Search Trees

CSE 2320 – Algorithms and Data Structures
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Based on slides and notes from:
Vassilis Athitsos and Bob Weems
University of Texas at Arlington

Search Trees

- Preliminary note: "search trees" as a term does **NOT** refer to a specific implementation of symbol tables.

- The term refers to a **family of implementations**, that may have **different properties**.
- We will discuss:
  - Binary search trees (BST).
  - 2-3-4 trees (a special type of a B-tree).
  - Other trees: red-black trees, AVL trees, splay trees, B-trees and other variations.

Search Trees

- All search trees support search, insertion and deletion operations.
- Insertions and deletions can differ among trees, and have important implications on overall performance.
- The main goal is to have insertions and deletions that:
  - Are efficient (at most logarithmic time).
  - Leave the tree balanced, to support efficient search (at most logarithmic time).

Binary Search Trees (BST)

Binary Search Tree (BST)

- Resources:
  - BST in general – CLRS
  - BST in general and solved problems: http://cslibrary.stanford.edu/110/BinaryTrees.html#s
  - Insertion at root (using insertion at a leaf and rotations)
    - Sedgewick
    - Dr. Bob Weems: Notes 11, parts: ‘11.D. Rotations’ and ’11.E. Insertion At Root’
  - Randomizing the tree by inserting at a random position:
    - Sedgewick

Tree Properties - Review

- Full tree
- Complete tree (e.g. heap tree)
- Perfect binary tree

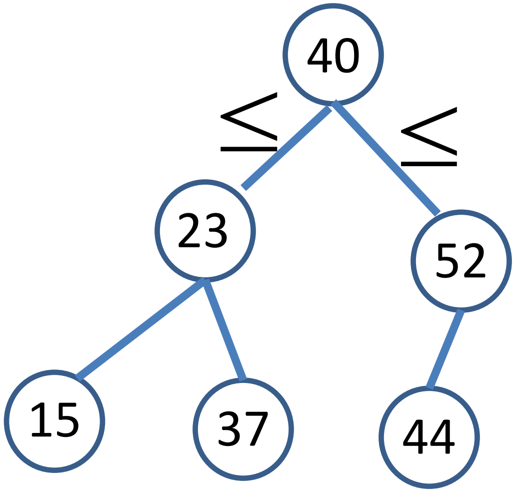
- Alternative definitions: complete (for perfect) and almost complete or nearly complete (complete).

- Tree – connected graph with no cycles, or connected graph with N-1 edges (and N vertices).

Binary Search Trees

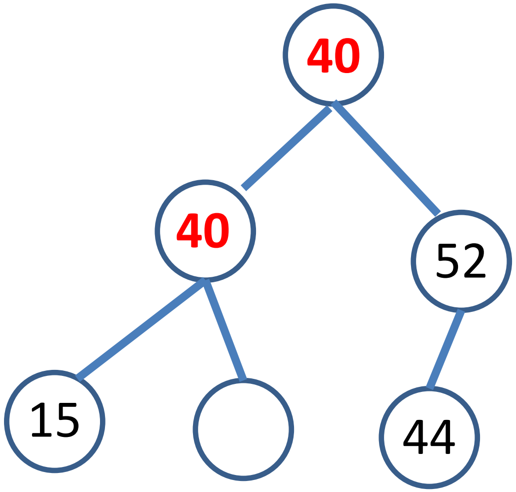
- Definition: a binary search tree is a binary tree where the item at each node is:
  - Greater than or equal to all items on the left subtree.
  - Less than all items in the right subtree.

- How do we search?
  - 30? 44?



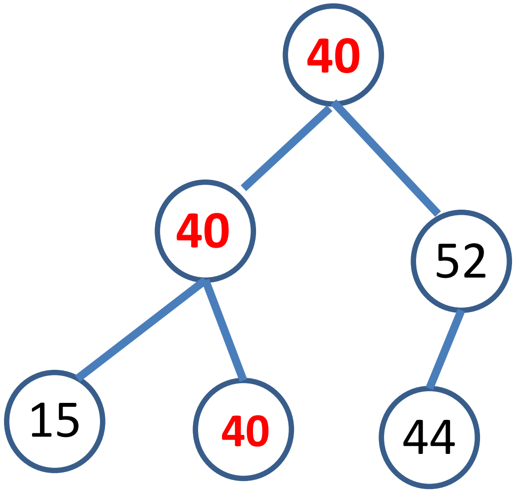
Example 1

- What values could the empty leaf have?



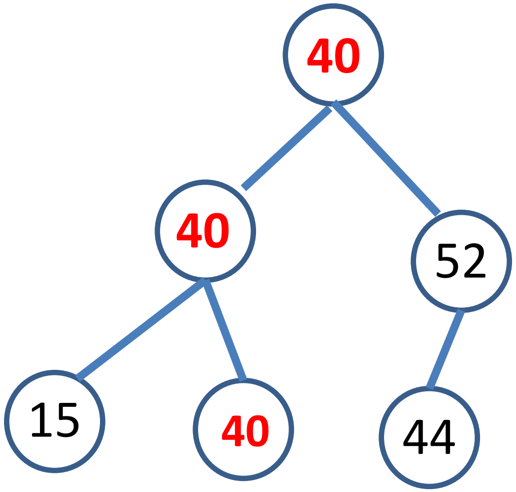
Example 1

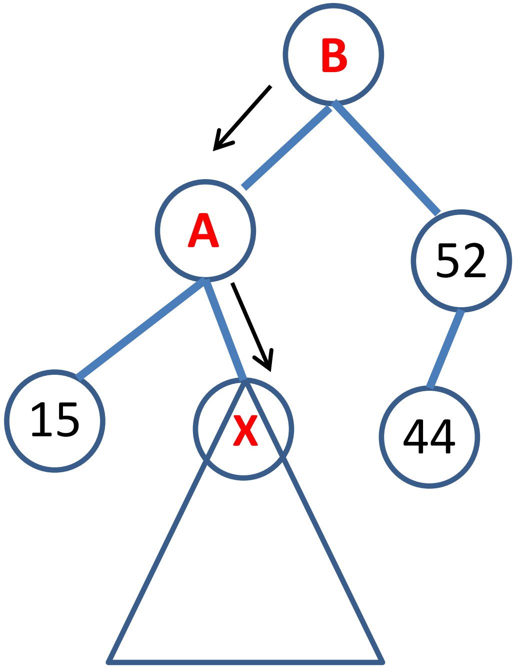
- What values could the empty leaf have?
- Only value 40



Example 1

- If you change direction twice, (go to A, left child, and then go to X, right child) all the nodes in the subtree rooted X will be in the range [A,B].

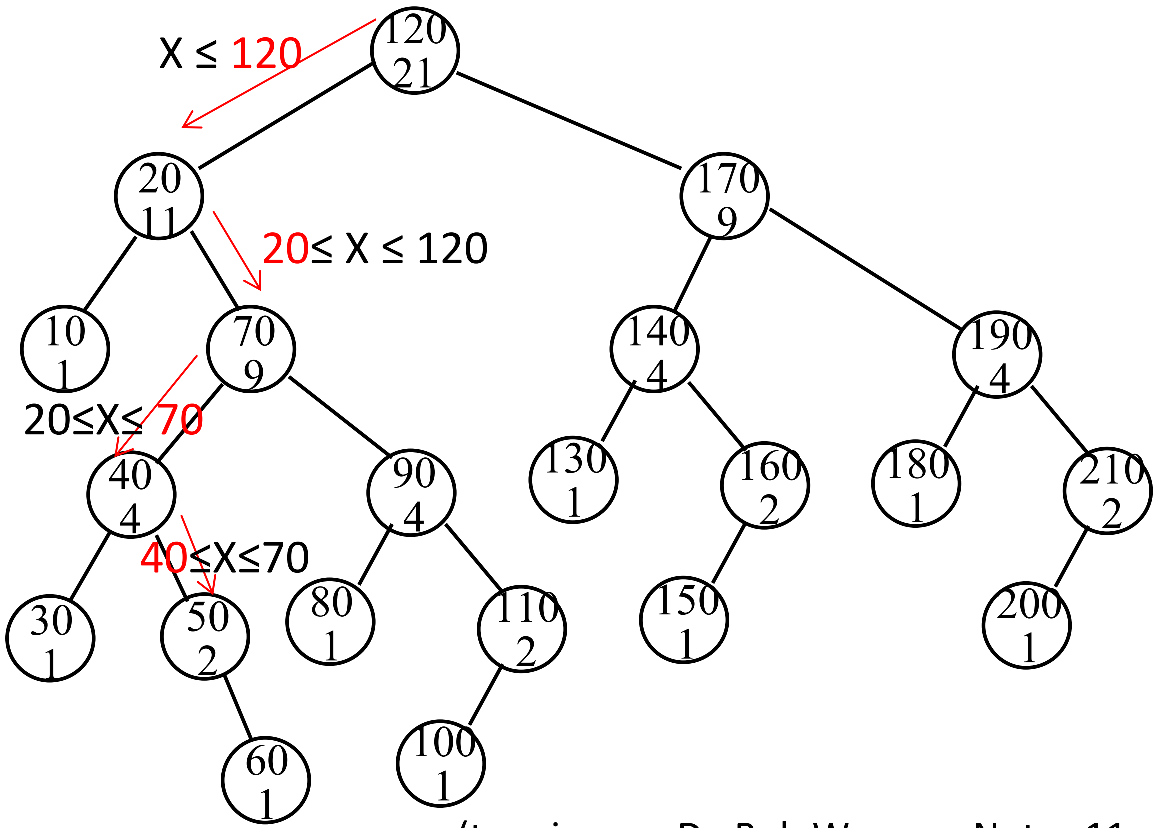




A ≤ X ≤ B

Range of possible values

- As we travel towards node 50, we find the interval of possible values in the tree rooted at 50 to be: [40,70]



The search for 50 gave the sequence:
120, 20, 70, 40, 50

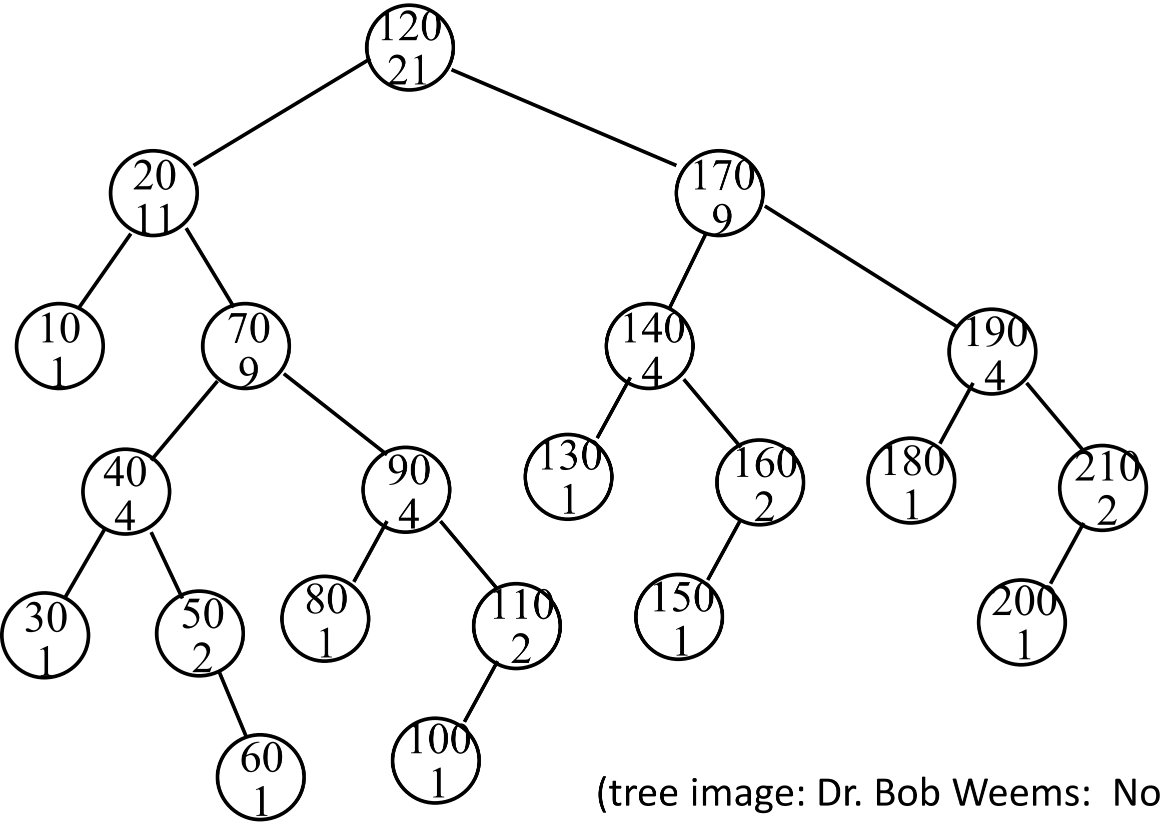
E.g. of impossible search sequence for 50 in a BST:
120, 20, 70 , 80, 50

The range is now [20,70] and so 80 is impossible (all nodes in the tree at 70 will be in the [20,70] range).

(tree image: Dr. Bob Weems: Notes 11, parts: ‘11.C. Binary Search Trees’)

Properties

- Where is the item with the **smallest** key?
- Where is the item with the **largest** key?
- What traversal prints the data in **increasing** order?
  - How about **decreasin**g order?



Consider the special cases where the root has:

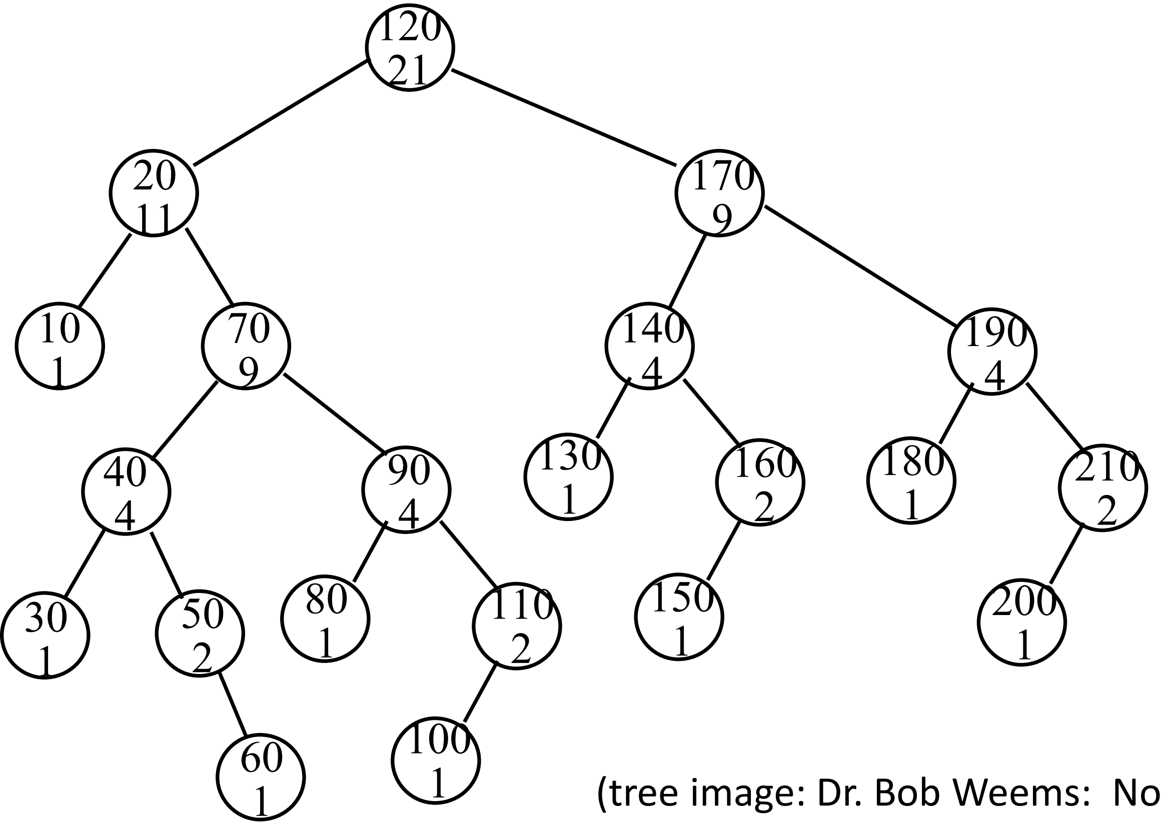
- No left child
- No right child

(tree image: Dr. Bob Weems: Notes 11, parts: ‘11.C. Binary Search Trees’)

Predecessor and Successor
(according to key order)

- When the node has 2 children.
- When the node has only one child (missing the child you need).

Here, in the nodes, the first number is the item/key and the second number is the tree size. Root has key 120 and size 21 (the whole tree has 21 nodes).

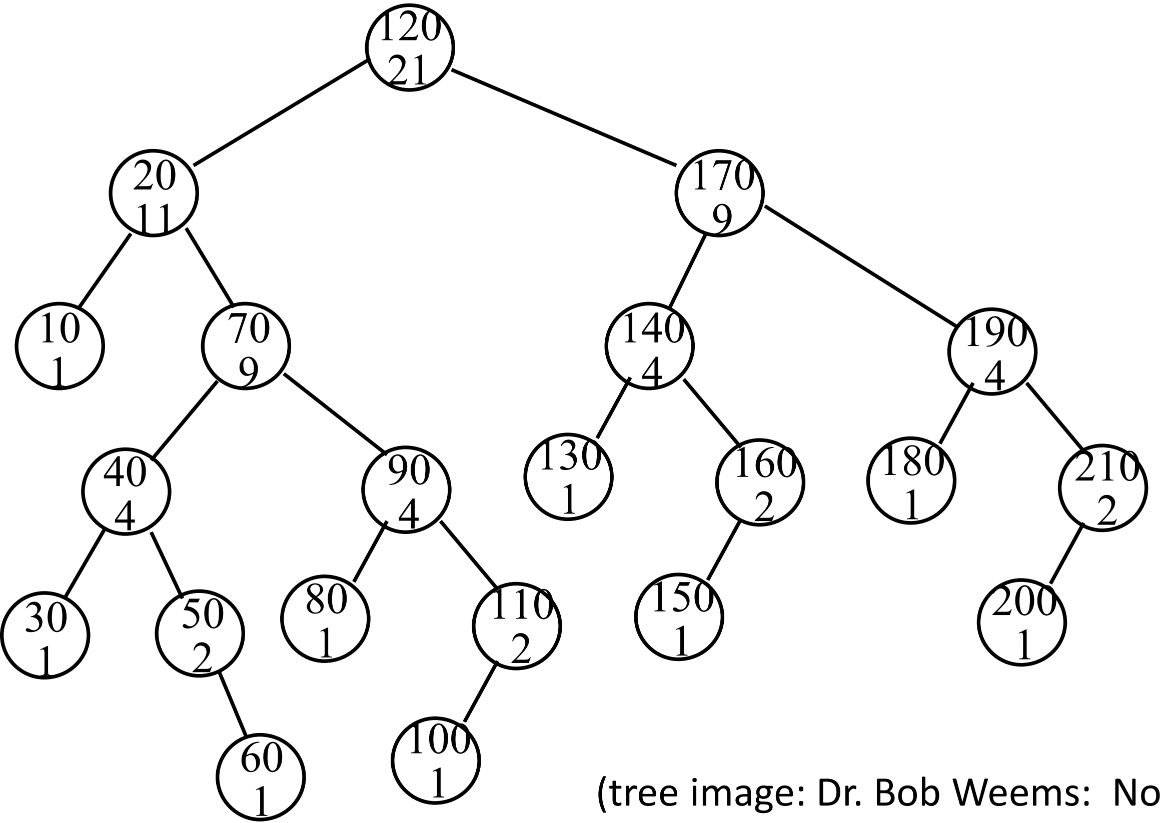


Node	Predecessor	Successor
120		
70		
170		
160		*
60		*
130	*	
50	*	
180	*	

(tree image: Dr. Bob Weems: Notes 11, parts: ‘11.C. Binary Search Trees’)

Predecessor and Successor (according to key order)

- Successor of node x with key k (go right):
  - Smallest node in the right subtree
  - Special case: no right subtree: first parent from the right
- Predecessor of node x with key k (go left):
  - Largest node in the left subtree
  - Special case: no left subtree: first parent from the left



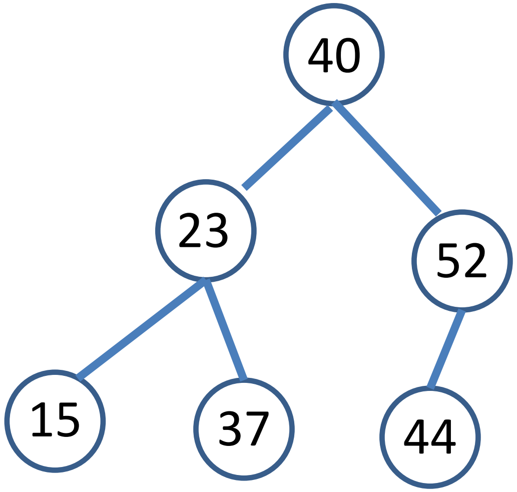
Node	**Predecessor**	**Successor**
120	110	130
70	60	80
170	160	180
160		***170**
60		***70**
130	***120**	
50	***40**	
180	***170**	

(tree image: Dr. Bob Weems: Notes 11, parts: '11.C. Binary Search Trees')

- Min: leftmost node (from the root keep going left)
  - Special case: no left child => root
- Max: rightmost node (from the root keep going right.
  - Special case: no right child => root
- Print in order:
  - Increasing: Left, Root, Right (inorder traversal)
  - Decreasing: Right, Root, Left
- Successor of node x with key k (go right):
  - Smallest node in the right subtree
  - Special case: no right subtree: first parent from the right
- Predecessor of node x with key k (go left):
  - Largest node in the left subtree
  - Special case: no left subtree: first parent from the left

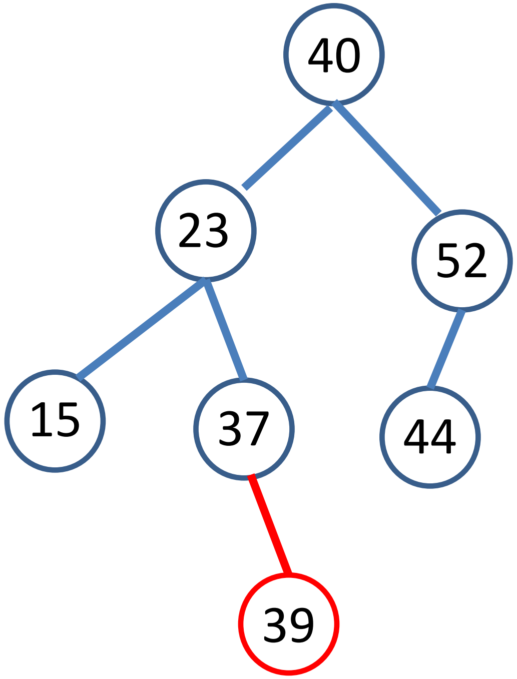
Naïve Insertion

- Inserting 39:



Naïve Insertion

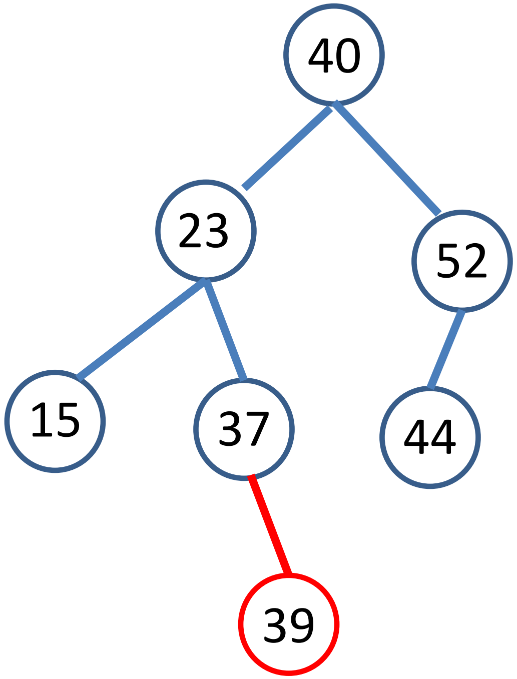
- Inserting 39:



To insert an item, the simplest approach is to travel down in the tree until finding a leaf position where it is appropriate to insert the item.

Naïve Insertion

```
link insert(link h, Item n_item)
   if (h == null) return new_tree(n_item);
   else if (n_item < h->item)
      h->left = insert(h->left, n_item);
   else if (n_item > h->item)
      h->right = insert(h->right, n_item);
   return h;
```

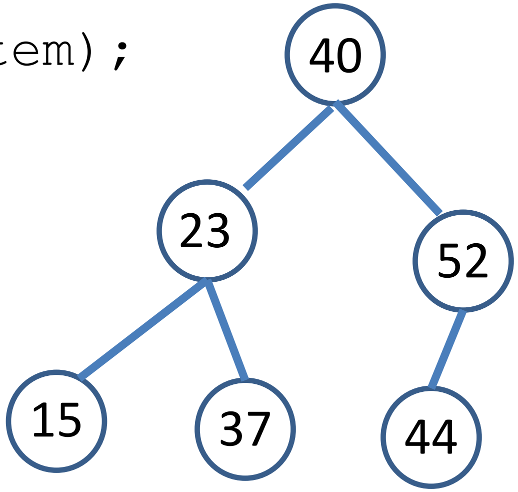


How will we call this method?
root = insert(root, item)

Note that we use:
h->left = insert(h->left, item)
to handle the base case, where we return a new node, and the parent must make this new node a child.

Binary Search Trees - Search

```
link search(link tree, Item s_item) {
   if (tree == null) return null;
   else if (s_item == tree->item)
      return tree;
   else if (s_item < tree->item)
      return search(tree->left, s_item);
   else return search(tree->right, s_item);
 }
```



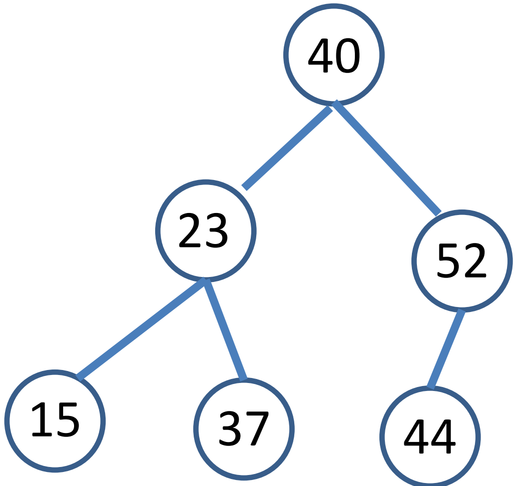
Runtime

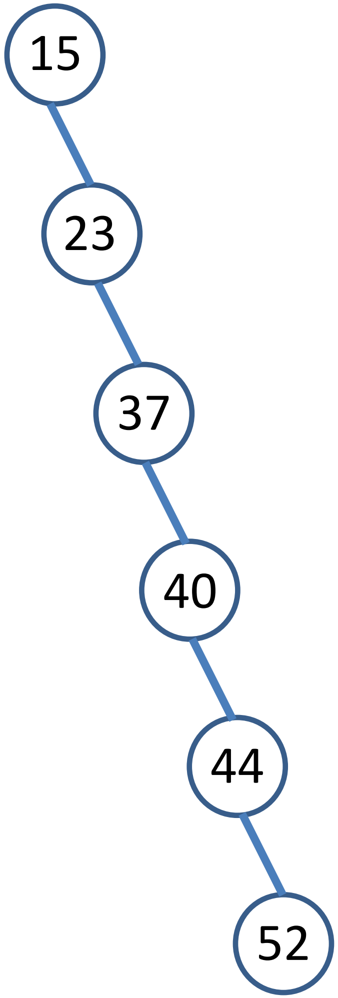
(in terms of ,N, number of nodes in the tree or tree height)

- Best case:
- Worst case:

Performance of BST

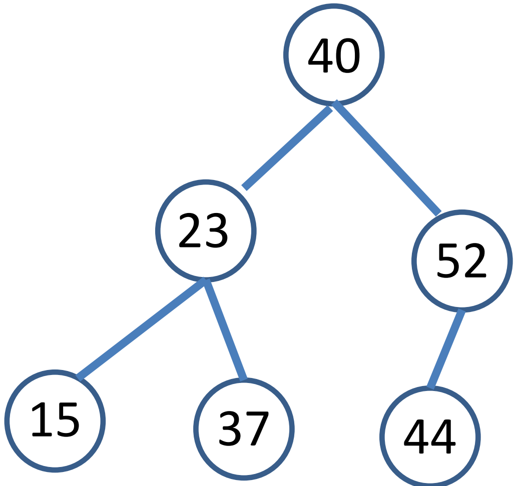
- Are these trees valid BST?
- Give two sequences of nodes s.t. when inserted in an empty tree will produce the two trees shown here (each sequence produces a different tree).





Performance of BST

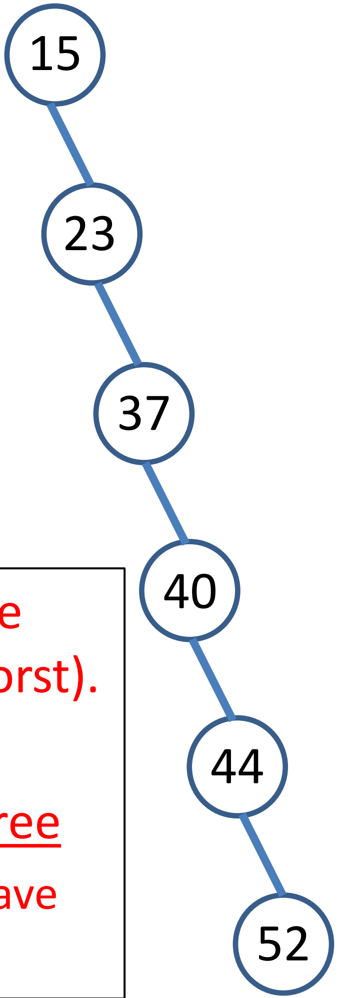
- Are these trees valid BST?
  - Yes
- Give two sequences of nodes s.t. when inserted in an empty tree will produce the two trees shown here (each sequence produces a different tree).
  - **40, 23, 37, 52, 44, 15**
  - **15, 23, 37, 40, 44, 52**



Search, Insert and Delete take time linear to the height of the tree (worst).

Ideal: build and keep a balanced tree

- insertions and deletions should leave the tree balanced.



Performance of BST

- If items are inserted in:
  - ascending order, the resulting tree is maximally imbalanced.
  - random order, the resulting trees are reasonably balanced.

- Can we insert the items in random order?
  - If we build the tree from a batch of items.
    - Shuffle them first, or grab them from random positions.
  - If they come online (we do not have them all as a batch).
    - **Insert in the tree at a random position**

BST - Randomized

- If the data can be inserted in random order, on average, the tree will be balanced.

- If we do not have control over the data: insert at a RANDOM position in the tree
  - When inserting in tree of size N,
    - Insert at the root with probability 1/(N+1)
      - New tree size will be (N+1)
    - If not at the root, go to the appropriate subtree (e.g. go left if the key of new item is smaller or equal than that of the root item) and repeat the process.

Insertion at a Random Position

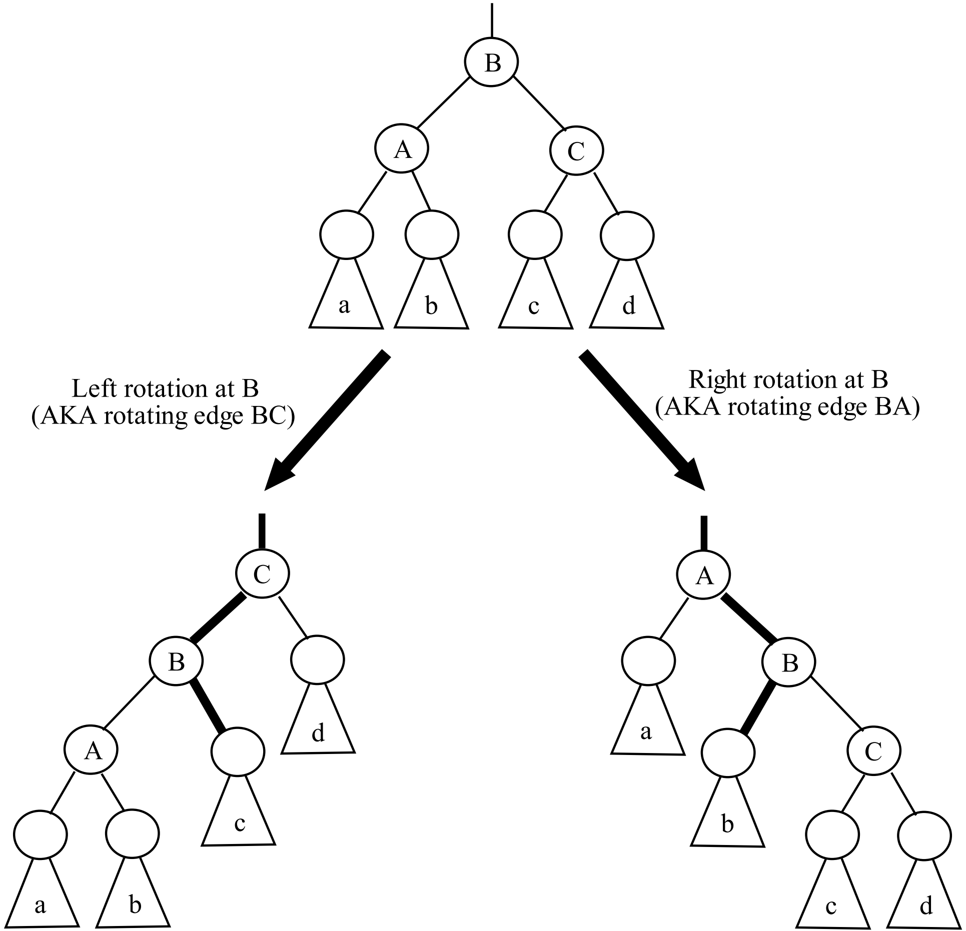
(small changes to Sedgewick code)

```
// Here each node also keeps the size of the tree rooted there
// See Example above from Dr. Weems.

link insertR(link h, Item n_item)
  { if (h == NULL) return new_tree(n_item, NULL, NULL, 1);
    if (rand()< RAND_MAX/(h->N+1)) //rand()->int in [0,RAND_MAX]
       return insertT(h, n_item);  // insert at the root
    if (n_item < h->item)
       h->left = insertR(h->left, n_item);
    else
       h->right = insertR(h->right, n_item);
   (h->N)++;
    return h;
  }

void STinsert(Item item)
  { head = insertR(head, item); }
```

BST - Rotations

- Left and right rotations

(image source: Dr. Bob Weems: Notes 11, parts: ‘11.D. Rotations’)



```
// Sedgewick code:

// rotate to the right
link rotR(link B)
  { link A = B->left;
    B->left = A->right;
    A->right = B;
    return A; }

// rotate to the left
link rotL(link B)
  { link C = B->right;
    B->right = C->left;
    C->left = B;
    return C; }
```

BST – Insertion at Root

(small changes to Sedgewick code)

```
link rotR(link h)
  { link x = h->left; h->left = x->right; x->right = h;
    return x; }
link rotL(link h)
  { link x = h->right; h->right = x->left; x->left = h;
    return x; }
-----
link insertT(link h, Item item)
  { if (h == NULL) return new_tree(item, NULL, NULL, 1);
    if (item < h->item)
      { h->left = insertT(h->left, item); h = rotR(h); }
    else
      { h->right = insertT(h->right, item); h = rotL(h); }
    return h;
  }
void STinsert(Item item)
  { head = insertT(head, item); } // Sedgewick code adaptation
```

BST - Deletion

Delete a node, ***z***, in a BST

- If z is a leaf, delete it,
- If z has only one child, replace z with the child
- If z has 2 children, *replace* it with its order-wise successor, **y**, and delete old **y**. (Note: y will be a leaf or have only one child.)

1. Method 1 (Simple: ***copy*** the data)
    1. Copy the data from y to z
    2. Delete node **y**.
    3. Problem if other components of the program maintain pointers to nodes in the tree they would not know that the tree was changed and their data cannot be trusted anymore.

2. Method 2 (***move*** the nodes)
    1. Replaces the **node (not content)** z with node y in the tree.
    2. Delete node **z** (y is now linked in place of z)
    3. Does not have the pointer referencing problem.
    4. 2 implementations: Sedgewick and CLRS.

BST – Deletion – Method 1 (Copy the data)

Delete(z) - delete a node *z* in a BST - Method 1.

1. If z is a leaf, delete it
2. If z has only one child, delete it and readjust the links (the child ‘moves’ in the place of z).
3. If z has 2 children:
   a) Find the successor, y, of z.
      1. Where is the successor of z?

   b) Copy only the data from y to z

   c) Call Delete(y) node y. Note that y can only be:
      1. Leaf (case 1 above)
      2. A node with only one child (the right child) (This is case 2 above.)

BST – Deletion – Method 2 (Move nodes)

Delete a node *z* in a BST - Method 2.

1. If z is a leaf, delete it
2. If z has only one child, delete it and readjust the links (the child ‘moves’ in the place of z).
3. If z has 2 children, find the successor, y, of z.

   Is y the right child of z?

   a) YES: Transplant y over z (y will have only the right child)

   b) NO:

Draw image

Sedgewick: Delete Any and Delete the k-th element

- Return k-th element:
    - Bring it to the root (partition the tree) and remove it (join the subtrees)
    - To bring it to the root recursively: bring the k-th element at the root of the left or right subtree and follow by a right (or left) rotation to move it in the root. (If the left subtree has t < k elements, bring the k'-th, k' = k-t-1 on the right subtree (the root itself will be one of the first k elements)).
- Delete an element:
    - If it is in the left subtree, replace the subtree with the subtree obtained by recursively deleting the node from it. Similar for the right one.
    - If the node is at the root, delete and combine the subtrees.
        - Use the above partitioning operation that brings the k-th node to the root (in this case, bring the smallest node of the right subtree to its root, it will not have a left subtree and so can link the original left subtree there).
- Will need to keep track of the count of nodes in each tree.

(Sedgewick, *Algorithms in C*, 3-rd edition: 12.9, page 519.)

Selection of k-th - Sedgewick

```
Item selectR(link h, int k)   //return item with k-th smallest key (do not remove it)
  { int t = h->l->N;
    if (h == z) return NULLitem;
    if (t > k) return selectR(h->l, k);
    if (t < k) return selectR(h->r, k-t-1);
    return h->item;
  }
Item STselect(int k)
  { return selectR(head, k); }
-----
link partR(link h, int k)    // bring to root the node with k-th item
  { int t = h->l->N;
    if (t > k )
      { h->l = partR(h->l, k); h = rotR(h); }
    if (t < k )
      { h->r = partR(h->r, k-t-1); h = rotL(h); }
    return h;
  }
```

Deletion using Join - Sedgewick

```
// Code from Sedgewick
link joinLR(link a, link b)
  {
    if (b == z) return a;
    b = partR(b, 0); b->l = a;
    return b;
  }
link deleteR(link h, Key v)
  { link x; Key t = key(h->item);
    if (h == z) return z;
    if (less(v, t)) h->l = deleteR(h->l, v);
    if (less(t, v)) h->r = deleteR(h->r, v);
    if (eq(v, t))
      { x = h; h = joinLR(h->l, h->r); free(x); }
    return h;
  }
void STdelete(Key v)
  { head = deleteR(head, v); }
```

2-3-4 Tree

- Next we will see 2-3-4 tree, which is guaranteed to stay balanced regardless of the order of insertions.

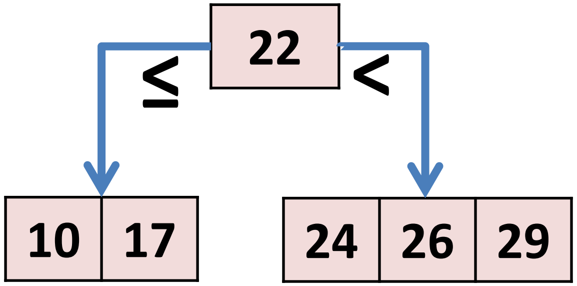
2-3-4 Trees

- **All leaves are at the same level.**
- Has three types of nodes:
- 2-nodes, which contain:
  - An item with key K.
  - A left subtree with keys <= K.
  - A right subtree with keys > K.
- 3-nodes, which contain:
  - Two items with keys K1 and K2, K1 <= K2.
  - A left subtree with keys <= K1.
  - A middle subtree with K1 < keys <= K2.
  - A right subtree with keys > K2.
- 4-nodes, which contain:
  - Three items with keys K1, K2, K3, K1 <= K2 <= K3.
  - A left subtree with keys <= K1.
  - A middle-left subtree with K1 < keys <= K2.
  - A middle-right subtree with K2 < keys <= K3.
  - A right subtree with keys > K3.

- The tree is guaranteed to stay balanced regardless of the order of insertions

Types of Nodes

2-node



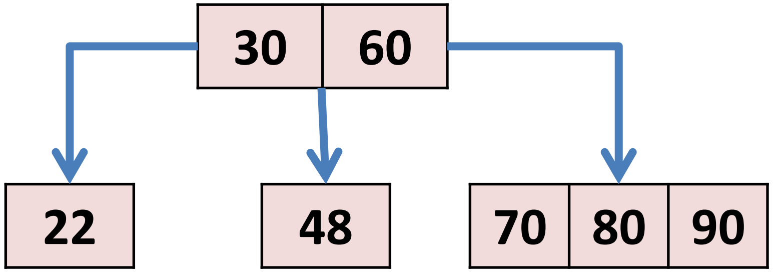
22

≤ <

10 17

24 26 29

3-node



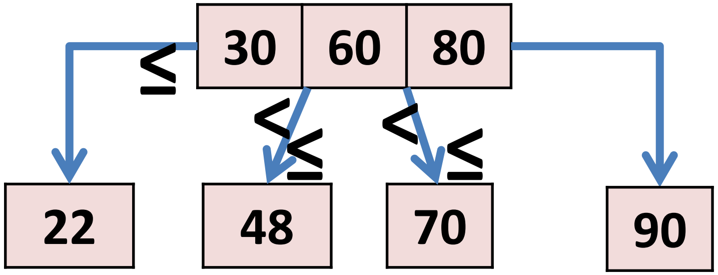
30 60

22

48

70 80 90

4-node



30 60 80

≤ < ≤ < ≤

22

48

70

90

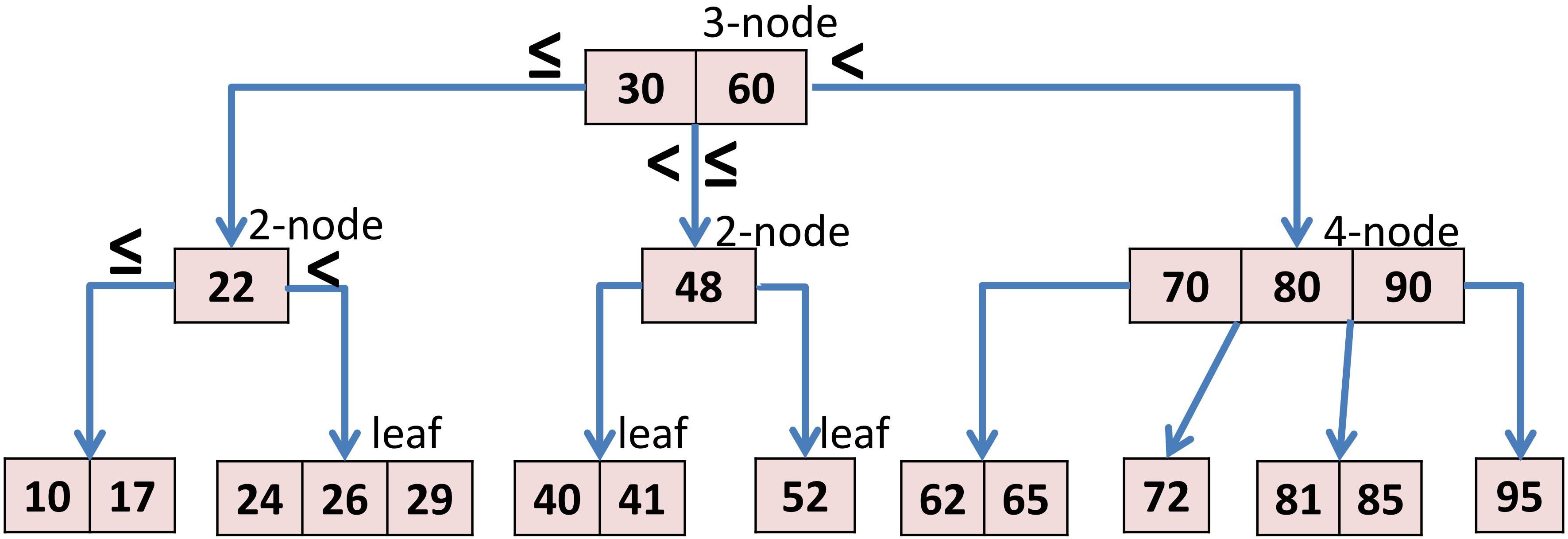
2-3-4 Trees

Nodes:

- 2-node : 1 item, 2 children
- 3-node: 2 items, 3 children
- 4-node: 3 items, 4 children

- Items in a node are in order of keys
- Given item with key k:
  - Keys in left subtree: ≤ k
  - Keys in right subtree: > k

All leaves must be at the same level. (It grows and shrinks from the root.)



Difference between items and nodes. How many nodes? Items? Types of nodes?

Search in 2-3-4 Trees

- For simplicity, **we assume that all keys are unique**.
- Search in 2-3-4 trees is a generalization of search in binary search trees.
  - select one of the subtrees by comparing the search key with the 1, 2, or 3 keys that are present at the node.
- Search time is logarithmic to the number of items.
  - The time is at linear to the height of the tree.
- Next:
  - how to implement insertions and deletions so that the tree keeps its property: all leaves are at the same level.

Insertion in 2-3-4 Trees

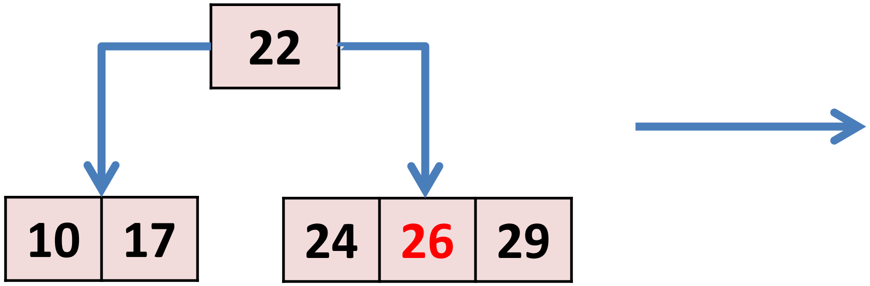
- We follow the same path as if we are searching for the item.
- We cannot just insert the item at the end of that path:
  - Case 1: If the leaf is a 2-node or 3-node, there is room to insert the new item with its key - OK
  - Case 2: If the leaf is a 4-node, there is NO room for the new item. In order to insert it here , we would have to create a new leaf that would be on a different level than all the other leaves – PROBLEM.
    - Fix nodes on the way to avoid this case.
- The tree will grow from the root.

Insertion in 2-3-4 Trees

- Given our key K: we follow the same path as in search.
- On the way we “fix“ all the 4 nodes we meet:
  - If the parent is a 2-node, transform the pair into a 3-node connected to two 2-nodes.
  - If the parent is a 3-node, we transform the pair into a 4-node connected to two 2-nodes.
  - If there is no parent (the root itself is a 4-node), split it into three 2-nodes (root and children). - This is how the tree height grows.
- These transformations:
  - Are local (they only affect the nodes in question).
  - Do not affect the overall height or balance of the tree (except for splitting a 4-node root).
- This way, when we get to the bottom of the tree, we know that the node we arrived at is not a 4-node, and thus it has room to insert the new item.

Transformation Examples

- If we find a 2-node being parent to a 4-node, we transform the pair into a 3-node connected to two 2-nodes, by pushing up the **middle** key of the 4-node.



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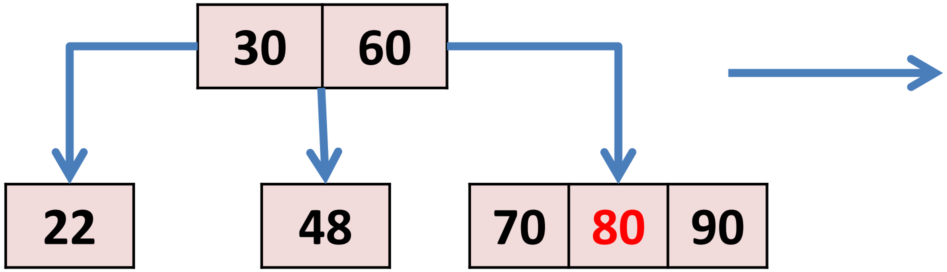

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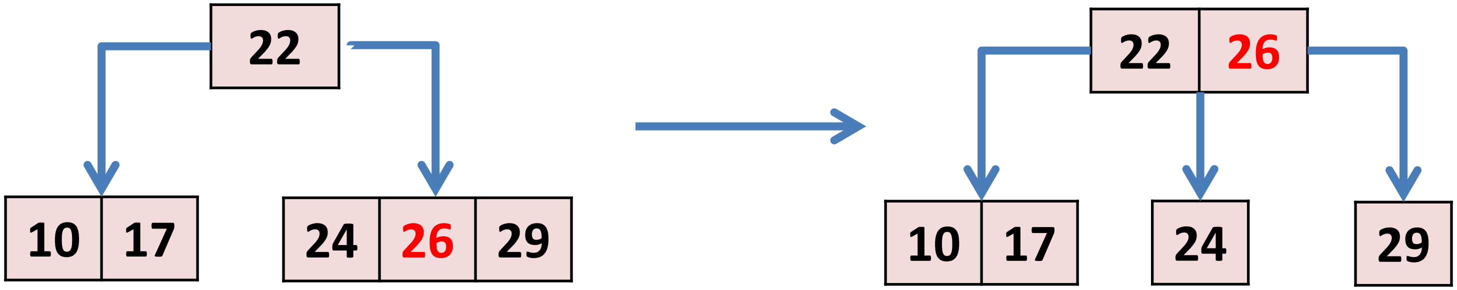
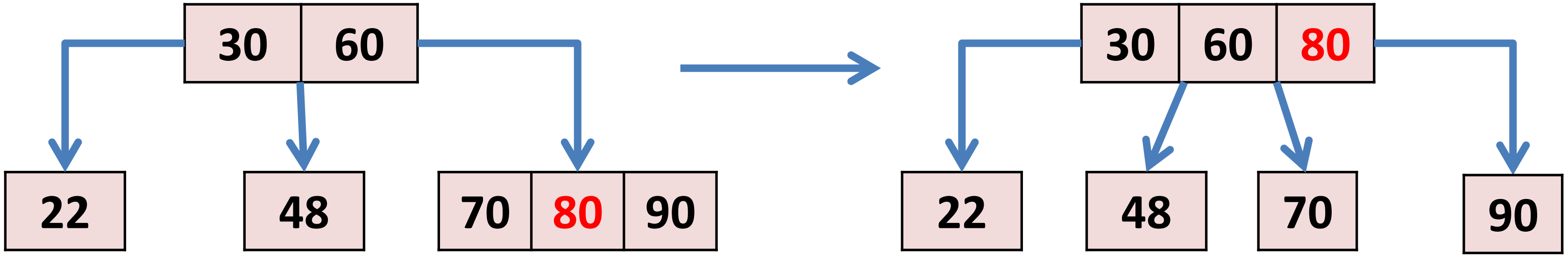


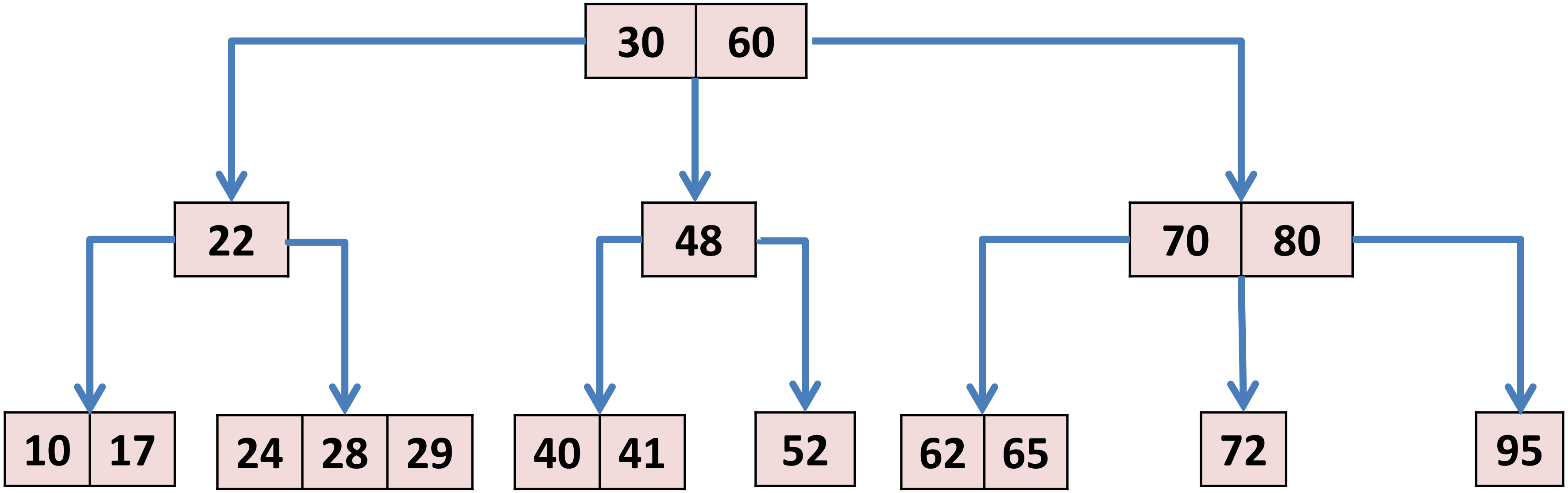
- If we find a 3-node being parent to a 4-node, we transform the pair into a 4-node connected to two 2-nodes, by pushing up the **middle** key of the 4-node.



Insertion Examples

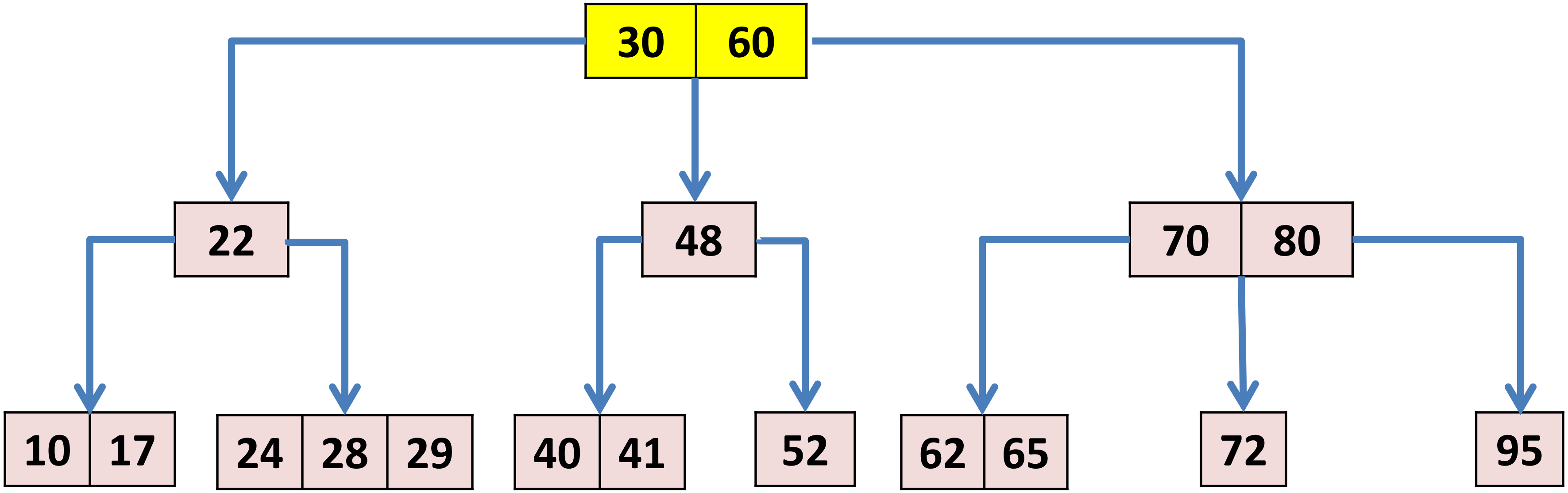
Insert 25

- Inserting an item with key 25:



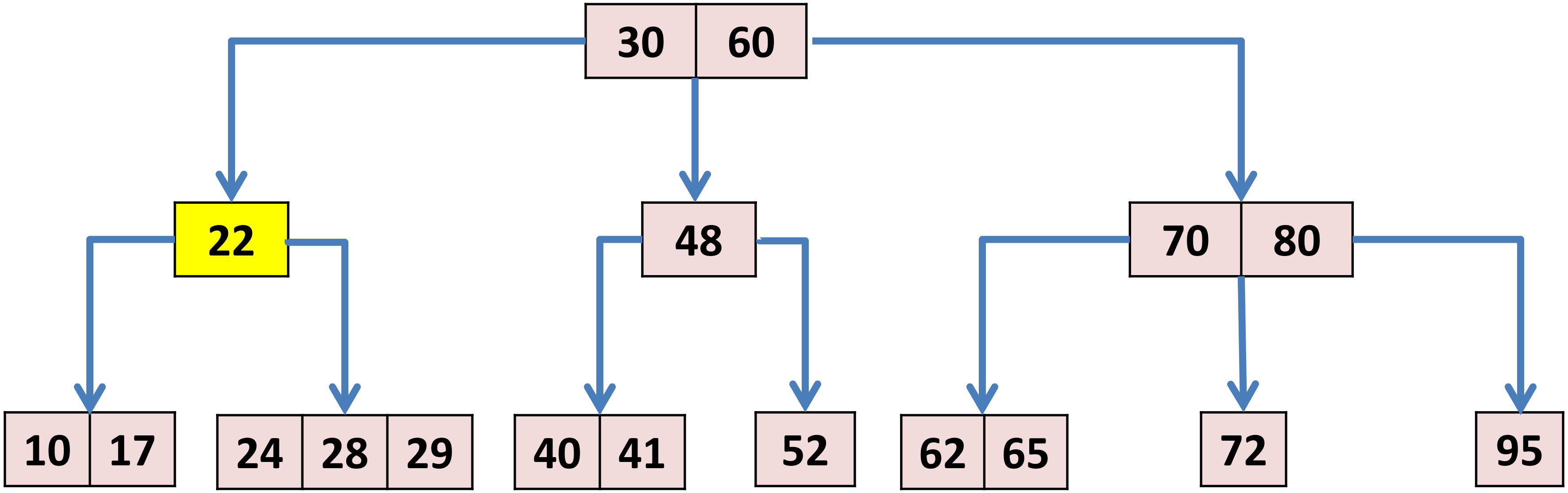
Insert 25

- Inserting an item with key 25:



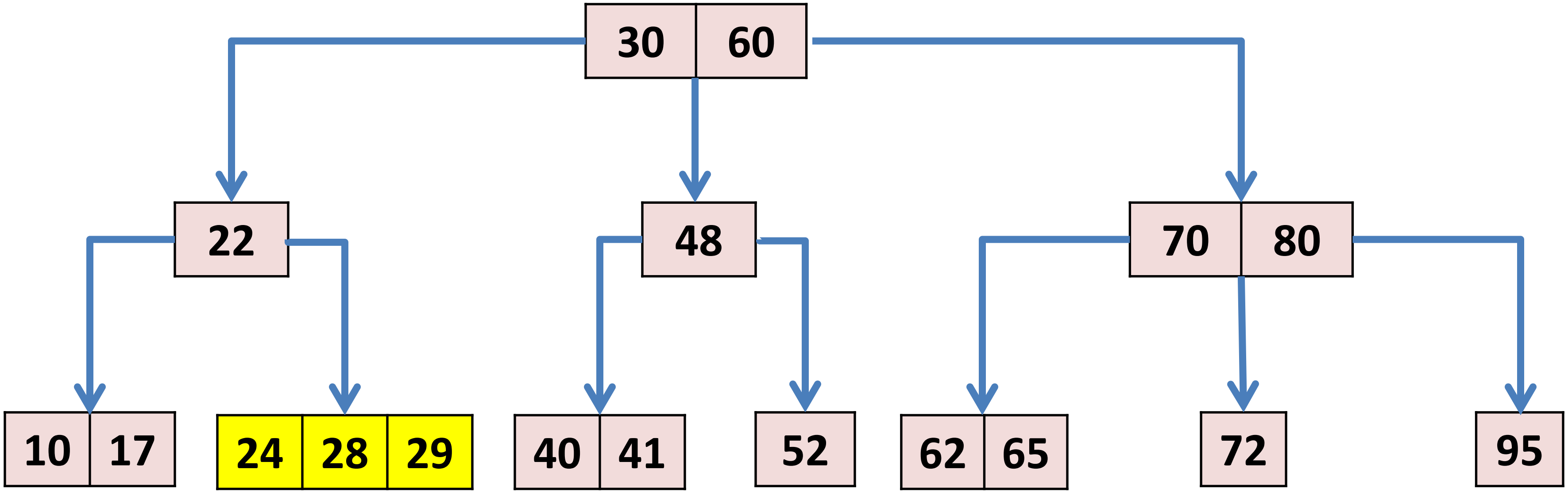
Insert 25

- Inserting an item with key 25:



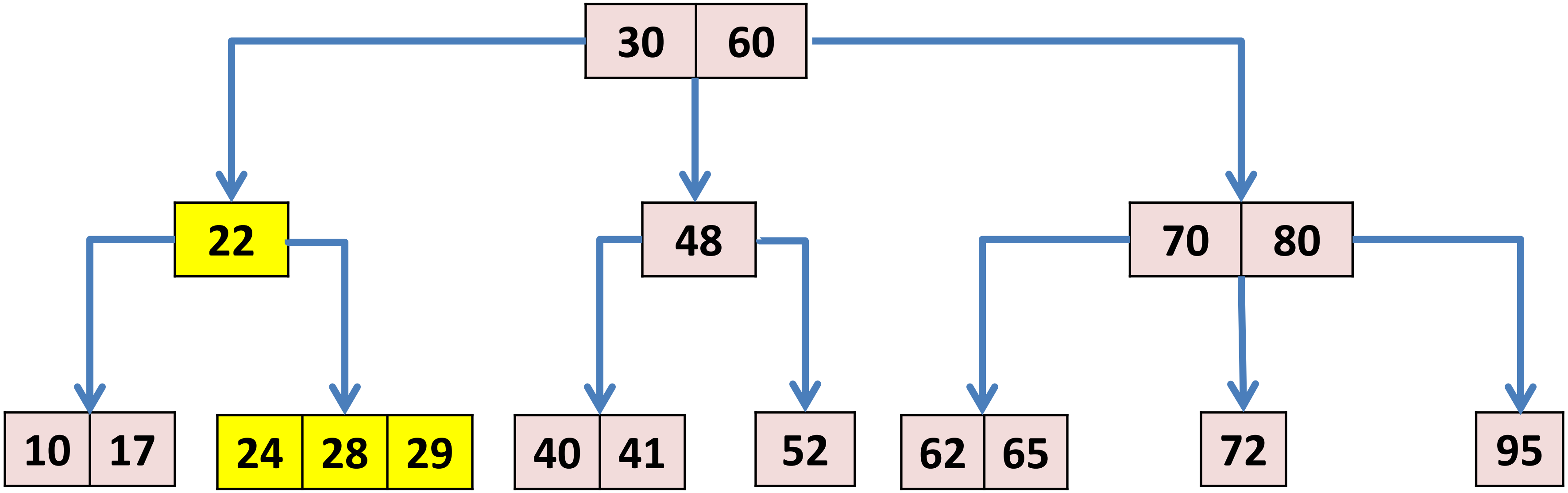
Insert 25

- Inserting an item with key 25:



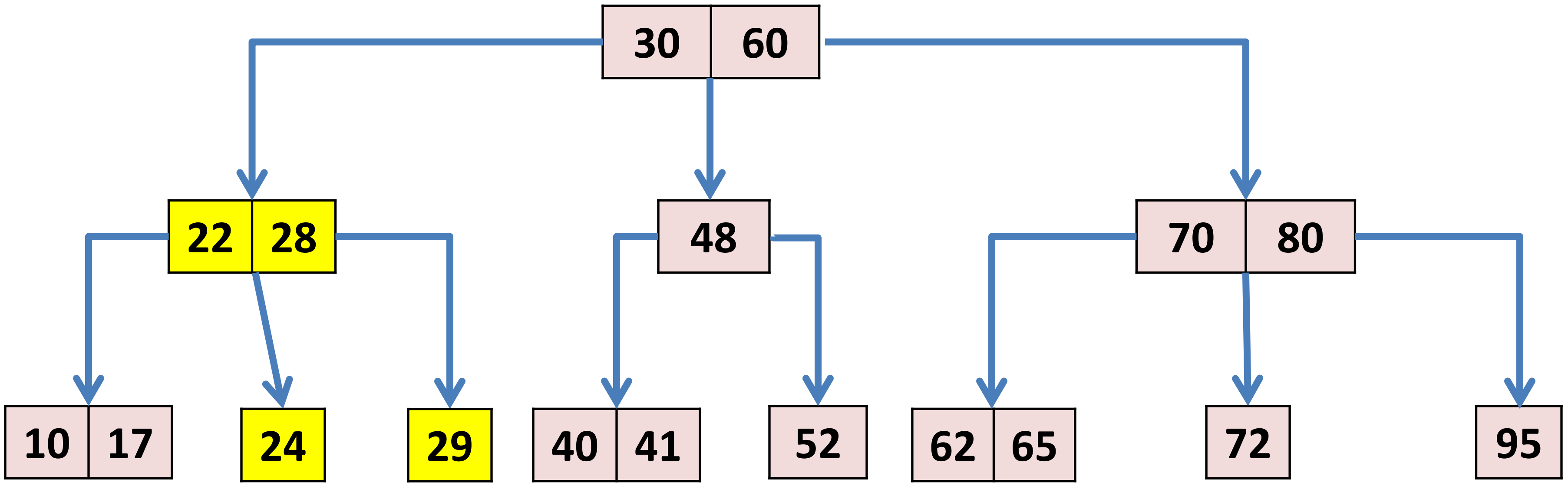
Insert 25

- We found a 4-node, we must split it and send an item up to the parent (2-node) which will become a 3-node.



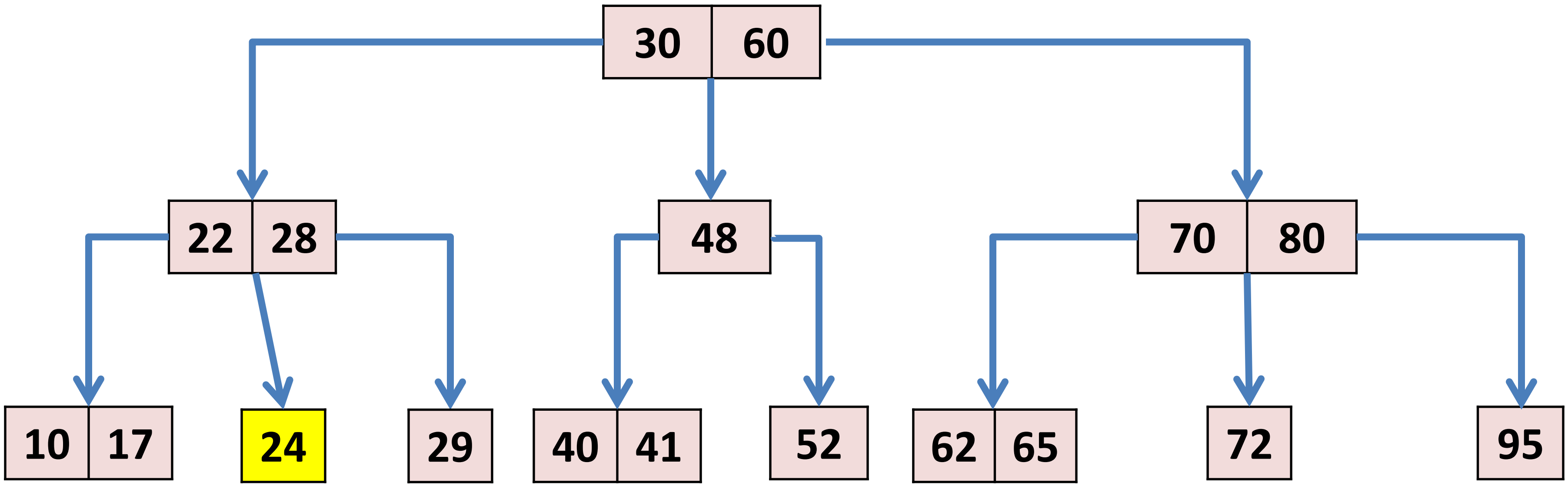
Insert 25

- Continue search for 25 from the updated (22,28) node.



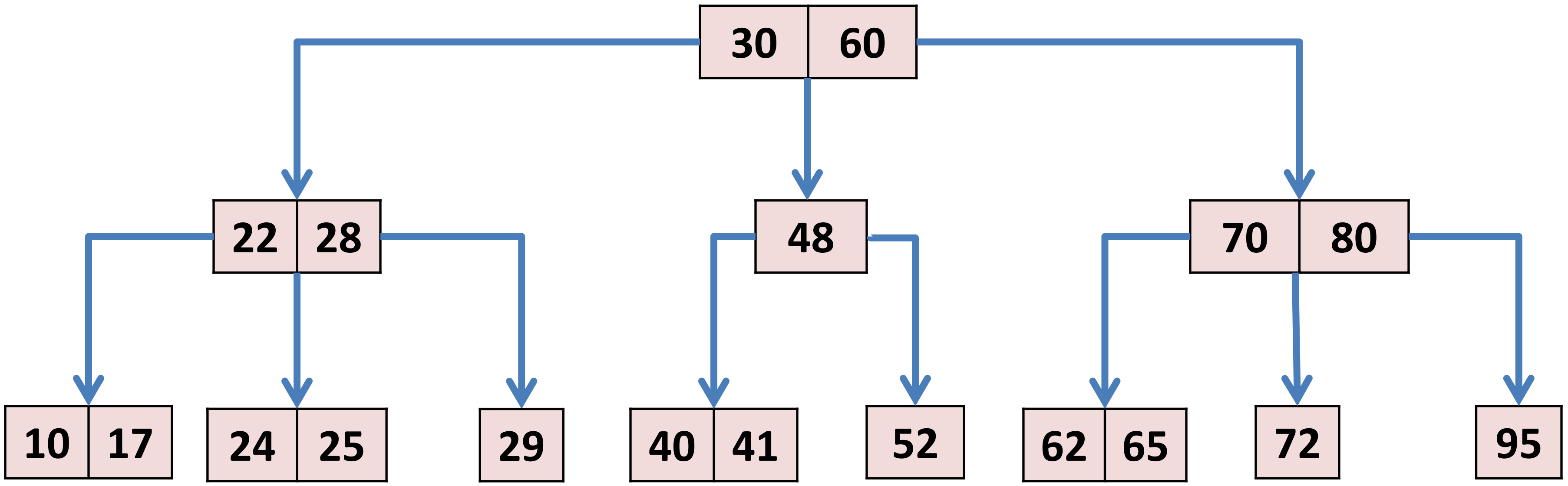
Insert 25

- Reached a leaf with less than 3 items. Add the item.

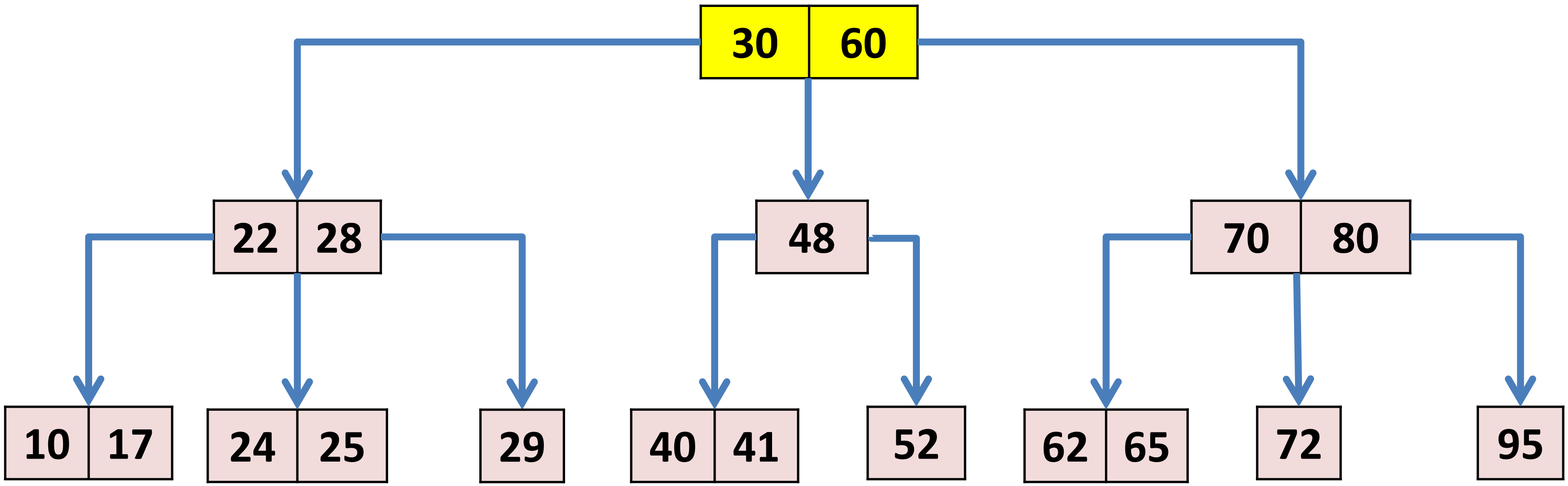


Insert 27

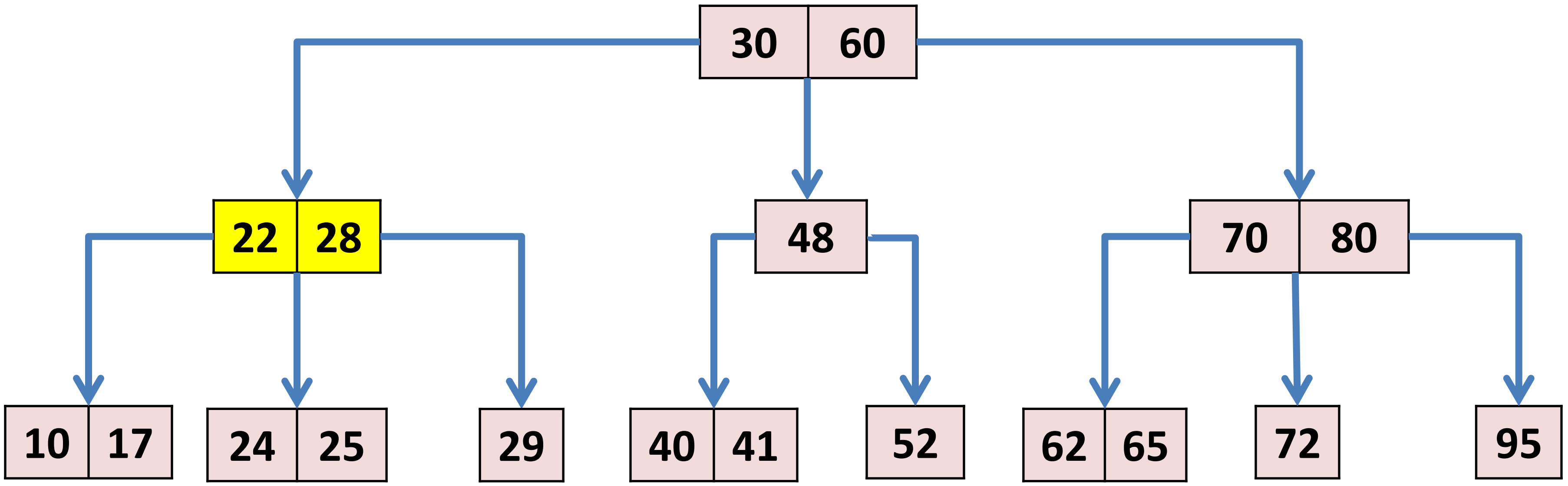
- Next: insert an item with key = 27.



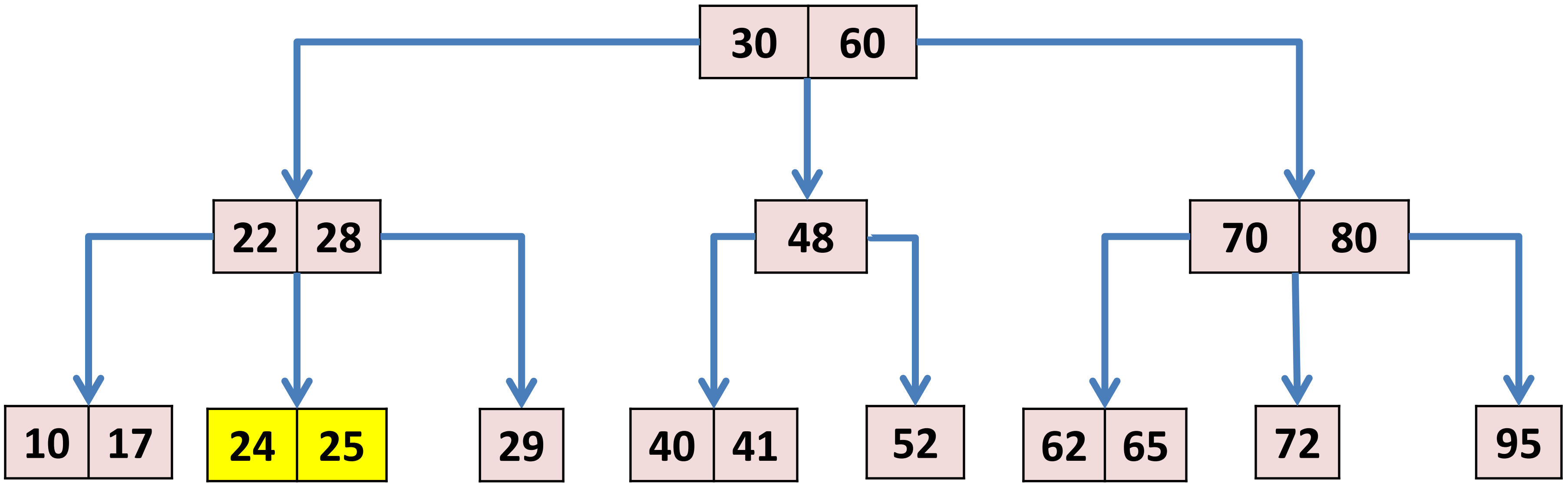
Insert 27



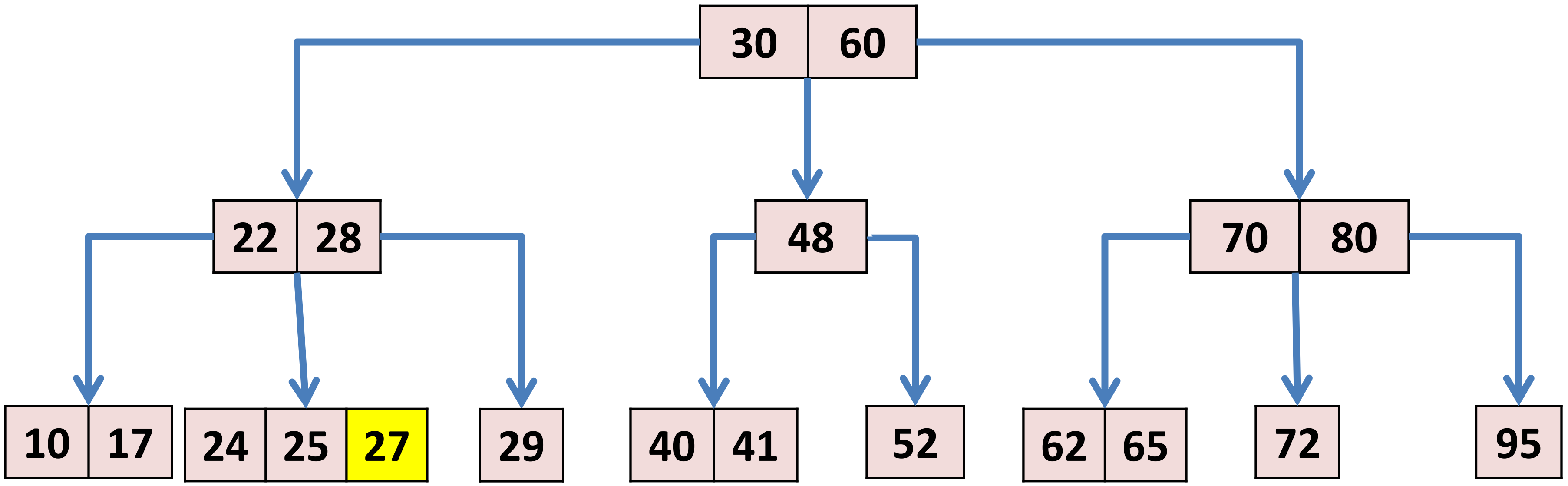
Insert 27



Insert 27

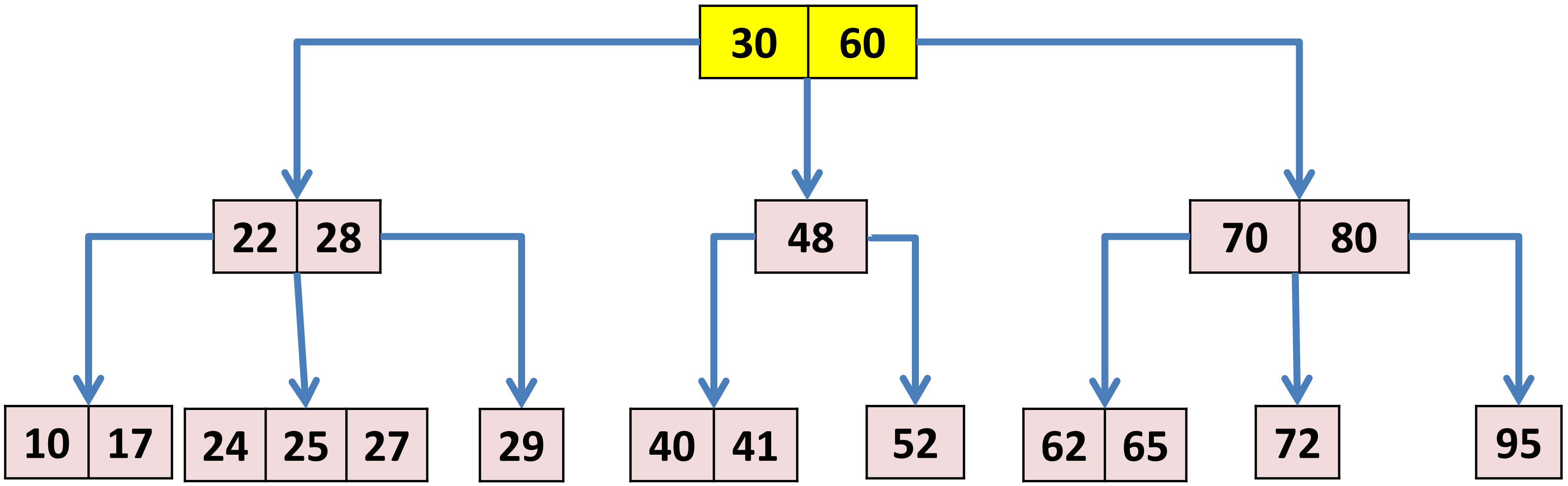


Insert 27

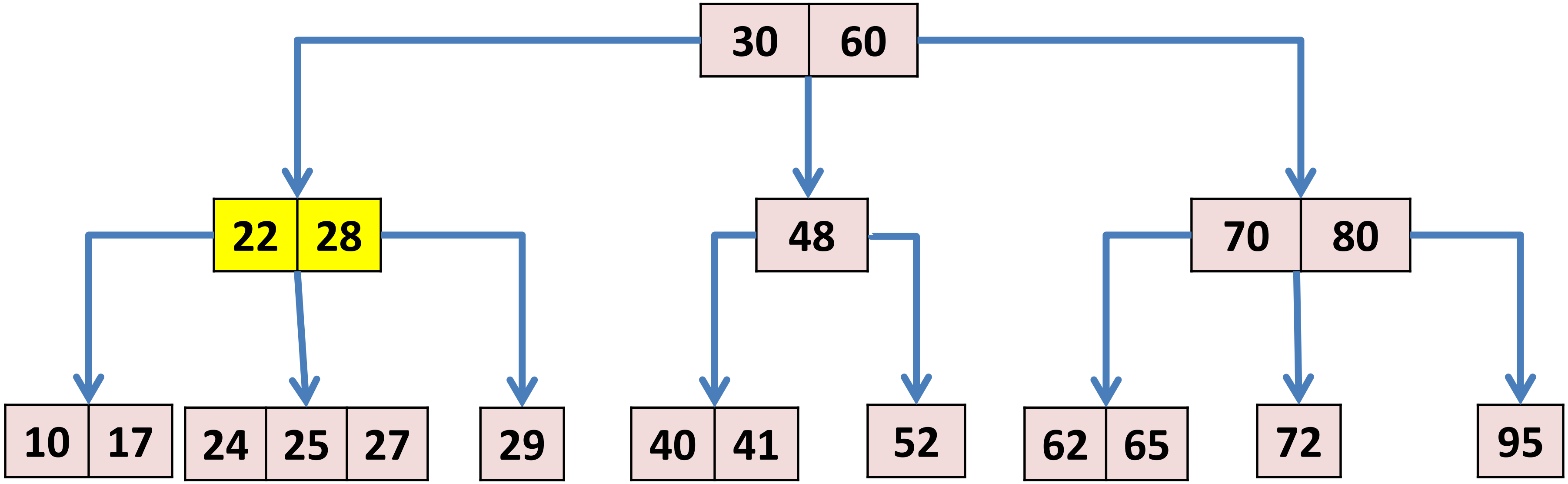


Insert 26

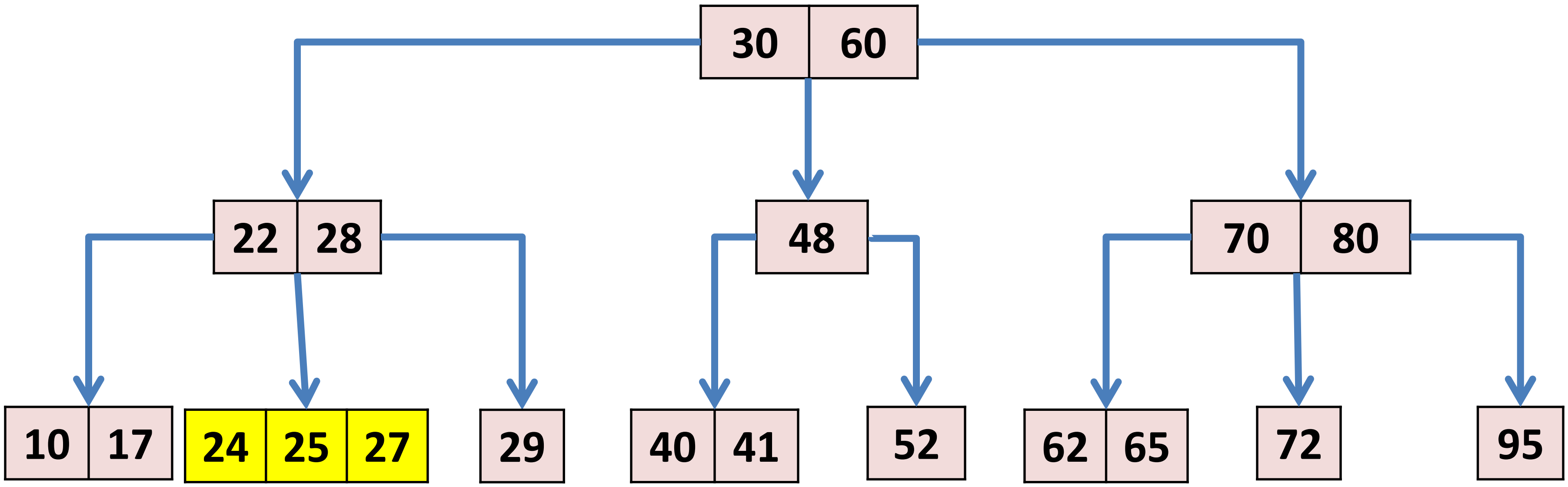
- Next: insert an item with key = 26.



Insert 26

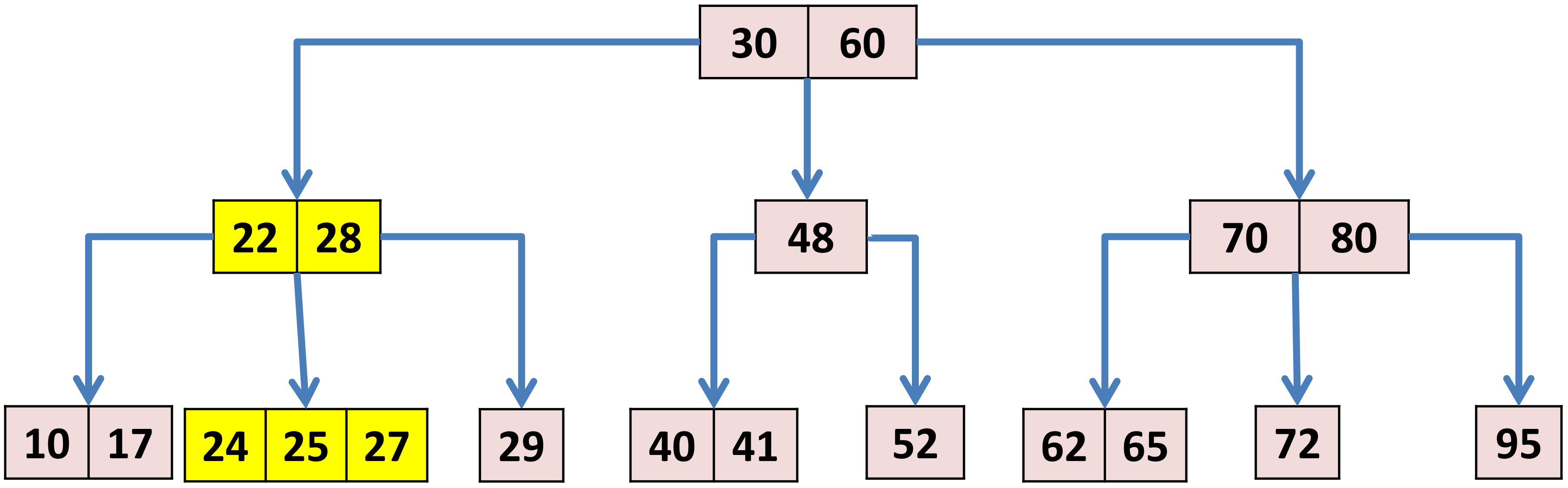


Insert 26



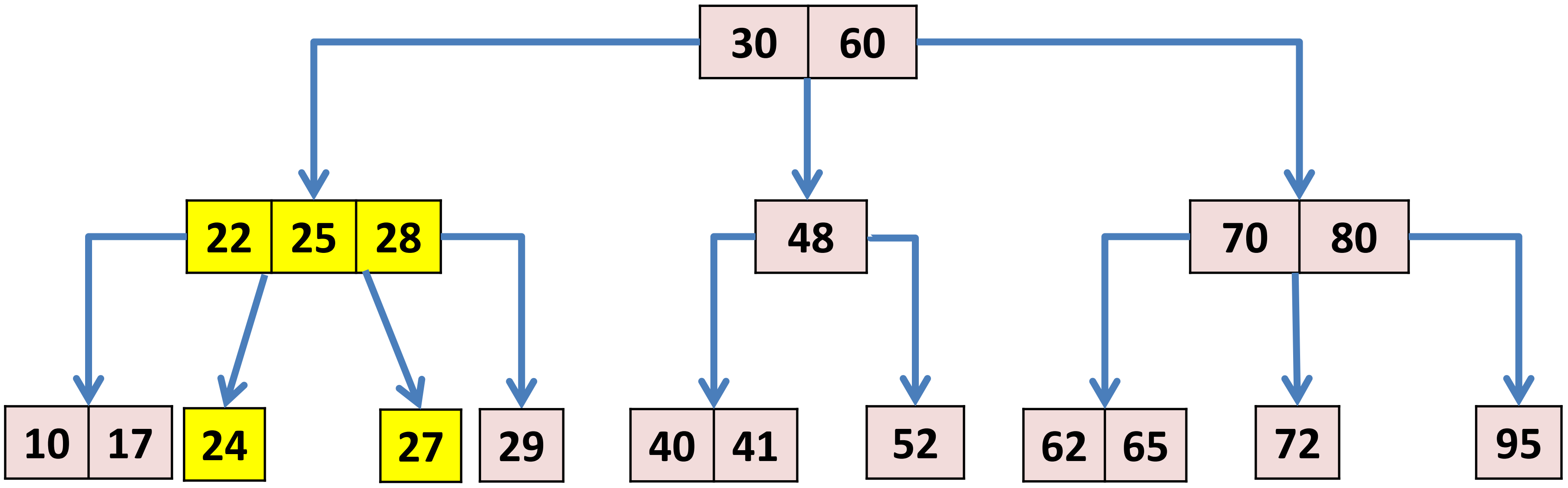
Insert 26

- Found a 3-node being parent to a 4-node, we must transform the pair into a 4-node connected to two 2-nodes.



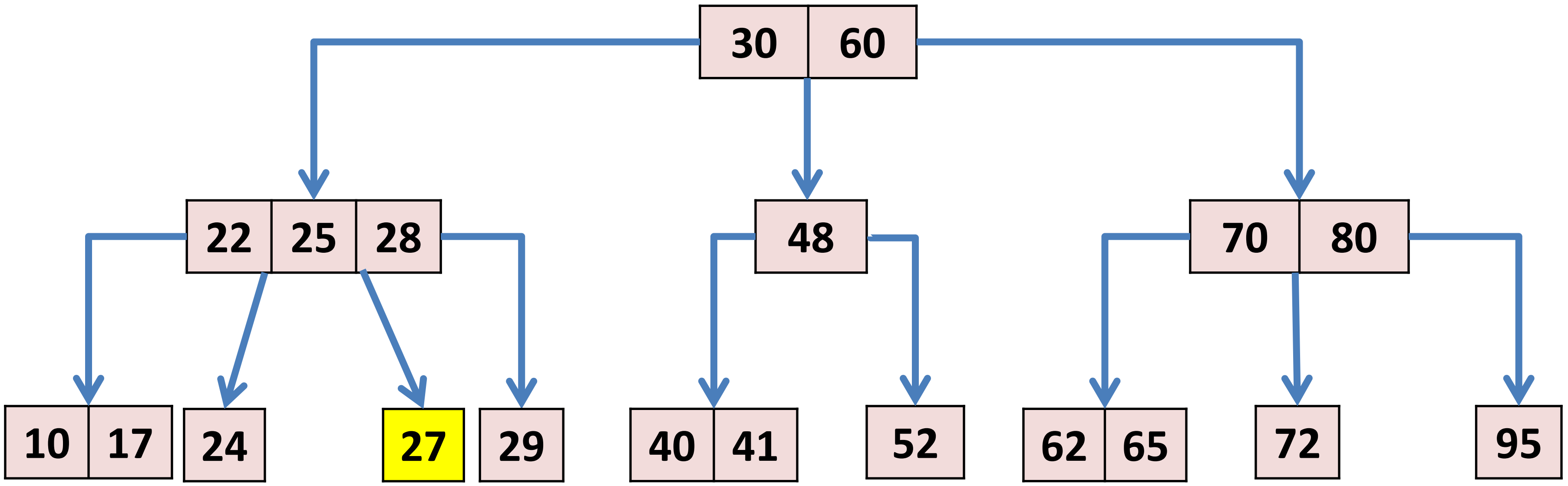
Insert 26

- Found a 3-node being parent to a 4-node, we must transform the pair into a 4-node connected to two 2-nodes.



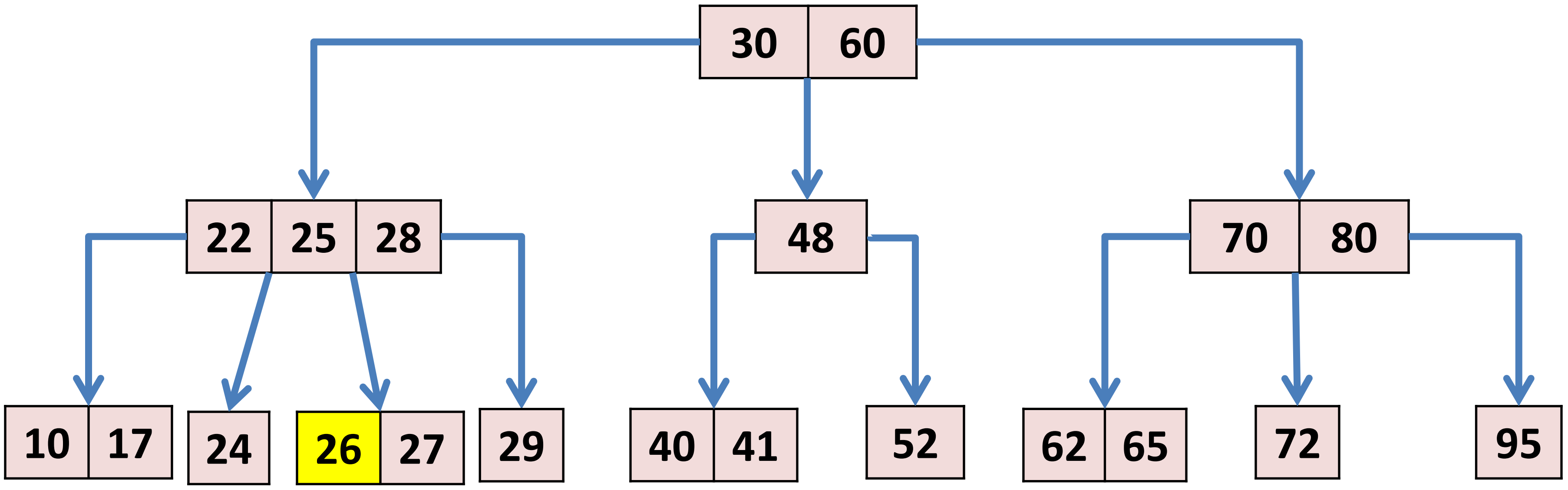
Insert 26

- Reached the bottom. Make insertion of item with key 26.



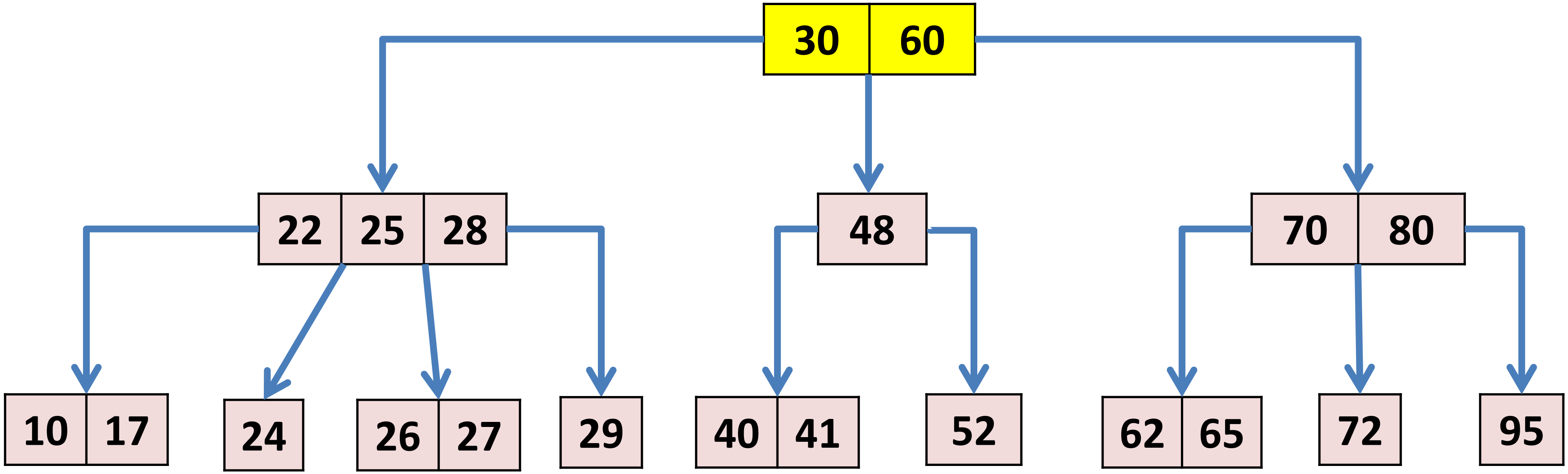
Insert 26

- Reached the bottom. Make insertion of item with key 26.



Insert 13

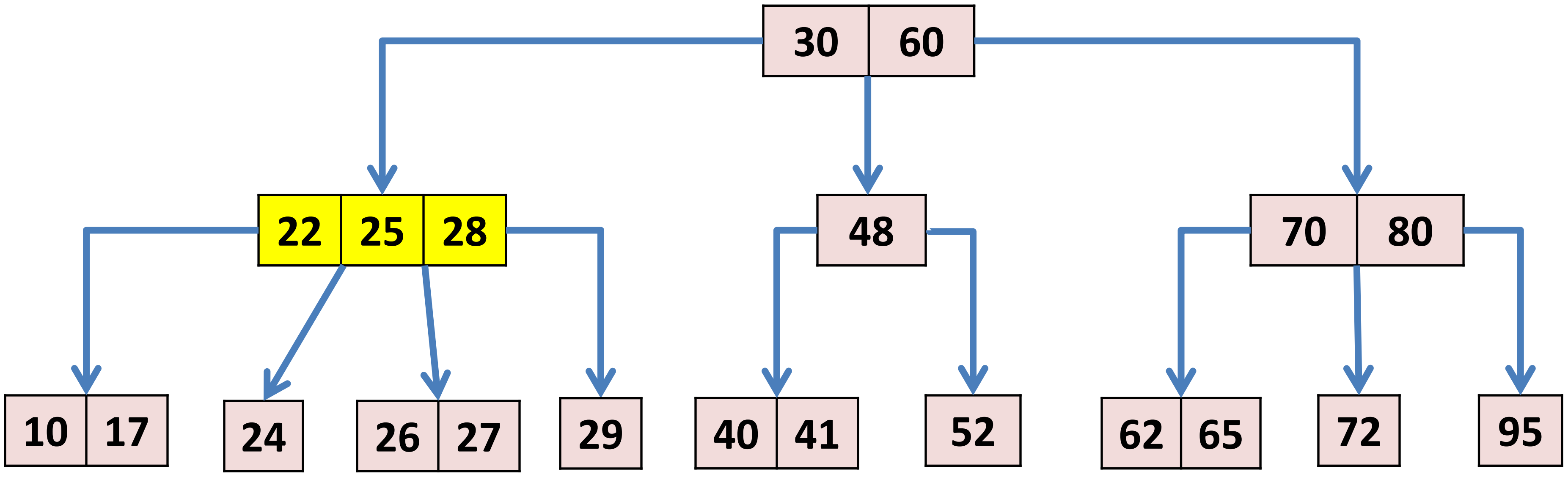
- Insert an item with key = 13.



Insert 13

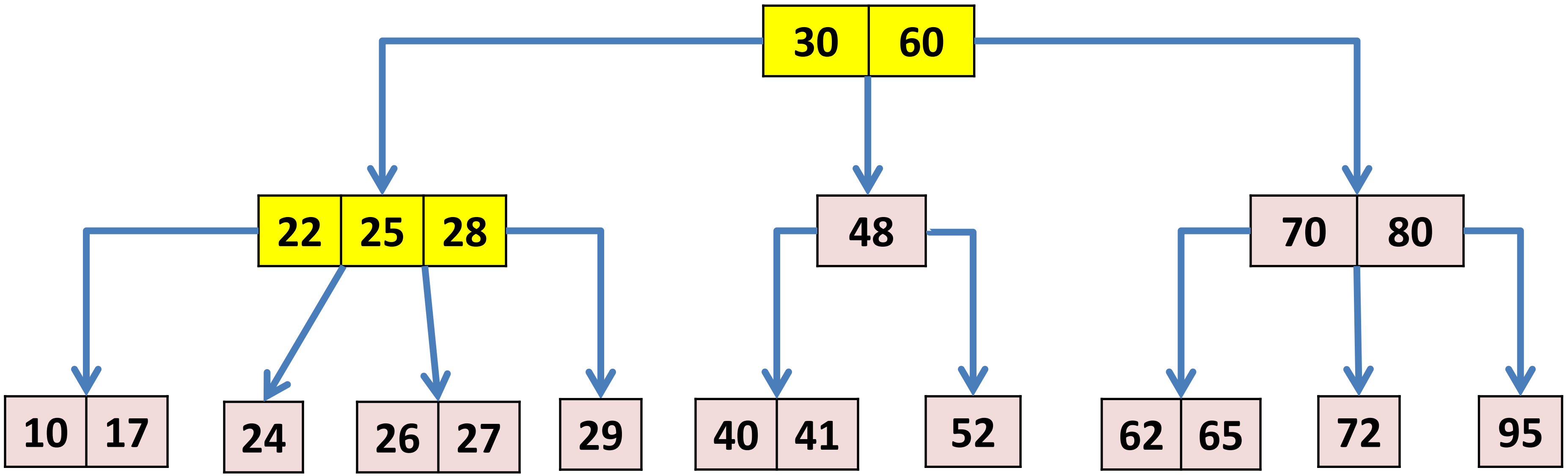
Our convention: Split this node! (It is full)

(Even though there is room for 13 in the leaf)



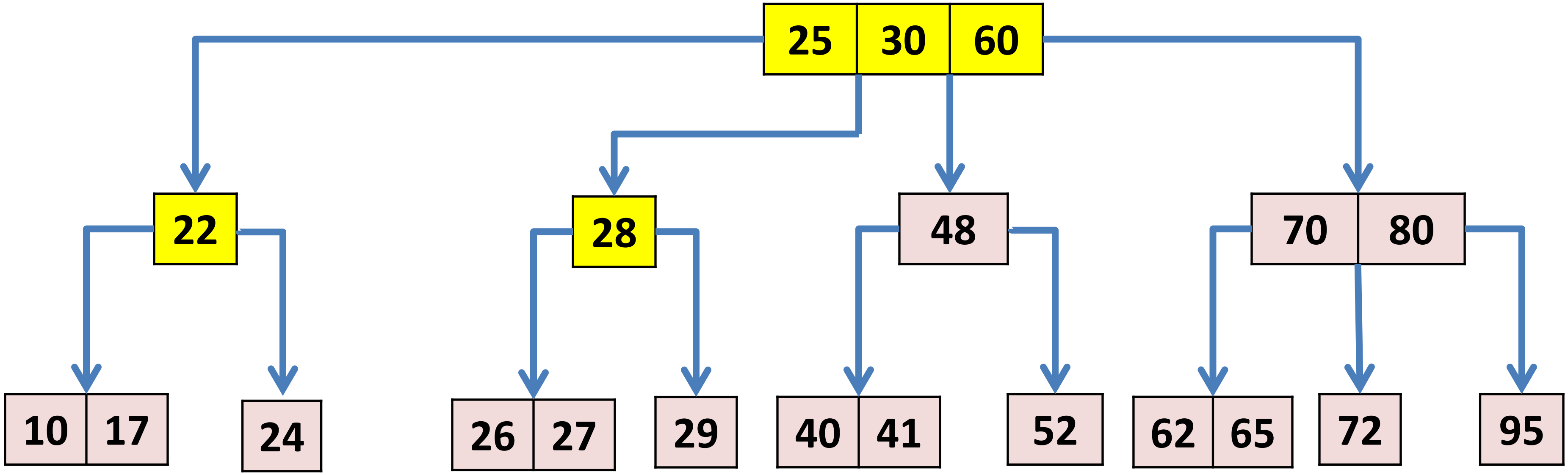
Insert 13

- Found a 3-node being parent to a 4-node, we must transform the pair into a 4-node connected to two 2-nodes.



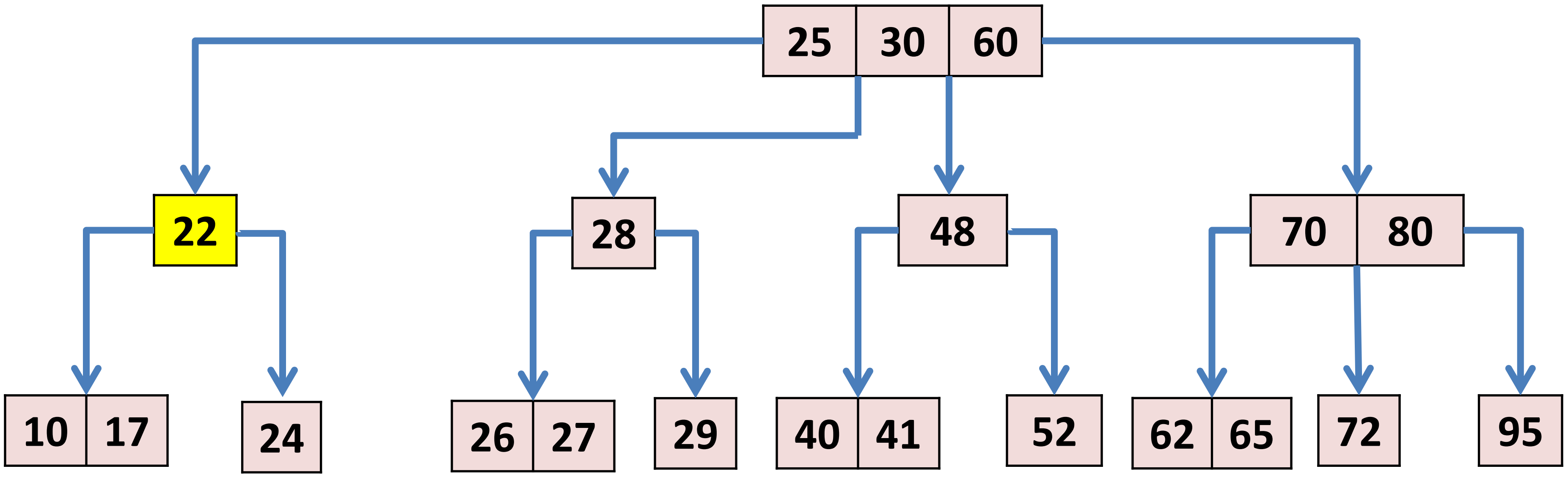
Insert 13

- The root became a 4 node, but we will not split it. (In some implementations the root is split at this point).



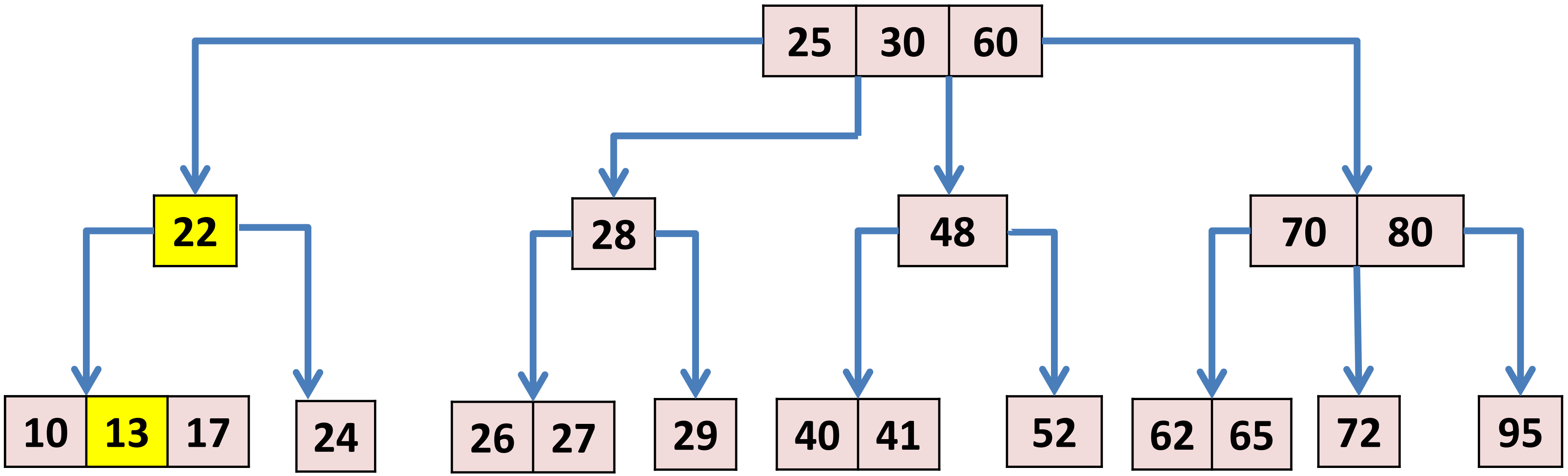
Insert 13

- Continue the search.



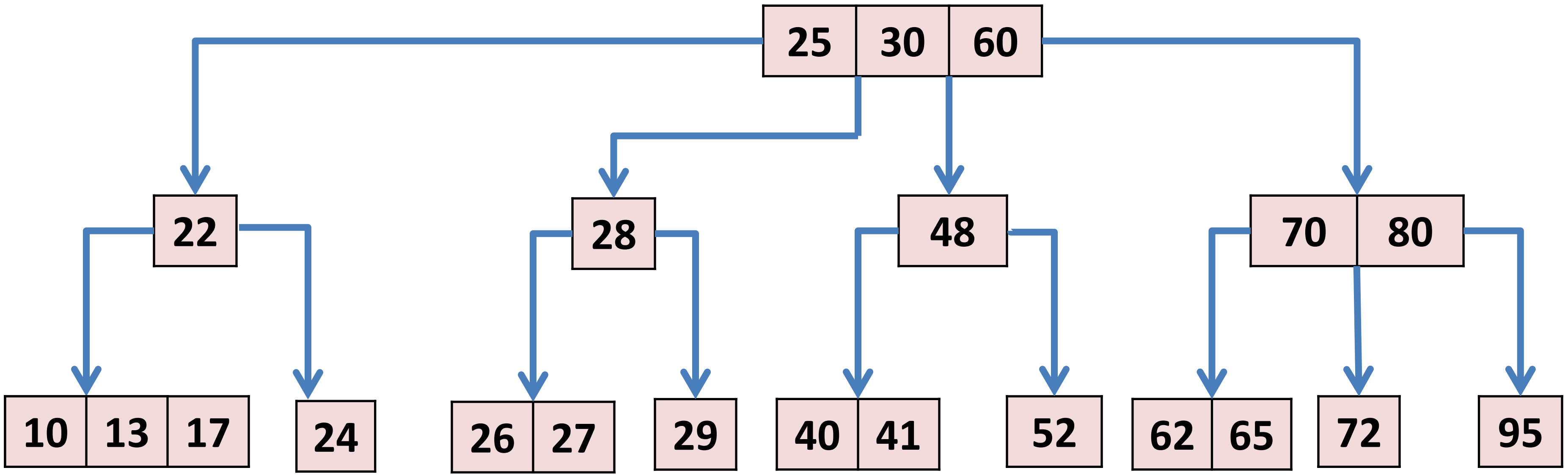
Insert 13

- Insert in leaf node.



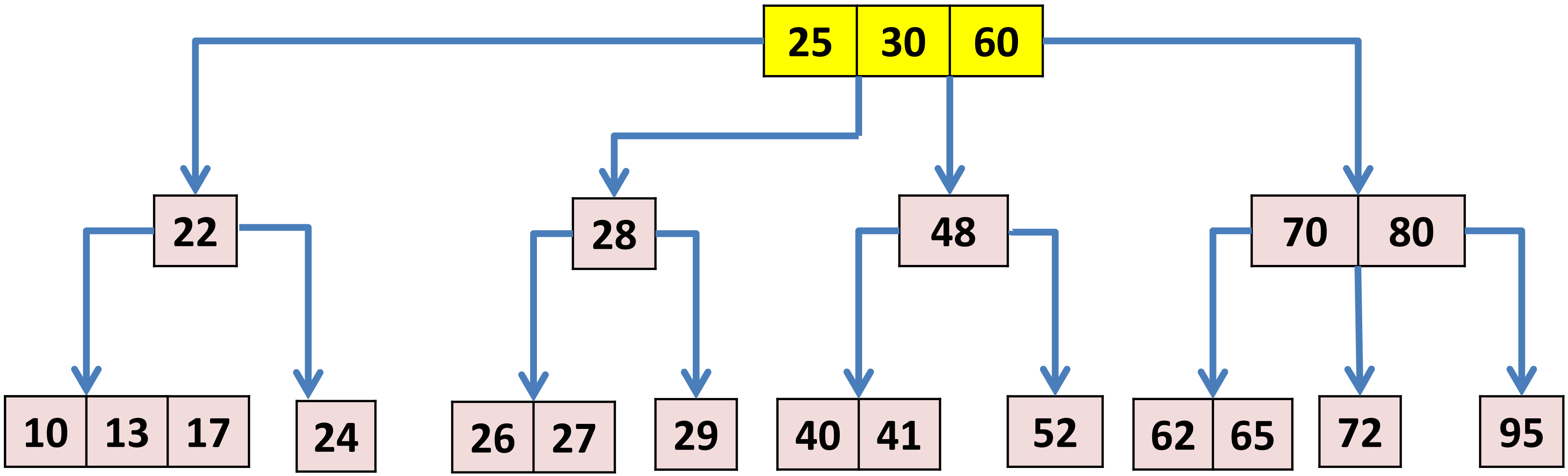
Insert 90

- Insert 90.



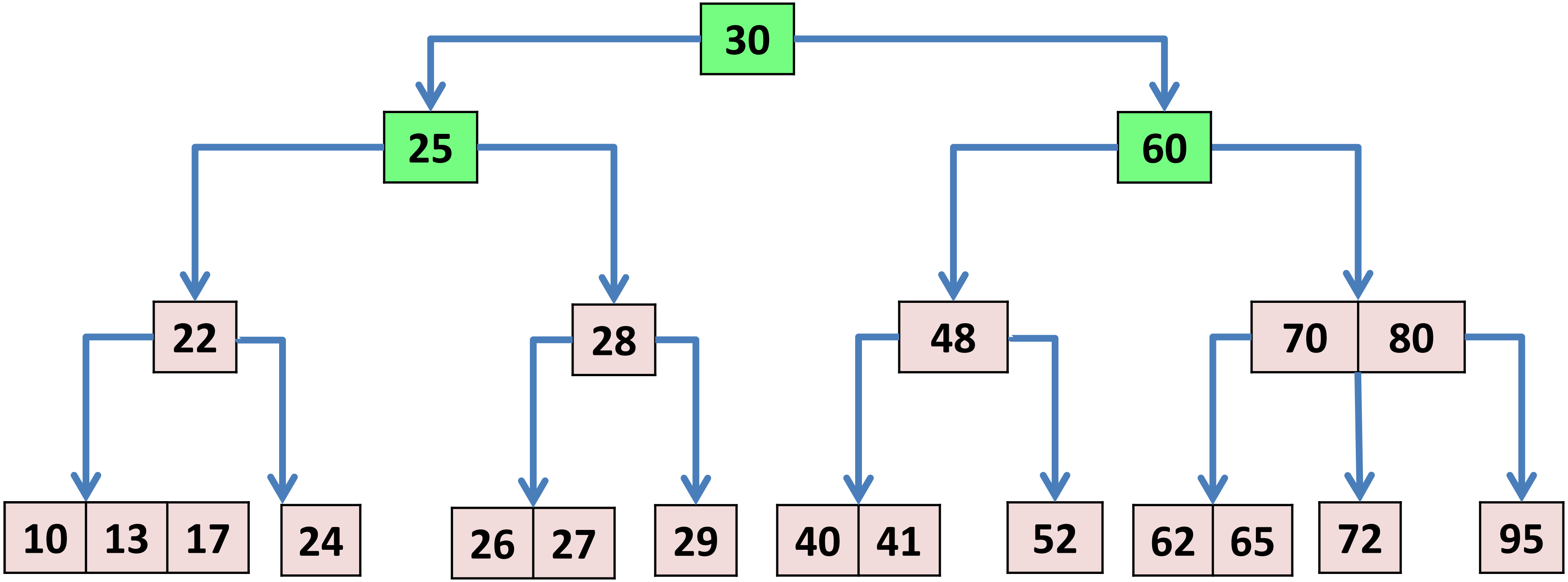
Insert 90

- Insert 90. The root is a 4-node. Split it.



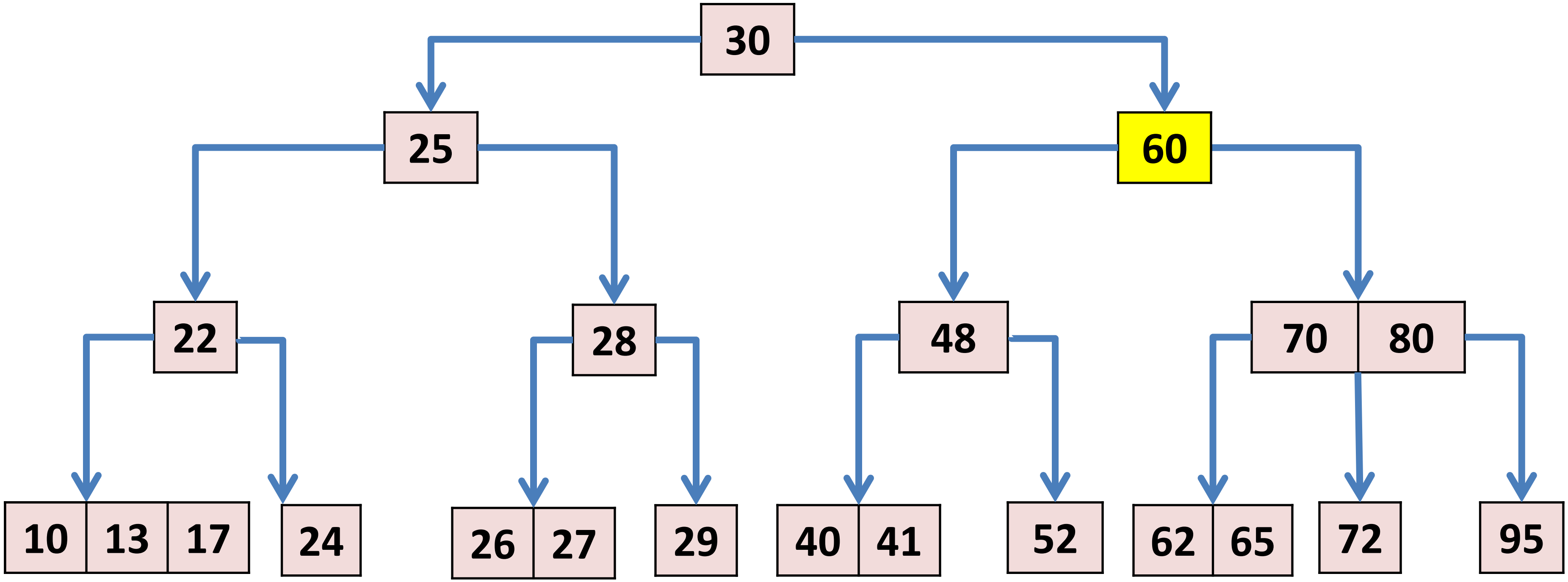
Insert 90

- Root is 4-node, must split!
- THIS IS HOW THE TREE HEIGHT GROWS!



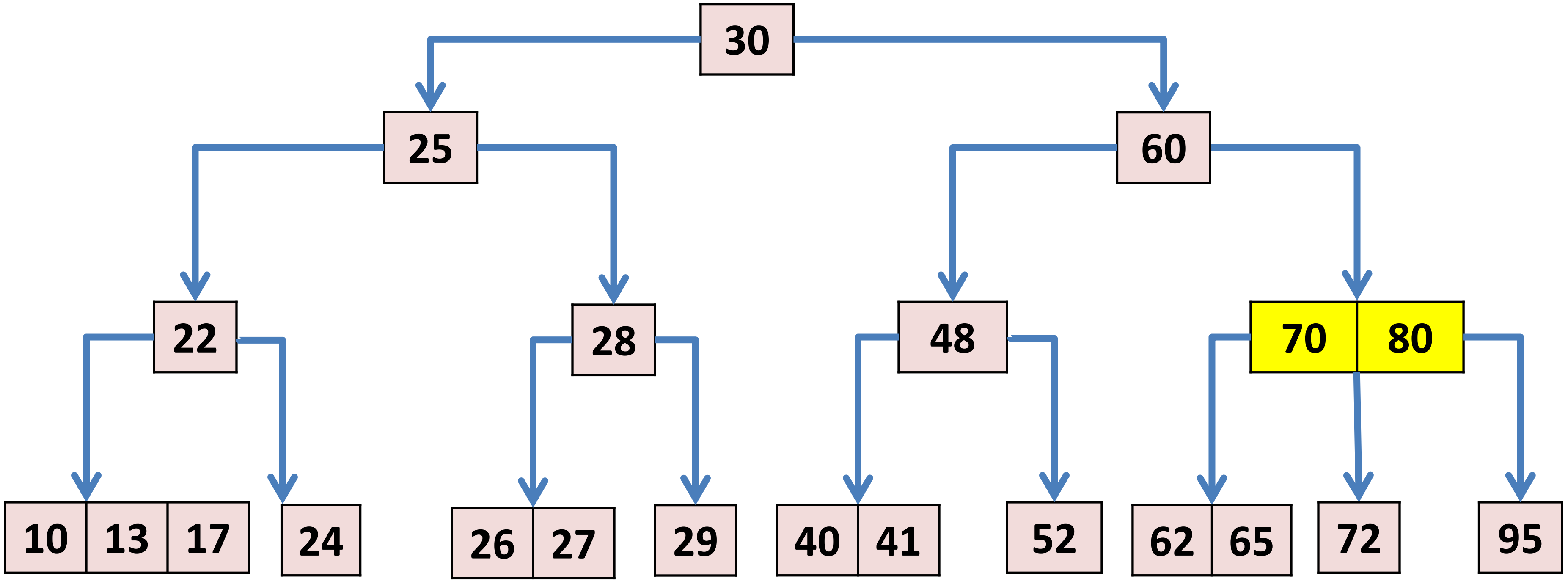
Insert 90

- Continue to search for 90.



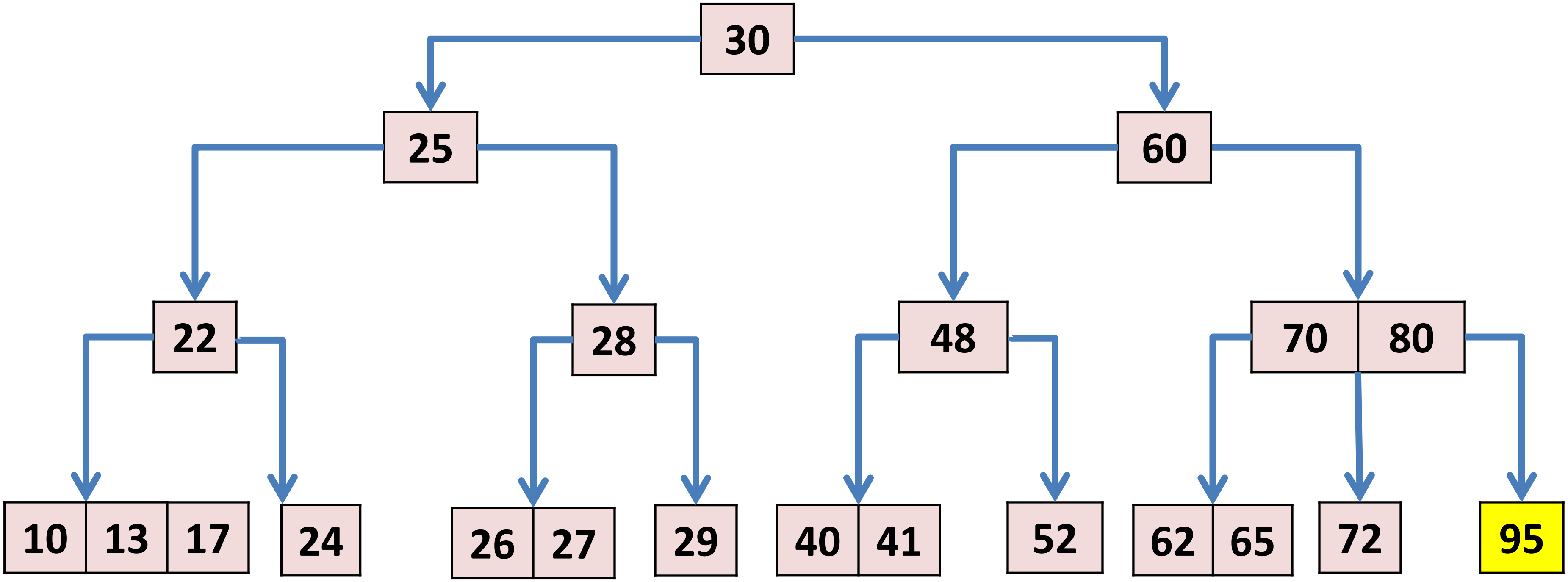
Insert 90

- Continue to search for 90.



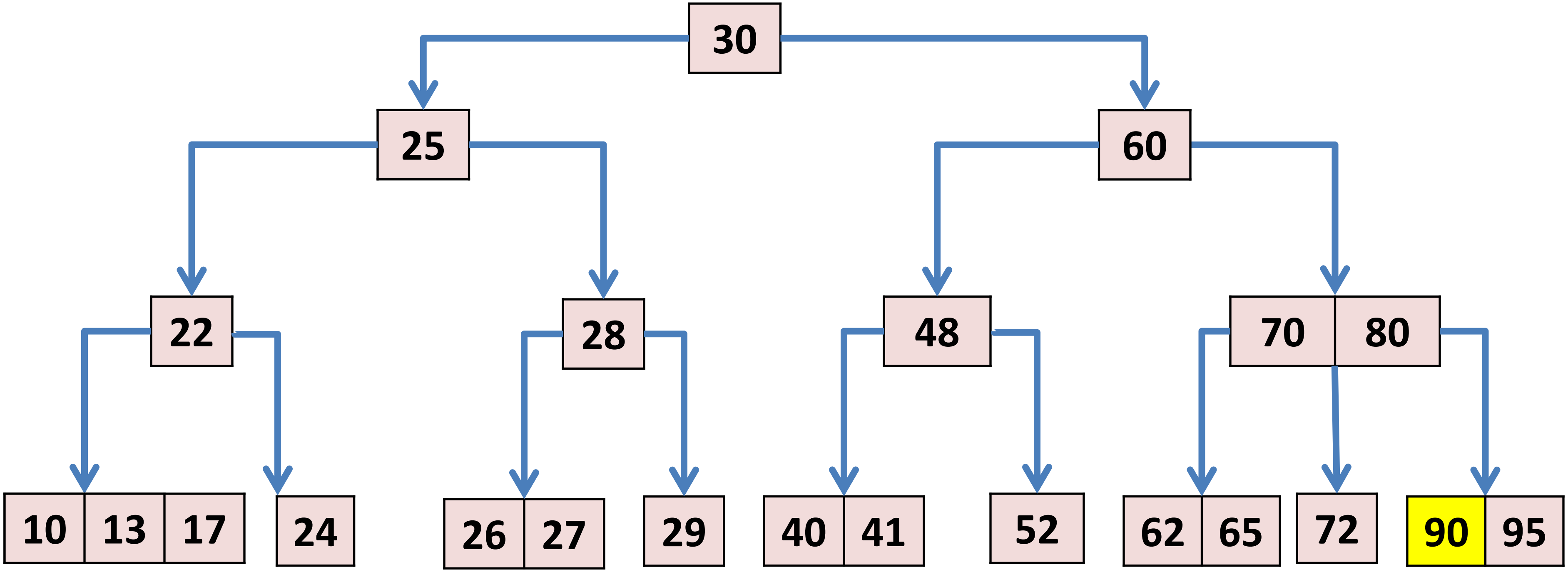
Insert 90

- Leaf, has space, insert 90.



Insert 90

- Leaf, has space, insert 90.



REMEMBER
our convention

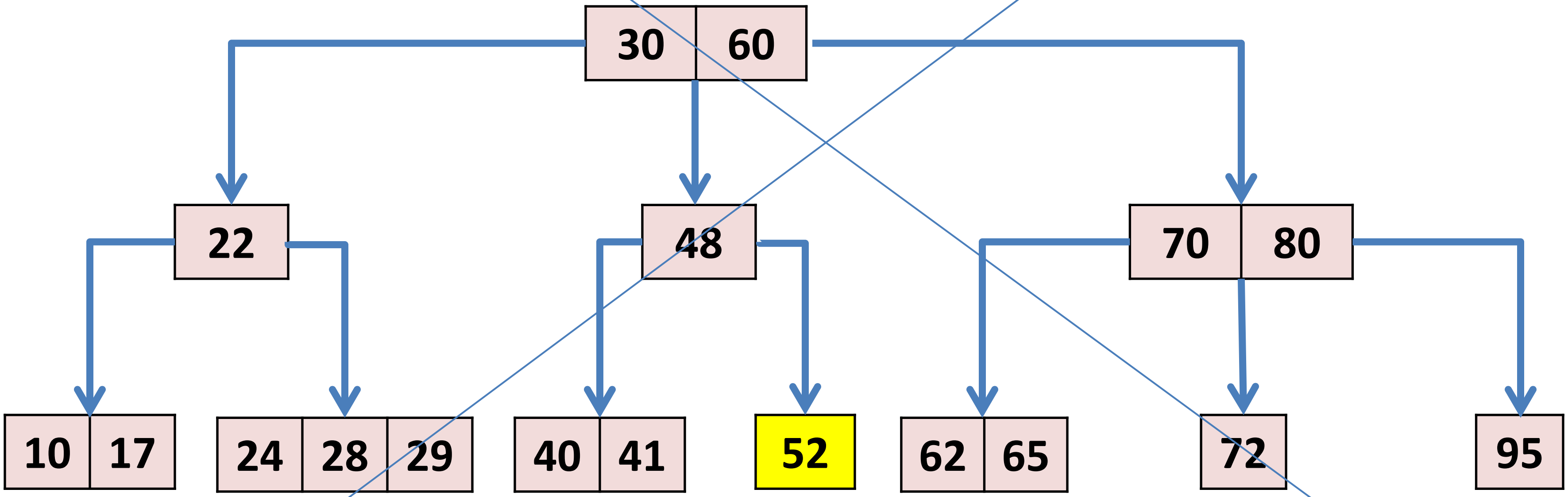
- If on your path to insert you see a 4 node, you split it!
- You do that even if there is room in the leaf and you can insert without splitting this node.

Deletion in 2-3-4 Trees

- More complicated.
  - Sedwick book does not cover it.
- Idea: in order to delete item *x* (with key *k*) search for *k*. When find it:
  - If in a leaf remove it,
  - Else replace it with the successor of *x, y.* (*y* is the item with the first key larger than *k*.) Note: y will be in a leaf.
    - remove *y* and put it in place of *x*.
- When removing *y* we have problems as with insert, but now the nodes may not have enough keys (need 2 or 3 keys) => fix nodes that have only one key on the path from root to *y*.

Delete 52

- Delete item with key 52:



- How about deleting item with key 95:

Deletion in 2-3-4 Trees

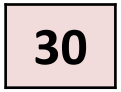
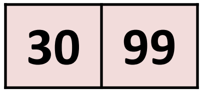
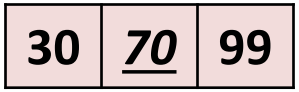
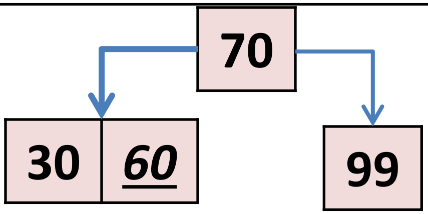
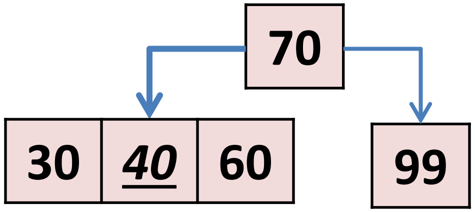
- Case 1: leaf has 2 or more items: remove y
- Case 2: node on the path has 2 or more items: fine
- Case 3: node on the path has only 1 item
  - A) Try to get a key from the sibling – must rotate with the parent key to preserve the order property
  - B) If no sibling has 2 or more keys, get a key from the parent and merge with your sibling neighboring that key.
  - C) The parent is the root and has only one key (and therefore exactly 2 children): merge the root and the 2 siblings together.

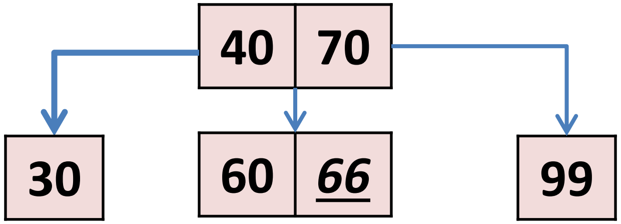
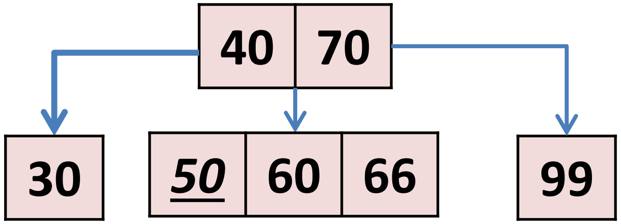
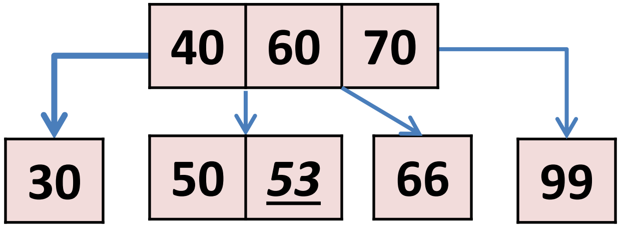
Example: Build a Tree

- In an empty tree, insert items given in order:

30, 99, 70, 60, 40, 66, 50, 53, 45, 42

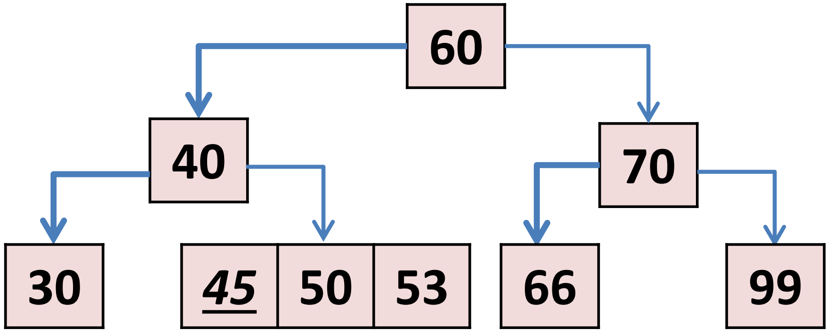
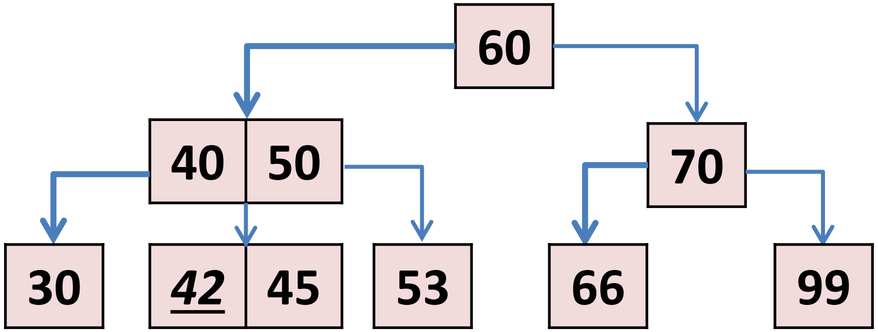
Building a Tree

Node to insert	Tree
30	
99	
70	
60	
40	

Node to insert	Tree
66	
50	
53	

Continues on next page …

Building a Tree

Node to insert	Tree
45	
42	

Self Balancing Binary Trees

- Red-Black trees
- AVL trees
- Splay trees
- ….

Red-Black, AVL

- Red-Black trees
  - Red & black nodes
    - Root is black,
    - a red node will have both his children black,
    - all leaves are black
    - For every node, any path from it to a leaf will have the same number of black nodes (i.e. same ‘black height’) => actual path lengths differ by at most a factor of two (cannot have 2 consec red nodes)
  - ** 2-3-4 trees can be mapped to red-black trees: 2-node = 1 black node, 3-node = 1 black & 1 red (left or right) 4 node = 1 black & 2 red children
- AVL trees
  - Height of two children differs by 1 at most.

Splay trees

- Splay trees
  - Splay insertion: the new item is inserted at the root by a rotation that replaces a node with his grandchild.
    - The grandchild node moves two levels up => the path on the way to the location (as a leaf) of the new node is cut in half.
  - Can apply the splay operation when searching for an item as well.
    - See Sedgewick page 545, for this effect.