Probability

<u>NOTATION</u>

Ω	$=$	sample space
$\varnothing$	$=$	empty event
$\boldsymbol{P}\{E\}$	$=$	probability of event E

DEFINITION 2.10

Assume a sample space Ω and a sigma-algebra of events $\mathfrak{M}$ on it. **Probability**

$$\boldsymbol{P} \; : \; \mathfrak{M} \rightarrow [0,1]$$

is a function of events with the domain $\mathfrak{M}$ and the range $[0,1]$ that satisfies the following two conditions,

(Unit measure) The sample space has unit probability, $\boldsymbol{P}(\Omega) = 1$.

(Sigma-additivity) For any finite or countable collection of *mutually exclusive* events $E_1, E_2, \ldots \in \mathfrak{M}$,

$$\boldsymbol{P}\{E_1 \cup E_2 \cup \ldots\} = \boldsymbol{P}(E_1) + \boldsymbol{P}(E_2) + \ldots.$$

Assuming all outcomes of an event are equally likely

$$\boldsymbol{P}\{E\} = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}} = \frac{\mathcal{N}_F}{\mathcal{N}_T}$$

Counting Outcomes:

Permutations with replacement

$$P_r(n,k) = \overbrace{n \cdot n \cdot \ldots \cdot n}^{k \text{ terms}} = n^k$$

Permutations without replacement

$$P(n,k) = \overbrace{n(n-1)(n-2) \cdot \ldots \cdot (n-k+1)}^{k \text{ terms}} = \frac{n!}{(n-k)!}$$

Combinations without replacement

$$C(n,k) = \binom{n}{k} = \frac{P(n,k)}{P(k,k)} = \frac{n!}{k!(n-k)!}$$

Combinations with replacement

$$C_r(n,k) = \binom{k+n-1}{k} = \frac{(k+n-1)!}{k!(n-1)!}$$

Union of events:

Probability of a union	$\boldsymbol{P}\{A \cup B\} = \boldsymbol{P}\{A\} + \boldsymbol{P}\{B\} - \boldsymbol{P}\{A \cap B\}$ For mutually exclusive events, $\boldsymbol{P}\{A \cup B\} = \boldsymbol{P}\{A\} + \boldsymbol{P}\{B\}$

To scale this up for mutually exclusive events just add their probabilities together. To scale this up for non-mutually exclusive events, use principle of inclusion and exclusion

Intersection of independent events

Independent events	$\boldsymbol{P}\{E_1 \cap \ldots \cap E_n\} = \boldsymbol{P}\{E_1\} \cdot \ldots \cdot \boldsymbol{P}\{E_n\}$

Intersection of dependent events are dealt with by conditional probability.

Intersection, general case	$\boldsymbol{P}\{A \cap B\} = \boldsymbol{P}\{B\}\,\boldsymbol{P}\{A \mid B\}$

Conditional Probability

DEFINITION 2.15

Conditional probability of event A given event B is the probability that A occurs when B is *known to occur.*

Conditional probability	$\boldsymbol{P}\{A \mid B\} = \dfrac{\boldsymbol{P}\{A \cap B\}}{\boldsymbol{P}\{B\}}$

For independent events, P { A | B } = P { A }

Bayes Rule

Bayes Rule	$\boldsymbol{P}\{B \mid A\} = \dfrac{\boldsymbol{P}\{A \mid B\}\,\boldsymbol{P}\{B\}}{\boldsymbol{P}\{A\}}$

Total Probability

Law of Total Probability	$\boldsymbol{P}\{A\} = \sum_{j=1}^{k} \boldsymbol{P}\{A \mid B_j\}\, \boldsymbol{P}\{B_j\}$ In case of two events ($k = 2$), $\boldsymbol{P}\{A\} = \boldsymbol{P}\{A \mid B\}\, \boldsymbol{P}\{B\} + \boldsymbol{P}\{A \mid \overline{B}\}\, \boldsymbol{P}\{\overline{B}\}$
Bayes Rule for two events	$\boldsymbol{P}\{B \mid A\} = \dfrac{\boldsymbol{P}\{A \mid B\}\, \boldsymbol{P}\{B\}}{\boldsymbol{P}\{A \mid B\}\, \boldsymbol{P}\{B\} + \boldsymbol{P}\{A \mid \overline{B}\}\, \boldsymbol{P}\{\overline{B}\}}$