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Homework - 4

10.31) $\bar{x} = 4$, $n = 1$

Number of accidents has Poisson distribution

$f(x/\theta)$ has Poisson (θ)

Prior distribution of θ is Gamma $(5, 1)$

$\pi(\theta)$ is Gamma$(5,1)$ $\Rightarrow \alpha = 5, \ \lambda = 1$

Baye's estimator of θ under squared error loss = ?

Posterior risk = ?

Gamma family is conjugate to the Poisson model.

Posterior is

$$\pi(\theta/x) = \text{Gamma}(\alpha + n\bar{x}\ ;\ \lambda + n)$$

$$= \text{Gamma}(5 + n\bar{x}\ ,\ 1 + n) \qquad \Rightarrow n = 1, \bar{x} = 4$$

$$= \text{Gamma}(5 + (1 \times 4)\ ;\ 1 + 1)$$

$$= \text{Gamma}(9, 2)$$

Baye's estimator of θ under squared error loss is

$$\hat{\theta}_B = E(\theta/x) = \frac{9}{2} = \underline{\underline{4.5}}$$

Posterior risk is $f(\hat{\theta}_B) = Var(\theta/x)$

$$= \frac{9}{2^2} = \frac{9}{4} = \underline{2.25}$$

10.36)

a) Prior experiment mean should be between 5.0 and 6.0 with probability 0.95

Conjugate prior distribution = ?

Let's assume θ follows Normal (μ, τ)

$Z_{0.025} = 1.96$

Confidence interval here is

$$\mu \pm Z_{\alpha/2}\tau = [5, 6] \Rightarrow \mu \pm Z_{0.025}\tau = [5, 6]$$

$$\mu = \frac{5+6}{2} = 5.5 \quad , \quad \tau = \frac{6-5}{2(1.96)} = 0.255$$

Hence, $\pi(\theta) = \text{Normal}(5.5, 0.255)$

b) Baye's estimator of μ = ? Posterior risk = ?

Given, $\bar{X} = 6.5$, $n = 6$, $\sigma = 2.2$

Posterior $\pi(\theta | x)$ for Normal (μ, τ) is given by

$$\text{Normal}\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}} , \frac{1}{\sqrt{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}} \right) = \text{Normal}(\mu_x, \tau)$$

$$\mu_x = \frac{\cancel{\text{total}}\ \frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}$$

$$= \frac{\frac{6 \times 6.5}{2.2^2} + \frac{5.5}{0.255^2}}{\frac{6}{2.2^2} + \frac{1}{0.255^2}} = 5.575$$

$$\sqrt{\tau^2} = \frac{1}{\sqrt{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}} = \frac{1}{\sqrt{\frac{6}{2.2^2} + \frac{1}{0.255^2}}} \doteq 0.06$$

Posterior distribution $\pi(\theta/x) = \text{Normal}(5.575, 0.245$

Baye's estimator $\hat{\theta}_B = \cancel{E(\theta/x)} = \mu_x = 5.575$

$P(\hat{\theta}_B) = \tau^2 = 0.06$

c) 95% HPD credible set for μ is

$$\mu_x \pm z_{\alpha/2}\, \tau_x = 5.575 \pm (1.96)(0.245)$$

$$= 5.575 \pm 0.480$$

$$= [5.095, 6.055]$$

10.37) Fair $\pi(\theta) = 0.99$

Biased $\pi(\theta) = 0.01$

Fair $f(x/\theta) = f\left(\frac{x}{\theta = \text{fair}}\right)$ is binomial distribution with

$n = 10, p = 0.5$

Biased $f(x/\theta) = f\left(\frac{x}{\theta = \text{biased}}\right)$ is uniform distribution

$= \text{Uniform}(0, 1)$

$$\pi(\theta = \text{fair}/x) = \frac{f(x/\theta = \text{fair})\ \pi(\text{fair})}{m(x)}$$

$$m(x) = \int_0^1 f(x/\theta)\ \pi(\theta)\ d\theta$$

$$= \int_0^1 \theta^{10}\ (0.01)\ d\theta$$

$$= 0.01 \left[\frac{\theta^{11}}{11}\right]_0^1$$

$$m(x) = 0.01 \left[\frac{1}{11}\right]$$

$$\pi(\theta = \text{fair}/x) = \frac{0.5^{10} \times 0.99}{0.5^{10} \times 0.99 + \frac{0.01}{11}} = 0.5154$$

$$\pi(\theta = \text{biased}/x) = 1 - 0.5154 = 0.4846$$