LU Decomposition

Numerical Analysis Presentation

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What is Numerical Analysis and Why do we need it?

- Numerical analysis is a branch of mathematics that focuses on developing efficient techniques to obtain numerical solutions for complex mathematical problems.
- Most of the Mathematical problems that arise in science and engineering are very hard and sometime impossible to solve exactly.
- Thus, an approximation to a difficult Mathematical problem is very important to make it more easy to solve.

Example

If the Population of a country has been growing an average of 4% per year and was 100 Crores Last year, What could be the Population this year?

A Reasonable guess would be 104 Crores, but we got to this by extrapolation.

Introduction

- The LU decomposition was introduced by the Polish astronomer T.Banachiewicz in 1938
- Lower–Upper (LU) decomposition factors a matrix as the product of a lower triangular matrix and an upper triangular matrix.
- A is a square matrix of size $n \times n$, then A can be represented as

$$A = LU$$

  where U is an Upper Triangular matrix and L is an Lower Triangular matrix
- Computers typically solve square systems of linear equations using LU decomposition, which is also an essential step in matrix inversion and determinant computation.

Preliminaries

- Lower Triangular Matrix

$$L = \begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix}$$

here a lower triangular matrix $L = [a_{ij}]_{n \times n}$ is one in which $a_{ij} = 0$ for all $i < j$

- Upper Triangular Matrix

$$U = \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}$$

here a upper triangular matrix $L = [a_{ij}]_{n \times n}$ is one in which $a_{ij} = 0$ for all $i > j$

A Minor Road Block

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}}_{n^2} = \underbrace{\begin{pmatrix} l_{11} & 0 & 0 & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ l_{31} & l_{32} & l_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ l_{n1} & l_{n2} & l_{n3} & \cdots & l_{nn} \end{pmatrix} \cdot \begin{pmatrix} u_{11} & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & u_{22} & u_{23} & \cdots & u_{2n} \\ 0 & 0 & u_{33} & \cdots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & u_{nn} \end{pmatrix}}_{n^2+n}$$

Doolittle and Crout's Schemes

When we choose

- $\forall i : l_{ii} = 1$, the method is known as **Doolittle** Method
- $\forall i : u_{ii} = 1$, the method is known as **Crout** Method

LU decomposition using Doolittle Method

let's use the Doolittle method to get the decomposition of

$$\begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Upon solving we get,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applications

1. Solving linear equations
2. Inverting a matrix
3. Computing the determinant

Solving a System of Linear Equations

Given a system of linear equations in matrix form:

$$A\mathbf{x} = \mathbf{b}$$

We want to solve for $\mathbf{x}$ given A and $\mathbf{b}$. Suppose we have already obtained the LU decomposition of A such that:

$$A = LU$$

So,

$$LU\mathbf{x} = \mathbf{b}$$

To find the solution, we proceed in two logical steps:

1. Solve the equation $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$.
2. Solve the equation $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

Example

Consider a 3×3 matrix

$$A = \begin{bmatrix} 6 & 18 & 3 \\ 2 & 12 & 1 \\ 4 & 15 & 3 \end{bmatrix}$$

We can decompose A into L and U where $A = LU$.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

Given $Ax = b$, solve $Ly = b$ and then $Ux = y$.

For example, if $b = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix} U = \begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1. Solve $Ly = b$:

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 19 \\ 0 \end{bmatrix}$$

2. Solve $Ux = y$:

$$\begin{bmatrix} 6 & 18 & 3 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Example

Upon solving we get, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ -11 \end{bmatrix}$

Computing the determinant

Given the LU decomposition $A = LU$ of a square matrix A, the determinant of A can be computed as follows:

$$\det(A) = \det(L)\det(U) = \left(\prod_{i=1}^{n} l_{ii}\right)\left(\prod_{i=1}^{n} u_{ii}\right)$$

The second equation arises from the fact that the determinant of a triangular matrix is the product of its diagonal entries.

Computing the determinant

The determinant of A can be computed from U:

$$\det(A) = \det(L) \cdot \det(U)$$

Since $\det(L) = 1$, we have:

$$\det(A) = 1 \cdot (6) \cdot 6 = 36$$

Drawbacks

1. Not all square matrices have an LU decomposition
   - LU decomposition is applicable only to square matrices (i.e., matrices with an equal number of rows and columns). Existence of LU decomposition,The square matrix A should be positive definite
2. Rounding off errors get carried forward
   - LU decomposition can be sensitive to round-off errors, which may accumulate and affect the accuracy of the solution, especially for large matrices.
3. Storage Requirements:
   - LU decomposition requires storing two matrices (L and U), which increases the memory requirements compared to some other decomposition methods.

References

1. Linear algebra done right by Sheldon Axler
2. Introduction to Numerical Analysis by S. Baskar

Thanks for your Time!

Doolittle Algorithm

For a general $n \times n$ non-singular matrix, the terms of the U matrix and L matrix in Doolittle's Algorithm are given by:

Upper Triangular Matrix U:

$$U_{ij} = \begin{cases} A_{ij} - \sum_{k=0}^{i-1} L_{ik} U_{kj} & \text{if } i < j \wedge i \neq 0, \\ A_{0j} & \text{if } i = 0, \\ 0 & \text{otherwise} \end{cases}$$

Lower Triangular Matrix L:

$$L_{ij} = \begin{cases} 1 & \text{if } i = j, \\ \frac{1}{U_{jj}} \left(A_{ij} - \sum_{k=0}^{j-1} L_{ik} U_{kj} \right) & \text{if } i > j, \\ 0 & \text{otherwise} \end{cases}$$