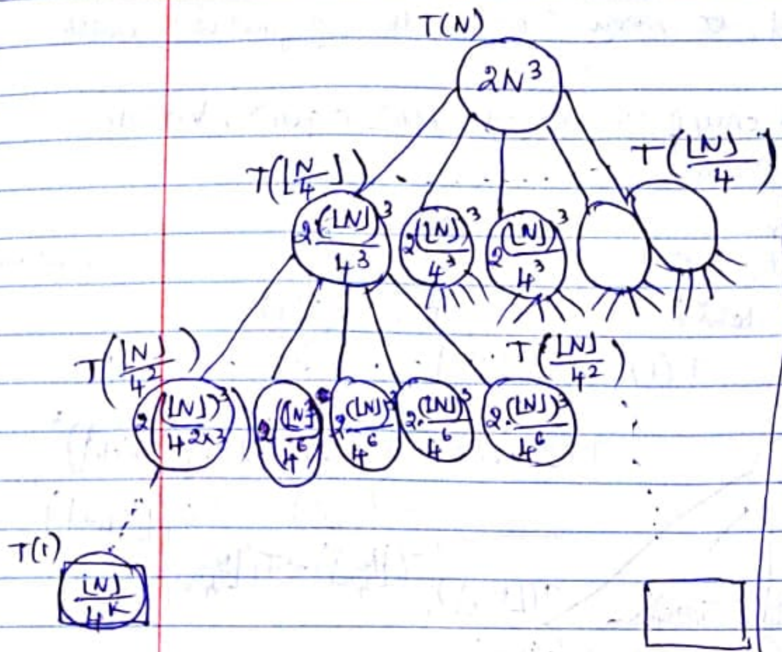
P1) $T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 \quad , T(0) = 1, \quad T(1) = 1$

$$T\left(\left\lfloor \frac{N}{4} \right\rfloor\right) = 5T\left(\left\lfloor \frac{N}{4^2} \right\rfloor\right) + 2\left(\left\lfloor \frac{N}{4} \right\rfloor\right)^3 = 5T\left(\frac{\lfloor N \rfloor}{4^2}\right) + 2\frac{(\lfloor N \rfloor)^3}{4^3}$$

$$T\left(\left\lfloor \frac{N}{4^2} \right\rfloor\right) = 5T\left(\left\lfloor \frac{N}{4^3} \right\rfloor\right) + 2\left(\frac{\lfloor N \rfloor}{4^2}\right)^3 = 5T\left(\frac{\lfloor N \rfloor}{4^3}\right) + 2\frac{(\lfloor N \rfloor)^3}{4^{2\times 3}}$$

$$T\left(\frac{\lfloor N \rfloor}{4^3}\right) = 5T\left(\frac{\lfloor N \rfloor}{4^4}\right) + 2\left(\frac{\lfloor N \rfloor}{4^3}\right)^3 = 5T\left(\frac{\lfloor N \rfloor}{4^4}\right) + 2\left(\frac{(\lfloor N \rfloor)^3}{4^{3\times 3}}\right)$$




$$\frac{\lfloor N \rfloor}{4^K} = 1 \Rightarrow \boxed{K = \log_4 N}$$

$$\frac{(\lfloor N \rfloor)^3}{4^{3K}} = \frac{\lfloor N \rfloor^2}{4^3} \times \frac{\lfloor N \rfloor}{4^K} = \frac{(\lfloor N \rfloor)^2}{4^3}$$

$$5^K * 2 * \frac{(\lfloor N \rfloor)^3}{4^{3K}} = 2\frac{(\lfloor N \rfloor)^3}{4^3}\left(\frac{5}{4}\right)^K$$

$$5^i * 2 * \frac{(\lfloor N \rfloor)^3}{4^{3i}} = 2\frac{(\lfloor N \rfloor)^3}{4^3}\left(\frac{5}{4}\right)^i = 2\left(\frac{\lfloor N \rfloor}{4}\right)^3\left(\frac{5}{4}\right)^i$$

| Level | Pb. size | Cost of 1 node | Node per level | Level cost |
|---|---|---|---|---|
| 0 | $N$ | $2N^3$ | 1 | $1 * 2N^3$ |
| 1 | $\frac{\lfloor N \rfloor}{4}$ | $2 * \frac{(\lfloor N \rfloor)^3}{4^3}$ | 5 | $5 * 2 * \frac{(\lfloor N \rfloor)^3}{4^3}$ |
| 2 | $\frac{\lfloor N \rfloor}{4^2}$ | $2 * \frac{(\lfloor N \rfloor)^3}{4^{2\times 3}}$ | $5^2$ | $5^2 * 2 * \frac{(\lfloor N \rfloor)^3}{4^{2\times 3}}$ |
| 3 | $\frac{\lfloor N \rfloor}{4^3}$ | $2 * \frac{(\lfloor N \rfloor)^3}{4^{3*3}}$ | $5^3$ | $5^3 * 2 * \frac{(\lfloor N \rfloor)^3}{4^{3*3}}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $i$ | $\frac{\lfloor N \rfloor}{4^i}$ | $2 * \frac{(\lfloor N \rfloor)^3}{4^{3i}}$ | $5^i$ | $5^i * 2 * \frac{(\lfloor N \rfloor)^3}{4^{3i}}$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $k$ | $\frac{\lfloor N \rfloor}{4^k} = 1$ | $2 * \frac{(\lfloor N \rfloor)^3}{4^{3k}} = 2 * \frac{(\lfloor N \rfloor)^2}{4^3}$ | $5^k$ | $5^k * 2 * \frac{(\lfloor N \rfloor)^3}{4^{3k}} = 2\left(\frac{\lfloor N \rfloor}{4}\right)^3\left(\frac{5}{4}\right)^k$ |

$$T(N) = \sum_{i=0}^{k} 2\left(\frac{\lfloor N \rfloor}{4}\right)^3 \left(\frac{5}{4}\right)^i$$

$$T(N) = 2\left(\frac{\lfloor N \rfloor}{4}\right)^3 \sum_{i=0}^{k} \left(\frac{5}{4}\right)^i$$

$\frac{5}{4} > 1$, so this is of the form $\sum_{i=0}^{k} x^i = \frac{x^{i+1} - 1}{x - 1}$

$$\sum_{i=0}^{k} \left(\frac{5}{4}\right)^i = \frac{\left(\frac{5}{4}\right)^{i+1} - 1}{\frac{5}{4} - 1} = \text{constant}$$

$$\boxed{T(N) = \theta\left(N^3\right)}$$

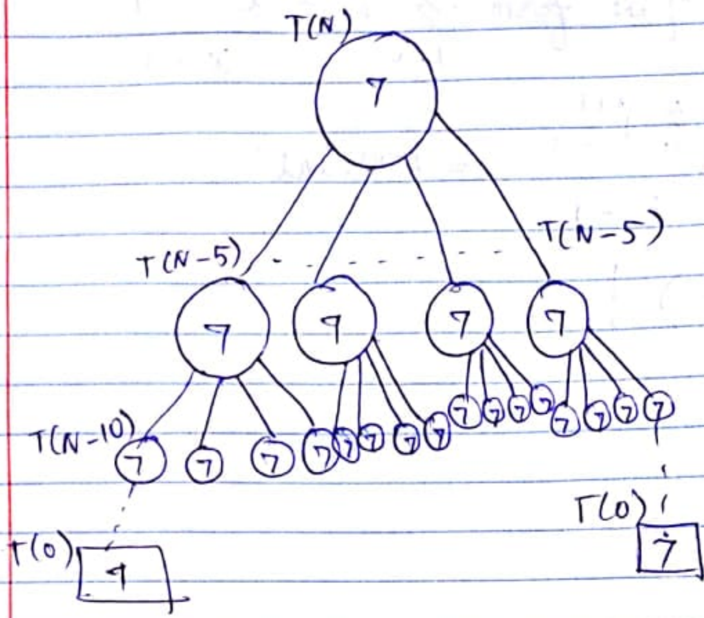
P2) $T(N) = 4T(N-5)+7$ , $T(N)=1$ for all $0 \le N \le 4$

LET $N=0$ , $T(0)=1$

$T(N-5) = 4T(N-10)+7$

$T(N-10) = 4T(N-15)+7$

$T(N-15) = 4T(N-20)+7$




| Level | Pb size | Cost per 1 node | Node per level | Level cost |
|---|---|---|---|---|
| 0 | $N$ | 7 | 1 | $1 \times 7$ |
| 1 | $N-5$ | 7 | 4 | $4 \times 7$ |
| 2 | $N-10$ | 7 | $4^2$ | $4^2 \times 7$ |
| ⋮ | | ⋮ | ⋮ | ⋮ |
| $i$ | $N-5i$ | 7 | $4^i$ | $4^i \times 7$ |
| ⋮ | ⋮ | ⋮ | ⋮ | ⋮ |
| $k = N/5$ | $N-5K=0$ | 7 | $4^k$ | $4^k \times 7$ |

$N - 5k = 0$

$N = 5k$

$\boxed{\frac{N}{5} = k}$

$T(N) = 4^k \times 7$

$\boxed{T(N) = \theta(1)}$

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P3) a) $T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3$. Assume $T(0) = 1$ and $T(1) = 1$

$a = 5$, $b = 4 \Rightarrow n^{\log_b a} = n^{\log_4 5} = n^{1.1}$

$f(n) = 2N^3 \overset{like}{=} \Omega(n^{1.1})$

If $\varepsilon = 1.9$

$f(n) = 2N^3 = \Omega(n^{1.1+1.9})$

cannot be $\Omega$, it's $\theta$ since it is equal.

$f(n) = 2N^3 = \theta(n^3)$

If $\boxed{\varepsilon = 0.1}$

$f(n) = 2N^3 = \Omega(n^{1.1+0.1}) = \Omega(n^{1.2})$

$a f\left(\frac{n}{b}\right) \leq k f(n)$

$a f\left(\frac{n}{b}\right) = 5 \times 2 \left(\frac{n}{4}\right)^3 \leq k 2n^3$

$\frac{10}{4^3} n^3 \leq k \times 2n^3$

$\frac{10}{4^3} n^3 \leq k\, 2n^3$

$\frac{10}{4^3} n^3 < \frac{10}{4^3} 2 \times n^3 \leq k \cdot 2n^3$

$k = \frac{10}{4^3} = \frac{5 \times \cancel{2}}{4^2 \times 2 \times \cancel{2}} = \frac{5}{32}$

$\boxed{k = \frac{5}{32}} < 1$

Case 3 $\Rightarrow$ $T(n) = \theta(2N^3) \Rightarrow \boxed{T(n) = \theta(N^3)}$

P3b) $T(N) = 4T(\lceil N/4 \rceil) + d,\ d > 0$. Assume $T(0) = 1, T(1) = 1$

$\left.\begin{matrix} a = 4 \\ b = 4 \end{matrix}\right\} n^{\log_b a} = n^{\log_4 4} = n$

$f(n) = d = \text{constant} = O(n)$

LET $\varepsilon = 0.5$

constant $d = O(n^{1-\varepsilon})$.

constant $d = O(n^{1-0.5}) = O(n^{0.5})$.

Case 1 $\Rightarrow \boxed{T(n) = \Theta(n)}$,

P3c) $T(N) = 6T(N/6) + 5N$. Assume $T(0) = 1$ and $T(1) = 1$

$\left.\begin{matrix} a = 6 \\ b = 6 \end{matrix}\right\} n^{\log_b a} = n^{\log_6 6} = n$

$f(n) = 5n = \Theta(n)$

case 2 $\Rightarrow \boxed{T(n) = \Theta(n \lg n)}$

p3d) $T(n) = 8T(n/2) + c\,n^3 \lg n$. Assume $T(0) = 1$ and $T(1) = 1$

$\left.\begin{matrix} a = 8 \\ b = 2 \end{matrix}\right\} n^{\log_b a} = n^{\log_2 8} = n^3$

$f(n) = cn^3 \lg n = \Omega(n^3)$

LET $\boxed{\varepsilon = 0.1}$

$cn^3 \lg n = \Omega(n^{3+0.1})$

$cn^3 \lg n = \Omega(n^{3.1})$

$a f(\frac{n}{b}) \le k f(n)$

$8c\left(\frac{n}{2}\right)^3 \lg \frac{n}{2} \le k\, cn^3 \lg n$

$\frac{8}{8} cn^3 \lg \frac{n}{2} \le \frac{8}{8} cn^3 \lg n \le k\, cn^3 \lg n$

$\boxed{K = 0.5}$

Case 3 $\Rightarrow T(n) = \Theta(n^3 \lg n)$

P4)

P4) Inadmissable equations.

* $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

a is not a constant; the number of subproblems should be fixed.

→ $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

non polynomial difference $f(n)$ and $n^{\log_b a}$.

the difference between $f(n)$ and $n^{\log_b a}$ can be expressed with the ratio $\frac{f(n)}{n^{\log_b a}} = \frac{n/\log n}{n^{\log_2 2}} = \frac{n}{n \log n} = \frac{1}{\log n}$.

It is clear that $\frac{1}{\log n} < n^{\varepsilon}$ for any constant $\varepsilon > 0$.

Therefore, the difference is not polynomial and the basic form of the Master theorem does not apply

* $T(n) = 0.5T\left(\frac{n}{2}\right) + n$

$a<1$ cannot have less than one sub problem

* $T(n) = 64T\left(\frac{n}{8}\right) - n^2 \log n$

$f(n)$, which is the combination time, is not positive.

* $T(n) = T\left(\frac{n}{2}\right) + n(2 - \cos n)$

case 3 but regularity violation.

P5) Show

$T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 = \theta(N^3)$, $T(0)=1, T(1)=1$

Assume

$T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 = \Theta(N^3)$

Using inductive step to show $\Theta$ with definition with constants

$f(n) = \Theta(g(n))$ iff $\exists c, n_0$ s.t $f(n) \leq c\, g(n)$, $\forall n \geq n_0$

$f(n) = T(N)$, $g(n) = N^3$

To show: $T(N) \leq c N^3$

Base case:

$N=1$, $T(1) = 1 \leq c(1)^3$

$T(1) = 5T(\lfloor \frac{1}{4} \rfloor) + 2(1)^3 = 5T(0) + 2 = 5+2 = 7 \leq c$

$n_0 > 0$

$N=2$, $T(2) = 5T(\lfloor \frac{2}{4} \rfloor) + 2(2)^3 = 5T(0) + 16 = 5+16 = 21$

$T(2) = 21 \leq c(2)^3 = 8c$

$T(2) \leq 8c \Rightarrow c \geq 21/8$, $c \geq 2.625$, $c = 3$

$N=3$, $T(3) = 5T(\lfloor \frac{3}{4} \rfloor) + 2(3)^3 = 5T(0) + 54 = 59$

$T(3) = 59 \leq c(3)^3 \Rightarrow c \geq 59/27$, $c \geq 2.18$, $c = 3$

$N=4$, $T(4) = 5T(\lfloor \frac{4}{4} \rfloor) + 2(4)^3 = 5T(1) + 128 = 5+128 = 133$

$T(4) = 133 \leq c(4)^3 \Rightarrow c \geq 133/64$, $c \geq 2.07$, $c = 3$

$N=5$, $T(5) = 5T(\lfloor \frac{5}{4} \rfloor) + 2(5)^3 = 5T(1) + 250 = 255$

$T(5) = 255 \leq c(5)^3 \Rightarrow c125 \geq 255$, $c \geq 2.04$, $c = 3$

$N=6$, $T(6) = 5T(\lfloor \frac{6}{4} \rfloor) + 2(6)^3 = 5T(1) + 432 = 5+432 = 437$

$T(6) = 437 \leq c(6)^3 \Rightarrow c216 \geq 437 \Rightarrow c \geq 2.02$, $c = 3$

$N=7$, $T(7) = 5T(\lfloor \frac{7}{4} \rfloor) + 2(7)^3 = 5T(1) + 686 = 691$

$T(7) = 691 \leq c(7)^3$, $c \geq 2.01$, $c = 3$

$N=8$, $T(8) = 5T(\lfloor \frac{8}{4} \rfloor) + 2(8)^3 = 5T(2) + 1024 = 1024 + 105 = 1129$

$T(8) = 1129 \leq c \cdot 512 \Rightarrow c \geq 2.2 \Rightarrow c = 3$

and so on.

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Assume $T(k) \leq ck^3 \quad , \forall k < n$

$$T(n) = 5T\left(\frac{\lfloor n \rfloor}{4}\right) + 2n^3 \longrightarrow (1)$$

$$T\left(\frac{\lfloor n \rfloor}{4}\right) \leq c\left(\frac{\lfloor n \rfloor}{4}\right)^3 \longrightarrow (2)$$

substituting (2) in (1)

$$T(n) = 5T\left(\frac{\lfloor n \rfloor}{4}\right) + 2n^3 \leq 5\left[c\left(\frac{\lfloor n \rfloor}{4}\right)^3\right] + 2n^3$$

$$\leq 5c\,\frac{n^3}{4^3} + 2n^3$$

$$\leq n^3\left[\frac{5c}{4^3} + 2\right]$$

$$cn^3 \quad < n^3\left[\underbrace{0.02c + 2}_{\leq c}\right]$$

$$0.02c + 2 \leq c$$

$$2 \leq \cancel{c} \cdot c - 0.02c$$

$$2 \leq 0.98c$$

$$c \geq 2.0408$$

$$\boxed{c = 3}$$

$n_0 = 1, \; c = 3$

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Assume

$T(n) = 5T(\lfloor N/4 \rfloor) + 2N^3 = \Omega(N^3)$

Using inductive method to show $\Omega$ with definition with constants

$f(n) = \Omega(g(n))$ iff $\exists c, n_0$ s.t $cg(n) \leq f(n)$, $\forall n \geq n_0$

$f(n) = T(n) \quad g(n) = n^3$

To show : $cn^3 \leq T(n)$ or $T(n) \geq cn^3$

$N=1$, $T(1) = 5T(\lfloor 1/4 \rfloor) + 2(1)^3 = 5T(0) + 2 = 5+2 = 7 \geq c(1)^3$

$T(n) = 7 \geq c(1)^3 \Rightarrow T(n) = 7 \geq c$

$n_0 > 0$

$N=2$, $T(2) = 5T\left(\left\lfloor \frac{2}{4} \right\rfloor\right) + 2(2^3) = 5T(0) + 16 = 5+16 = 21$

$T(2) = 21 \geq c(2^3) \Rightarrow T(2) = 21 \geq 8c \Rightarrow c \leq 2.625$, $c = 2$

$N=3$, $T(3) = 5T\left(\left\lfloor \frac{3}{4} \right\rfloor\right) + 2(3)^3 = 5T(0) + 54 = 59$

$T(3) = 59 \geq c(3)^3 \Rightarrow c \leq 2.18 \Rightarrow c = 2$

$N=4$, $T(4) = 5T\left(\left\lfloor \frac{4}{4} \right\rfloor\right) + 2(4)^3 = 5T(1) + 128 = 5 + 128 = 133$

$T(4) = 133 \geq c(4)^3 \Rightarrow c \leq 2.07 \Rightarrow c = 2$

$N=5$, $T(5) = 5T\left(\left\lfloor \frac{5}{4} \right\rfloor\right) + 2(5)^3 = 5T(1) + 250 = 5+250 = 255$

$T(5) = 255 \geq c(5)^3 \Rightarrow c \leq 255/125 \Rightarrow c \leq 2.04 \Rightarrow c = 2$

$N=6$, $T(6) = 5T\left(\left\lfloor \frac{6}{4} \right\rfloor\right) + 2(6)^3 = 5T(1) + 432 = 5+432 = 437$

$T(6) = 437 \leq c(6)^3 \Rightarrow c \leq 437/216 \Rightarrow c \leq 2.02 \Rightarrow c = 2$

$N=7$, $T(7) = 5T\left(\left\lfloor \frac{7}{4} \right\rfloor\right) + 2(7)^3 = 5T(1) + 686 = 691$

$T(7) = 691 \geq c(7)^3 \Rightarrow c \leq 691/343 \Rightarrow c \leq 2.01 \Rightarrow c = 2$

$N=8$, $T(8) = 5T\left(\left\lfloor \frac{8}{4} \right\rfloor\right) + 2(8)^3 = 5T(2) + 1024 = 105 + 1024 = 1129$

$T(8) = 1129 \geq c(8)^3 \Rightarrow c \leq 2.2 \Rightarrow c = 2$

and so on.

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Assume $T(k) \geq ck^3$ , $\forall k < n$

$T(n) = 5T\left(\lfloor \frac{n}{4} \rfloor\right) + 2n^3 \longrightarrow$ 1

$T\left(\lfloor \frac{n}{4} \rfloor\right) \geq c\left(\lfloor \frac{n}{4} \rfloor\right)^3 \longrightarrow$ 2

Substituting 2 in 1

$T(n) = 5T\left(\lfloor \frac{n}{4} \rfloor\right) + 2n^3 \geq 5\left[c\left(\lfloor \frac{n}{4} \rfloor\right)^3\right] + 2n^3$

$\geq 5c\frac{n^3}{4^3} + 2n^3$

$\geq n^3\left[\frac{5c}{4^3} + 2\right]$

$cn^3 \geq n^3[0.02c + 2]$

$0.02c + 2 \geq c$

$2 \geq c - 0.02$

$c \leq 2.0408$

$\boxed{c = 2}$

$n_0 = 1$ , $c = 2$