Integer Solutions to Linear Equations

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Let's start with a simple equation,

$$x_1 + x_2 = 5 \; ; \; x_1 \; \in Z \; , x_2 \; \in Z$$
$$\text{s.t. } x_1 \; \geq 0 \; ; \; x_2 \; \geq \; 0$$

We can simply this problem significantly, if we take up another approach;

Here it is easy to see that the x_i's which are of our interest are interchangeble i.e., there seems to be no order assigned to them.

Consider the following scenario,

Scenario showed in the graph, corresponds to the solution $x_1 = 2$; $x_2 = 3$

Here there are five identical balls and there is one stick, total number of such permuations, is really an elementary question. To be exact it is

$$\frac{6!}{5! \; 1!} = 6 = \binom{5+2-1}{2-1} = \binom{6}{1}$$

We can infact argue that both the above seemingly difficult problem and the simpler one are just the same. In the ball and stick scenario, we have one stick partitioning the 5 balls into two sections and these two sections are interchangable(Unordered) and the number of such permutations is 6.

Hence The total number[1] of postive integer solution pairs to the above given equation is 6.

We can generalize this to the k variables quite easily,

$$x_1 + x_2 + ... + x_k = n \; ; \; x_1 \; \in Z \; , x_2 \; \in Z, \; ... \; x_k \; \in Z$$

$$\text{s.t. } x_1 \; \geq 0 \;\; x_2 \; \geq \; 0; \; ... \; x_k \; \geq 0$$

We get the number of non negative solutions tuples to the above equation as,

$$\binom{n+k-1}{k-1}$$

[1] Explicit Solution Pairs are $(5, 0), (4, 1), (3, 2), (2, 3), (1, 4), (0, 5)$