

Summations

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Math background - Review

- Series (the terms) and their summations:
 - Geometric
 - Arithmetic
 - (This information is in the following slides and the class cheat sheet)
- Integrals
 - From the Integrals cheat sheet see:
 - The fundamental theorem of calculus Part II:
 - Common integrals
- Cheat sheets and other useful links are on the Slides and Resources webpage.

Book Reference

- Book (CLRS) references for this lecture: Appendix A, especially
 - pages 1145-1147,
 - page 1150 (bounding the terms),
 - Pages 1154-1156 (approximation by integrals)

Overview

- Summations

- Summation of *arithmetic series*: $\sum_{k=0}^t (a_1 + kd)$

- Where a_1 is the first term and d is the step.

- Summation of *geometric series*: $\sum_{k=0}^n x^k$

- $0 < x < 1$, $\Theta(1)$

- $x > 1$, $\Theta(x^n)$

- $x = 1$, $\Theta(n)$ (easy to check: $1+1+\dots+1$)

- Approximation by integrals

- Induction

- You suspect/guess that a summation $S = \Theta(f(N))$ and prove/verify it using induction.

- See Dr. Bob Weems, “Notes 3: Summations”, section 3.D.

- Very useful exercise in strengthening one’s math skills.

Summations

- Summations are formulas of the sort:

$$\sum_{k=0}^n f(k)$$

- Used to solve recurrences.
- Oftentimes, establishing **upper bounds** is sufficient, since we use big-Oh notation.
 - If $f(k) \geq 0$, then: $\sum_{k=0}^n f(k) \leq \sum_{k=0}^{\infty} f(k)$
 - Summing to infinity may give a simpler formula.

Summation of Arithmetic Series

- Examples:
 - 1,5,9,13,17,21,.....
 - 'Step' between terms: $d = 4$
 - First term: $a_1 = 1$
 - Note that 1,2,3,...,n is a special case where the step is +1.
 - 3,13,23,33,43,.....
 - Step? First term (a_1)?
- General formula: $a_i = a_1 + (i-1)d$
 - Here d is the step and a_1 is the first term
- Summation: $\sum_{i=1}^n a_i = n \frac{(a_1 + a_n)}{2} = \frac{n}{2} [2a_1 + (n-1)d]$
- Summation of consecutive terms: $1 + 2 + 3 + \dots + n = \sum_{k=1}^n k = \frac{n(n+1)}{2}$
- Do not confuse the summation of terms with the count of terms! (See a future slide)

Summation of Geometric Series

$$0 < x < 1$$

- Suppose that $0 < x < 1$:
- Finite summations: $\sum_{k=0}^n x^k = \frac{1 - x^{n+1}}{1 - x} = \Theta(??)$
 - Infinite summations: $\sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$
 - Important to note: $\sum_{k=0}^n x^k \leq \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$
 - Therefore, $\sum_{k=0}^n x^k = \Theta(1)$. Why?

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 - Important to note: $\sum_{k=0}^n x^k \leq \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$
 - Therefore, $\sum_{k=0}^n x^k = \Theta(1)$. Why?
 - Because $\frac{1}{1 - x}$ is independent of n .
 - Strictly speaking we showed that it is $O(1)$, but the sum is also $\neq 0$ and so it is $\Omega(1)$.

Summation of Geometric Series

$$x \geq 1$$

- $x > 1$ The formula is the same, and can be rewritten as:

$$\sum_{k=0}^n x^k = \frac{x^{n+1} - 1}{x - 1}$$

– Remember this formula!

– For example:

$$1 + 5 + 5^2 + 5^3 + \dots + 5^n = \frac{5^{n+1} - 1}{5 - 1} = \theta(5^n)$$

5 is a constant and so: $5^{n+1} = 5 * 5^n = \theta(5^n)$ (Exponential!)

- $x = 1 \Rightarrow \sum_{k=0}^n x^k = \sum_{k=0}^n 1 = n + 1 = \theta(n)$

- Do not confuse the summation of terms with the count of terms!
- Progression: 1, 2, 3, ..., n
- Summation of terms: $1+2+\dots+n = n(n+1)/2 = \Theta(n^2)$
- Count of terms: 1, 2, 3, ..., n \Rightarrow n terms $\Rightarrow \Theta(n)$
- Progression: 1, 2, 4, ..., 2^i , ..., 2^n
- Summation: $\sum_{i=0}^n 2^i = \frac{2^{n+1}-1}{2-1} = \Theta(2^{n+1})$
- Count of terms: n+1
- Progression: 1, 2, 4, ..., 2^i , ..., $2^p \leq n$ (last power of 2 less or equal to n)
- Summation: $\sum_{i=0}^p 2^i = \frac{2^{p+1}-1}{2-1} = \frac{2n-1}{2-1} = \Theta(n)$ b.c. ($2^p = n$)
- Count of terms: $1+\lg n$ (more precisely: $1+\text{floor}(\lg n)$)

Approximation by Integrals

$$\int_{m-1}^n f(x) dx \leq \sum_{k=m}^n f(k) \leq \int_m^{n+1} f(x) dx$$

when $f(x)$ is a **monotonically increasing** function.

In most cases***, our end result will be:

$$\sum_{k=1}^n f(k) = \Theta\left(\int_0^n f(x) dx\right) \text{ when } f(x) \text{ is monotonically increasing}$$

*** For the problems we look at, you can adjust the borders since the differences should result in lower order terms (see the worked-out example). But always be prepared to check the correctness.

Approximation by Integrals - Theory

$$\int_{m-1}^n f(x)dx \leq \sum_{k=m}^n f(k) \leq \int_m^{n+1} f(x)dx$$

- Suppose that $f(x)$ is a monotonically increasing function:
 - This means that $x \leq y \Rightarrow f(x) \leq f(y)$.
- Then, we can approximate summation $\sum_{k=m}^n f(k)$ using integral $\int_m^{n+1} f(x)dx$.
- Why? Because $\int_{k-1}^k f(x)dx \leq f(k) \leq \int_k^{k+1} f(x)dx$.
 - Proof for: $f(k) \leq \int_k^{k+1} f(x)dx$
For every x in the interval $[k, k+1]$, $x \geq k$. Since $f(x)$ is increasing, if $x \geq k$ then $f(x) \geq f(k)$. (Remember that the integral is the sum of $f(x)$ on the interval $[k, k+1]$ of size 1.)
 - Similar proof for $\int_{k-1}^k f(x)dx \leq f(k)$

Approximation by Integrals - Example

$$\int_{m-1}^n f(x)dx \leq \sum_{k=m}^n f(k) \leq \int_m^{n+1} f(x)dx$$

- Example: approximate $T(n) = \sum_{k=1}^n k^2$ using an integral:
 - Notice that $T(n)$ is a sum of k^2 terms. We will use $f(x) = x^2$ to approximate this summation.
 - $f(x) = x^2$ is a monotonically increasing function.
- O: $T(n) = \sum_{k=1}^n k^2 \leq \int_1^{n+1} x^2 dx = \frac{(n+1)^3 - 1^3}{3} = \frac{n^3 + 2n^2 + 2n + 1 - 1}{3} = O(n^3)$
- Ω : $T(n) = \sum_{k=1}^n k^2 \geq \int_0^n x^2 dx = \frac{n^3 - 0^3}{3} = \frac{n^3}{3} = \Omega(n^3)$
- $\Rightarrow f(n) = \Theta(n^3)$

Approximation by Integrals

$$\int_{m-1}^n f(x)dx \leq \sum_{k=m}^n f(k) \leq \int_m^{n+1} f(x)dx$$

when $f(x)$ is a monotonically **increasing** function.

$$\int_m^{n+1} f(x)dx \leq \sum_{k=m}^n f(k) \leq \int_{m-1}^n f(x)dx$$

when $f(x)$ is a monotonically **decreasing** function.

Worksheet

1) Find Θ for

$$a) \sum_{k=1}^n k^d$$

$$b) \sum_{k=1}^n k^{-1}$$

Hint: pay attention to the borders for b).

2) Given summation: $1 + 2^5 + 3^5 + \dots + N^5$

Can you solve this in terms of Θ , ~~Ω or Θ~~ ?