CSE 2320 - Homework 5 NAME:____________SOLUTION______________________________________

Total points: 100 Topics: Recurrences , solved with methods: Master Theorem, Tree, Substitution (induction)

P1. (**23 points**) Use the tree and table method to compute the **Θ** time complexity for $T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3$.

Assume T(0) = 1 and T(1) = 1. Fill in the table below and finish the computations outside of it:

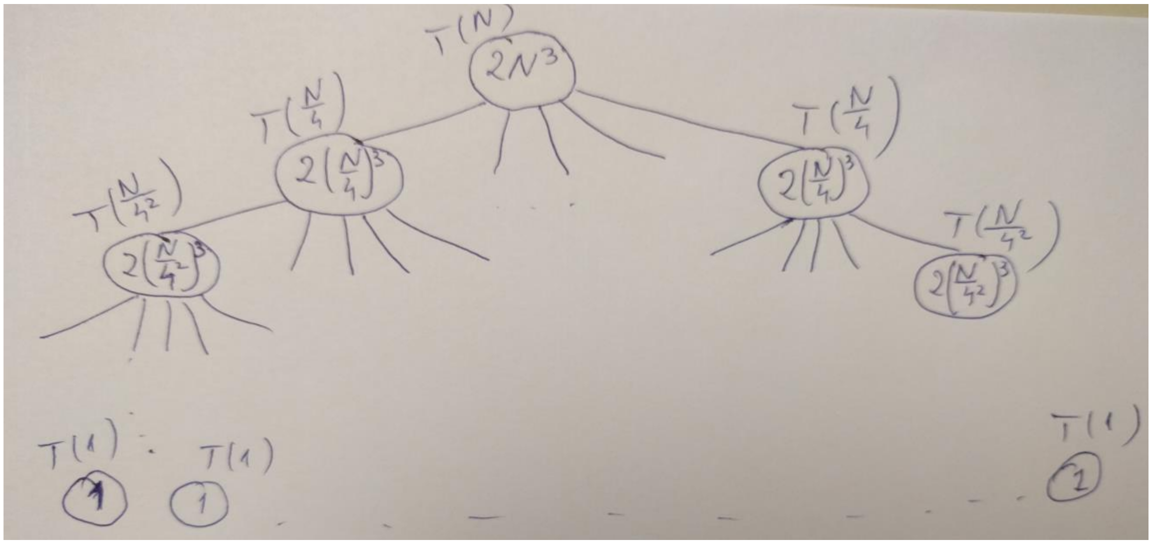
Level	Argument/ Problem size	Cost of one node	Nodes per level	Cost of whole level
0	N	$2N^3$	1	$2N^3$
1	N/4	$2(N/4)^3$	5	$5*2(N/4)^3 = 2N^3(5/4^3)^1$
2	$N/4^2$	$2(N/4^2)^3$	5^2	$5^2*2(N/4^2)^3 = 2N^3(5/4^3)^2$
i	$N/4^i$	$2(N/4^i)^3$	5^i	$5^i*2(N/4^i)^3 = 2N^3(5/4^3)^i$
$k=\log_4 N$ *Leaf level. Write k as a function of N.*	$N/4^k = 1$	1 Acceptable to use 2 instead of 1, if stated that using 2 instead of the 1 (correct), results in a lower order term or a constant and does not affect Θ.	5^k	$=1*5^k$ It is also: $5^k*(N/4^k)^3 = N^3(5/4^3)^k$

Total tree cost calculation: ……

$$\left(\sum_{i=0}^{k} 2N^3(5/4^3)^i\right) - 5^k = \left(2N^3\sum_{i=0}^{k}(5/4^3)^i\right) - 5^k \le 2N^3 - 5^k = 2N^3 - 5^{\log_4 N} = 2N^3 - N^{\log_4 5}$$

T(N) = Θ(……………N^3……) (cubic – polynomial)

Draw the tree. Show **levels 0,1,2** and the **leaves level**. Show the problem size T(…) as a label next to the node and inside the node show the local cost (cost of one node) as done in class. For the leaf level and level 2 it suffices to show a few nodes.



P2. (**23 points**) Use the tree and table method to compute the **Θ** time complexity for $T(N) = 4T(N-5) + 7$. Assume T(N) = ~~1~~ 7 for all 0≤N≤4. Assume N is a convenient value for your computations. $\mathrm{T(N)}=5\mathrm{T(N-4)}+7$

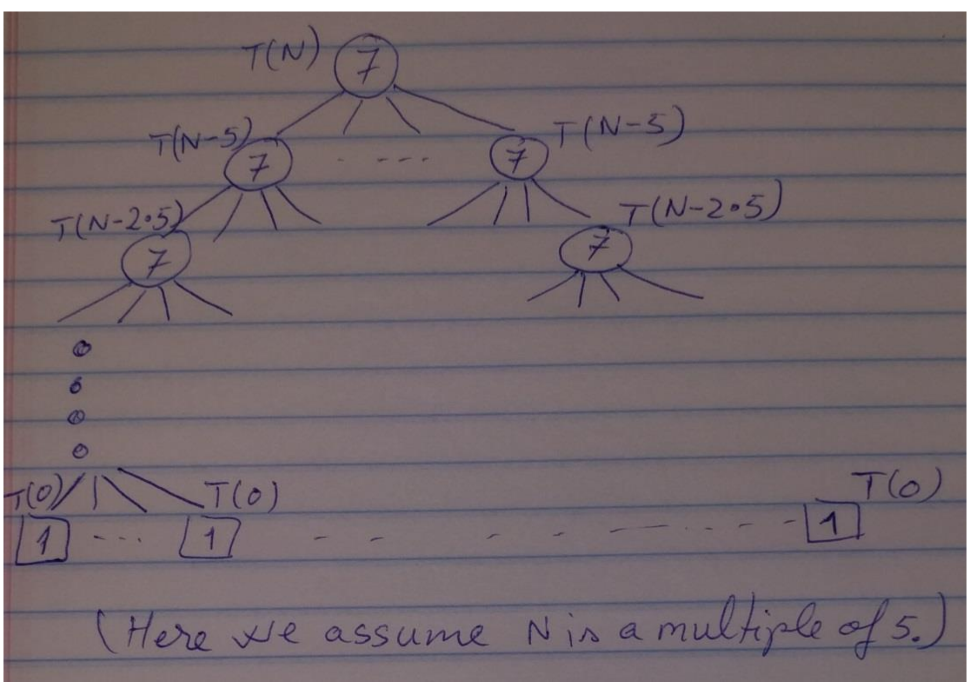
Fill in the table below and finish the computations outside of it:

Level	Argument/ Problem size	Cost of one node	Nodes per level	Cost of whole level
0	N	7	1	7
1	N-5	7	4	4*7
2	N-5*2	7	4^2	4^2*7
i	N-5i	7	4^i	4^i*7
k=N/5 *Leaf level. Write k as a function of N.*	N-5k	7	4^k	4^k*7

Total tree cost calculation: $\sum_{i=0}^{k} 4^i * 7 = 7\sum_{i=0}^{k} 4^i = 7\frac{4^{k+1}-1}{4-1} = 7\frac{4*4^k-1}{3} = \frac{7}{3}\left(4 * 4^{\frac{N}{5}} - 1\right) = \frac{7}{3}(4 * \sqrt[5]{4}^N - 1)$

T(N) = Θ(… ($\sqrt[5]{4}^N$) ……) (exponential)

Draw the tree. Show **levels 0,1,2** and the **leaves level**. Show the problem size T(…) as a label next to the node and inside the node show the local cost (cost of one node) as done in class. For the leaf level and level 2 it suffices to show a few nodes.



P3. (**36 points**) Can you use the Master Theorem to solve the recurrences below? If yes, solve it with this method, if no, show why you cannot use it.

a) $\mathrm{T(N)} = 5\mathrm{T}(\lfloor N/4 \rfloor) + 2N^3$. Assume T(0) = 1 and T(1) = 1.

$$a = 5, b = 4, N^{\log_b a} = N^{\log_4 5}$$
$$f(N) = 2N^3 = \Omega(N^{\log_4 5 + 1}) \Rightarrow \text{case } 3$$
$$af(N/b) = 5 * 2(N/4)^3 = 2N^3 * (\frac{5}{64}) \leq (\frac{5}{25}) * 2N^3 = (\frac{5}{25}) f(N), Pick: c = (\frac{5}{25}) < 1, \text{verified} \Rightarrow$$
$$T(N) = \Theta(f(N)) = \Theta(N^3)$$

b) $\mathrm{T(N)} = 4\mathrm{T}(\lceil N/4 \rceil) + d$, for some constant d>0. Assume T(0) = 1 and T(1) = 1.

$$a = 4, b = 4, N^{\log_b a} = N^{\log_4 4} = N$$
$$f(N) = d = O(N^{1-0.1}) \Rightarrow \text{case } 1 \Rightarrow T(N) = \Theta(N^{\log_b a}) = \Theta(N)$$

c) $T(N) = 6T(N/6) + 5N$, Assume T(0) = 1 and T(1) = 1

$$a = 6, b = 6, N^{log_b a} = N^{log_b 6} = N^1 = N$$
$$f(N) = 5N = \Theta(N) \Rightarrow case\ 2 \Rightarrow T(N) = \Theta(NlgN)$$

d) $T(N) = 8T(N/2) + cN^3 lgN$, Assume T(0) = 1 and T(1) = 1

$$a = 8, b = 2, N^{log_b a} = N^{log_2 8} = N^3$$
$$f(N) = cN^3 lgN = O(N^3)$$
$$\Rightarrow it\ may\ be\ case\ 3\ but\ it\ fails\ it\ because\ lgn\ grows\ slower\ than\ ANY\ polinomial$$
$$\Rightarrow Cannot\ find\ any\ \varepsilon\ s.t.\ lgn = \Omega(N^\varepsilon)$$
$$(because\ in\ fact\ \forall\varepsilon, lgN = O(N^\varepsilon)\Theta(NlgN)$$
$$\Rightarrow cannot\ apply\ case\ 3\ or\ any\ other\ case$$
$$\Longrightarrow cannot\ solve\ it\ with\ the\ Master\ Theorem$$

P4. (**4 points**) Go to the Wikipedia webpage https://en.wikipedia.org/wiki/Master_theorem_(analysis_of_algorithms) . See section "Inadmissible equations" and list the equation and the reason why it does not satisfy the Master Theorem requirements.

NOTE: Here it is copy/pasted, but ideally student homework answer was handwritten or retyped to help retain the information.

The following equations cannot be solved using the master theorem:[3]

- $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

  a is not a constant; the number of subproblems should be fixed
- $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

  non-polynomial difference between f(n) and $n^{\log_b a}$ (see below)
- $T(n) = 0.5T\left(\frac{n}{2}\right) + n$

  $a < 1$ cannot have less than one sub problem
- $T(n) = 64T\left(\frac{n}{8}\right) - n^2 \log n$

  f(n), which is the combination time, is not positive
- $T(n) = T\left(\frac{n}{2}\right) + n(2 - \cos n)$

  case 3 but regularity violation.

In the second inadmissible example above, the difference between $f(n)$ and $n^{\log_b a}$ can be expressed with the ratio $\frac{f(n)}{n^{\log_b a}} = \frac{n/\log n}{n^{\log_2 2}} = \frac{n}{n \log n} = \frac{1}{\log n}$. It is clear that $\frac{1}{\log n} < n^{\epsilon}$ for any constant $\epsilon > 0$. Therefore, the difference is not polynomial and the Master Theorem does not apply.

P5. (**14 points**) Show that $\mathrm{T(N)}=5\mathrm{T}(\lfloor N/4 \rfloor) + 2N^3 = \Theta(N^3)$ by showing that it is $O(N^3)$ and also $\Omega(N^3)$. Assume T(0) = 1 and T(1) = 1

a) (9 pts) Use the induction method to show $O(N^3)$. As done in class, start with the inductive step and then check and refine for enough low values of N until the inductive step can be applied (See lecture from Wed, Oct 11).

Prove that $T(N)=O(N^3)$, using the definition: find c and N_1 s.t. $T(N) \le cN^3$ (here:f(N) = T(N) , g(N) = N^3)

Show with induction: $T(N) \le cN^3$ (for some c>0, for all N≥ N_1)

Base cases:

N = 0: T(0)=1, but cN^3 = 3*0 = 0 => *fails*

N = 1: T(1)=1 ≤ $c*1^3$ = c , for all ***c≥1 (1*)***

T(2) and T(3) use T(0) (in the recurrence formula), but T(0) fails the hypothesis we are proving, so we cannot prove them using the inductive step => must treat T(2) and T(3) as base cases (same way as we did for T(1)).

N = 2: T(2)= 5 * T(0) + $2*2^3$ = 5*1+16 = 21. We want 21 ≤ $c*2^3$. Holds for ***c≥21/8 (2*)***

N = 3: T(3)= 5 * T(0) + $2*3^3$ = 5*1+54 = 59. We want 59 ≤ $c*3^3$. Holds for ***c≥59/27 (3*)***

We want all of the above to hold, so we will use: c≥max{1, 21/8, 59/27}

Inductive step:

Now we have proved enough base cases. Every value of N≥4, will use T(1) or higher and the hypothesis holds for T of 1 or higher for all c≥ max{1, 21/8, 59/27}. We can **prove the recursive case** for all N≥4.

We will show that $T(N) \le cN^3$ (for some c≥max{1,21/8,59/27} for all N≥ 4)

$$T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 \le$$

$$\downarrow \textit{by inductive hypothesis applied to } T(\lfloor N/4 \rfloor)$$

$$\leq 5c\left\lfloor\frac{N}{4}\right\rfloor^3 + 2N^3 \leq$$

$$\leq 5c\left(\frac{N}{4}\right)^3 + 2N^3 =$$

$$= cN^3\left(\frac{5}{64}\right) + 2N^3 =$$

$$= N^3\left(\frac{5c}{64} + 2\right)$$

We want:

$$N^3\left(\frac{5c}{64} + 2\right) \leq \mathrm{c}N^3 =>$$

$$N^3\left(c - 2 - \frac{5c}{64}\right) \geq 0 =>$$

$$c\left(1 - \frac{5}{64}\right) \geq 2 =>$$

$$c \geq \frac{2}{\frac{59}{64}} =>$$

$$c \geq \frac{128}{59}$$

Pick c = 3 (b.c. 3 ≥max{1, 21/8, 59/27, 128/59}

We have shown that $\boldsymbol{T(N) \leq cN^3}$, for $\boldsymbol{c=3}$, for all $\boldsymbol{N \geq 1}$. Therefor pick $\boldsymbol{N_1=1}$.

b) (5 pts) Use just the definition with c and n_0 to show that it is $\Omega(N^3)$. Assume that $T(N) \geq 0$, for all $N \geq 0$. You should not need to use induction.

Let $c_0 = 1$ and $N_0 = 1, 1 * N^3 \leq 5T(\lfloor N/4 \rfloor) + 2N^3, \forall N \geq N_0 => T(N) = \Omega(N^3)$

Write your answers in a document called **2320_H5.pdf**. It can be hand-written and scanned, but it must be uploaded electronically. Follow the same conventions: place it in a folder called **HW5**, zip that and send it.

Remember to include your name at the top.