Gov 50: 17. Sampling Distributions

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Roadmap

1. Poll example
2. Random variables and probability distributions
3. Sampling distribution
4. Normal variables and the Central Limit Theorem

1/ Poll example

How popular is Joe Biden?

- What proportion of the public approves of Biden's job as president?
- Latest Gallup poll:
  - Sept 1st-16th
  - 812 adult Americans
  - Telephone interviews
  - Approve (42%), Disapprove (56%)

Poll in our framework

- **Population**: adults 18+ living in 50 US states and DC.
- **Population parameter**: population proportion of all US adults that approve of Biden.
    - Census: not possible.
- **Sample:** random digit dialing phone numbers (cell and landline).
- **Point estimate**: sample proportion that approve of Biden

2/ Random variables and probability distributions

Random variables

Random variables are numerical summaries of chance processes:

$$X_i = \begin{cases} 1 & \text{if respondent } i \text{ supports Biden,} \\ 0 & \text{otherwise} \end{cases}$$

With a simple random sample, chance of $X_i = 1$ is equal to the population proportion of people that support Biden.

Types of random variables

- **Discrete**: X can take a finite (or countably infinite) number of values.
  - Number of heads in 5 coin flips
  - Sampled senator is a woman ($X = 1$) or not ($X = 0$)
  - Number of battle deaths in a civil war
- **Continuous**: X can take any real value (usually within an interval).
  - GDP per capita (average income) in a country.
  - Share of population that approves of Biden.
  - Amount of time spent on a website.

Probability distributions

Probability distributions tell us the chances of different values of a r.v. occurring

Discrete variables: like a frequency barplot for the population distribution.

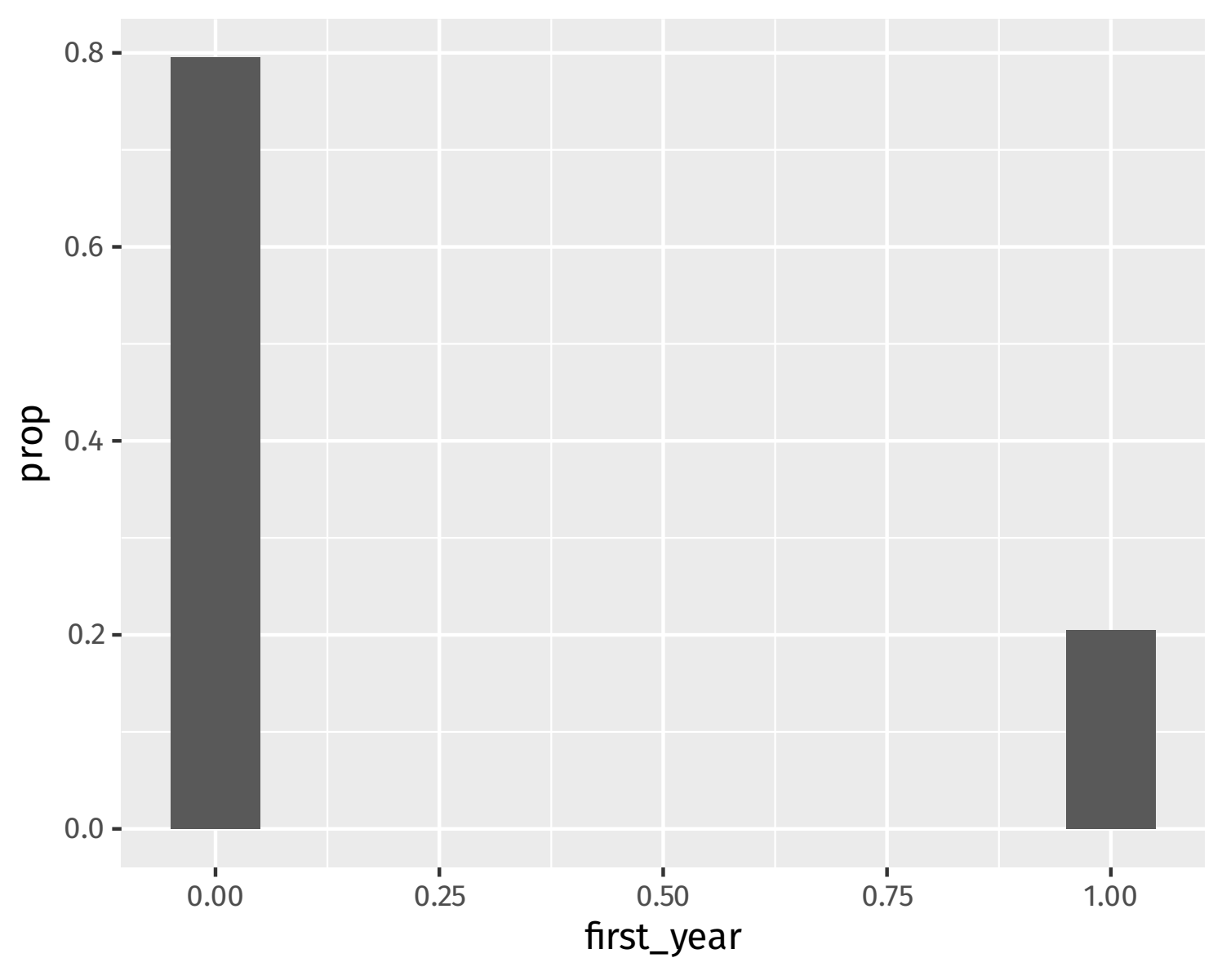
Continuous variables: like a continuous version of population histogram.

Discrete probability distribution

We can use the `y = ..prop..` aesthetic to get a barplot with proportions instead of count to show us the chance/probability of selecting a first-year student:

```
library(gov50data)
class_years |>
  mutate(first_year = as.numeric(year == "First-Year")) |>
  ggplot(aes(x = first_year)) +
  geom_bar(mapping = aes(y = ..prop..), width = 0.1)
```

Discrete probability distribution



Midwest data

```
library(ggplot2)
midwest
```

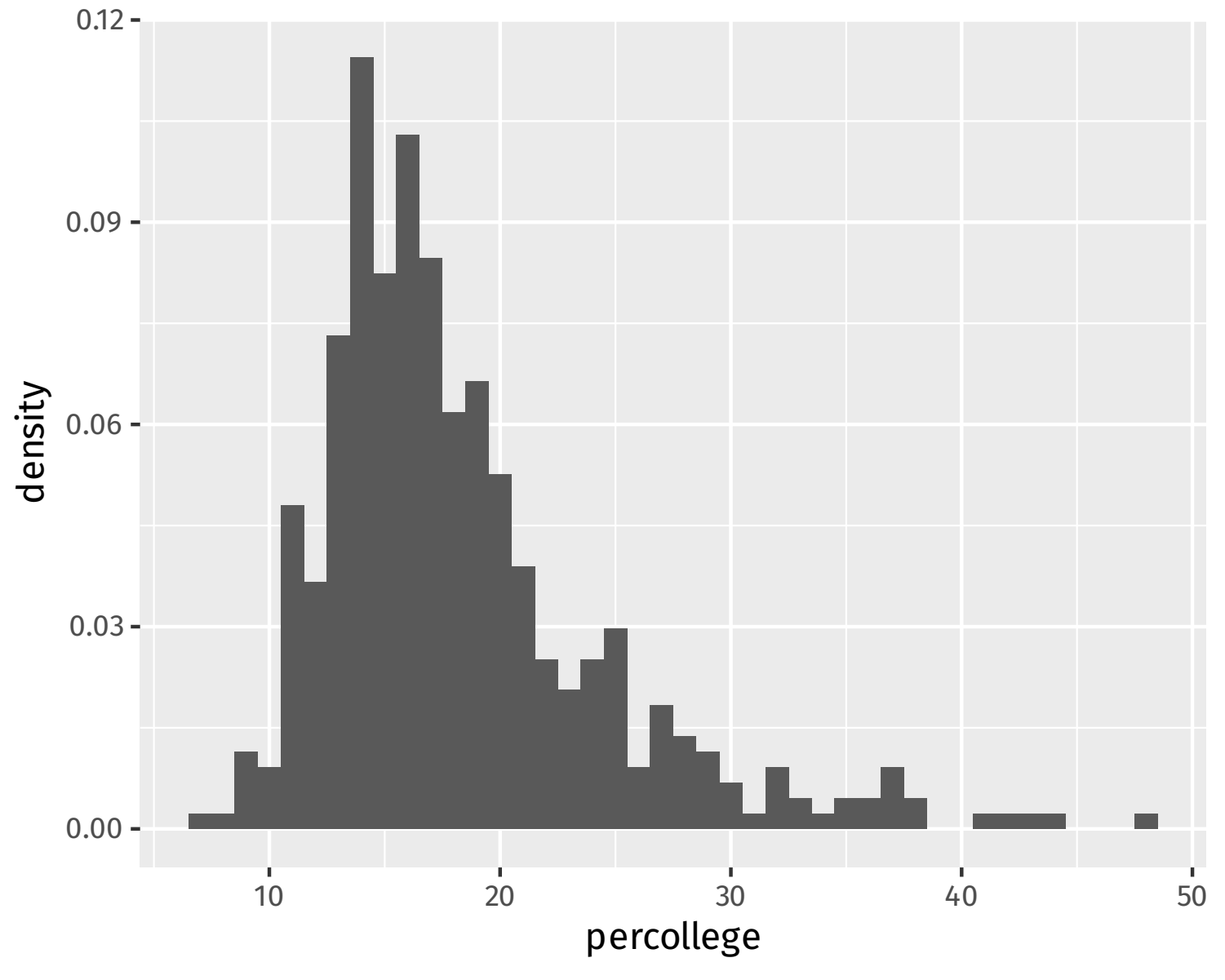
```
## # A tibble: 437 x 28
##      PID county    state  area poptotal popdensity popwhite
##    <int> <chr>     <chr> <dbl>    <int>      <dbl>    <int>
##  1   561 ADAMS     IL    0.052    66090      1271.    63917
##  2   562 ALEXANDER IL    0.014    10626       759      7054
##  3   563 BOND      IL    0.022    14991       681.    14477
##  4   564 BOONE     IL    0.017    30806      1812.    29344
##  5   565 BROWN     IL    0.018     5836       324.     5264
##  6   566 BUREAU    IL    0.05     35688       714.    35157
##  7   567 CALHOUN   IL    0.017     5322       313.     5298
##  8   568 CARROLL   IL    0.027    16805       622.    16519
##  9   569 CASS      IL    0.024    13437       560.    13384
## 10   570 CHAMPAIGN IL    0.058   173025      2983.   146506
## # i 427 more rows
## # i 21 more variables: popblack <int>, popamerindian <int>,
## #   popasian <int>, popother <int>, percwhite <dbl>,
## #   percblack <dbl>, percamerindan <dbl>, percasian <dbl>,
## #   percother <dbl>, popadults <int>, perchsd <dbl>,
## #   percollege <dbl>, percprof <dbl>,
```

Continuous probability distribution

We can use the `y = ..density..` to create a **density histogram** instead of a count histogram so that the area of the histogram boxes are equal to the chance of randomly selecting a unit in that bin:

```
midwest |>
  ggplot(aes(x = percollege)) +
  geom_histogram(aes(y = ..density..), binwidth = 1)
```

Continuous probability distribution



Why density?

Histograms with **density** on the y-axis are drawn so that the area of each box is equal to the proportion of units in the sample in that horizontal bin.

Easier to compare distributions across sample sizes.

Sum up all the area = 1 (but heights can go above 1)

3/ Sampling distribution

Key properties of sums and means

Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from a population distribution with mean μ ("mu") and variance σ^2 ("sigma squared")

Sample mean: $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$

$$\overline{X}_n = \frac{X_1 + X_2 + \cdots + X_n}{n}$$

. . .

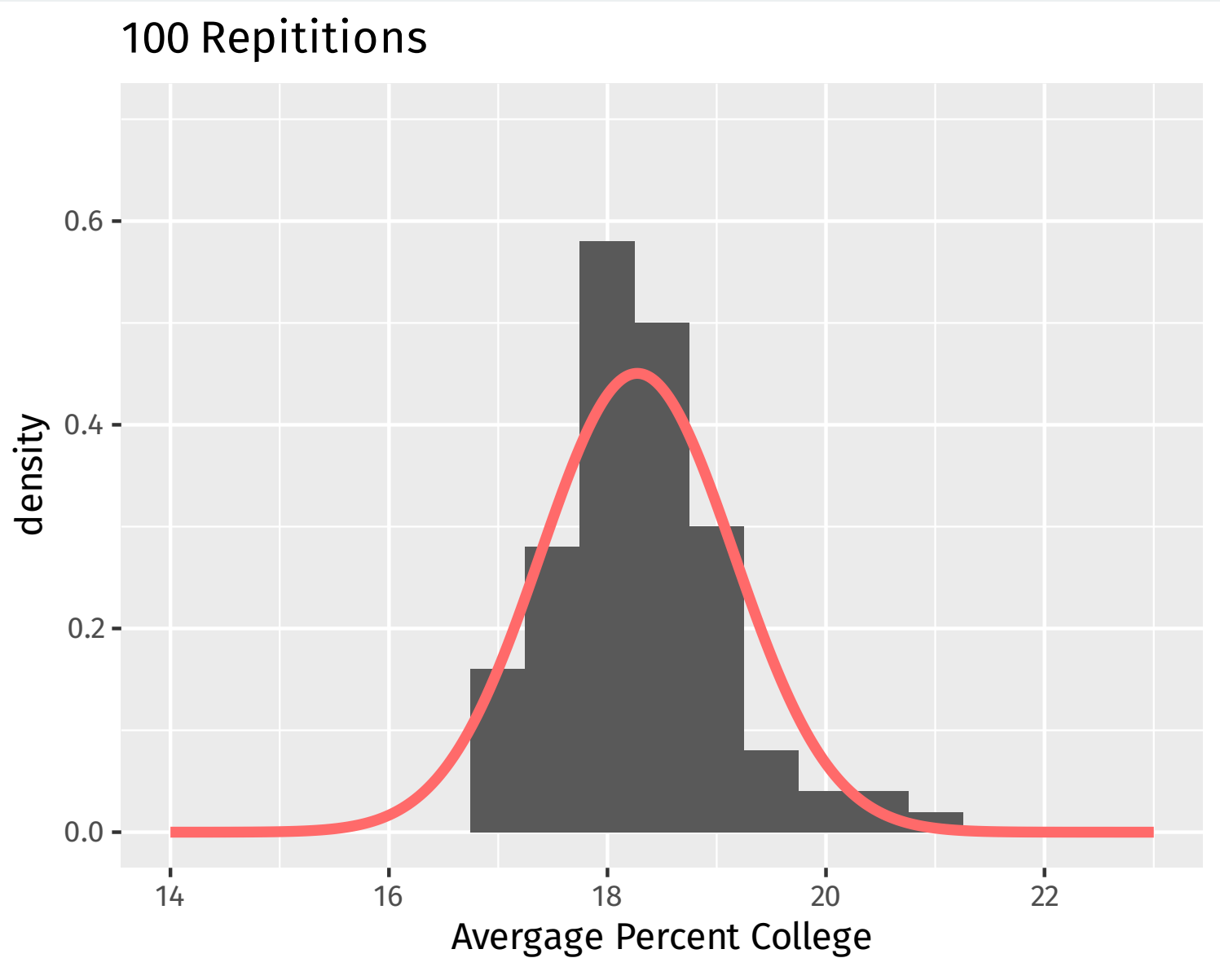
$\overline{X}_n$ is a random variable with a distribution!!

Sample means/proportions distribution

Sampling distributions are the probability distributions of an estimator like $\overline{X}_n$

When we have access to the full population, we can approximate the sampling distribution with repeated sampling.

```
library(infer)
midwest |>
  rep_slice_sample(n = 50, reps = 100) |>
  group_by(replicate) |>
  summarize(`Avergage Percent College` = mean(percollege)) |>
  ggplot(aes(x = `Avergage Percent College`)) +
  geom_histogram(mapping = aes(y = ..density..), binwidth = 0.5) +
  coord_cartesian(xlim = c(14, 23), ylim = c(0, 0.7)) +
  labs(title = "100 Repititions") +
  stat_function(fun = dnorm, args = c(mean(midwest$percollege), sd(midwest$
                color = "indianred1", size = 1.5, xlim = c(14, 23))
```


100 Repititions
density
0.6
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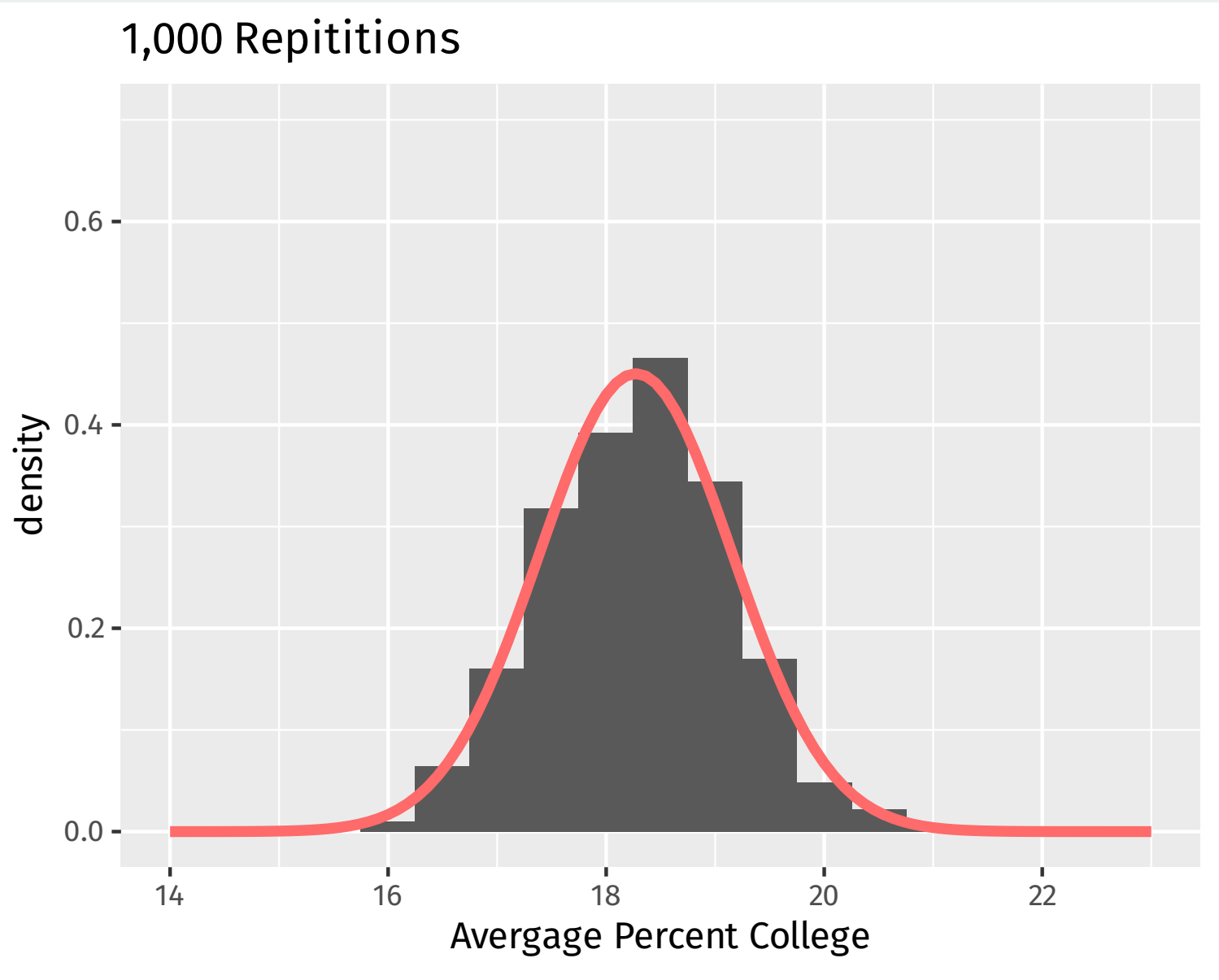
14
16
18
20
22
Avergage Percent College


1,000 Repititions
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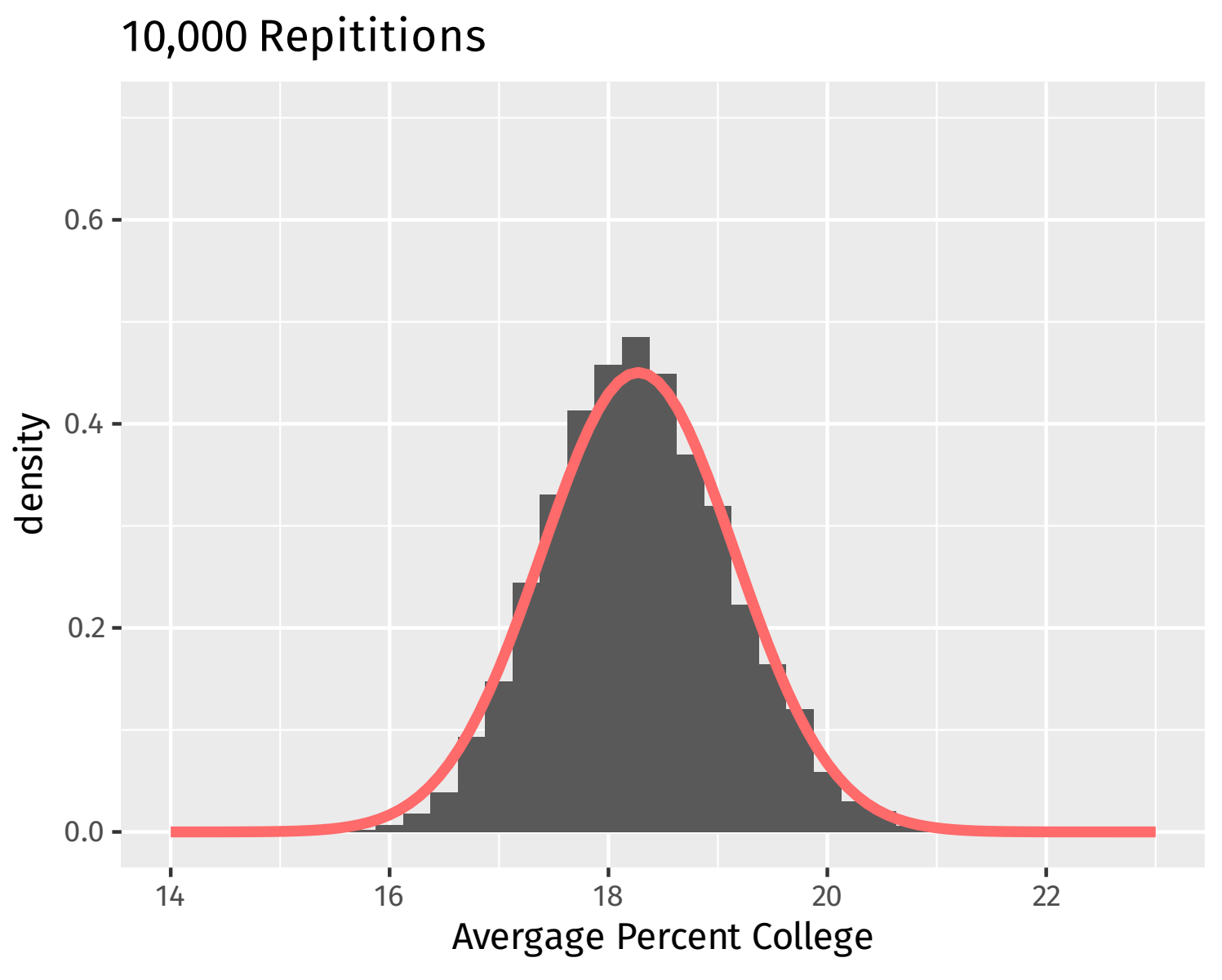
Avergage Percent College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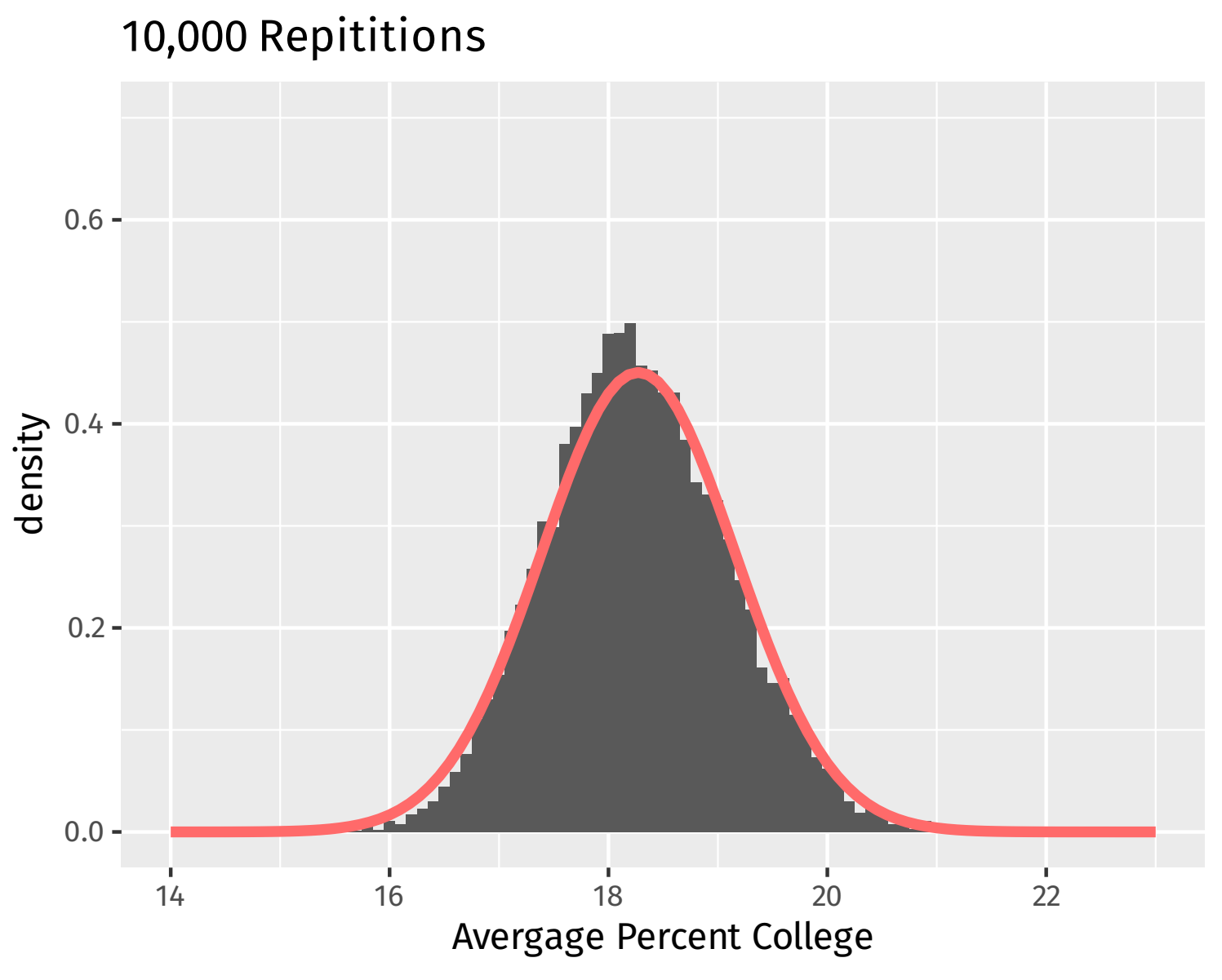
10,000 Repititions
density
0.6
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Avergage Percent College

Sampling distribution of the sample mean

Suppose $X_1, X_2, \ldots, X_n$ is a simple random sample from a population distribution with mean μ and variance σ^2.

Expected value of the distribution of $\overline{X}_n$ is the population mean, μ.

Standard error of the distribution of $\overline{X}_n$ is approximately $\sigma/\sqrt{n}$:

$$SE \approx \frac{\text{population standard deviation}}{\sqrt{\text{sample size}}}$$

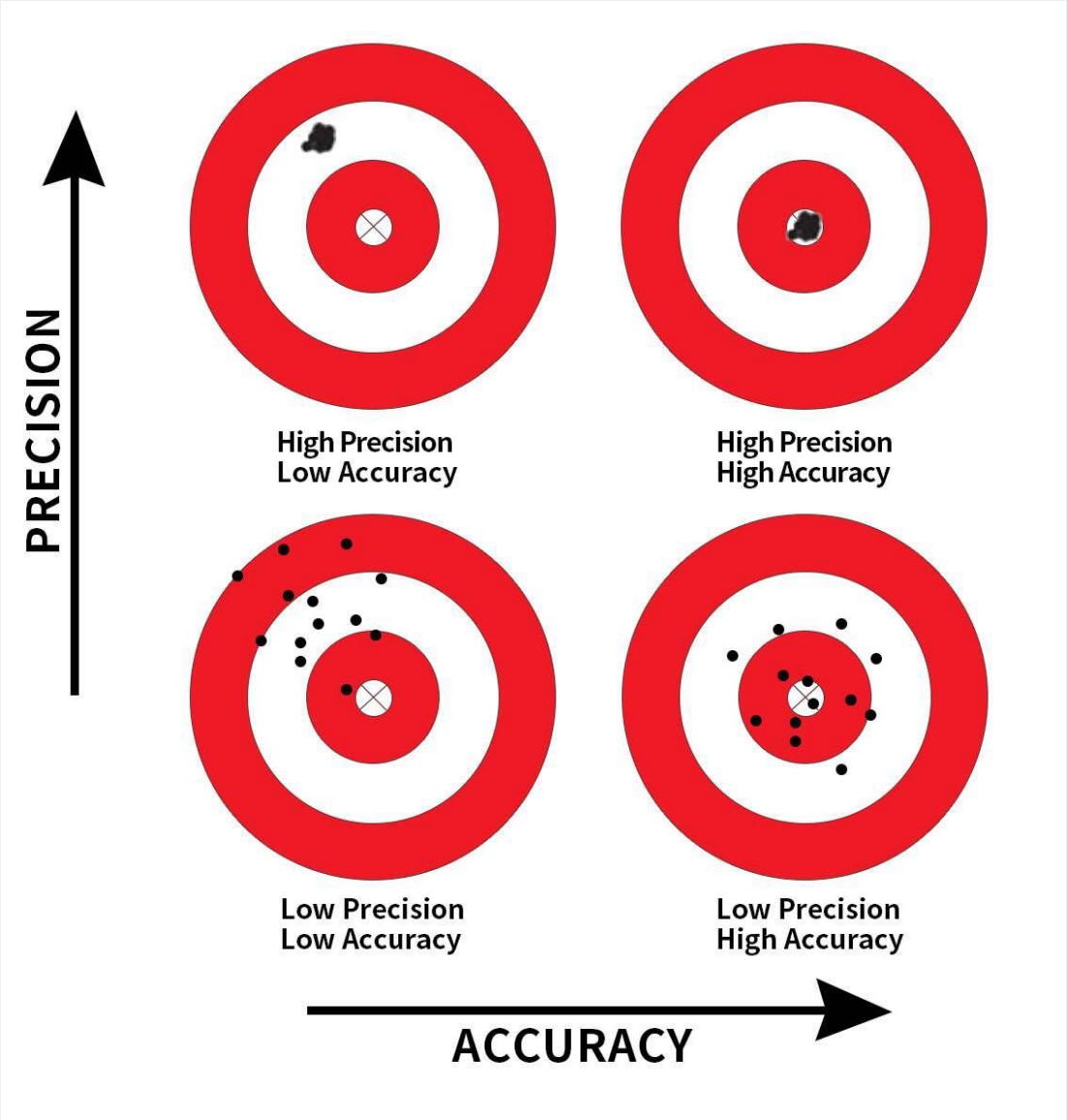
Unbiasedness

An estimator is **unbiased** when its expected value across repeated samples equals the population parameter of interest.

Sample mean of a simple random sample is **unbiased** for the population mean, $\mathbb{E}[\overline{X}_n] = \mu$

An estimator that isn't unbiased is called **biased**.

Precision vs accuracy



Law of large numbers

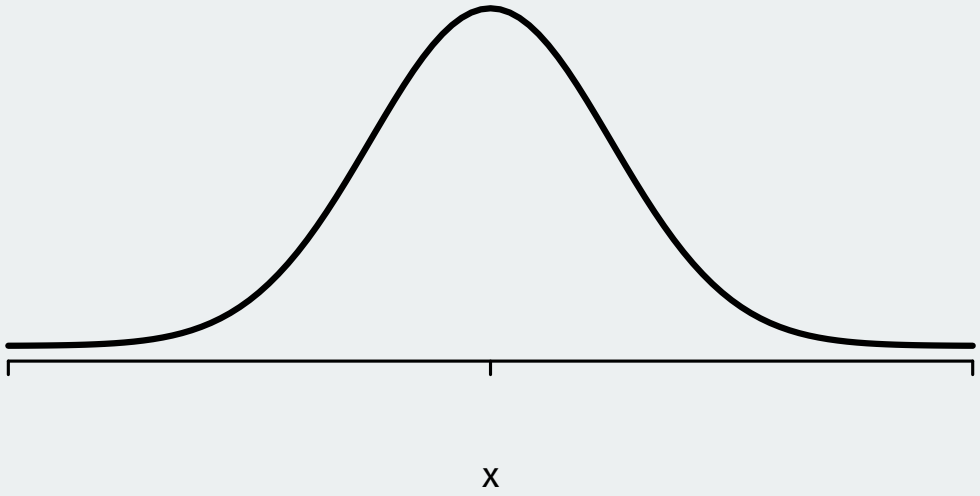
Law of large numbers

Let $X_1, \ldots, X_n$ be a simple random sample from a population with mean μ and finite variance σ^2. Then, $\overline{X}_n$ converges to μ as n gets large.

- Probability of $\overline{X}_n$ being "far away" from μ goes to 0 as n gets big.
- The distribution of sample mean "collapses" to population mean.
- Can see this from the SE of $\overline{X}_n$: $SE = \sigma/\sqrt{n}$.
- Not necessarily true with a biased sample!

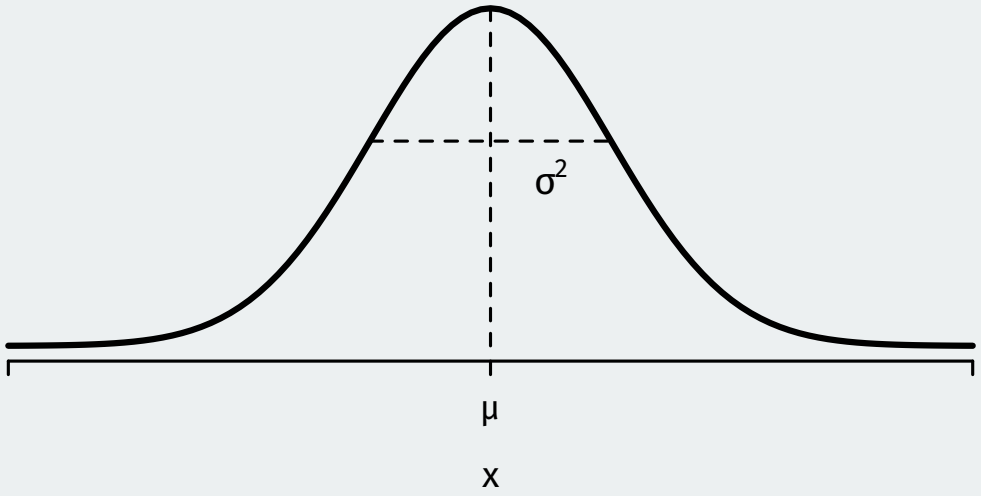
4/ Normal variables and the Central Limit Theorem

Normal random variable



- A **normal distribution** has a PDF that is the classic "bell-shaped" curve.
  - Extremely ubiquitous in statistics.
  - An r.v. is more likely to be in the center, rather than the tails.
- Three key properties of this PDF:
  - **Unimodal**: one peak at the mean.
  - **Symmetric** around the mean.
  - **Everywhere positive**: any real value can possibly occur.

Normal distribution



- A normal distribution can be affect by two values:
  - **mean/expected value** usually written as μ
  - **variance** written as σ^2 (standard deviation is σ)
  - Written $X \sim N(\mu, \sigma^2)$.
- **Standard normal distribution**: mean 0 and standard deviation 1.

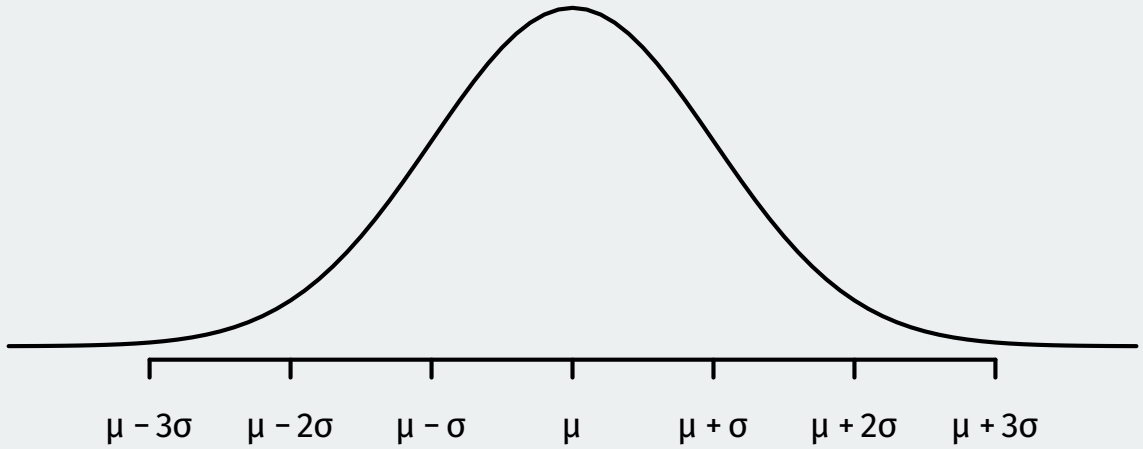
Central limit theorem

Central limit theorem

Let $X_1, \ldots, X_n$ be a simple random sample from a population with mean μ and finite variance σ^2. Then, $\overline{X}_n$ will be approximately distributed $N(\mu, \sigma^2/n)$ in large samples.

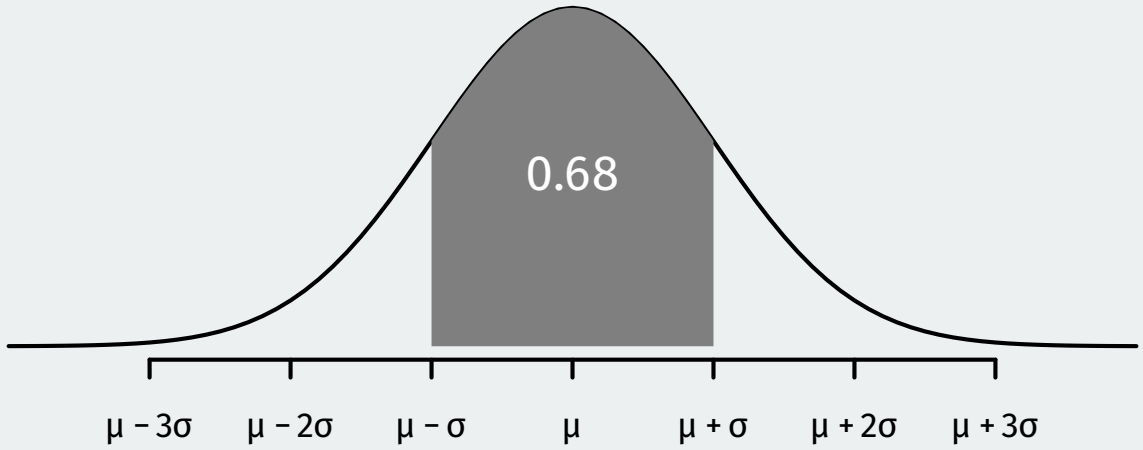
- "Sample means tend to be normally distributed as samples get large."
- $\rightsquigarrow$ we know (an approx. of) the entire probability distribution of $\overline{X}_n$
  - Approximation is better as n goes up.
  - Does not depend on the distribution of X_i!

Empirical Rule for the Normal Distribution



- If $X \sim N(\mu, \sigma^2)$, then:

Empirical Rule for the Normal Distribution



- If $X \sim N(\mu, \sigma^2)$, then:
  - $\approx$ 68% of the distribution of X is within 1 SD of the mean.

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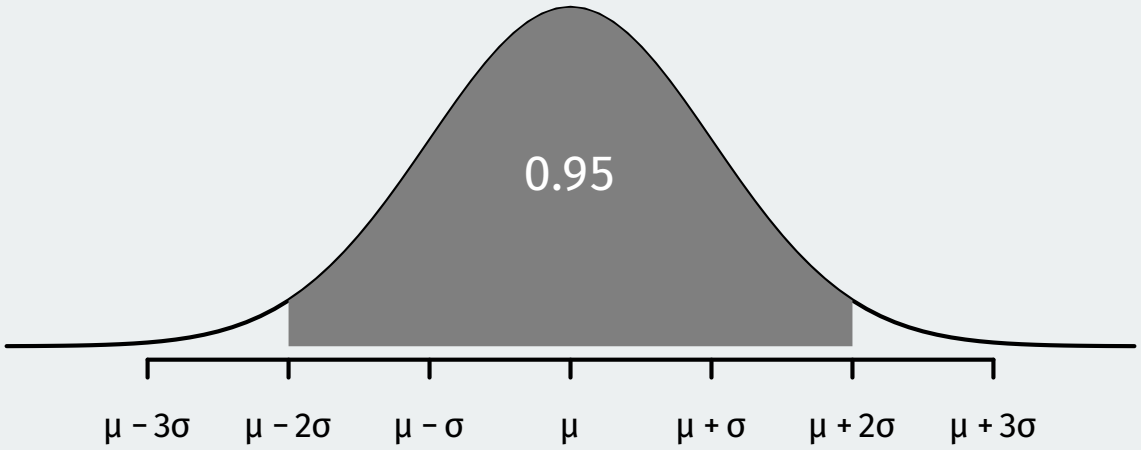
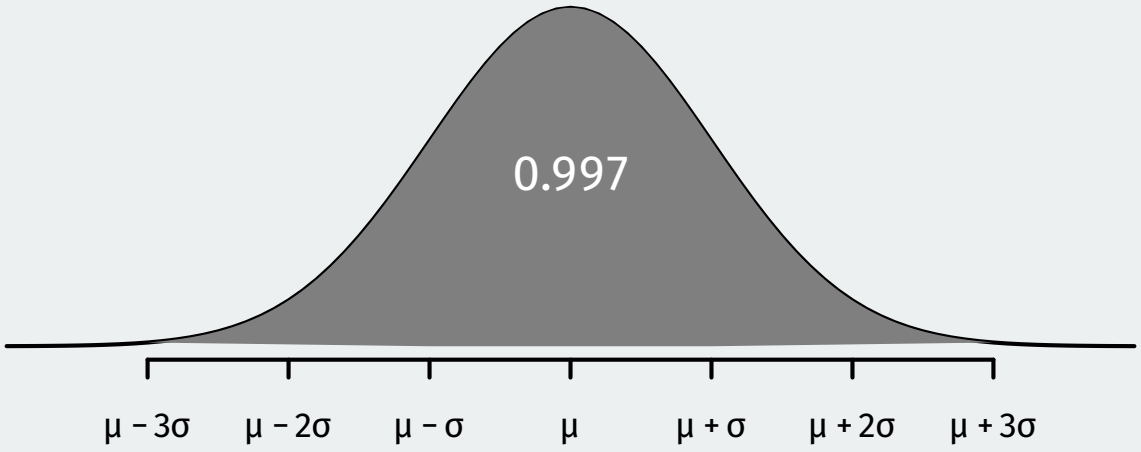


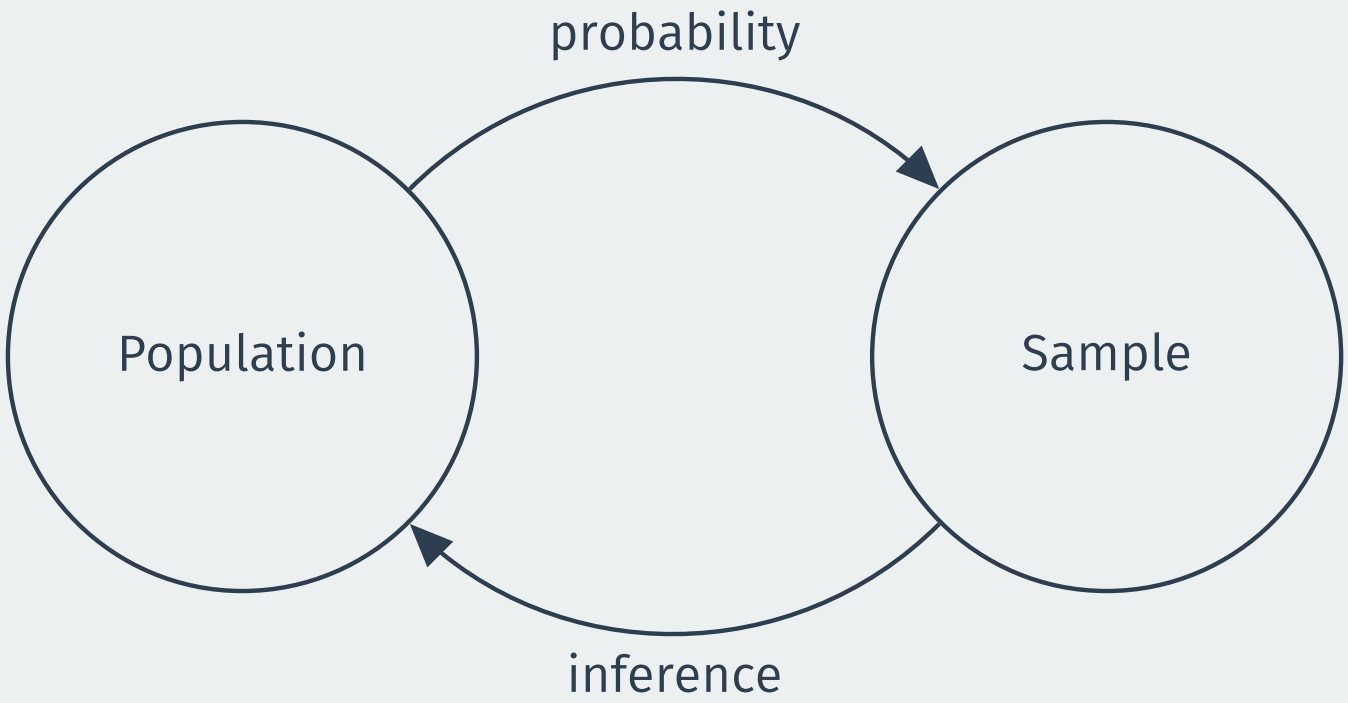
- If $X \sim N(\mu, \sigma^2)$, then:
  - $\approx$ 68% of the distribution of X is within 1 SD of the mean.
  - $\approx$ 95% of the distribution of X is within 2 SDs of the mean.

Empirical Rule for the Normal Distribution



- If $X \sim N(\mu, \sigma^2)$, then:
  - $\approx$ 68% of the distribution of X is within 1 SD of the mean.
  - $\approx$ 95% of the distribution of X is within 2 SDs of the mean.
  - $\approx$ 99.7% of the distribution of X is within 3 SDs of the mean.
- CLT + empirical rule: we'll know the rough distribution of estimation errors we should expect.

Where are we going?



We only get 1 sample. Can we learn about the population from that sample?