Confidence Intervals and Hypothesis Testing

Confidence Intervals:

Confidence interval, Normal distribution	If parameter θ has an unbiased, Normally distributed estimator $\hat{\theta}$, then $\hat{\theta} \pm z_{\alpha/2} \cdot \sigma(\hat{\theta}) = \left[\hat{\theta} - z_{\alpha/2} \cdot \sigma(\hat{\theta}),\ \hat{\theta} + z_{\alpha/2} \cdot \sigma(\hat{\theta})\right]$ is a $(1-\alpha)100\%$ confidence interval for θ. If the distribution of $\hat{\theta}$ is *approximately* Normal, we get an *approximately* $(1-\alpha)100\%$ confidence interval.

If we do not know the population std. dev. but we know the n is large, then σ(θ) can be replaced by s(θ)

Confidence interval for the difference of means; known standard deviations	$\bar{X} - \bar{Y} \pm z_{\alpha/2}\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}$
Confidence interval for a population proportion	$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
Confidence interval for the difference of proportions	$\hat{p}_1 - \hat{p}_2 \pm z_{\alpha/2}\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} + \frac{\hat{p}_2(1-\hat{p}_2)}{n_2}}$

If n is small,

Confidence interval for the mean; σ is unknown	$\bar{X} \pm t_{\alpha/2}\frac{s}{\sqrt{n}}$ where $t_{\alpha/2}$ is a critical value from T-distribution with $n-1$ degrees of freedom
Confidence interval for the difference of means; equal, unknown standard deviations	$\bar{X} - \bar{Y} \pm t_{\alpha/2}\, s_p \sqrt{\frac{1}{n} + \frac{1}{m}}$ where s_p is the *pooled standard deviation*, a root of the pooled variance in (9.11) and $t_{\alpha/2}$ is a critical value from T-distribution with $(n+m-2)$ degrees of freedom

Pooled Std. Deviation:

$$s_p^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2}{n+m-2} = \frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}. \qquad (9.11)$$

Confidence interval for the difference of means; unequal, unknown standard deviations	$\bar{X} - \bar{Y} \pm t_{\alpha/2}\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}$ where $t_{\alpha/2}$ is a critical value from T-distribution with ν degrees of freedom given by formula (9.12)

Statterthwaite approximation of degrees of freedom (9.12):

$$\nu = \frac{\left(\frac{s_X^2}{n} + \frac{s_Y^2}{m}\right)^2}{\frac{s_X^4}{n^2(n-1)} + \frac{s_Y^4}{m^2(m-1)}}$$

Hypothesis Testing (Z-tests):

Right Tailed Test (H_A: $\theta > \theta_0$): $\begin{cases} \text{reject } H_0 & \text{if} \quad Z \geq z_\alpha \\ \text{accept } H_0 & \text{if} \quad Z < z_\alpha \end{cases}$

Left Tailed Test (H_A: $\theta < \theta_0$): $\begin{cases} \text{reject } H_0 & \text{if} \quad Z \leq -z_\alpha \\ \text{accept } H_0 & \text{if} \quad Z > -z_\alpha \end{cases}$

Two Tailed Test (H_A: $\theta \neq \theta_0$): $\begin{cases} \text{reject } H_0 & \text{if} \quad |Z| \geq z_{\alpha/2} \\ \text{accept } H_0 & \text{if} \quad |Z| < z_{\alpha/2} \end{cases}$

Hypothesis Testing (t-tests):

For a **right-tail alternative**,

$$\begin{cases} \text{reject } H_0 & \text{if} \quad t \geq t_\alpha \\ \text{accept } H_0 & \text{if} \quad t < t_\alpha \end{cases}$$

For a **left-tail alternative**,

$$\begin{cases} \text{reject } H_0 & \text{if} \quad t \leq -t_\alpha \\ \text{accept } H_0 & \text{if} \quad t > -t_\alpha \end{cases}$$

For a **two-sided alternative**,

$$\begin{cases} \text{reject } H_0 & \text{if} \quad |t| \geq t_{\alpha/2} \\ \text{accept } H_0 & \text{if} \quad |t| < t_{\alpha/2} \end{cases}$$

Note: 2 sided tests using Confidence Intervals

A level α Z-test of $H_0 : \theta = \theta_0$ vs $H_A : \theta \neq \theta_0$
accepts the null hypothesis

if and only if

a symmetric $(1-\alpha)100\%$ confidence Z-interval for θ contains θ_0.

Summary of Z - Tests

Null hypothesis H_0	Parameter, estimator θ, $\hat{\theta}$	If H_0 is true: $\mathbf{E}(\hat{\theta})$	If H_0 is true: $\mathrm{Var}(\hat{\theta})$	Test statistic $Z = \dfrac{\hat{\theta} - \theta_0}{\sqrt{\mathrm{Var}(\hat{\theta})}}$
One-sample Z-tests for means and proportions, based on a sample of size n				
$\mu = \mu_0$	μ, $\bar{X}$	μ_0	$\dfrac{\sigma^2}{n}$	$\dfrac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$
$p = p_0$	p, $\hat{p}$	p_0	$\dfrac{p_0(1-p_0)}{n}$	$\dfrac{\hat{p} - p_0}{\sqrt{\frac{p_0(1-p_0)}{n}}}$
Two-sample Z-tests comparing means and proportions of two populations, based on independent samples of size n and m				
$\mu_X - \mu_Y = D$	$\mu_X - \mu_Y$, $\bar{X} - \bar{Y}$	D	$\dfrac{\sigma_X^2}{n} + \dfrac{\sigma_Y^2}{m}$	$\dfrac{\bar{X} - \bar{Y} - D}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}}$
$p_1 - p_2 = D$	$p_1 - p_2$, $\hat{p}_1 - \hat{p}_2$	D	$\dfrac{p_1(1-p_1)}{n} + \dfrac{p_2(1-p_2)}{m}$	$\dfrac{\hat{p}_1 - \hat{p}_2 - D}{\sqrt{\frac{\hat{p}_1(1-\hat{p}_1)}{n} + \frac{\hat{p}_2(1-\hat{p}_2)}{m}}}$
$p_1 = p_2$	$p_1 - p_2$, $\hat{p}_1 - \hat{p}_2$	0	$p(1-p)\left(\dfrac{1}{n} + \dfrac{1}{m}\right)$, where $p = p_1 = p_2$	$\dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\left(\frac{1}{n} + \frac{1}{m}\right)}}$ where $\hat{p} = \dfrac{n\hat{p}_1 + m\hat{p}_2}{n+m}$

Summary of t-tests

Hypothesis H_0	Conditions	Test statistic t	Degrees of freedom
$\mu = \mu_0$	Sample size n; unknown σ	$t = \dfrac{\bar{X} - \mu_0}{s/\sqrt{n}}$	$n - 1$
$\mu_X - \mu_Y = D$	Sample sizes n, m; unknown but equal standard deviations, $\sigma_X = \sigma_Y$	$t = \dfrac{\bar{X} - \bar{Y} - D}{s_p\sqrt{\frac{1}{n} + \frac{1}{m}}}$	$n + m - 2$
$\mu_X - \mu_Y = D$	Sample sizes n, m; unknown, unequal standard deviations, $\sigma_X \neq \sigma_Y$	$t = \dfrac{\bar{X} - \bar{Y} - D}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}$	Satterthwaite approximation, formula (9.12)

P Values (Reject H_0 if P < 0.01, Accept H_0 if P > 0.1, Not enough evidence otherwise):

P values for Z tests:

Hypothesis H_0	Alternative H_A	P-value	Computation
	right-tail $\theta > \theta_0$	$\boldsymbol{P}\{Z \geq Z_{\text{obs}}\}$	$1 - \Phi(Z_{\text{obs}})$
$\theta = \theta_0$	left-tail $\theta < \theta_0$	$\boldsymbol{P}\{Z \leq Z_{\text{obs}}\}$	$\Phi(Z_{\text{obs}})$
	two-sided $\theta \neq \theta_0$	$\boldsymbol{P}\{\lvert Z\rvert \geq \lvert Z_{\text{obs}}\rvert\}$	$2(1 - \Phi(\lvert Z_{\text{obs}}\rvert))$

P values for t tests:

Hypothesis H_0	Alternative H_A	P-value	Computation
	right-tail $\theta > \theta_0$	$\boldsymbol{P}\{t \geq t_{\text{obs}}\}$	$1 - F_\nu(t_{\text{obs}})$
$\theta = \theta_0$	left-tail $\theta < \theta_0$	$\boldsymbol{P}\{t \leq t_{\text{obs}}\}$	$F_\nu(t_{\text{obs}})$
	two-sided $\theta \neq \theta_0$	$\boldsymbol{P}\{\lvert t\rvert \geq \lvert t_{\text{obs}}\rvert\}$	$2(1 - F_\nu(\lvert t_{\text{obs}}\rvert))$

Confidence Intervals (variance):

Confidence interval for the variance

$$\left[\frac{(n-1)s^2}{\chi^2_{\alpha/2}},\ \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}\right]$$

Confidence interval for the standard deviation

$$\left[\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}},\ \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}\right]$$

Hypothesis tests for variance (can also be used for Std Dev by conv question to variance):

Null Hypothesis	Alternative Hypothesis	Test statistic	Rejection region	P-value
	$\sigma^2 > \sigma_0^2$		$\chi^2_{\text{obs}} > \chi^2_\alpha$	$\boldsymbol{P}\left\{\chi^2 \geq \chi^2_{\text{obs}}\right\}$
$\sigma^2 = \sigma_0^2$	$\sigma^2 < \sigma_0^2$	$\dfrac{(n-1)s^2}{\sigma_0^2}$	$\chi^2_{\text{obs}} < \chi^2_\alpha$	$\boldsymbol{P}\left\{\chi^2 \leq \chi^2_{\text{obs}}\right\}$
	$\sigma^2 \neq \sigma_0^2$		$\chi^2_{\text{obs}} \geq \chi^2_{\alpha/2}$ or $\chi^2_{\text{obs}} \leq \chi^2_{1-\alpha/2}$	$2\min\left(\boldsymbol{P}\left\{\chi^2 \geq \chi^2_{\text{obs}}\right\},\ \boldsymbol{P}\left\{\chi^2 \leq \chi^2_{\text{obs}}\right\}\right)$

Testing ratio of Variances (can also be used for Std Dev by conv question to variance):

Null Hypothesis $H_0 : \frac{\sigma_X^2}{\sigma_Y^2} = \theta_0$		Test statistic $F_{\text{obs}} = \frac{s_X^2}{s_Y^2} / \theta_0$
Alternative Hypothesis	Rejection region	P-value Use F$(n-1, m-1)$ distribution
$\frac{\sigma_X^2}{\sigma_Y^2} > \theta_0$	$F_{\text{obs}} \geq F_\alpha(n-1, m-1)$	$\boldsymbol{P}\{F \geq F_{\text{obs}}\}$
$\frac{\sigma_X^2}{\sigma_Y^2} < \theta_0$	$F_{\text{obs}} \leq F_\alpha(n-1, m-1)$	$\boldsymbol{P}\{F \leq F_{\text{obs}}\}$
$\frac{\sigma_X^2}{\sigma_Y^2} \neq \theta_0$	$F_{\text{obs}} \geq F_{\alpha/2}(n-1, m-1)$ or $F_{\text{obs}} < 1/F_{\alpha/2}(m-1, n-1)$	$2\min\left(\boldsymbol{P}\{F \geq F_{\text{obs}}\}, \boldsymbol{P}\{F \leq F_{\text{obs}}\}\right)$