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Math 374
Homework on chapter 1.2

1. The Maclaurin series for $(1+x)^n$ is also known as the **binomial series**. It states that

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots \quad (x^2 < 1)$$

Derive this series. Then give its particular forms in summation notation by letting $n = 2$, $n = 3$, and $n = \frac{1}{2}$.

When $n=2$

$$(1+x)^2 = 1 + 2x + \frac{2(2-1)}{2!}x^2 + \ldots$$

$$= 1 + 2x \ \frac{2}{2}x^2 + \ldots$$

$$= \boxed{1 + 2x + x^2}$$

When $n=3$

$$(1+x)^3 = 1 + 3x + \frac{3(3-1)}{2!}x^2 + \frac{3(3-1)(3-2)}{3!}x^3 + \ldots$$

$$= 1 + 3x + \frac{3(2)}{2}x^2 + \frac{3\cdot 2\cdot 1}{3\cdot 2\cdot 1}x^3 + \ldots$$

$$= \boxed{1 + 3x + 3x^2 + x^3}$$

When $n = 1/2$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!}x^3 + \ldots$$

$$= 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3\cdot 2\cdot 1}x^3 + \ldots$$

$$\boxed{= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots}$$

Next use the last form to compute $\sqrt{1.0001}$ correct to 15 decimal places (rounded).

For $\sqrt{1.0001}$:

$$(1+x)^{1/2} \Rightarrow (1.0001)^{1/2} \Rightarrow x = 0.0001$$

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \dots$$

$$= 1 + \frac{0.0001}{2} - \frac{(0.0001)^2}{8} + \frac{(0.0001)^3}{16} - \dots$$

$$\approx 1 + 0.00005 - 0.00000000125 + 0.00000000000000625 \dots$$

$$\approx 1.000049999875006$$

$$\boxed{\therefore \sqrt{1.0001} \approx 1.000049999875006}$$

13. Use the Alternating Series Theorem to determine the number of terms in series (5) needed for computing ln 1.1 with error less than $\frac{1}{2} \times 10^{-8}$.

(5) $\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{x^k}{k}$ $(-1 \leq x \leq 1)$

$\ln(1.1) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} (0.1)^k}{k} = \boxed{\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k \cdot 10^k}}$

Using Alternating Series:

$$\left| \ln(1.1) - \sum_{k=1}^{n} a_k \right| \leq |a_{n+1}|$$

$$|a_{n+1}| < \frac{1}{2} \cdot 10^{-8}$$

$$\frac{(-1)^{k+2}}{(k+1)\, 10^{k+1}} < \frac{1}{2 \cdot 10^8} \Rightarrow \frac{1}{(k+1) \cdot 10^{k-7}} < \frac{1}{2}$$

$\Rightarrow n \geq 7$

$\therefore$ to compute $\ln(1.1)$ with error less than $\frac{1}{2} \cdot 10^{-8}$, we need to use up to 7 terms.

14. Write the Taylor series for the function $f(x) = x^3 - 2x^2 + 4x - 1$, using $x = 2$ as the point of expansion; that is, write a formula for $f(2+h)$.

$c = 2$

$f(x) = x^3 - 2x^2 + 4x - 1 \qquad f(2) = 7$

$f'(x) = 3x^2 - 4x + 4 \qquad f(2) = 8$

$f''(x) = 6x - 4 \qquad f(2) = 8$

$f'''(x) = 6 \qquad f(2) = 6$

Formal Taylor Series for f about c

$$\boxed{f(x) \sim f(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)(x-c)^3}{3!} + \cdots}$$

Therefore:

$$x^3 - 2x^2 + 4x - 1 \sim 7 + 8(x-2) + \frac{8(x-2)^2}{2!} + \frac{6(x-2)^3}{3!} + \cdots$$

$$\sim 7 + 8x - 16 + \frac{8x^2 - 32x + 32}{2} + \frac{6x^3 - 36x^2 + 72x - 48}{6} + \cdots$$

$$\sim 8x - 9 + 4x^2 - 16x + 16 + x^3 - 6x^2 + 12x - 8 + \cdots$$

$$\sim x^3 + 4x^2 - 6x^2 + 8x - 16x + 12x - 8 - 9 + 16 + \cdots$$

$$\sim \boxed{x^3 - 2x^2 + 4x - 1}$$

now we plug in $2+h$

$$f(2+h) = (2+h)^3 - 2(2+h)^2 + 4(2+h) - 1$$

$$= 8 + 12h + 6h^2 + h^3 - 8 - 8h - 2h^2 + 8 + 4h - 1$$

$$= h^3 + 6h^2 - 2h^2 + 12h - 8h + 4h + 8 - 8 + 8 - 1$$

$$\boxed{\Rightarrow h^3 + 4h^2 + 8h + 7}$$