# Gov 50: 17. Sampling Distributions

Matthew Blackwell

Harvard University

# Roadmap

1. Poll example
2. Random variables and probability distributions
3. Sampling distribution
4. Normal variables and the Central Limit Theorem

# 1/ Poll example

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    - Approve (42%), Disapprove (56%)

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- **Point estimate**: sample proportion that approve of Biden

# 2/ Random variables and probability distributions

# Random variables

**Random variables** are numerical summaries of chance processes:

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With a simple random sample, chance of $X_i = 1$ is equal to the population proportion of people that support Biden.

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    - GDP per capita (average income) in a country.
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    - Amount of time spent on a website.

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**Discrete variables**: like a frequency barplot for the population distribution.

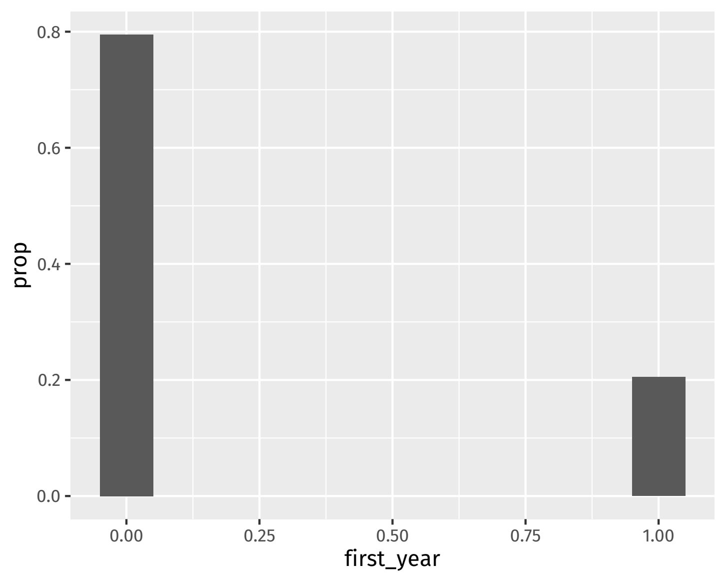
**Continuous variables**: like a continuous version of population histogram.

# Discrete probability distribution

We can use the `y = ..prop..` aesthetic to get a barplot with proportions instead of count to show us the chance/probability of selecting a first-year student:

```
library(gov50data)
class_years |>
  mutate(first_year = as.numeric(year == "First-Year")) |>
  ggplot(aes(x = first_year)) +
  geom_bar(mapping = aes(y = ..prop..), width = 0.1)
```

# Discrete probability distribution



# Midwest data

```r
library(ggplot2)
midwest
```

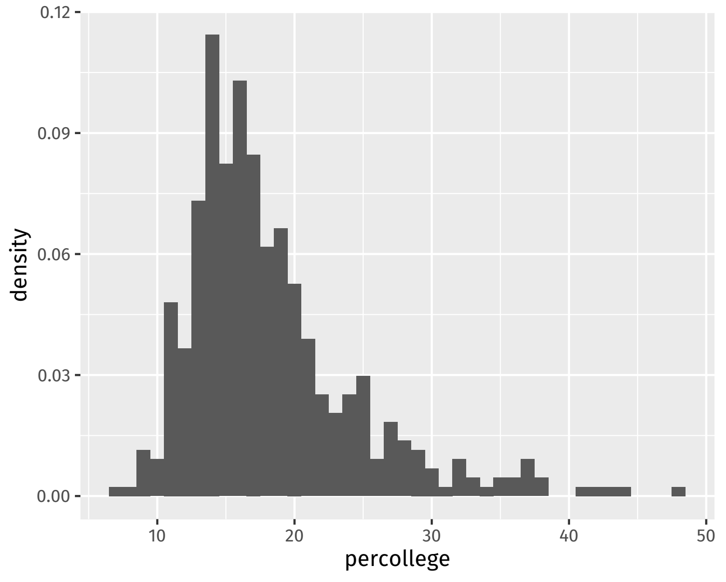
```
## # A tibble: 437 x 28
##      PID county  state  area popto~1 popde~2 popwh~3 popbl~4
##    <int> <chr>   <chr> <dbl>   <int>   <dbl>   <int>   <int>
##  1   561 ADAMS   IL    0.052   66090   1271.   63917    1702
##  2   562 ALEXAN~ IL    0.014   10626    759     7054    3496
##  3   563 BOND    IL    0.022   14991    681.   14477     429
##  4   564 BOONE   IL    0.017   30806   1812.   29344     127
##  5   565 BROWN   IL    0.018    5836    324.    5264     547
##  6   566 BUREAU  IL    0.05    35688    714.   35157      50
##  7   567 CALHOUN IL    0.017    5322    313.    5298       1
##  8   568 CARROLL IL    0.027   16805    622.   16519     111
##  9   569 CASS    IL    0.024   13437    560.   13384      16
## 10   570 CHAMPA~ IL    0.058  173025   2983.  146506   16559
## # ... with 427 more rows, 20 more variables:
## #   popamerindian <int>, popasian <int>, popother <int>,
## #   percwhite <dbl>, percblack <dbl>, percamerindan <dbl>,
## #   percasian <dbl>, percother <dbl>, popadults <int>,
## #   perchsd <dbl>, percollege <dbl>, percprof <dbl>,
## #   poppovertyknown <int>, percpovertyknown <dbl>,
```

# Continuous probability distribution

We can use the `y = ..density..` to create a **density histogram** instead of a count histogram so that the area of the histogram boxes are equal to the chance of randomly selecting a unit in that bin:

```
midwest |>
  ggplot(aes(x = percollege)) +
  geom_histogram(aes(y = ..density..), binwidth = 1)
```

# Continuous probability distribution



# Why density?

Histograms with **density** on the y-axis are drawn so that the area of each box is equal to the proportion of units in the sample in that horizontal bin.

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Sum up all the area = 1 (but heights can go above 1)

# **3/** Sampling distribution

# Key properties of sums and means

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**Sample mean**: $\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i$

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

. . .

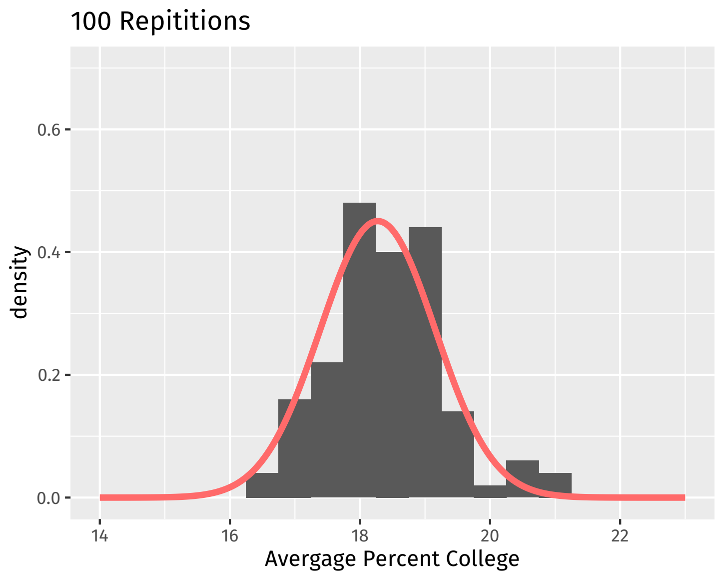
$\overline{X}_n$ is a random variable with a distribution!!

# Sample means/proportions distribution

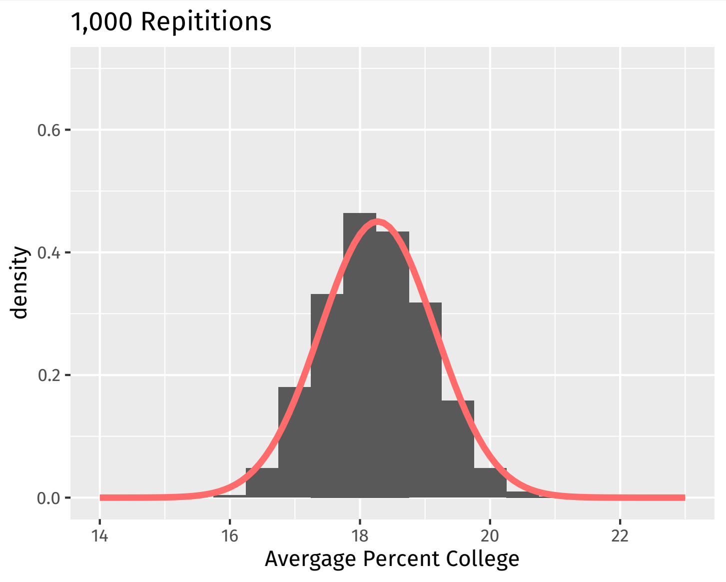
**Sampling distributions** are the probability distributions of an estimator like $\overline{X}_n$

When we have access to the full population, we can approximate the sampling distribution with repeated sampling.

```
library(infer)
midwest |>
  rep_slice_sample(n = 50, reps = 100) |>
  group_by(replicate) |>
  summarize(`Avergage Percent College` = mean(percollege)) |>
  ggplot(aes(x = `Avergage Percent College`)) +
  geom_histogram(mapping = aes(y = ..density..), binwidth = 0.5) +
  coord_cartesian(xlim = c(14, 23), ylim = c(0, 0.7)) +
  labs(title = "100 Repititions") +
  stat_function(fun = dnorm, args = c(mean(midwest$percollege), sd(midwest$
                color = "indianred1", size = 1.5, xlim = c(14, 23))
```



100 Repititions
density
0.0
0.2
0.4
0.6
14
16
18
20
22
Avergage Percent College


1,000 Repititions
density
0.0
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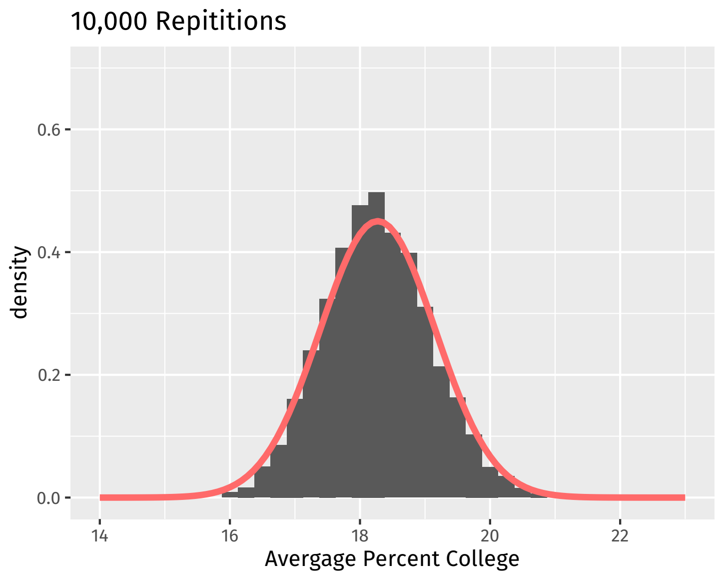
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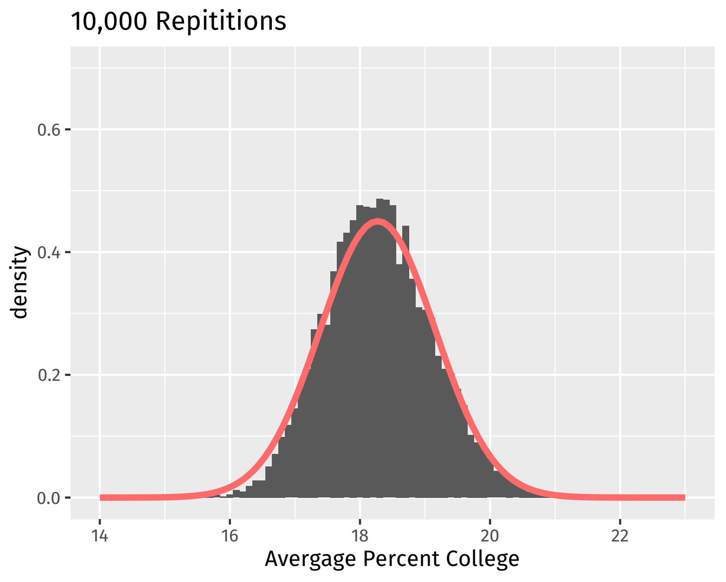
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**Expected value** of the distribution of $\overline{X}_n$ is the population mean, $\mu$.

**Standard error** of the distribution of $\overline{X}_n$ is approximately $\sigma/\sqrt{n}$:

$$SE \approx \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

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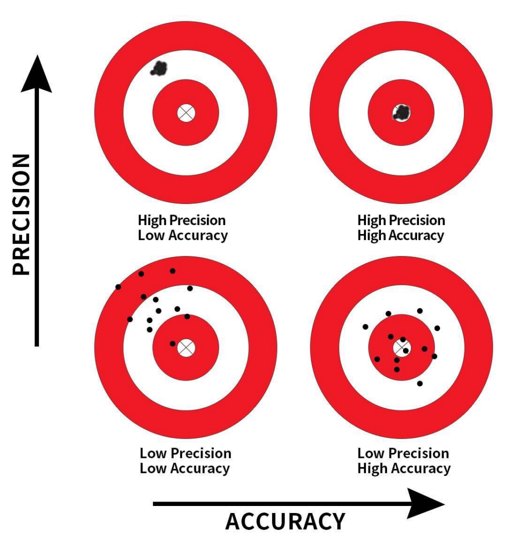
An estimator that isn't unbiased is called **biased**.

# Precision vs accuracy



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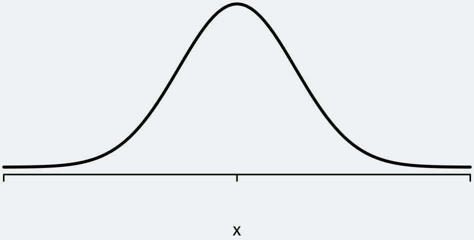
- Not necessarily true with a biased sample!

# 4/ Normal variables and the Central Limit Theorem

# Normal random variable

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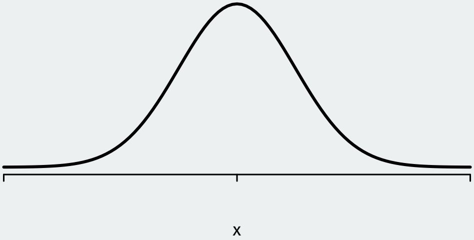


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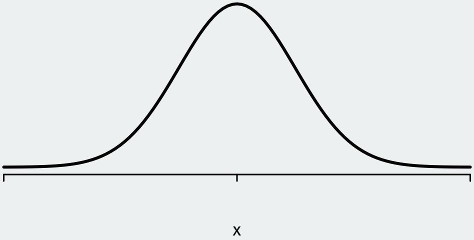


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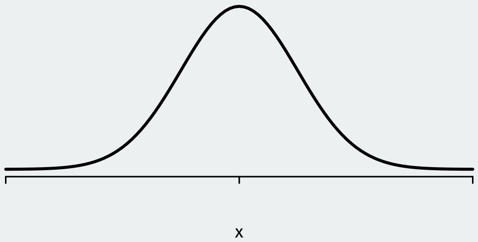


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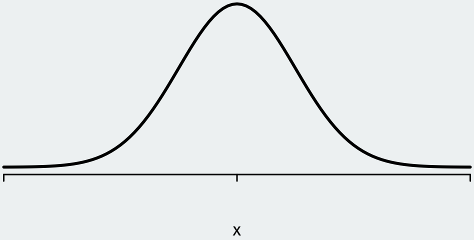


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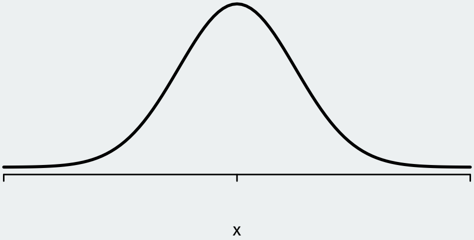


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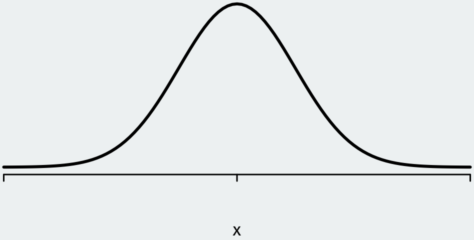


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  - **Everywhere positive**: any real value can possibly occur.

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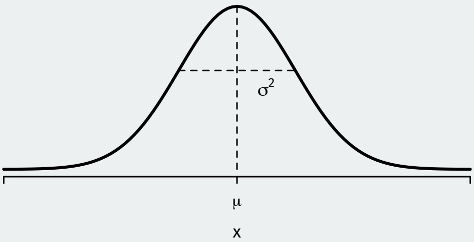


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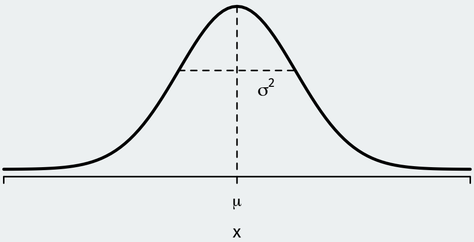


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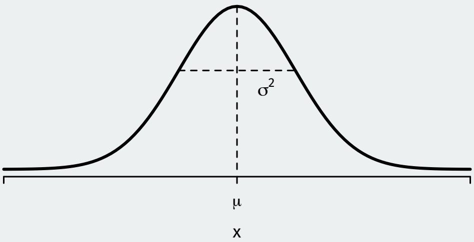


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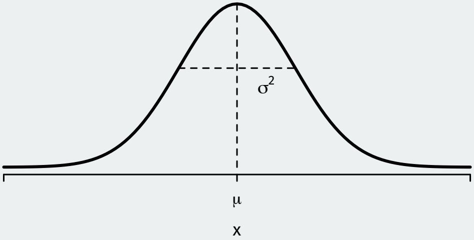


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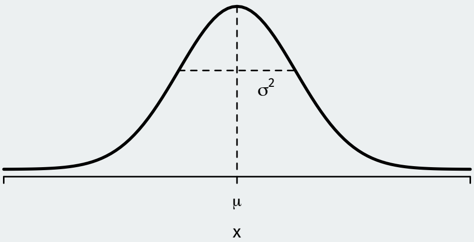


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- **Standard normal distribution**: mean 0 and standard deviation 1.

# Central limit theorem

## Central limit theorem

Let $X_1, \ldots, X_n$ be a simple random sample from a population with mean $\mu$ and finite variance $\sigma^2$. Then, $\overline{X}_n$ will be approximately distributed $N(\mu, \sigma^2/n)$ in large samples.

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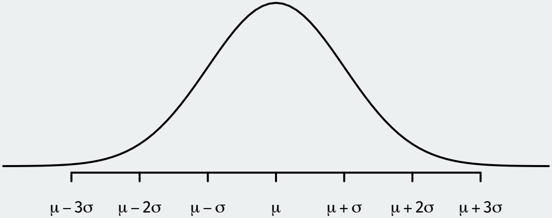
- $\rightsquigarrow$ we know (an approx. of) the entire probability distribution of $\overline{X}_n$
    - Approximation is better as $n$ goes up.
    - Does not depend on the distribution of $X_i$!

# Empirical Rule for the Normal Distribution

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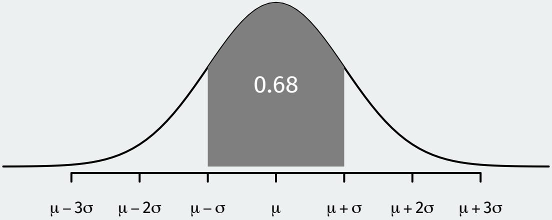


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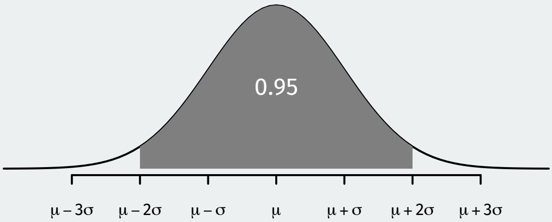


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  - $\approx$ 95% of the distribution of $X$ is within 2 SDs of the mean.

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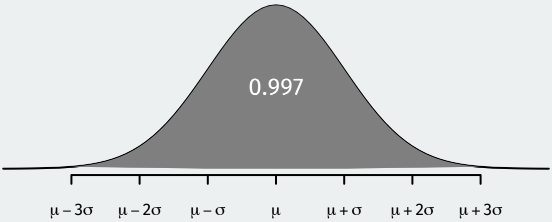
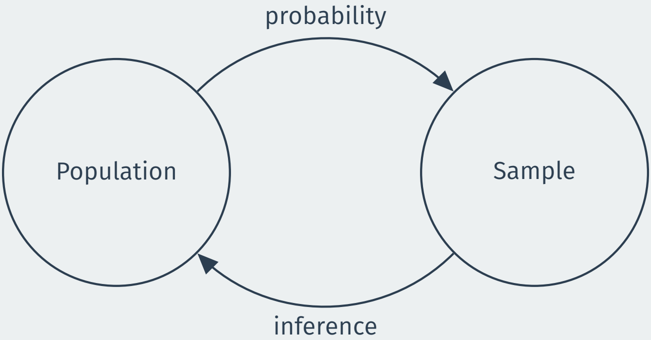



- If $X \sim N(\mu, \sigma^2)$, then:
  - $\approx$ 68% of the distribution of $X$ is within 1 SD of the mean.
  - $\approx$ 95% of the distribution of $X$ is within 2 SDs of the mean.
  - $\approx$ 99.7% of the distribution of $X$ is within 3 SDs of the mean.
- CLT + empirical rule: we'll know the rough distribution of estimation errors we should expect.

# Where are we going?




We only get 1 sample. Can we learn about the population from that sample?