**Handout**

**Confidence Intervals:**

| | |
|---|---|
| **Confidence interval, Normal distribution** | If parameter $\theta$ has an unbiased, Normally distributed estimator $\hat{\theta}$, then $\hat{\theta} \pm z_{\alpha/2} \cdot \sigma(\hat{\theta}) = \left[\hat{\theta} - z_{\alpha/2} \cdot \sigma(\hat{\theta}),\ \hat{\theta} + z_{\alpha/2} \cdot \sigma(\hat{\theta})\right]$ is a $(1-\alpha)100\%$ confidence interval for $\theta$. If the distribution of $\hat{\theta}$ is *approximately* Normal, we get an *approximately* $(1-\alpha)100\%$ confidence interval. |

If we do not know the population std. dev. but we know the n is large, then σ(θ) can be replaced by s(θ)

| | |
|---|---|
| **Confidence interval for the difference of means; known standard deviations** | $\bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$ |

| | |
|---|---|
| **Confidence interval for a population proportion** | $\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$ |

| | |
|---|---|
| **Confidence interval for the difference of proportions** | $\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$ |

Common Z values:

$$z_{0.10} = 1.282, \qquad z_{0.05} = 1.645, \qquad z_{0.025} = 1.960$$
$$z_{0.01} = 2.326, \qquad z_{0.005} = 2.576.$$

Can also obtained from Z-table or from T-table with v = ∞

If n is small,

| | |
|---|---|
| **Confidence interval for the mean; $\sigma$ is unknown** | $\bar{X} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$ where $t_{\alpha/2}$ is a critical value from T-distribution with $n-1$ degrees of freedom |

**Confidence interval for the difference of means; equal, unknown standard deviations**

$$\bar{X} - \bar{Y} \pm t_{\alpha/2}\, s_p \sqrt{\frac{1}{n} + \frac{1}{m}}$$

where $s_p$ is the *pooled standard deviation*, a root of the pooled variance in (9.11)

and $t_{\alpha/2}$ is a critical value from T-distribution with $(n + m - 2)$ degrees of freedom

Pooled Std. Deviation:

$$s_p^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2}{n + m - 2} = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{n + m - 2}. \tag{9.11}$$

**Confidence interval for the difference of means; unequal, unknown standard deviations**

$$\bar{X} - \bar{Y} \pm t_{\alpha/2} \sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}$$

where $t_{\alpha/2}$ is a critical value from T-distribution with $\nu$ degrees of freedom given by formula (9.12)

Statterthwaite approximation of degrees of freedom (9.12):

$$\nu = \frac{\left(\frac{s_X^2}{n} + \frac{s_Y^2}{m}\right)^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}}$$

**Hypothesis Testing (Z-tests)**:

Right Tailed Test ($H_A$: $\theta > \theta_0$): $\begin{cases} \text{reject } H_0 & \text{if} \quad Z \geq z_\alpha \\ \text{accept } H_0 & \text{if} \quad Z < z_\alpha \end{cases}$

Left Tailed Test ($H_A$: $\theta < \theta_0$): $\begin{cases} \text{reject } H_0 & \text{if} \quad Z \leq -z_\alpha \\ \text{accept } H_0 & \text{if} \quad Z > -z_\alpha \end{cases}$

Two Tailed Test ($H_A$: $\theta \neq \theta_0$): $\begin{cases} \text{reject } H_0 & \text{if} \quad |Z| \geq z_{\alpha/2} \\ \text{accept } H_0 & \text{if} \quad |Z| < z_{\alpha/2} \end{cases}$

**Hypothesis Testing (t-tests):**

For a **right-tail alternative**,

$$\begin{cases} \text{reject } H_0 & \text{if} \quad t \geq t_\alpha \\ \text{accept } H_0 & \text{if} \quad t < t_\alpha \end{cases}$$

For a **left-tail alternative**,

$$\begin{cases} \text{reject } H_0 & \text{if} \quad t \leq -t_\alpha \\ \text{accept } H_0 & \text{if} \quad t > -t_\alpha \end{cases}$$

For a **two-sided alternative**,

$$\begin{cases} \text{reject } H_0 & \text{if} \quad |t| \geq t_{\alpha/2} \\ \text{accept } H_0 & \text{if} \quad |t| < t_{\alpha/2} \end{cases}$$

## Summary of Z - Tests

| Null hypothesis $H_0$ | Parameter, estimator $\theta$, $\hat{\theta}$ | If $H_0$ is true: $\mathbf{E}(\hat{\theta})$ | If $H_0$ is true: $\mathrm{Var}(\hat{\theta})$ | Test statistic $Z = \dfrac{\hat{\theta} - \theta_0}{\sqrt{\mathrm{Var}(\hat{\theta})}}$ |
|---|---|---|---|---|
| One-sample Z-tests for means and proportions, based on a sample of size $n$ | | | | |
| $\mu = \mu_0$ | $\mu$, $\bar{X}$ | $\mu_0$ | $\dfrac{\sigma^2}{n}$ | $\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$ |
| $p = p_0$ | $p$, $\hat{p}$ | $p_0$ | $\dfrac{p_0(1-p_0)}{n}$ | $\dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$ |
| Two-sample Z-tests comparing means and proportions of two populations, based on independent samples of size $n$ and $m$ | | | | |
| $\mu_X - \mu_Y = D$ | $\mu_X - \mu_Y$, $\bar{X} - \bar{Y}$ | $D$ | $\dfrac{\sigma_X^2}{n} + \dfrac{\sigma_Y^2}{m}$ | $\dfrac{\bar{X} - \bar{Y} - D}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$ |
| $p_1 - p_2 = D$ | $p_1 - p_2$, $\hat{p}_1 - \hat{p}_2$ | $D$ | $\dfrac{p_1(1-p_1)}{n} + \dfrac{p_2(1-p_2)}{m}$ | $\dfrac{\hat{p}_1 - \hat{p}_2 - D}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}_2(1-\hat{p}_2)}{m}}}$ |
| $p_1 = p_2$ | $p_1 - p_2$, $\hat{p}_1 - \hat{p}_2$ | $0$ | $p(1-p)\left(\dfrac{1}{n} + \dfrac{1}{m}\right)$, where $p = p_1 = p_2$ | $\dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}}$ where $\hat{p} = \dfrac{n\hat{p}_1 + m\hat{p}_2}{n+m}$ |

## Summary of t-tests

| Hypothesis $H_0$ | Conditions | Test statistic $t$ | Degrees of freedom |
|---|---|---|---|
| $\mu = \mu_0$ | Sample size $n$; unknown $\sigma$ | $t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$ | $n - 1$ |
| $\mu_X - \mu_Y = D$ | Sample sizes $n$, $m$; unknown but equal standard deviations, $\sigma_X = \sigma_Y$ | $t = \dfrac{\bar{X} - \bar{Y} - D}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$ | $n + m - 2$ |
| $\mu_X - \mu_Y = D$ | Sample sizes $n$, $m$; unknown, unequal standard deviations, $\sigma_X \neq \sigma_Y$ | $t = \dfrac{\bar{X} - \bar{Y} - D}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}$ | Satterthwaite approximation, formula (9.12) |

**P Values (Reject $H_0$ if P < 0.01, Accept $H_0$ if P > 0.1, Not enough evidence otherwise):**

P values for Z tests:

| Hypothesis $H_0$ | Alternative $H_A$ | P-value | Computation |
|---|---|---|---|
| $\theta = \theta_0$ | right-tail $\theta > \theta_0$ | $\boldsymbol{P}\{Z \geq Z_{\text{obs}}\}$ | $1 - \Phi(Z_{\text{obs}})$ |
| | left-tail $\theta < \theta_0$ | $\boldsymbol{P}\{Z \leq Z_{\text{obs}}\}$ | $\Phi(Z_{\text{obs}})$ |
| | two-sided $\theta \neq \theta_0$ | $\boldsymbol{P}\{\lvert Z\rvert \geq \lvert Z_{\text{obs}}\rvert\}$ | $2(1 - \Phi(\lvert Z_{\text{obs}}\rvert))$ |

P values for t tests:

| Hypothesis $H_0$ | Alternative $H_A$ | P-value | Computation |
|---|---|---|---|
| $\theta = \theta_0$ | right-tail $\theta > \theta_0$ | $\boldsymbol{P}\{t \geq t_{\text{obs}}\}$ | $1 - F_\nu(t_{\text{obs}})$ |
| | left-tail $\theta < \theta_0$ | $\boldsymbol{P}\{t \leq t_{\text{obs}}\}$ | $F_\nu(t_{\text{obs}})$ |
| | two-sided $\theta \neq \theta_0$ | $\boldsymbol{P}\{\lvert t\rvert \geq \lvert t_{\text{obs}}\rvert\}$ | $2(1 - F_\nu(\lvert t_{\text{obs}}\rvert))$ |

**Confidence Intervals (variance):**

**Confidence interval for the variance**

$$\boxed{\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}}, \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right]}$$

**Confidence interval for the standard deviation**

$$\boxed{\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}}, \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}\right]}$$

**Hypothesis tests for variance (can also be used for Std Dev by conv question to variance):**

| Null Hypothesis | Alternative Hypothesis | Test statistic | Rejection region | P-value |
|---|---|---|---|---|
| $\sigma^2 = \sigma_0^2$ | $\sigma^2 > \sigma_0^2$ | $\frac{(n-1)s^2}{\sigma_0^2}$ | $\chi^2_{\text{obs}} > \chi^2_\alpha$ | $\boldsymbol{P}\left\{\chi^2 \geq \chi^2_{\text{obs}}\right\}$ |
| | $\sigma^2 < \sigma_0^2$ | | $\chi^2_{\text{obs}} < \chi^2_\alpha$ | $\boldsymbol{P}\left\{\chi^2 \leq \chi^2_{\text{obs}}\right\}$ |
| | $\sigma^2 \neq \sigma_0^2$ | | $\chi^2_{\text{obs}} \geq \chi^2_{\alpha/2}$ or $\chi^2_{\text{obs}} \leq \chi^2_{1-\alpha/2}$ | $2\min\left(\boldsymbol{P}\left\{\chi^2 \geq \chi^2_{\text{obs}}\right\}, \boldsymbol{P}\left\{\chi^2 \leq \chi^2_{\text{obs}}\right\}\right)$ |

**Testing ratio of Variances (can also be used for Std Dev by conv question to variance):**

| Null Hypothesis $H_0 : \frac{\sigma_X^2}{\sigma_Y^2} = \theta_0$ | | Test statistic $F_{\text{obs}} = \frac{s_X^2}{s_Y^2} / \theta_0$ |
|---|---|---|
| Alternative Hypothesis | Rejection region | P-value<br>Use F$(n-1, m-1)$ distribution |
| $\frac{\sigma_X^2}{\sigma_Y^2} > \theta_0$ | $F_{\text{obs}} \geq F_\alpha(n-1, m-1)$ | $\boldsymbol{P}\{F \geq F_{\text{obs}}\}$ |
| $\frac{\sigma_X^2}{\sigma_Y^2} < \theta_0$ | $F_{\text{obs}} \leq F_\alpha(n-1, m-1)$ | $\boldsymbol{P}\{F \leq F_{\text{obs}}\}$ |
| $\frac{\sigma_X^2}{\sigma_Y^2} \neq \theta_0$ | $F_{\text{obs}} \geq F_{\alpha/2}(n-1, m-1)$ or<br>$F_{\text{obs}} < 1/F_{\alpha/2}(m-1, n-1)$ | $2 \min\left(\boldsymbol{P}\{F \geq F_{\text{obs}}\}, \boldsymbol{P}\{F \leq F_{\text{obs}}\}\right)$ |