Gov 50: 17. Sampling Distributions

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Roadmap

1. Poll example
2. Random variables and probability distributions
3. Sampling distribution
4. Normal variables and the Central Limit Theorem

1/ Poll example

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    - Approve (42%), Disapprove (56%)

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- **Sample:** random digit dialing phone numbers (cell and landline).
- **Point estimate**: sample proportion that approve of Biden

2/ Random variables and probability distributions

Random variables

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With a simple random sample, chance of $X_i = 1$ is equal to the population proportion of people that support Biden.

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  - Amount of time spent on a website.

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Discrete variables: like a frequency barplot for the population distribution.

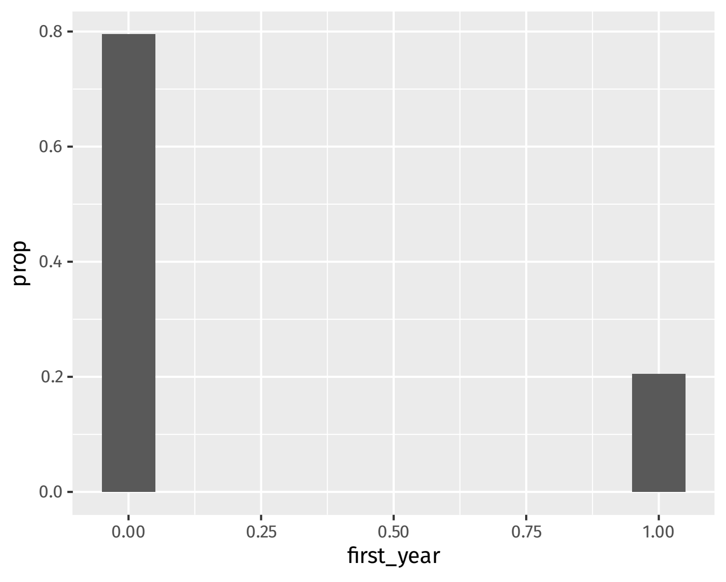
Continuous variables: like a continuous version of population histogram.

Discrete probability distribution

We can use the `y = ..prop..` aesthetic to get a barplot with proportions instead of count to show us the chance/probability of selecting a first-year student:

```
library(gov50data)
class_years |>
  mutate(first_year = as.numeric(year == "First-Year")) |>
  ggplot(aes(x = first_year)) +
  geom_bar(mapping = aes(y = ..prop..), width = 0.1)
```

Discrete probability distribution



Midwest data

```r
library(ggplot2)
midwest
```

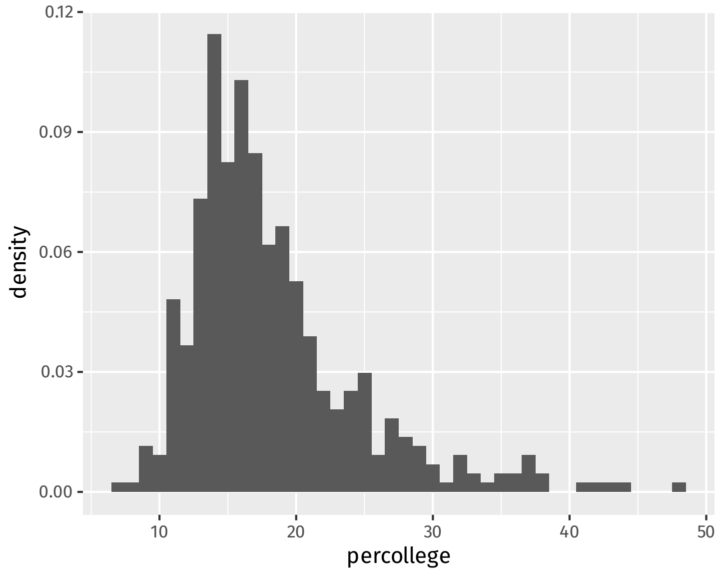
```
## # A tibble: 437 x 28
##      PID county    state  area poptotal popdensity popwhite
##    <int> <chr>     <chr> <dbl>    <int>      <dbl>    <int>
##  1   561 ADAMS     IL    0.052    66090      1271.    63917
##  2   562 ALEXANDER IL    0.014    10626       759      7054
##  3   563 BOND      IL    0.022    14991       681.    14477
##  4   564 BOONE     IL    0.017    30806      1812.    29344
##  5   565 BROWN     IL    0.018     5836       324.     5264
##  6   566 BUREAU    IL    0.05     35688       714.    35157
##  7   567 CALHOUN   IL    0.017     5322       313.     5298
##  8   568 CARROLL   IL    0.027    16805       622.    16519
##  9   569 CASS      IL    0.024    13437       560.    13384
## 10   570 CHAMPAIGN IL    0.058   173025      2983.   146506
## # i 427 more rows
## # i 21 more variables: popblack <int>, popamerindian <int>,
## #   popasian <int>, popother <int>, percwhite <dbl>,
## #   percblack <dbl>, percamerindan <dbl>, percasian <dbl>,
## #   percother <dbl>, popadults <int>, perchsd <dbl>,
## #   percollege <dbl>, percprof <dbl>,
```

Continuous probability distribution

We can use the `y = ..density..` to create a **density histogram** instead of a count histogram so that the area of the histogram boxes are equal to the chance of randomly selecting a unit in that bin:

```
midwest |>
  ggplot(aes(x = percollege)) +
  geom_histogram(aes(y = ..density..), binwidth = 1)
```

Continuous probability distribution



Why density?

Histograms with **density** on the y-axis are drawn so that the area of each box is equal to the proportion of units in the sample in that horizontal bin.

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Sum up all the area = 1 (but heights can go above 1)

3/ Sampling distribution

Key properties of sums and means

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Sample mean: $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

. . .

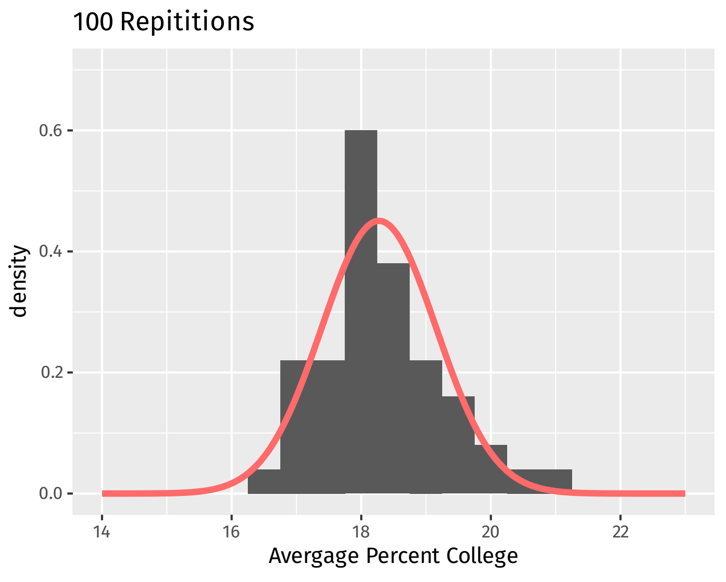
$\overline{X}_n$ is a random variable with a distribution!!

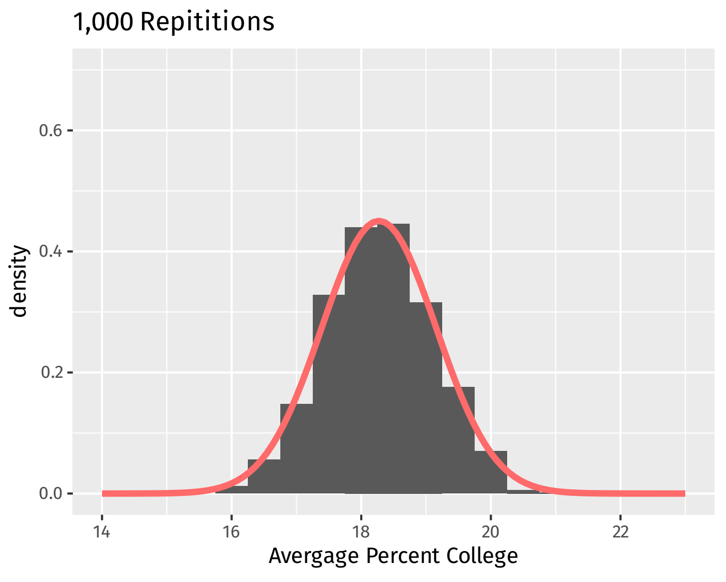
Sample means/proportions distribution

Sampling distributions are the probability distributions of an estimator like $\overline{X}_n$

When we have access to the full population, we can approximate the sampling distribution with repeated sampling.

```
library(infer)
midwest |>
  rep_slice_sample(n = 50, reps = 100) |>
  group_by(replicate) |>
  summarize(`Avergage Percent College` = mean(percollege)) |>
  ggplot(aes(x = `Avergage Percent College`)) +
  geom_histogram(mapping = aes(y = ..density..), binwidth = 0.5) +
  coord_cartesian(xlim = c(14, 23), ylim = c(0, 0.7)) +
  labs(title = "100 Repititions") +
  stat_function(fun = dnorm, args = c(mean(midwest$percollege), sd(midwest$
                color = "indianred1", size = 1.5, xlim = c(14, 23))
```


100 Repititions
density
0.0
0.2
0.4
0.6
14
16
18
20
22
Avergage Percent College


1,000 Repititions
density
0.0
0.2
0.4
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14
16
18
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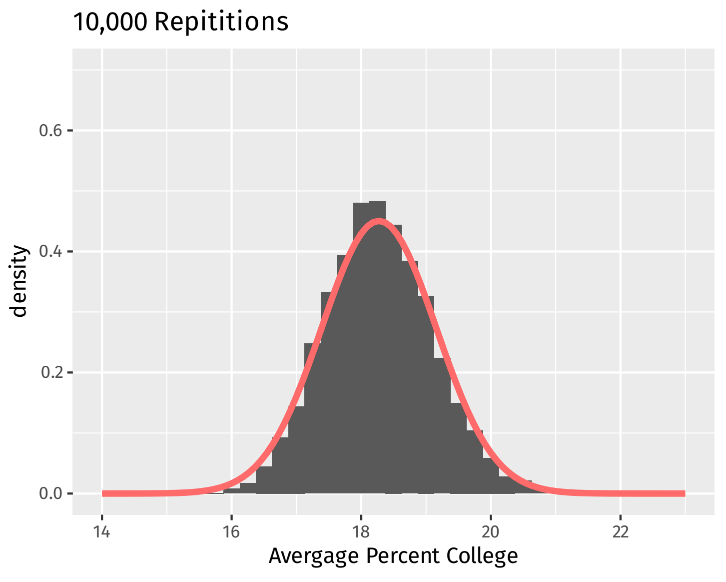
10,000 Repititions
density
0.6
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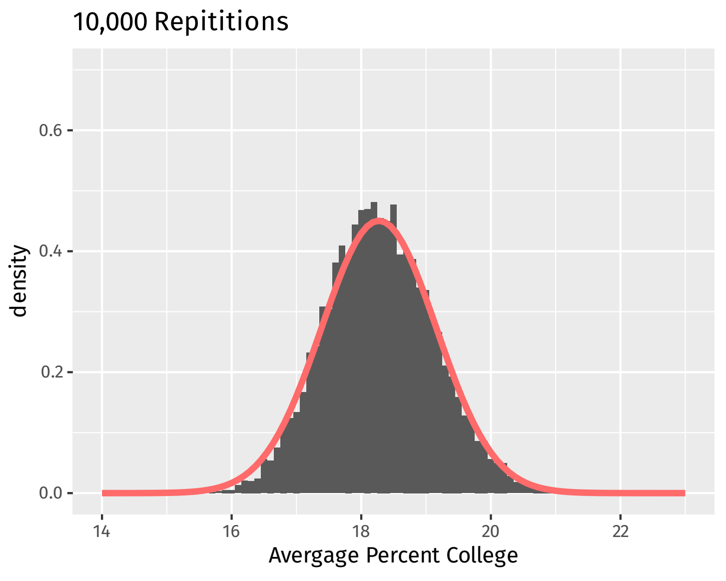
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Standard error of the distribution of $\overline{X}_n$ is approximately $\sigma/\sqrt{n}$:

$$SE \approx \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

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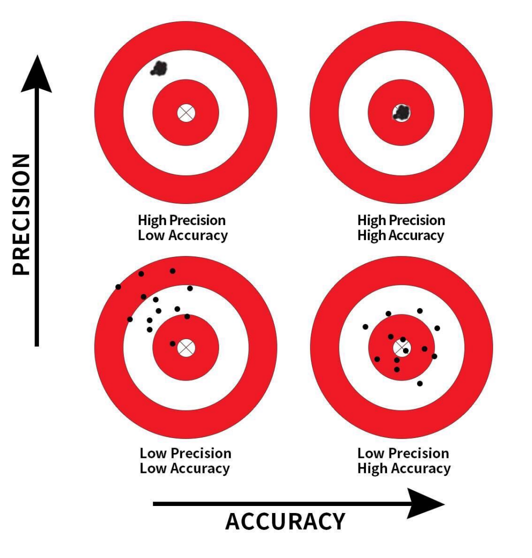
An estimator that isn't unbiased is called **biased**.

Precision vs accuracy



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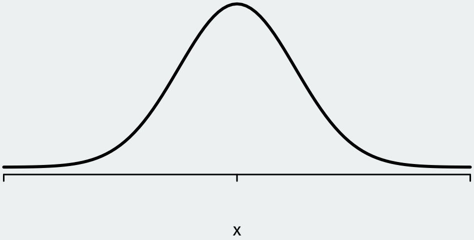
- Not necessarily true with a biased sample!

4/ Normal variables and the Central Limit Theorem

Normal random variable



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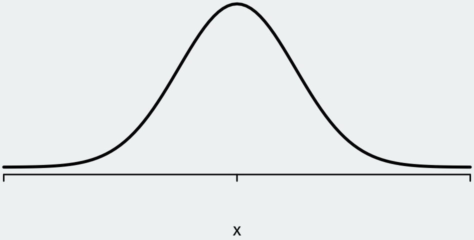

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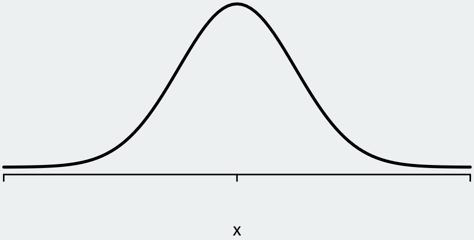

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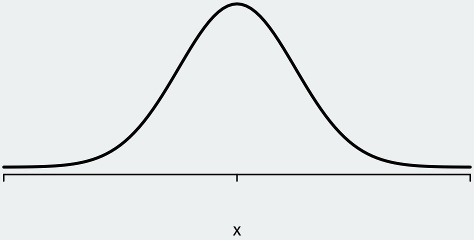

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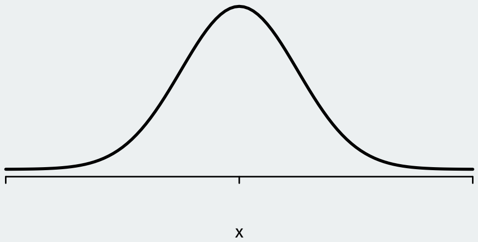

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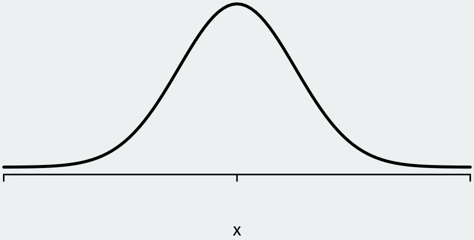

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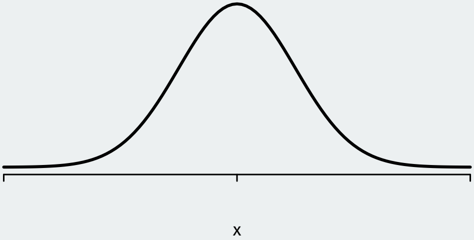

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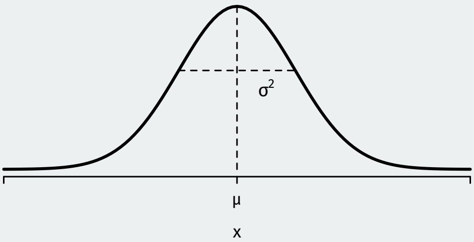
    - **Everywhere positive**: any real value can possibly occur.

Normal distribution



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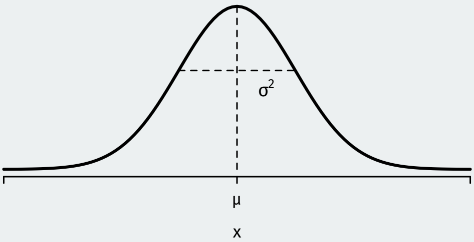

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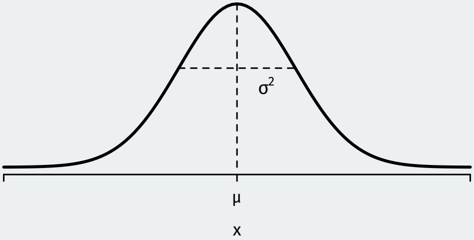

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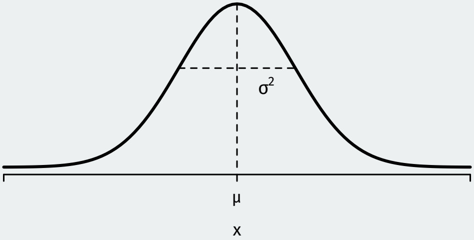

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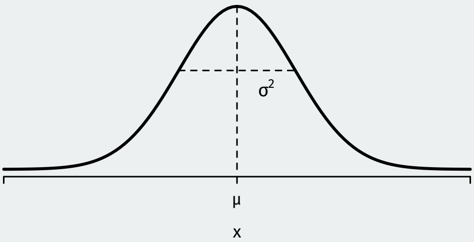

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- **Standard normal distribution**: mean 0 and standard deviation 1.

Central limit theorem

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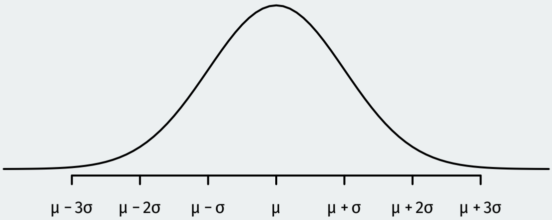
- $\rightsquigarrow$ we know (an approx. of) the entire probability distribution of $\overline{X}_n$
  - Approximation is better as n goes up.
  - Does not depend on the distribution of X_i!

Empirical Rule for the Normal Distribution



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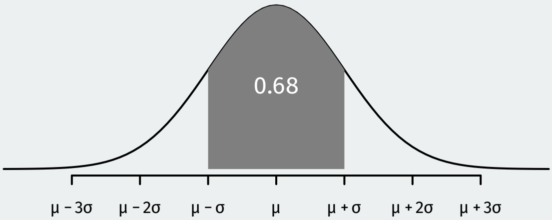

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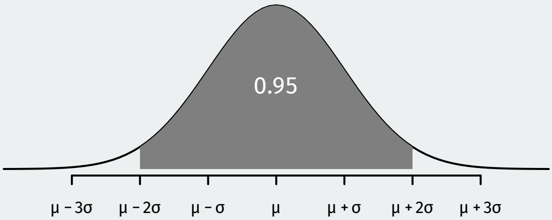

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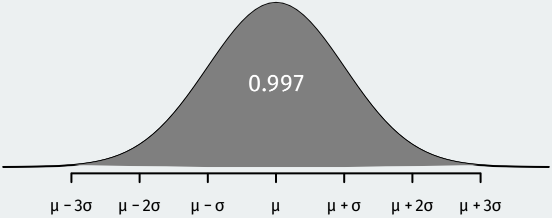

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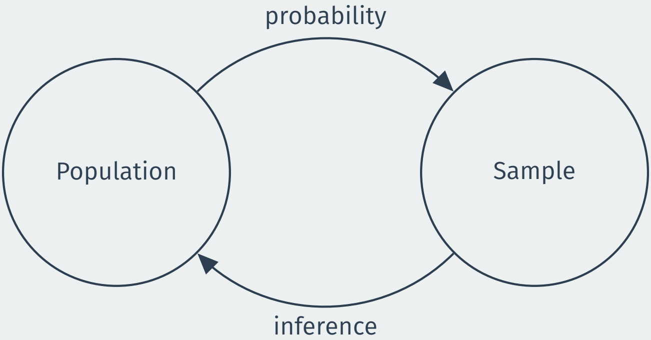
    - $\approx$ 68% of the distribution of X is within 1 SD of the mean.
    - $\approx$ 95% of the distribution of X is within 2 SDs of the mean.
    - $\approx$ 99.7% of the distribution of X is within 3 SDs of the mean.
- CLT + empirical rule: we'll know the rough distribution of estimation errors we should expect.

Where are we going?



We only get 1 sample. Can we learn about the population from that sample?