Bernoulli distribution

- Used to model experiments with a binary outcome (yes/no, pass/fail, true/false)
  - Called Bernoulli Trials
- Random variable can take values 0 (fail) or 1 (pass)
- p = probability of success

$$P(x) = \begin{cases} 1-p & if \quad x=0 \\ p & if \quad x=1 \end{cases}$$
$$E(X) = p$$
$$Var(X) = p(1-p)$$

Binomial Distribution

- It is used to model number of success in a sequence of **independent** Bernoulli trials
- Models the probability of x successes in n trials
- p = probability of success; n = number of trials

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$
$$E(X) = np$$
$$Var(X) = np(1-p)$$

Geometric Distribution

- It is used to model number trials needed to achieve the first success in a sequence of **independent** Bernoulli trials
- Models the probability of x^{th} successive trial resulting in a success
- p = probability of success

$$P(x) = (1-p)^{x-1}p$$
$$E(X) = \frac{1}{p}$$
$$Var(X) = \frac{1-p}{p^2}$$

Negative Binomial Distribution

- It is used to model number of trials needed to obtain k success in a sequence of **independent** Bernoulli trials
- Models the probability of x^{th} successive trial resulting in the k^{th} success
- p = probability of success; k = number of success

$$P(x) = \binom{x-1}{k-1}(1-p)^{x-k}p^k$$
$$E(X) = \frac{k}{p}$$
$$Var(X) = \frac{k(1-p)}{p^2}$$

Side Note: Calculating Neg. Binomial in Practice

- If X follows Negative Binomial(k, p)
  - P(X = x) = prob of needing x trials for k success = prob of kth trial being success * prob of getting k-1 success in x-1 trials = p * P(Y = k-1)
    - Where Y follows Binomial(x-1, p)
  - P(X ≥ x) = prob of needing ≥ x trials for k success = prob of x-1 trials not having ≤ k-1 success = P(Y ≤ k-1)
    - Where Y follows Binomial(x-1, p)

Poisson Distribution

- It is used to model number rare events occurring within a fixed period of time
- Models the probability of x rare events occurring in a fixed period of time if we know the frequency at which the events occur on average
- λ = frequency (average number of events in a fixed time period)

$$P(x) = e^{-\lambda}\frac{\lambda^x}{x!}$$
$$E(X) = \lambda$$
$$Var(X) = \lambda$$

Side Note: Poisson Approx. of Binomial

- If the number of trials is large and the probability of success is low, then we can use Poisson Distribution to approximate the Binomial Distribution
  - Also works if probability of failure is very low
- $\lim_{\substack{n\to\infty \\ p\to 0 \\ np\to\lambda}} \binom{n}{x} p^x (1-p)^x = e^{-\lambda}\frac{\lambda^x}{x!}$
- Can use this approximation if n >= 30 and p <= 0.05

Uniform Distribution

- Used to model scenarios where outcome lies within a given interval (a,b) and all outcomes are equally likely.
- If interval is (0,1) it is called standard uniform distribution

$$f(x) = \frac{1}{b-a}, a < x < b$$
$$E(X) = \frac{a+b}{2}$$
$$Var(X) = \frac{(b-a)^2}{12}$$

Exponential Distribution

- Used to model time (or separation) between events occurring at frequency λ (rate at which the events occur)

$$f(x) = \lambda e^{-\lambda x}, x > 0$$
$$E(X) = \frac{1}{\lambda}$$
$$Var(X) = \frac{1}{\lambda^2}$$

Gamma Distribution

- Used to model total time of multistage processes with α steps (shape parameter) where time of each step can be modeled as a Exponential distribution with frequency λ.

$$f(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \quad if\ \alpha > 0\ x > 0$$

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx \quad if\ \alpha > 0$$

$$\text{also } \Gamma(\alpha) = (n-1)! \quad if\ \alpha\ is\ a\ positve\ integer$$

$$E(X) = \frac{\alpha}{\lambda}$$

$$Var(X) = \frac{\alpha}{\lambda^2}$$

- Please note that Gamma(1, λ) = Exponential(λ)

Side-Note: Gamma-Poisson Formula

- Can be used to simplify calculation of probabilities of RV T with Gamma Distribution.

$$\{T > t\} = \{X < \alpha\}$$

- Where is T has Gamma distribution with parameters α (number of events) and λ (frequency of each event).
- X models the number of events that occurs before time t. It has Poisson distribution with parameter λt.

  So,

$$P\{T > t\} = P\{X < \alpha\}$$
$$P\{T \leq t\} = P\{X \geq \alpha\}$$

  Where T has Gamma(α, λ) distribution and X has Poisson(λt) distribution

Normal Distribution

- Used to model a large number of scenarios
  - Sums, averages or errors: Mainly due to CLT
  - Naturally occurring phenomena
- Allows you to model a scenario on the basis of expectation μ (location parameter) and standard deviation σ (scale parameter)

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\left(\frac{-(x-\mu)^2}{2\sigma^2}\right)} - \infty < x < \infty$$
$$E(X) = \mu$$
$$Var(X) = \sigma^2$$

Central Limit Theorem

- If you a Random variable that is expressed as a sum of large number of independent random variables (usually >= 30), then you can use this theorem to model them as a Normal Distribution.

Theorem 1 (Central Limit Theorem) *Let $X_1, X_2, \ldots$ be independent random variables with the same expectation $\mu = \mathbf{E}(X_i)$ and the same standard deviation $\sigma = \mathrm{Std}(X_i)$, and let*

$$S_n = \sum_{i=1}^{n} X_i = X_1 + \ldots + X_n.$$

As $n \to \infty$, the standardized sum

$$Z_n = \frac{S_n - \mathbf{E}(S_n)}{\mathrm{Std}(S_n)} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$

converges in distribution to a Standard Normal random variable, that is,

$$F_{Z_n}(z) = \boldsymbol{P}\left\{\frac{S_n - n\mu}{\sigma\sqrt{n}} \le z\right\} \to \Phi(z)$$

for all z.

Side-Note: Using Normal to Approx. Binomial

- A binomial distribution is a sum of n Bernoulli trials. So if n is large (>=30) but p is not small enough (or large enough) to use Poisson approximation (0.05 <= p <= 0.95) then we can model the binomial as a sum of Bernoulli distributions with mean p and variance p(1 - p).
- So by Central Limit Theorem,

$$\mathrm{Binomial}(n, p) \approx Normal\left(\mu = np, \sigma = \sqrt{np(1-p)}\right)$$

- This normal distribution can be calculated by converting it to a standard normal distribution