Useful constants, functions, properties

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Some Useful Constants and Functions

symbol	value
e	2.71828…
γ (gamma)	0.57721…
ϕ (phi)	$(1 + \sqrt{5}) / 2 = 1.61803…$

These tables are for reference. We may use such symbols and functions as we discuss specific algorithms.

function	name	approximation
$\lfloor x \rfloor$	floor function	x
$\lceil x \rceil$	ceiling function	x
F_N	Fibonacci numbers	$\phi^N / \sqrt{5}$
H_N	harmonic numbers	ln(N) + γ
N!	factorial function	$(N / e)^N$
lg(N!)		N lg(N) - 1.44N

Summation of Geometric Series

- Summation of consecutive terms: $1 + 2 + 3 + \ldots + n$
  - $\sum_{k=1}^{n} k = \frac{(n+1)*n}{2} = \Theta(n^2)$
- Summation of Geometric Series: $1 + x + x^2 + \ldots + x^n$
  - $\boldsymbol{x > 1}$:
    - $\sum_{k=0}^{n} x^k = \frac{x^{n+1}-1}{x-1} = \Theta(x^n)$
  - $\boldsymbol{0 < x < 1}$:
    - $\sum_{k=0}^{n} x^k \leq \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} = \Theta(1)$
  - $\boldsymbol{x = 1}$
    - $\sum_{k=0}^{n} 1^k = n + 1 = \Theta(n)$

Approximation by Integrals

- If $f(x)$ is a monotonically increasing function:
  - This means that $x \leq y \Rightarrow f(x) \leq f(y)$.
- Then, we can approximate the summation $\sum_{k=m}^{n} f(k)$ using integral $\int_{m}^{n+1} f(x)dx$.

- Reason:

$$\int_{m-1}^{n} f(x)dx \leq \sum_{k=m}^{n} f(k) \leq \int_{m}^{n+1} f(x)dx$$

Other

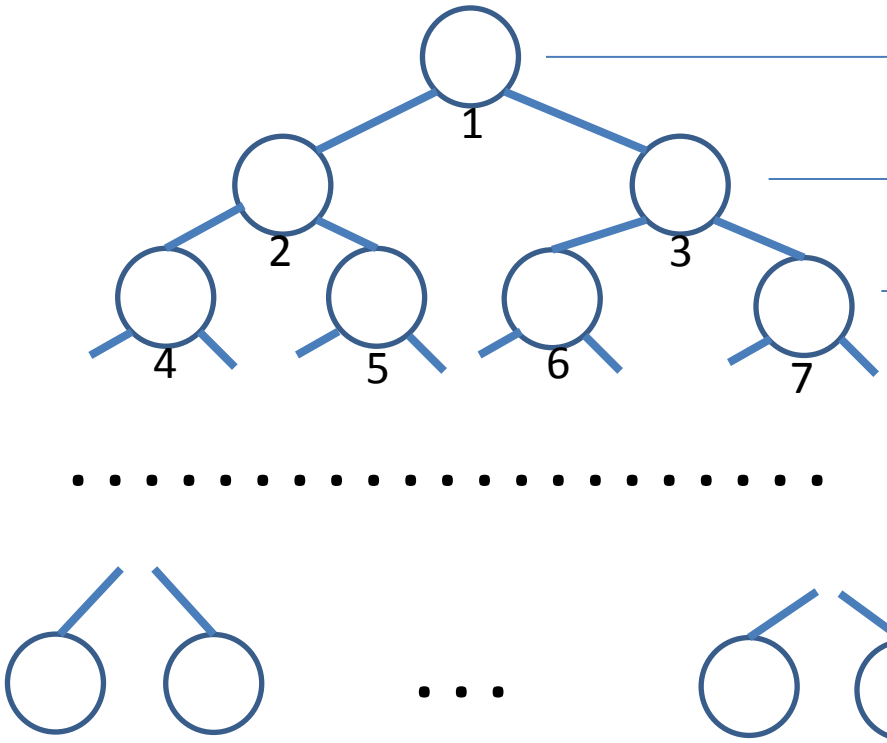
- $a^{log_b(n)} = n^{log_b(a)}$ (note $\log_b$ in the exponent)
- $a^n \neq n^a$

Perfect Binary Trees

A **perfect binary tree** with N nodes has:

- $\lfloor \lg N \rfloor$ +1 levels
- height $\lfloor \lg N \rfloor$
- $\lceil N/2 \rceil$ leaves (half the nodes are on the last level)
- $\lfloor N/2 \rfloor$ internal nodes (half the nodes are internal)

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1$$



Level	Nodes per level	Sum of nodes from root up to this level
0	2^0 (=1)	$2^1 - 1$ (=1)
1	2^1 (=2)	$2^2 - 1$ (=3)
2	2^2 (=4)	$2^3 - 1$ (=7)
…	…	
i	2^i	$2^{i+1} - 1$
…	…	
n	2^n	$2^{n+1} - 1$

Properties of Full Trees

- A **full** binary tree (0/2 children) with N internal nodes has:
  - N+1 external nodes.
  - 2N edges (links).
  - height at least lg N and at most N:
    - lg N if all external nodes are at the same level (complete or perfect tree)
    - N if each internal node has one external child.