= \frac{1}{2ab}\int_{u=0}^{t} [\cos(at - u(a+b)] - \cos(at - u(a-b)]\, du$$

$$= \frac{1}{2ab}\left[\frac{\sin at - u(a+b)}{-(a+b)} - \frac{\sin at - u(a-b)}{-(a-b)}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin at - u(a-b)}{a-b} - \frac{\sin at - u(a+b)}{a+b}\right]_0^t$$

$$= \frac{1}{2ab}\left[\frac{\sin at - at + bt}{a-b} - \frac{\sin at - at - bt}{a+b}\right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b}\right]$$

$$= \frac{1}{2ab}\left\{\left[\frac{\sin bt}{a-b} - \frac{\sin(-bt)}{a+b}\right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\left[\frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right] - \left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\sin bt\left[\frac{1}{a-b} + \frac{1}{a+b}\right] - \sin at\left[\frac{1}{a-b} - \frac{1}{a+b}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\sin bt\left[\frac{2a}{a^2-b^2}\right] - \sin at\left[\frac{2b}{a^2-b^2}\right]\right\}$$

$$= \frac{1}{2ab}\left\{\frac{\sin bt}{b} - \frac{\sin at}{a}\right\}$$

(8) $\dfrac{s}{(s^2+a^2)(s^2+b^2)}$

$$\bar{f}(s)\cdot\bar{g}(s) = \frac{1}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}$$

$$L^{-1}[\bar{f}(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at = f(t)$$

$$L^{-1}[\bar{g}(s)] = L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos bt = g(t)$$

WKT

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = f(t) * g(t)$$

$$= \int_{u=0}^{t} f(t-u)\,g(u)\,du$$

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$$= \int_{u=0}^{t} \frac{1}{a} \sin(at-au) \cos bu\, du$$

$$= \frac{1}{2a} \int_{u=0}^{t} 2 \sin(at-au) \cos bu\, du$$

$$= \frac{1}{2a} \int_{u=0}^{t} [\sin(at-au+bu) + \sin(at-au-bu)] du$$

$$= \frac{1}{2a} \int_{u=0}^{t} \sin[at+u(b-a)] + \sin[at-u(a+b)] du$$

$$= \frac{1}{2a} \left[\frac{-\cos at-u(a-b)}{-(a-b)} - \frac{\cos(at-u(a+b))}{-(a+b)} \right]_0^t$$

$$= \frac{1}{2a} \left\{ \left[\frac{\cos at-u(a+b)}{a+b} - \frac{\cos at-u(a-b)}{-a-b} \right]_0^t - \left[\frac{\cos at}{a+b} + \frac{\cos at}{a-b} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \left[\frac{\cos(-bt)}{a+b} + \frac{\cos bt}{a-b} \right] - \left[\frac{\cos at}{a+b} + \frac{\cos at}{a-b} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \cos bt \left[\frac{1}{a+b} + \frac{1}{a-b} \right] - \cos at \left[\frac{1}{a+b} + \frac{1}{a-b} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \cos bt \left[\frac{a-b+a+b}{a^2-b^2} \right] - \cos at \left[\frac{a-b+a+b}{a^2-b^2} \right] \right\}$$

$$= \frac{1}{2a} \left\{ \cos bt \frac{2a}{a^2-b^2} - \cos at \frac{2a}{a^2-b^2} \right\}$$

$$= \frac{2a}{2a(a^2-b^2)} [\cos bt - \cos at]$$

$$= \frac{1}{a^2-b^2} [\cos bt - \cos at]$$

Applications of Laplace transform :-

Working rule :-

Step 1 :- Express the given differential eq. with the notations $y(t), y'(t), y''(t)$ ----.

Step 2 :- Apply Laplace transform on both sides and substitute $L[y(t)] = \bar{y}(s)$

$$L[y'(t)] = s\bar{y}(s) - y(0)$$

$$L[y''(t)] = s^2\bar{y}(s) - sy(0) - y'(0)$$

$$L[y'''(t)] = s^3\bar{y}(s) - s^2y(0) - sy'(0) - y''(0).$$

for the initial conditions $y(0), y'(0), y''(0)$.

Step 3 :- Write the function of the variable 's'.

Step 4 :- Take the inverse Laplace transform on the both side and hence find $y(t)$.

① Using Laplace transform $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t}$ where $y(0) = 0 \quad y'(0) = 0$.

Given $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 4y = e^{-t} \quad y(0) = 0 \quad y'(0) = 0$

$$\Rightarrow y''(t) + 4y'(t) + 4y(t) = e^{-t}$$

$$\Rightarrow L[y''(t)] + 4L[y'(t)] + 4L[y(t)] = L[e^{-t}]$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 4\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) + 4s\bar{y}(s) + 4\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 4s + 4)\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s+2)^2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s+1)(s+2)^2} \rightarrow ①$$

$$\Rightarrow \frac{1}{(s+1)(s+2)^2} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s+1)(s+2) + C(s+1) \rightarrow ②$$

when $s = -1$ when $s = -2$

② $\Rightarrow 1 = A(1)^2$ ② $\Rightarrow 1 = C(-2+1)$ $A+B=0$

$A = 1$ $C = -1$ $B = -A$

$B = -1$

$$\therefore ① \Rightarrow \bar{y}(s) = \frac{1}{s+1} - \frac{1}{s+2} - \frac{1}{(s+2)^2}$$

$$\Rightarrow L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] - L^{-1}\left[\frac{1}{(s+2)^2}\right]$$

$$\Rightarrow y(t) = e^{-t} - e^{-2t} - t\,e^{-2t}$$

② Solve the differential equation $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$ with the initial conditions $y(0) = 0,\ y'(0) = 1$.

Given $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{-t}$

$$\Rightarrow y''(t) + 3y'(t) + 2y(t) = e^{-t}$$

$$\Rightarrow L[y''(t)] + 3L[y'(t)] + 2L[y(t)] = L[e^{-t}]$$

$$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow s^2\bar{y}(s) - 1 + 3s\bar{y}(s) + 2\bar{y}(s) = \frac{1}{s+1}$$

$$\Rightarrow (s^2 + 3s + 2)\bar{y}(s) = \frac{1}{s+1} + 1$$

$$\Rightarrow (s+1)(s+2)\bar{y}(s) = \frac{1}{s+1} + 1$$

$$= \frac{s+2}{s+1}$$

$$\Rightarrow \bar{y}(s) = \frac{s+2}{(s+1)^2(s+2)}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s+1)^2}$$

$$\Rightarrow L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$\Rightarrow y(t) = e^{-t} L^{-1}\left[\frac{1}{s^2}\right]$$

$$\Rightarrow y(t) = t\, e^{-t}$$

③ Using laplace transform solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$ given $y(0) = 0$ $\frac{dy}{dt} = 0$ where $t = 0$.

Given $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$ $y(0) = 0$ $y'(0) = 1$

$$\Rightarrow y''(t) + 2y'(t) - 3y(t) = \sin t$$

$$\Rightarrow L[y''(t)] + 2L[y'(t)] - 3L[y(t)] = L[\sin t]$$

$$\Rightarrow [s^2\bar{y}(s) - s\,y(0) - y'(0)] + 2[s\bar{y}(s) - y(0)] - 3\bar{y}(s) = \frac{1}{s^2+1}$$

$$\Rightarrow (s^2 + 2s - 3)\,\bar{y}(s) = \frac{1}{s^2+1}$$

$$\Rightarrow (s-1)(s+3)\,\bar{y}(s) = \frac{1}{s^2+1}$$

$$\Rightarrow \bar{y}(s) = \frac{1}{(s-1)(s+3)(s^2+1)}$$

$$\Rightarrow \frac{1}{(s-1)(s+3)(s^2+1)} = \frac{A}{(s-1)} + \frac{B}{(s+3)} + \frac{Cs+D}{(s^2+1)}$$

$$\Rightarrow 1 = A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s-1)(s+3) \rightarrow ②$$

$$\Rightarrow 1 = A(s^3 + s + 3s^2 + 3) + B(s^3 + s - s^2 - 1) + (Cs+D)(s^2 + 2s - 3)$$

$$\Rightarrow 1 = A(s^3 + 3s^2 + s + 3) + B(s^3 - s^2 + s - 1) + Cs^3 + 2Cs^2 - 3Cs + Ds^2 + 2Ds - 3D$$

$$\Rightarrow 1 = (A + B + C)s^3 + (3A - B + 2C + D)s^2 + (A + B - 3C + 2D)s + (3A - B - 3D) \rightarrow ③$$

when $s = 1$

② $\Rightarrow 1 = 8A$

$A = 1/8$

when $s = -3$

$1 = -40B$

$B = \frac{-1}{40}$

$A + B + C = 0$

$C = -A - B$

$C = \frac{-1}{10}$

$3A - B - 3D = 1$

$3D - 3A - B = 1$

$3D = \frac{3}{8} + \frac{1}{40} - 1$

$D = -\frac{3}{5}$

$$\therefore \bar{y}(s) = \frac{1}{8}\frac{1}{s-1} - \frac{1}{40}\frac{1}{s+3} - \frac{1}{10}\frac{s}{s^2+1} - \frac{1}{5}\frac{1}{s^2+1}$$

$$\Rightarrow y(t) = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}\cos t - \frac{1}{5}\sin t$$

(4) solve $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = e^{-t}$ given $y(0) = y'(0) = 1$

Given $y''(t) + 4y'(t) + 3y(t) = e^{-t}$

$\therefore L[y''(t)] + 4L[y'(t)] + 3L[y(t)] = L[e^{-t}]$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 4[s\bar{y}(s) - y(0)] + 3\bar{y}(s) = \frac{1}{s+1}$

$\Rightarrow s^2\bar{y}(s) - s - 1 + 4s\bar{y}(s) - 4 + 3\bar{y}(s) = \frac{1}{s+1}$

$\Rightarrow (s^2+4s+3)\bar{y}(s) - (s+5) = \frac{1}{s+1}$

$\Rightarrow (s^2+4s+3)\bar{y}(s) = \frac{1}{s+1} + (s+5)$

$\Rightarrow (s+1)(s+3)\ \bar{y}(s) = \frac{1+(s+1)(s+5)}{s+1}$

$\Rightarrow (s+1)(s+3)\ \bar{y}(s) = \frac{s^2+6s+6}{s+1}$

$\Rightarrow \bar{y}(s) = \frac{s^2+6s+6}{(s+1)^2(s+3)} \rightarrow (1)$

$$\therefore \frac{s^2+6s+6}{(s+1)^2(s+3)} = \frac{A}{(s+1)} + \frac{B}{(s+1)^2} + \frac{C}{(s+3)}$$

$s^2+6s+6 = A(s+1)(s+3) + B(s+3) + C(s+1)^2 \rightarrow (2)$

when $s = -1$

$(2) \Rightarrow 1 = 2B$

$B = \frac{1}{2}$

when $s = -3$

$-3 = 4C$

$C = -3/4$

$A + C = 1$

$A = 1 - C$

$A = 7/4$

$① \Rightarrow \bar{y}(s) = \frac{7}{4} \cdot \frac{1}{s+1} + \frac{1}{2} \frac{1}{(s+1)^2} - \frac{3}{4} \frac{1}{s+3}$

$\Rightarrow L^{-1}[\bar{y}(s)] = \frac{7}{4} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{2} L^{-1}\left[\frac{1}{(s+1)^2}\right] - \frac{3}{4} L^{-1}\left[\frac{1}{s+3}\right]$

$\Rightarrow y(t) = \frac{7}{4} e^{-t} + \frac{1}{2} e^{-t} t - \frac{3}{4} e^{-3t}$

⑤ By using laplace transform $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$ with $x(0) = 0$, $\frac{dx}{dt}(0) = -1$.

Given $\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^{2t}$ $x(0) = 0$ $x'(0) = -1$

$\Rightarrow x''(t) - 2x'(t) + x(t) = e^{2t}$

$\Rightarrow L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^{2t}]$

$\Rightarrow [s^2\bar{x}(s) - s x(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \frac{1}{s-2}$

$\Rightarrow s^2\bar{x}(s) - 0 + 1 - 2s\,\bar{x}(s) + 0 + \bar{x}(s) = \frac{1}{s-2}$

$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) = \frac{1}{s-2} - 1$

$\Rightarrow (s-1)^2 \bar{x}(s) = \frac{1 - s + 2}{(s-2)}$

$\Rightarrow (s-1)^2 \bar{x}(s) = \frac{3-s}{(s-2)}$

$\Rightarrow \bar{x}(s) = \frac{3-s}{(s-1)^2(s-2)} \rightarrow ①$

$\therefore \frac{3-s}{(s-1)^2(s-2)} = \frac{A}{(s-1)} + \frac{B}{(s-1)^2} + \frac{C}{(s-2)}$

$3s = A(s-1) + B(s-2) + C(s-1)^2$

when $s = 1$ | when $s = 2$ | and $A + C = 0$

$② \Rightarrow 2 = -B$ | $② \Rightarrow 1 = C$ | $A = -C$

$B = -2$ | $C = 1$ | $A = 1$

$① \Rightarrow \bar{x}(s) = \frac{-1}{s-1} - \frac{2}{(s-1)^2} + \frac{1}{s-2}$

$\Rightarrow L^{-1}[\bar{x}(s)] = L^{-1}\left[\frac{1}{s-2}\right] - L^{-1}\left[\frac{1}{(s-1)}\right] - 2L^{-1}\left[\frac{1}{(s-1)^2}\right]$

$\Rightarrow x(t) = e^{2t} - e^{t} - 2te^{t}$

⑥ Solve the equation $y'' - 3y' + 2y = e^{3t}$, $y(0) = 1$, $y'(0) = 0$ using Laplace transform technique.

$\Rightarrow$ Given $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = e^{3t}$

$\Rightarrow y''(t) - 3y'(t) + 2y(t) = e^{3t}$

$\Rightarrow L[y''(t)] - 3L[y'(t)] + 2L[y(t)] = Le^{3t}$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] - 3s\bar{y}(s) + 3y(0) + 2\bar{y}(s) = \frac{1}{s-3}$

$\Rightarrow s^2\bar{y}(s) - s - 3s\bar{y}(s) + 3 + 2\bar{y}(s) = \frac{1}{s-3}$

$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) - s + 3 = \frac{1}{s-3}$

$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) - (s-3) = \frac{1}{s-3}$

$\Rightarrow (s^2 - 3s + 2)\bar{y}(s) = \frac{1}{(s-3)} + (s-3)$

$\Rightarrow (s-1)(s-2)\bar{y}(s) = \frac{1 + (s-3)^2}{(s-3)}$

$\Rightarrow \bar{y}(s) = \frac{1 + s^2 + 9 - 3 \times 2s}{(s-1)(s-2)(s-3)}$

$\Rightarrow \bar{y}(s) = \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} \rightarrow ①$

$\therefore \frac{s^2 - 6s + 10}{(s-1)(s-2)(s-3)} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$

$\Rightarrow s^2 - 6s + 10 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2) \rightarrow ②$

$\Rightarrow$ when $s = 1$

② $\Rightarrow 1 - 6 + 10 = A(-1)(-2)$

$\Rightarrow 5 = 2A$

$\Rightarrow A = 5/2$

when $s = 2$

② $\Rightarrow 4 - 12 + 10 = B(2-1)(2-3)$

$\Rightarrow 2 = B(-1)$

$\Rightarrow B = -1/2$

when $s = 3$

② $\Rightarrow 9 - 18 + 10 = C(3-1)(3-2)$

$1 = C(2)$

$C = \frac{1}{2}$

① $\Rightarrow \bar{y}(s) = \frac{5}{2}\frac{1}{s+1} + \left(\frac{-1}{2}\right)\frac{1}{s-2} + \frac{1}{2}\frac{1}{s-3}$

$\Rightarrow L^{-1}[\bar{y}(s)] = \frac{5}{2}L^{-1}\left[\frac{1}{s+1}\right] + \left(\frac{-1}{2}\right)L^{-1}\left[\frac{1}{s-2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s-3}\right]$

$y(t) = \frac{5}{2}e^{-t} - \frac{1}{2}e^{2t} + \frac{1}{2}e^{3t}$ //

⑦ solve the equation $y'' + 3y' + 2y = 0$, $y(0) = 1$, $y'(0) = 0$ using laplace transform technique.

$\Rightarrow$ Given $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$

$\Rightarrow y''(t) + 3y'(t) + 2y(t) = 0$

$\Rightarrow L[y''(t)] + 3L[y'(t)] + 2L[y(t)] = 0$

$\Rightarrow [s^2\bar{y}(s) - sy(0) - y'(0)] + 3[s\bar{y}(s) - y(0)] + 2\bar{y}(s) = 0$

$\Rightarrow s^2\bar{y}(s) - sy(0) - y'(0) + 3s\bar{y}(s) - 3 + 2\bar{y}(s) = 0$

$\Rightarrow s^2\bar{y}(s) - s + 3s\bar{y}(s) - 3 + 2\bar{y}(s) = 0$

$\Rightarrow (s^2 + 3s + 2)\bar{y}(s) - (s+3) = 0$

$\Rightarrow (s+1)(s+2)\bar{y}(s) = (s+3)$

$\Rightarrow \bar{y}(s) = \frac{(s+3)}{(s+1)(s+2)} \rightarrow$ ①

$\frac{(s+3)}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$

$s+3 = A(s+2) + B(s+1) \rightarrow$ ②

when $s = -2$

② $\Rightarrow 1 = -B$

$B = -1$

when $s = -1$

② $\Rightarrow 2 = A$

$A = 2$

(1) $\Rightarrow \bar{y}(s) = \dfrac{2}{(s+1)} + \dfrac{(-1)}{(s+2)}$

$L^{-1}[\bar{y}(s)] = 2L^{-1}\left[\dfrac{1}{s+1}\right] - 1L^{-1}\left[\dfrac{1}{s+2}\right]$

$y(t) = 2e^{-t} - e^{-2t}$

(8) Solve the equation $x'' - 2x' + x = e^{2t}$, $x(0) = 1$, $\dfrac{dx(0)}{dt} = -1$ using laplace transform technique.

$\Rightarrow$ Given $\dfrac{d^2x}{dt^2} - 2\dfrac{dx}{dt} + x = e^{2t}$ $\quad x(0) = 1 \quad x'(0) = -1$

$\Rightarrow x''(t) - 2x'(t) + x(t) = e^{2t}$

$\Rightarrow L[x''(t)] - 2L[x'(t)] + L[x(t)] = L[e^{2t}]$

$\Rightarrow [s^2\bar{x}(s) - s x(0) - x'(0)] - 2[s\bar{x}(s) - x(0)] + \bar{x}(s) = \dfrac{1}{(s-2)}$

$\Rightarrow s^2\bar{x}(s) - s + 1 - 2s\bar{x}(s) + 2 + \bar{x}(s) = \dfrac{1}{(s-2)}$

$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) - s + 3 = \dfrac{1}{(s-2)}$

$\Rightarrow (s^2 - 2s + 1)\bar{x}(s) - (s-3) = \dfrac{1}{(s-2)}$

$\Rightarrow (s-1)^2(\bar{x}(s)) = \dfrac{1}{(s-2)} + (s-3)$

$\Rightarrow (s-1)^2(\bar{x}(s)) = \dfrac{1 + (s-3)(s-2)}{(s-2)}$

$\Rightarrow (s-1)^2(\bar{x}(s)) = \dfrac{1 + s^2 - 2s - 3s + 6}{(s-2)}$

$\Rightarrow (s-1)^2(\bar{x}(s)) = \dfrac{s^2 - 5s + 7}{(s-2)}$

$\bar{x}(s) = \dfrac{s^2 - 5s + 7}{(s-2)(s-1)^2} \rightarrow (1)$

$\dfrac{s^2 - 5s + 7}{(s-2)(s-1)^2} = \dfrac{A}{(s-2)} + \dfrac{B}{(s-1)} + \dfrac{C}{(s-1)^2}$

$$s^2 - 5s + 7 = A(s-1)^2 + B(s-1)(s-2) + C(s-2) \rightarrow (3)$$

when $s = 2$

$(3) \Rightarrow 4 - 10 + 7 = A(2-1)^2$

$1 = A$

$A = 1$

when $s = 1$

$(3) \Rightarrow 1 - 5 + 7 = C(1-2)$

$3 = -C$

$C = -3$

$1 = A + B$

$B = 1 - A$

$= 1 - 1$

$\boxed{B = 0}$

$$(2) \Rightarrow \bar{y}(s) = \frac{1}{(s-2)} - \frac{3}{(s-1)^2}$$

$$\Rightarrow L^{-1}[\bar{y}(s)] = L^{-1}\left[\frac{1}{s-2}\right] - 3\left[\frac{1}{(s-1)^2}\right]$$

$$\Rightarrow y(t) = e^{2t} - 3e^t t$$