09/09/2020 # Assignment - 2 - ASU-ID - 1218506822

1) Given

* $a_{ij} = P(q_{t+1} = s_j \mid q_t = s_i)$ —— ①

* $b_j(O_{t+1}) = P(O_{t+1} \mid q_{t+1} = s_j)$ —— ②

* $\alpha_t(j) = P(O_1, O_2, O_3 \ldots O_t, q_t = s_j / \lambda)$ —— ③

* $\beta_t(j) = P(O_{t+1}, O_{t+2} \ldots O_T \mid q_t = s_j, \lambda)$ —— ④

To Show:

$$\xi_t(i,j) = \frac{\alpha_t(i)\, a_{ij}\, b_j(O_{t+1})\, \beta_{t+1}(j)}{P(O \mid \lambda)}$$

Given: $P(\underbrace{q_t = s_i, q_{t+1} = s_j}_{A} \mid \underbrace{O, \lambda}_{B,\, C})$

$\therefore P\left(\frac{A, B}{C}\right) = P\left(\frac{A}{B, C}\right) P\left(\frac{B}{C}\right)$

$\Rightarrow \boxed{P\left(\frac{A}{B, C}\right) = \frac{P\left(\frac{A, B}{C}\right)}{P\left(\frac{B}{C}\right)}}$

$\boxed{O = O_1 \ldots O_t\, O_{t+1} \ldots O_T}$

$$\Rightarrow \frac{P(q_t = s_i, q_{t+1} = s_j, O \mid \lambda)}{P(O \mid \lambda)}$$

$$\Rightarrow \frac{P(\underbrace{q_t = s_i, O_1 \ldots O_t}_{A}, \underbrace{q_{t+1} = s_j, O_{t+1} \ldots O_T}_{B} \mid \lambda)}{P(O \mid \lambda)}$$

$\Rightarrow$ $\alpha_t(i)$ — from ③

$P(\underbrace{q_t = s_i, O_1 \ldots O_t}_{A} \mid \lambda)\, P(\underbrace{O_{t+1} \ldots O_T}_{B}, \underbrace{q_{t+1} = s_j}_{A} \mid \underbrace{q_t = s_i, O_1 \ldots O_t}_{A}, \lambda)$

$$\Rightarrow \frac{\alpha_t(i)\, P(\overbrace{O_{t+1} \cdots O_T}^{B},\ \overbrace{q_{t+1} = s_j}^{A} \mid q_t = s_i, O_1 \cdots O_t, \lambda)}{P(O \mid \lambda)}$$

$$\boxed{P(A, B) = P(A)\, P(B/A)} \rightarrow ①$$

$$\Rightarrow \frac{\alpha_t(i)\, P\left(q_{t+1} = s_j \mid q_t = s_i, O_1 \cdots O_t, \lambda\right) P\left(O_{t+1} \cdots O_T \mid q_{t+1} = s_j, q_t = s_i, O_1 - O_t, \lambda\right)}{P(O \mid \lambda)}$$

$$\Rightarrow \boxed{\begin{array}{l} q_{t+1} \text{ is independent of } O_1 \cdots O_t \ \& \\ O_{t+1} \cdots O_T \text{ is independent of } q_t, O_1 \cdots O_t \end{array}}$$

$$\Rightarrow \frac{\alpha_t(i)\, \underbrace{P\left(q_{t+1} = s_j \mid q_t = s_i, \lambda\right)}_{a_{ij} \text{ from } ①} P\left(O_{t+1} \cdots O_T \mid q_{t+1} = s_j, \lambda\right)}{P(O \mid \lambda)}$$

$$\Rightarrow \frac{\alpha_t(i)\, a_{ij}\, P\left(\overbrace{O_{t+1}}^{B}, \overbrace{O_{t+2} \cdots O_T}^{A} \mid q_{t+1} = s_j, \lambda\right)}{P(O \mid \lambda)} \rightarrow \text{using } ①$$

$$\Rightarrow \frac{\alpha_t(i)\, a_{ij}\, \overbrace{P\left(O_{t+1} \mid O_{t+2} \cdots O_T, q_{t+1} = s_j, \lambda\right)}^{b_j(O_{t+1}) \text{ from } ③}\ \overbrace{P\left(O_{t+2} \cdots O_T \mid q_{t+1} = s_j, \lambda\right)}^{\beta_{t+1}(j) \text{ from } ④}}{P(O \mid \lambda)}$$

O_{t+1} is independent of $O_{t+2} \cdots O_T$

$$\Rightarrow \frac{\alpha_t(i)\, a_{ij}\, \beta_{t+1}(j)\, b_j(O_{t+1})}{P(O \mid \lambda)} \Rightarrow \underline{\text{Proved}}$$

Q2) $\beta_t(i) = \sum_{j=1}^{N} a_{ij} \cdot b_j(O_{t+1}) \cdot \beta_{t+1}(j)$ Prove $1 \le i \le N$ and $1 \le t \le T$

Given $\beta_t(i) = P(O_{t+1}, O_{t+2} \cdots O_T \mid q_t = s_i, \lambda)$

$$= \sum_{j=1}^{N} P(\underbrace{O_{t+1}, O_{t+2} \cdots O_T}_{A}, \underbrace{q_{t+1} = s_j}_{B} \mid q_t = s_i, \lambda)$$

Introducing the parent term, q_{t+1} in the equation, The value of $\beta_t(i)$ will be equal to the summation of probabilities over all possible values of q_{t+1}

$$\Rightarrow \sum_{j=1}^{N} P(\underbrace{O_{t+1}, q_{t+1} = s_j}_{A}, \underbrace{O_{t+2} \cdots O_T}_{B} \mid q_t = s_i, \lambda)$$

$$\boxed{\begin{array}{l} P(A) = \sum_B P(A, B) \\ P(A, B) = P(A) P\left(\frac{B}{A}\right) \end{array}}$$

$$\Rightarrow \sum_{j=1}^{N} P(\underbrace{O_{t+1}}_{B}, \underbrace{q_{t+1} = s_j}_{A} \mid q_t = s_i, \lambda) \cdot P(O_{t+2} \cdots O_T \mid O_{t+1}, q_{t+1} = s_j, q_t = s_i, \lambda)$$

$$\Rightarrow \sum_{j=1}^{N} \underbrace{P\left(\frac{q_{t+1} = s_j}{q_t = s_i, \lambda}\right)}_{a_{ij}} \cdot \underbrace{P\left(\frac{O_{t+1}}{q_{t+1} = s_j, q_t = s_i}\right)}_{b_j(O_{t+1})} \cdot \cancel{P(O_{t+2} \cdots} \; \beta_{t+1}(j)$$

$$\cdot \underbrace{P(O_{t+2} \cdots O_T \mid q_{t+1} = s_j)}_{\beta_{t+1}(j) \text{ from } (4)}$$

$$\boxed{\begin{array}{l} \because O_{t+2} \cdots O_T \text{ independent of } O_{t+1}, q_t = s_i \\ \because O_{t+1} \text{ is independent of } q_t \end{array}}$$

$$\Rightarrow \sum_{j=1}^{N} a_{ij} \cdot b_j(O_{t+1}) \beta_{t+1}(j) \Rightarrow \text{proved}$$

given

③ $\gamma_t(j) = P(q_t = s_j / O, \lambda)$ →

show that $\gamma_t(j) = \dfrac{\alpha_t(j) \cdot \beta_t(j)}{P(O/\lambda)}$

Given $\gamma_t(j) = P(\underbrace{q_t = s_j}_{A} / \underbrace{O, \lambda}_{B, C})$

$\boxed{P\left(\frac{A}{B, C}\right) = \frac{P(A, B \mid C)}{P(B \mid C)}}$

$\Rightarrow \dfrac{P(q_t = s_j, O_1, O_2 \, O_t, O_{t+1} \, O_T / \lambda)}{P(O/\lambda)}$ $O = O_1 \ldots O_t \ldots O_T$

$\boxed{P(A, B) = P(A) \, P\left(\frac{B}{A}\right)}$

$\Rightarrow \dfrac{P(q_t = s_j, O_1 \cdots O_t / \lambda) \cdot P\left(\dfrac{O_{t+1} \ldots O_T}{q_t = s_j, O_1, \ldots O_t, \lambda}\right)}{P(O/\lambda)}$

$\Rightarrow$ $\boxed{\because O_{t+1} \text{ is independent of } O_1, O_2 \ldots O_t.}$

$\Rightarrow$ $\alpha_t(j) \cdot \beta_t(j)$ → from ③ and ④

$\Rightarrow \dfrac{P(q_t = s_j, O_1 \ldots O_t / \lambda) \cdot P\left(\dfrac{O_{t+1} \ldots O_T}{q_t = s_j, \lambda}\right)}{P(O/\lambda)}$

$\Rightarrow$

$\Rightarrow \dfrac{\alpha_t(j) \cdot \beta_t(j)}{P(O/\lambda)}$ (proved)