

Homework #5

CSE 2320 – Section 002

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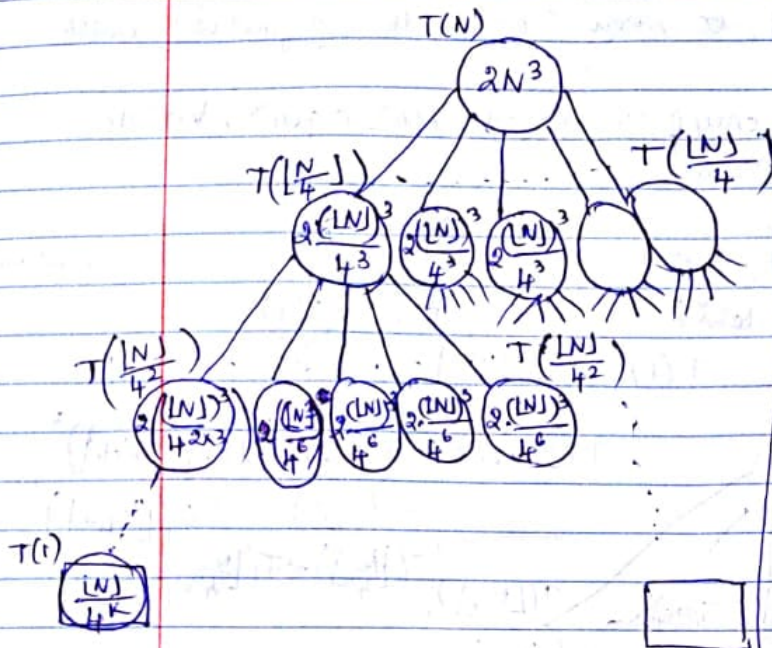
UTA ID: 1001669338

P1) $T(N) = 5T(N/4) + 2N^3$, $T(0) = 1$, $T(1) = 1$

$$T\left(\frac{N}{4}\right) = 5T\left(\frac{N}{4^2}\right) + 2\left(\frac{N}{4}\right)^3 = 5T\left(\frac{N}{4^2}\right) + 2\frac{(N)^3}{4^3}$$

$$T\left(\frac{N}{4^2}\right) = 5T\left(\frac{N}{4^3}\right) + 2\left(\frac{N}{4^2}\right)^3 = 5T\left(\frac{N}{4^3}\right) + 2\frac{(N)^3}{4^{2 \times 3}}$$

$$T\left(\frac{N}{4^3}\right) = 5T\left(\frac{N}{4^4}\right) + 2\left(\frac{N}{4^3}\right)^3 = 5T\left(\frac{N}{4^4}\right) + 2\frac{(N)^3}{4^{3 \times 3}}$$



$$\frac{N}{4^k} = 1 \Rightarrow \boxed{k = \log_4 N}$$

$$\frac{(N)^3}{4^{3k}} = \frac{(N)^2}{4^3} \times \frac{N}{4^k} = \frac{(N)^2}{4^3}$$

$$5^k \times 2 \times \frac{(N)^3}{4^{3k}} = 2 \frac{(N)^3}{4^3} \left(\frac{5}{4}\right)^k$$

$$5^i \times 2 \times \frac{(N)^3}{4^{3i}} = 2 \frac{(N)^3}{4^3} \left(\frac{5}{4}\right)^i = 2 \frac{(N)^3}{4^3} \left(\frac{5}{4}\right)^k$$

Level	Prob. Size	Cost of 1 node	Nodes per level	Level cost
0	N	$2N^3$	1	$1 \times 2N^3$
1	$\frac{N}{4}$	$2 \times \frac{(N)^3}{4^3}$	5	$5 \times 2 \times \frac{(N)^3}{4^3}$
2	$\frac{N}{4^2}$	$2 \times \frac{(N)^3}{4^{2 \times 3}}$	5^2	$5^2 \times 2 \times \frac{(N)^3}{4^{2 \times 3}}$
3	$\frac{N}{4^3}$	$2 \times \frac{(N)^3}{4^{3 \times 3}}$	5^3	$5^3 \times 2 \times \frac{(N)^3}{4^{3 \times 3}}$
\vdots	\vdots	\vdots	\vdots	\vdots
i	$\frac{N}{4^i}$	$2 \times \frac{(N)^3}{4^{3i}}$	5^i	$5^i \times 2 \times \frac{(N)^3}{4^{3i}}$
\vdots	\vdots	\vdots	\vdots	\vdots
k	$\frac{N}{4^k} = 1$	$2 \times \frac{(N)^3}{4^{3k}} = 2 \times \frac{(N)^2}{4^3}$	5^k	$5^k \times 2 \times \frac{(N)^3}{4^{3k}} = 2 \frac{(N)^3}{4^3} \left(\frac{5}{4}\right)^k$

$$T(N) = \sum_{i=0}^k 2 \left(\frac{LN}{H} \right)^3 \left(\frac{5}{4} \right)^i$$

$$T(N) = 2 \left(\frac{LN}{H} \right)^3 \sum_{i=0}^k \left(\frac{5}{4} \right)^i$$

$\frac{5}{4} > 1$, so this is of the form $\sum_{i=0}^k x^i = \frac{x^{k+1} - 1}{x - 1}$

$$\sum_{i=0}^k \left(\frac{5}{4} \right)^i = \frac{\left(\frac{5}{4} \right)^{k+1} - 1}{\frac{5}{4} - 1} = \text{constant}$$

$$T(N) = \Theta(N^3)$$

P2) $T(N) = 4T(N-5) + 7$, $T(N) = 1$ for all $0 \leq N \leq 4$

Let $N = 0$, $T(0) = 1$

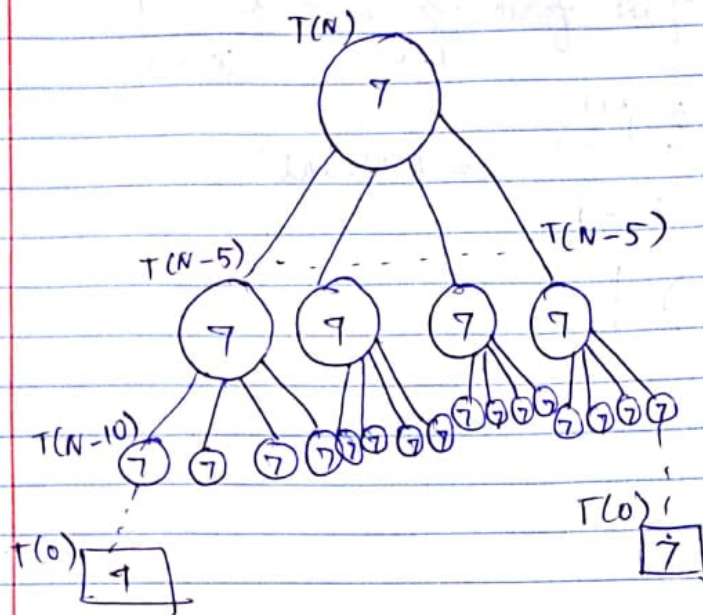
$T(N-5) = 4T(N-10) + 7$

Level 4

Prove

$T(N-10) = 4T(N-15) + 7$

$T(N-15) = 4T(N-20) + 7$



Level	Prob size	Cost per node	Node per level	Level cost
0	N	7	1	1×7
1	N-5	7	4	4×7
2	N-10	7	4^2	$4^2 \times 7$
\vdots	\vdots	\vdots	\vdots	\vdots
i	N-5i	7	4^i	$4^i \times 7$
\vdots	\vdots	\vdots	\vdots	\vdots
$k = N/5$	N-5k=0	7	4^k	$4^k \times 7$

$N - 5k = 0$

$N = 5k$

$\boxed{\frac{N}{5} = k}$

$T(N) = 4^k \times 7$

$\boxed{T(N) = \Theta(1)}$

P3) a) $T(N) = 5T(N/4) + 2N^3$. Assume $T(1) = 1$ and $T(1) = 1$
 $a = 5, b = 4 \Rightarrow n^{\log_b a} = n^{\log_4 5} = n^{1.1}$
 $f(n) = 2N^3 \stackrel{\text{like}}{=} \Omega(n^{1.1})$

$\epsilon = 1.9$
 $f(n) = 2N^3 = \Omega(n^{1.1+1.9})$
 cannot be Ω , it's θ since it is equal.

$$f(n) = 2N^3 = \theta(n^3)$$

$\epsilon = 0.1$
 $f(n) = 2N^3 = \Omega(n^{1.1+0.1}) = \Omega(n^{1.2})$

$$a f\left(\frac{n}{b}\right) \leq k f(n)$$

$$a f\left(\frac{n}{b}\right) = 5 \times 2 \left(\frac{n}{4}\right)^3 \leq k \cdot 2n^3$$

$$\frac{10}{4^3} n^3 \leq k \times 2n^3$$

$$\frac{10}{4^3} n^3 \leq k \cdot 2n^3$$

$$\frac{10}{4^3} n^3 < \frac{10}{4^3} 2 \times n^3 \leq k \cdot 2n^3$$

$$k = \frac{10}{4^3} = \frac{5 \times 2}{4^3 \times 2} = \frac{5}{32}$$

$$\boxed{k = \frac{5}{32}} < 1$$

Case 3 $\Rightarrow T(n) = \theta(2N^3) \Rightarrow \boxed{T(n) = \theta(N^3)}$

P3b) $T(N) = 4T(N/4) + d$, $d > 0$. Assume $T(0) = 1, T(1) = 1$

$$a=4 \left\{ \begin{array}{l} n^{\log_b a} = n^{\log_4 4} = n \\ b=4 \end{array} \right.$$

$$f(n) = d = \text{constant} = O(n)$$

$$\text{let } \varepsilon = 0.5$$

$$\text{constant } d = O(n^{1-\varepsilon})$$

$$\text{constant } d = O(n^{1-0.5}) = O(n^{0.5})$$

$$\text{case 1} \Rightarrow T(n) = \Theta(n)$$

P3c) $T(N) = 6T(N/6) + 5N$. Assume $T(0) = 1$ and $T(1) = 1$

$$a=6 \left\{ \begin{array}{l} n^{\log_b a} = n^{\log_6 6} = n \\ b=6 \end{array} \right.$$

$$f(n) = 5n = O(n)$$

$$\text{case 2} \Rightarrow T(n) = \Theta(n \lg n)$$

P3d) $T(n) = 8T(n/2) + c n^3 \lg n$. Assume $T(0) = 1$ and $T(1) = 1$

$$a=8 \left\{ \begin{array}{l} n^{\log_b a} = n^{\log_2 8} = n^3 \\ b=2 \end{array} \right.$$

$$f(n) = c n^3 \lg n = \Omega(n^3)$$

$$\text{let } \varepsilon = 0.1$$

$$c n^3 \lg n = \Omega(n^{3+0.1})$$

$$c n^3 \lg n = \Omega(n^{3.1})$$

$$a f(n/b) \leq k f(n)$$

$$8 c \left(\frac{n}{2}\right)^3 \lg \frac{n}{2} \leq k c n^3 \lg n$$

$$\frac{8}{8} c n^3 \lg \frac{n}{2} \leq \frac{8}{8} c n^3 \lg n \leq k c n^3 \lg n$$

$$k = 0.5$$

$$\text{case 3} \Rightarrow T(n) = \Theta(n^3 \lg n)$$

P4) Inadmissible equations.

* $T(n) = 2^n T\left(\frac{n}{2}\right) + n^n$

a is not a constant; the number of subproblems should be fixed.

* $T(n) = 2T\left(\frac{n}{2}\right) + \frac{n}{\log n}$

non polynomial difference $f(n)$ and $n^{\log_b a}$.

the difference between $f(n)$ and $n^{\log_b a}$ can be expressed with the ratio $\frac{f(n)}{n^{\log_b a}} = \frac{n/\log n}{n^{\log_2 2}} = \frac{n}{n \log n} = \frac{1}{\log n}$.

It is clear that $\frac{1}{\log n} < n^\epsilon$ for any constant $\epsilon > 0$.

Therefore, the difference is not polynomial and the basic form of the Master theorem does not apply.

* $T(n) = 0.5T\left(\frac{n}{2}\right) + n$

$a < 1$ cannot have less than one sub problem

* $T(n) = 64T\left(\frac{n}{8}\right) - n^2 \log n$

$f(n)$, which is the combination time, is not positive.

* $T(n) = T\left(\frac{n}{2}\right) + n(2 - \cos n)$

case 3 but regularity violation.

P5) Show $T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 = \Theta(N^3)$, $T(0) = 1, T(1) = 1$

Assume

$$T(N) = 5T(\lfloor N/4 \rfloor) + 2N^3 = \Theta(N^3)$$

Using inductive step to show Θ with definition of with constants
 $f(n) = \Theta(g(n))$ iff $\exists c, N_0$ s.t. $f(n) \leq c g(n), \forall n > N_0$

$$f(n) = T(N), g(n) = N^3$$

$$\text{To show: } T(N) \leq c N^3$$

Base case:

$$N=1, T(1) = 1 \leq c(1)^3$$

$$T(1) = 5 + \left(\left\lfloor \frac{1}{4} \right\rfloor\right) + 2(1)^3 = 5T(0) + 2 = 5 + 2 = 7 \leq c$$

$N_0 > 0$

$$N=2, T(2) = 5T\left(\left\lfloor \frac{2}{4} \right\rfloor\right) + 2(2)^3 = 5T(0) + 16 = 5 + 16 = 21$$

$$T(2) = 21 \leq c(2)^3 \Rightarrow c \geq 21/8$$

$$T(2) \leq 8c \Rightarrow c \geq 21/8, c \geq 2.625, c = 3$$

$$N=3, T(3) = 5T\left(\left\lfloor \frac{3}{4} \right\rfloor\right) + 2(3)^3 = 5T(0) + 54 = 59$$

$$T(3) = 59 \leq c(3)^3 \Rightarrow c \geq 59/27, c \geq 2.18, c = 3$$

$$N=4, T(4) = 5T\left(\left\lfloor \frac{4}{4} \right\rfloor\right) + 2(4)^3 = 5T(1) + 128 = 5 + 128 = 133$$

$$T(4) = 133 \leq c(4)^3 \Rightarrow c \geq 133/64, c \geq 2.07, c = 3$$

$$N=5, T(5) = 5T\left(\left\lfloor \frac{5}{4} \right\rfloor\right) + 2(5)^3 = 5T(1) + 250 = 255$$

$$T(5) = 255 \leq c(5)^3 \Rightarrow c \geq 255/125, c \geq 2.04, c = 3$$

$$N=6, T(6) = 5T\left(\left\lfloor \frac{6}{4} \right\rfloor\right) + 2(6)^3 = 5T(1) + 432 = 5 + 432 = 437$$

$$T(6) = 437 \leq c(6)^3 \Rightarrow c \geq 437/216, c \geq 2.02, c = 3$$

$$N=7, T(7) = 5T\left(\left\lfloor \frac{7}{4} \right\rfloor\right) + 2(7)^3 = 5T(1) + 686 = 691$$

$$T(7) = 691 \leq c(7)^3, c \geq 2.01, c = 3$$

$$N=8, T(8) = 5T\left(\left\lfloor \frac{8}{4} \right\rfloor\right) + 2(8)^3 = 5T(2) + 1024 = 1024 + 105 = 1129$$

$$T(8) = 1129 \leq c(8)^3 \Rightarrow c \geq 2.22, c = 3$$

and so on.

Assume $T(k) \leq ck^3$, $\forall k < n$

$$T(n) = 5T\left(\frac{\lfloor n \rfloor}{4}\right) + 2n^3 \rightarrow (1)$$

$$T\left(\frac{\lfloor n \rfloor}{4}\right) \leq c\left(\frac{\lfloor n \rfloor}{4}\right)^3 \rightarrow (2)$$

substituting (2) in (1)

$$T(n) = 5T\left(\frac{\lfloor n \rfloor}{4}\right) + 2n^3 \leq 5\left[c\left(\frac{\lfloor n \rfloor}{4}\right)^3\right] + 2n^3$$

$$\leq 5c \frac{n^3}{4^3} + 2n^3$$

$$\leq n^3 \left[\frac{5c}{4^3} + 2 \right]$$

$$cn^3 \leq n^3 \left[\frac{5c}{4^3} + 2 \right]$$

$$0.02c + 2 \leq c$$

$$2 \leq c - 0.02c$$

$$2 \leq 0.98c$$

$$c \geq 2.0408$$

$$\boxed{c=3}$$

$$n_0 = 1, c = 3$$

Assume

$$T(n) = 5T(\lfloor n/4 \rfloor) + 2n^3 = \Omega(n^3)$$

Using inductive method to show Ω with definition with constants

$$f(n) = \Omega(g(n)) \text{ iff } \exists c, n_0 \text{ s.t. } cg(n) \leq f(n), \forall n \geq n_0$$

$$f(n) = T(n) \quad g(n) = n^3$$

$$\text{To show: } cn^3 \leq T(n) \quad \text{or} \quad T(n) \geq cn^3$$

$$N=1, T(1) = 5T(\lfloor 1/4 \rfloor) + 2(1)^3 = 5T(0) + 2 = 5 + 2 = 7 \geq c(1)^3$$

$$T(1) = 7 \geq c(1)^3 \Rightarrow T(n) = 7 \geq c$$

$$N=2, T(2) = 5T(\lfloor \frac{2}{4} \rfloor) + 2(2^3) = 5T(0) + 16 = 5 + 16 = 21$$

$$T(2) = 21 \geq c(2^3) \Rightarrow T(2) = 21 \geq 8c \Rightarrow c \leq 2.625, c=2$$

$$N=3, T(3) = 5T(\lfloor \frac{3}{4} \rfloor) + 2(3)^3 = 5T(0) + 54 = 59$$

$$T(3) = 59 \geq c(3)^3 \Rightarrow c \leq 2.18 \Rightarrow c=2$$

$$N=4, T(4) = 5T(\lfloor \frac{4}{4} \rfloor) + 2(4)^3 = 5T(1) + 128 = 5 + 128 = 133$$

$$T(4) = 133 \geq c(4)^3 \Rightarrow c \leq 2.07 \Rightarrow c=2$$

$$N=5, T(5) = 5T(\lfloor \frac{5}{4} \rfloor) + 2(5)^3 = 5T(1) + 250 = 5 + 250 = 255$$

$$T(5) = 255 \geq c(5)^3 \Rightarrow c \leq 255/125 \Rightarrow c \leq 2.04 \Rightarrow c=2$$

$$N=6, T(6) = 5T(\lfloor \frac{6}{4} \rfloor) + 2(6)^3 = 5T(1) + 432 = 5 + 432 = 437$$

$$T(6) = 437 \leq c(6)^3 \Rightarrow c \leq 437/216 \Rightarrow c \leq 2.02 \Rightarrow c=2$$

$$N=7, T(7) = 5T(\lfloor \frac{7}{4} \rfloor) + 2(7)^3 = 5T(1) + 686 = 691$$

$$T(7) = 691 \geq c(7)^3 \Rightarrow c \leq 691/343 \Rightarrow c \leq 2.01 \Rightarrow c=2$$

$$N=8, T(8) = 5T(\lfloor \frac{8}{4} \rfloor) + 2(8)^3 = 5T(2) + 1024 = 105 + 1024 = 1129$$

$$T(8) = 1129 \geq c(8)^3 \Rightarrow c \leq 2.2 \Rightarrow c=2$$

and so on.

Assume $T(k) \geq ck^3$, $\forall k < n$

$$T(n) = 5T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + 2n^3 \rightarrow (1)$$

$$T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) \geq c\left(\left\lfloor \frac{n}{4} \right\rfloor\right)^3 \rightarrow (2')$$

Substituting (2') in (1)

$$T(n) = 5T\left(\left\lfloor \frac{n}{4} \right\rfloor\right) + 2n^3 \geq 5\left[c\left(\left\lfloor \frac{n}{4} \right\rfloor\right)^3\right] + 2n^3$$

$$\geq 5c \frac{n^3}{4^3} + 2n^3$$

$$\geq n^3 \left[\frac{5c}{4^3} + 2 \right]$$

$$cn^3 \geq n^3 [0.02c + 2]$$

$$0.02c + 2 \geq c$$

$$2 \geq c - 0.02$$

$$c \leq 2.0408$$

$$\boxed{c = 2}$$

$$n_0 = 1, c = 2$$