UTA ID: 1001669338 FALL 2019
CSE 5311 - 004 DESIGN & ANALYSIS OF ALGORITHMS
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Problem 2-2 THE CORRECTNESS OF BUBBLE SORT

BUBBLE SORT (A)

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for i = 1 to A.length - 1
    for j = A.length downto i+1
        if A[j] < A[j-1]
            exchange A[j] with A[j-1]
```

a) Let A' denote the output of Bubblesort(A).

To prove Bubblesort is correct, we need to prove that it terminates and that

$$A'[1] \le A'[2] \le \dots \le A'[n],$$

where $n = A.length$. In order to show that Bubblesort actually sorts

we need to prove that output A' contains the same elements as in A but in a sorted order

b) Loop invariant: Since j is the last position index everytime it loops, the smallest element in A will be atmost at index j. This holds true for initialization, maintenance and termination cases.

Initialization: Initially, before the first iteration, the maximum position of an element cannot be beyond j (A.length). Hence, the loop invariant holds good.

Maintenance: In the lines 3-4 of the pseudocode, the smallest element is being moved to the position j-1. Hence, we make sure that the smallest element lies in the range A[1.... j-1]. Hence, the loop invariant holds good.

Termination: In line 3 of the pseudocode, if the condition does not hold good i.e. if $A[j] \geq A[j-1]$, the execution just moves to the next iteration. By this, A[j-1] still has the smallest element. Hence, the loop invariant holds good.

c) Loop invariant: In part (b) when the loop terminates the first element is the smallest and the array is in sorted order. Since the smallest element is moved towards the starting positions of array, we can tell that the subarray A[1.... i-1] contains i-1 sorted elements.

Initialization: Before the first iteration, the value in the array is zero. So all zero elements are sorted already. Array is empty.

Maintenance: After the execution of the inner loop A[i] will be the smallest element of A[i...n]. In the beginning of outer loop, A[1...i-1] consists of elements that are smaller than the elements of A[i...n] in sorted order. After the execution of outer loop, elements in A[1...i] < elements in A[i+1 ... n] in sorted order

Termination: Since outer for loop repeats until the length of array (A.length or n), the array consists of n sorted elements.

Problem 3-2 RELATIVE ASYMPTOTIC GROWTHS

a) LET $f(n) = \log^k n$, $g(n) = n^\varepsilon$

Given $k \geqslant 1$, $\varepsilon > 0$ and $c > 1$.

By limit theorem,

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = c \neq 0$ then $f(n) = \Theta(g(n))$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ or c. then $f(n) = O(g(n))$

If $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$ or c then $f(n) = \Omega(g(n))$

If $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$ then $f(n) = o(g(n))$

If $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$ then $f(n) = \omega(g(n))$

· LET us consider values $k = 1$, $\varepsilon = 2$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log^k n}{n^\varepsilon} = \lim_{n\to\infty} \frac{\log^1 n}{n^2}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\log n}{n^2} = \lim_{n\to\infty} \frac{\frac{1}{n}}{2n} = \lim_{n\to\infty} \frac{1}{2n^2} = 0$$

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$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{\lg^k n}{n^{\varepsilon}} = \lim_{n\to\infty} \left(\frac{\ln_e n}{\ln_e 10}\right)^k \frac{1}{n^{\varepsilon}}.$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{1}{(\ln 10)^k} \lim_{n\to\infty} \frac{(\ln n)^k}{n^{\varepsilon}}$$ (By L'Hopital's rule)

$$= \frac{1}{(\ln 10)^k} \lim_{n\to\infty} \frac{k(\ln^{k-1} n)\,1/n}{\varepsilon\, n^{\varepsilon-1}}$$

$$= \frac{1}{\ln^k 10} \lim_{n\to\infty} \frac{k \ln^{k-1} n}{\varepsilon\, n^{\varepsilon}}$$

$$= \frac{1}{\varepsilon(\ln^k 10)} \lim_{n\to\infty} \frac{k(k-1)(\ln^{k-2} n)\,1/n}{\varepsilon\, n^{\varepsilon-1}}$$

$$= \frac{k(k-1)}{\varepsilon^2(\ln^k 10)} \lim_{n\to\infty} \frac{\ln^{k-2} n}{n^{\varepsilon}}$$

$\vdots$

$$= \frac{k!}{\varepsilon^k(\ln^k 10)} \lim_{n\to\infty} \frac{1}{n^{\varepsilon}} = 0$$

$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $g(n)$ reaches infinity quickly than $f(n)$

$f(n) = O(g(n))$ and $f(n) = o(g(n))$

$f(n) \neq \Theta(g(n))$ and $f(n) \neq \Omega(g(n))$ and $f(n) \neq \omega(g(n))$

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b) LET $f(n) = n^k$, $g(n) = c^n$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^k}{c^n}$$

$$= \lim_{n\to\infty} \frac{k n^{k-1}}{c^n \ln c}$$ (By L'Hopital's rule)

$$= \lim_{n\to\infty} \frac{k(k-1) n^{k-2}}{c^n (\ln c)^2}$$

$$\vdots$$

$$= \lim_{n\to\infty} \frac{k!}{(\ln c)^k c^n}$$

$$= \frac{k!}{(\ln c)^k} \lim_{n\to\infty} \frac{1}{c^n}$$

$$= \frac{k!}{(\ln c)^k}(0) = 0$$

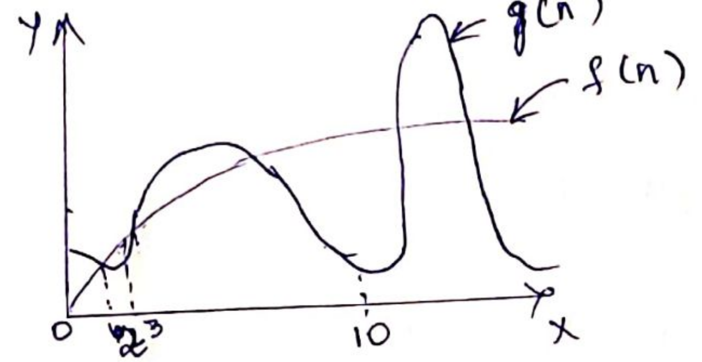
$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$, $g(n)$ reaches infinity quickly than $f(n)$

$f(n) = O(g(n))$ and $f(n) = o(g(n))$

$f(n) \neq \Theta(g(n))$ and $f(n) \neq \Omega(g(n))$ and $f(n) \neq \omega(g(n))$

c) LET $f(n) = \sqrt{n}$, $g(n) = n^{\sin n}$

Since sin is a periodic function, we cannot compare these functions for relative growth



From the graph, we can infer that there is no particular Θ, Ω since sin is a periodic function

d) LET $f(n) = 2^n$, $g(n) = 2^{n/2}$. By L'Hopital's rule

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{2^n}{2^{n/2}} = \lim_{n\to\infty} 2^{n/2} = 2^{\infty} = \infty$$

$f(n)$ reaches ∞ and $g(n)$ is nearing zero

Here $\lim_{n\to\infty} \frac{g(n)}{f(n)} = 0$, Hence

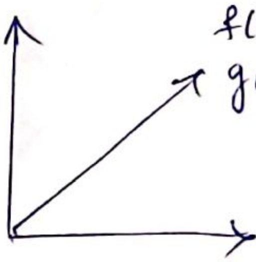
$f(n) = \Omega(g(n))$ and $f(n) = \omega(g(n))$

$f(n) \neq \Theta(g(n))$ and $f(n) \neq o(g(n))$ and $f(n) \neq O(g(n))$

e) LET $f(n) = n^{\lg c}$, $g(n) = c^{\lg n}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{n^{\log c}}{c^{\lg n}} = \lim_{n\to\infty} \frac{\lg n^{\log c}}{\lg c^{\lg n}}$$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{\lg c \lg n}{\lg n \lg c} = 1$$



$$c_1 g(n) \le f(n) \le c_2 g(n)$$

$$c_1 \le \frac{f(n)}{g(n)} \le c_2$$

$f(n) = \theta(g(n))$ and $f(n) = O(g(n))$, $f(n) = \Omega(g(n))$

f) LET $f(n) = \lg(n!)$, $g(n) = \lg(n^n)$

n! cannot be solved by L'Hopital's rule

Hence, by using Stirling's approximation

$f(n) = \lg n! = \lg(1 \cdot 2 \cdot \dots n)$, $g(n) = n \lg n$

$= \lg 1 + \lg 2 + \dots \lg n$

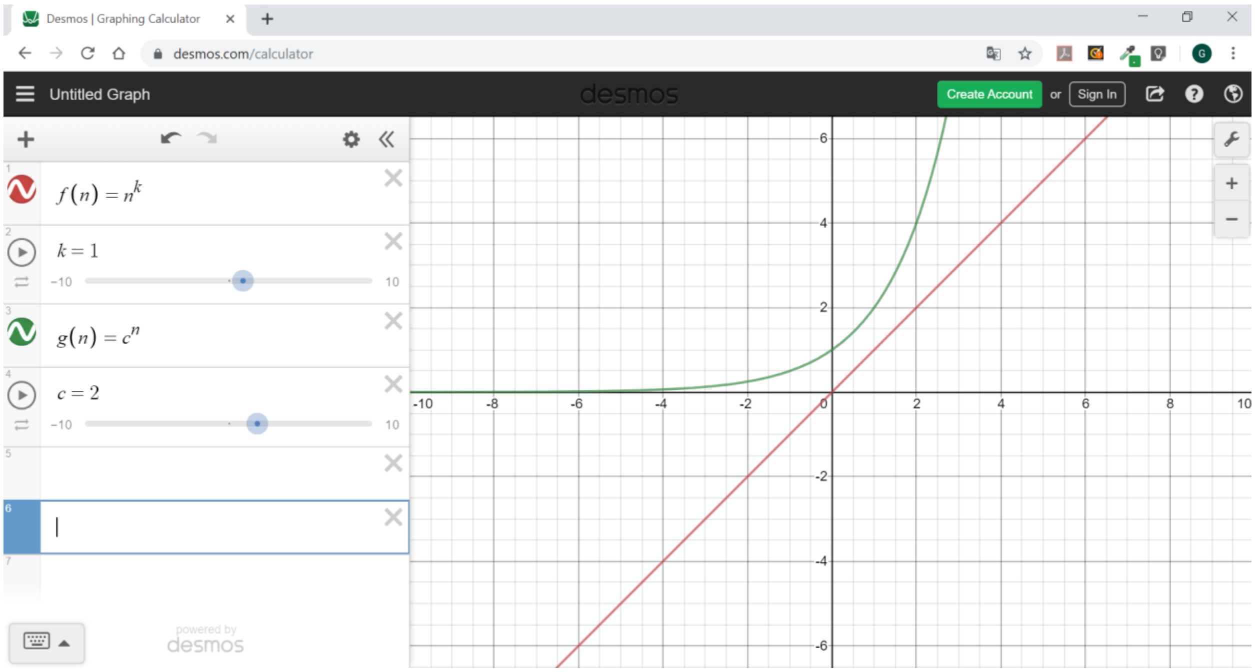
$= \sum_{i=1}^{n} \lg i \approx \int_1^n \lg x \, dx = [x \lg x - x]_1^n$

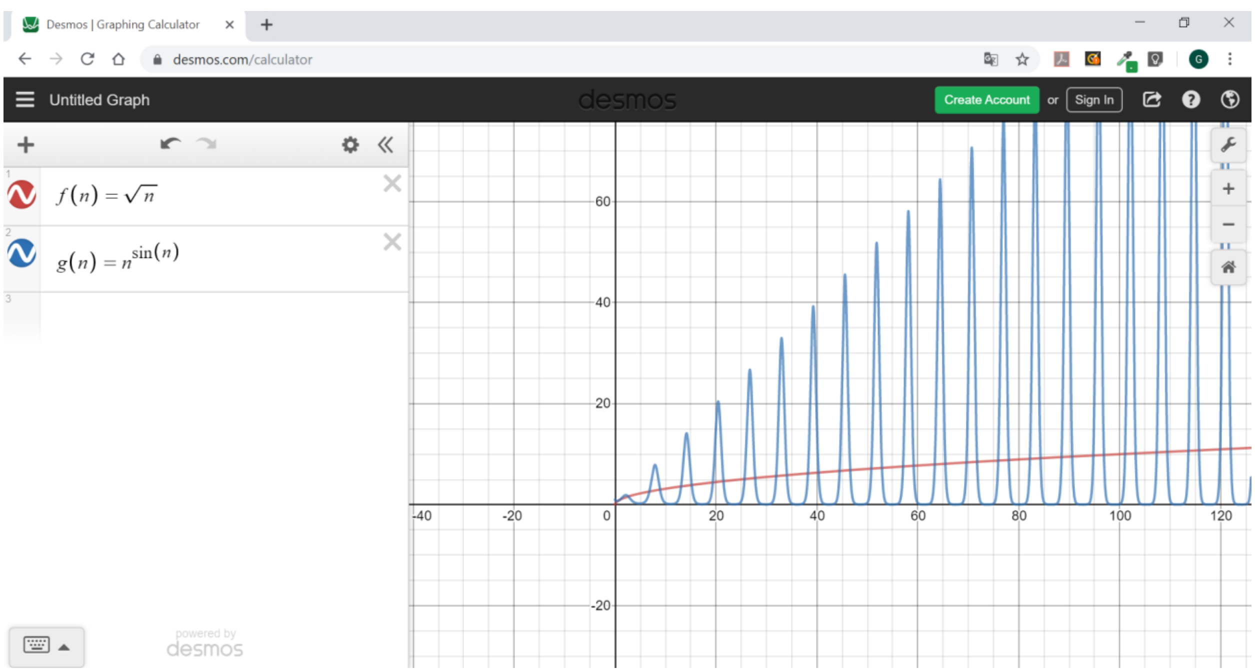
$f(n) := n \lg n - n + 1$

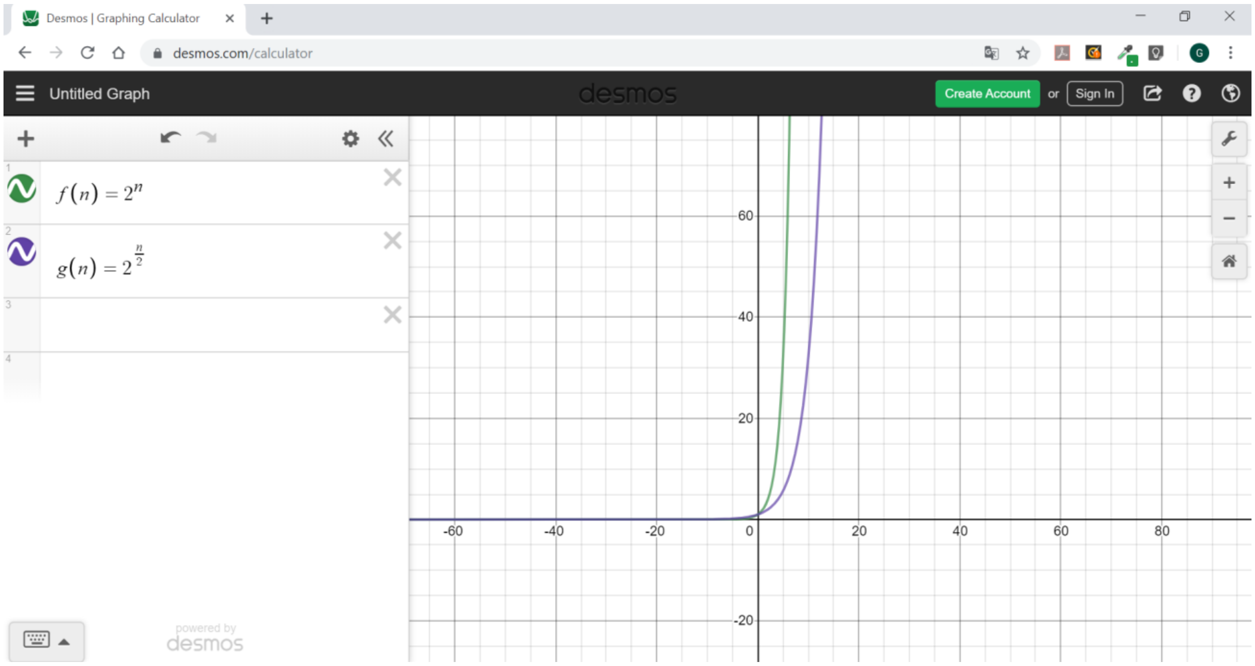
$f(n) \simeq n \lg n$

$f(n)$ has same rate of growth as $g(n) \Rightarrow$ $f(n) = \theta(g(n))$, $f(n) = O(g(n))$, $f(n) = \Omega g(n)$

Problem 3-2 – Relative Asymptotic growths – Graphs



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Create Account
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$f(n) = 2^n$
$g(n) = 2^{\frac{n}{2}}$
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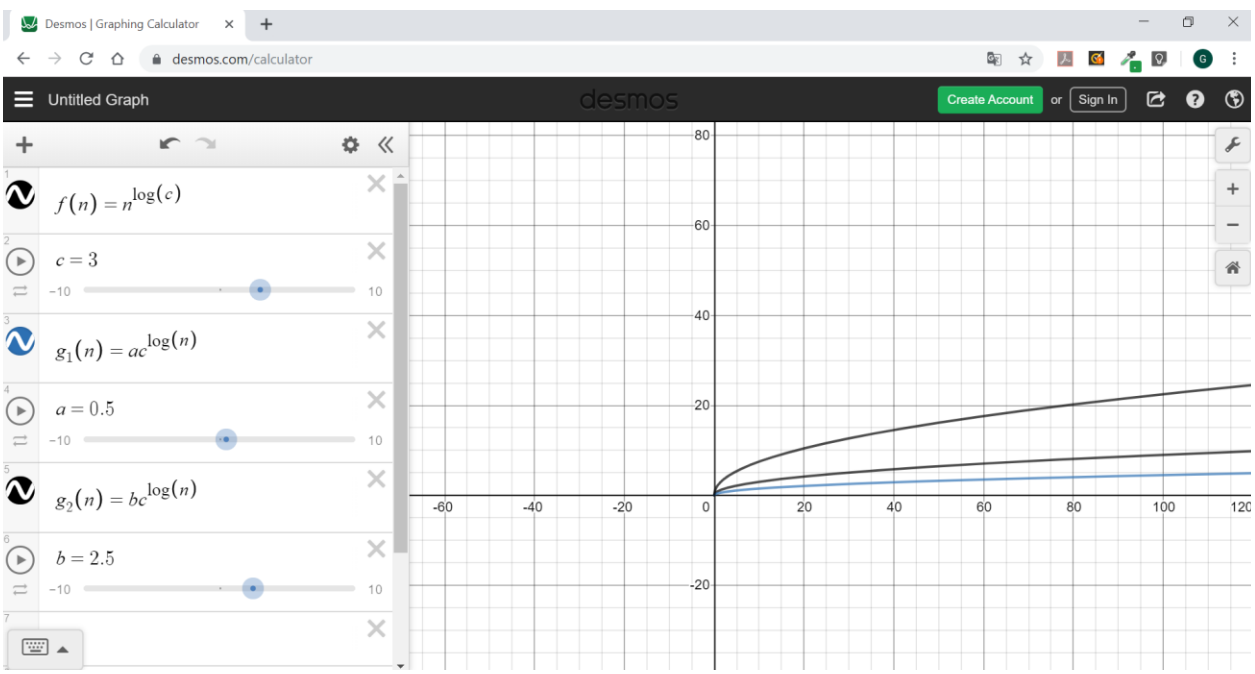
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$f(n) = n^{\log(c)}$
$c = 3$
$g_1(n) = ac^{\log(n)}$
$a = 0.5$
$g_2(n) = bc^{\log(n)}$
$b = 2.5$
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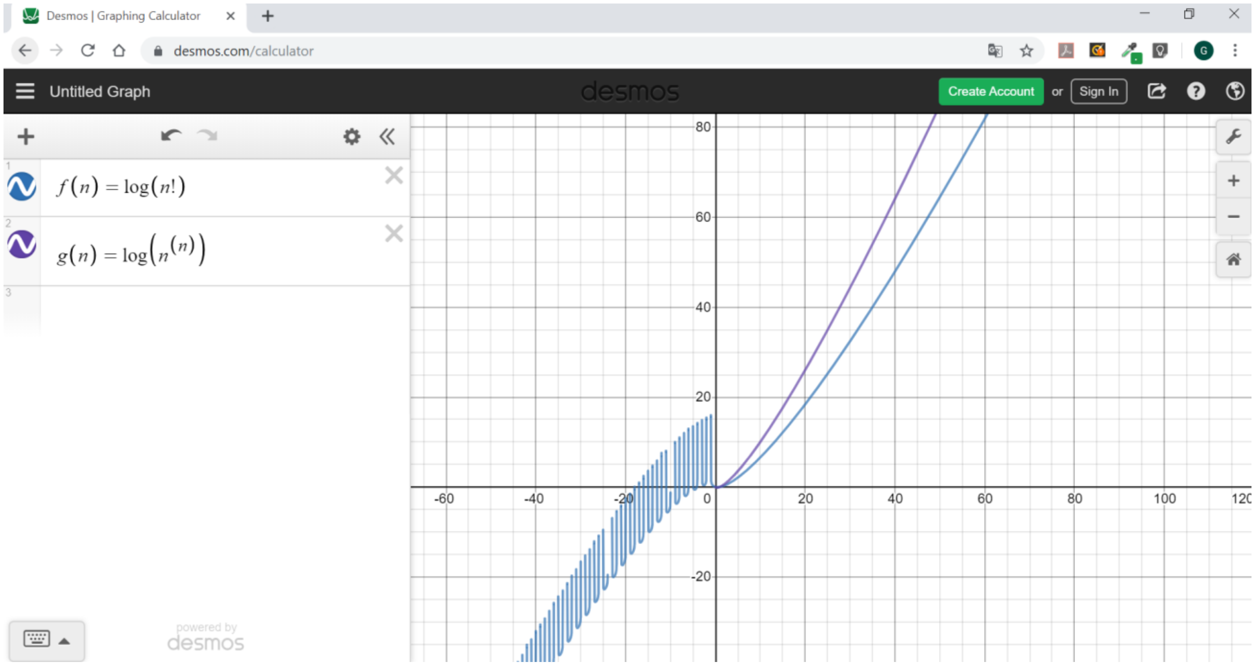
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