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7. Prove inequality (1)

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$

$$c_n = \frac{a_n + b_n}{2}$$

$$b_n - a_n = \frac{b_0 - a_0}{2^n}$$

$$a_n \leq r \leq b_n$$

$$|r_n - c_n| = \left| r - \frac{a_n + b_n}{2} \right|$$

$$\left| r - \frac{a_n + b_n}{2} \right| \leq \frac{b_n - a_n}{2} = \frac{b_0 - a_0}{2^{n+1}}$$

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$

8. If $a = 0.1$ and $b = 1.0$, how many steps of the bisection method are needed to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$?

$a = 0.1$, $b = 1.0$

$$\frac{b - a}{2^n} \leq \varepsilon$$

$$n \geq \frac{\ln(b-a) - \ln(\varepsilon)}{\ln(2)}$$

$$n \geq \frac{\ln(1.0 - 0.1) - \ln(\frac{1}{2} \cdot 10^{-8})}{\ln(2)}$$

$$n \geq \frac{\ln(0.9) - (-19.11)}{0.69315}$$

$$n \geq 27.4234$$

$n \approx 28$ steps

14. Denote the successive intervals that arise in the bisection method by $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, and so on. Show that

a. $a_0 \leqq a_1 \leqq a_2 \leqq \cdots$ and $b_0 \geqq b_1 \geqq b_2 \geqq \cdots$.

b. $b_n - a_n = 2^{-n}(b_0 - a_0)$.

c. $a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$, for all n.

(a) $a_n = \frac{a+b}{2}$

If $f(a_n) = 0$ then a_n is the root

If $f(a_n) \neq 0$ then we check!

If $f(a) * f(a_n) < 0$, then root lies $[a, a_n]$

If $f(a_n) * f(b) < 0$, then root lies $[a_n, b]$.

Then $a_0 \in [a,b]$ such that $f(a_0) = 0$

The interval is $[a_n, b_n]$

let the midpoint b c_n, then we have $f(c_n) \neq 0$ then

$f(a_n) * f(c_n) < 0$, in interval $[a_n, c_n]$

This gives us $a_{n+1} = a_n$

$b_{n+1} = c_n$

a_n remains the same, but b_n decreases to c_n

In another case

$f(c_n) * f(b_n) < 0$ and the root lies on interval $[c_n, b_n]$

Therefore $a_{n+1} = c_n$, $b_{n+1} = b_n$,

c_n increases and b_n stays same.

Since $a_n < c_n < b_n$

a_n increases or stays same

b_n decreases or stays same

which shows

$a_0 \leq a_1 \leq a_2 \leq \ldots$ $b_0 \geq b \geq b_2 \geq \ldots$

b) $c_0 = \frac{a_0 + b_0}{2}$

next interval is either $[a_0, c_0]$ or $[c_0, b_0]$

$b_1 - a_1 = \frac{b_0 - a_0}{2}$

further iterations:

$b_2 - a_2 = \frac{b_1 - a_1}{2}$

$= \frac{b_0 - a_0}{2^2}$

$b_3 - a_3 = \frac{b_2 - a_2}{2}$

$= \frac{b_0 - a_0}{2^3}$

which gives

$b_n - a_n = \frac{b_0 - a_0}{2^n} = 2^{-n}(b_0 - a_0)$

The length of an interval is the difference between the interval's upper and lower endpoints

C) Using the induction theorem,

Suppose $a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$

and $n = 1$

$a_1 b_1 + a_0 b_0 = a_0 b_1 + a_1 b_0$, so since

$c_0 = \frac{a_0 + b_0}{2}$

If $f(a_0) \cdot f(c_0) < 0$ then $a_1 = a_0$ and $b_1 = c_0$

So, $a_1 b_1 + a_0 b_0 = a_0 c_0 + a_1 b_0$

$= a_0 b_1 + a_1 b_0$

and if $f(c_0) \cdot f(b_0) < 0$ then $a_1 = c_0$, $b_1 = b_0$

then $a_1 b_1 + a_0 b_0 = c_0 b_0 + a_0 b_1$

$= a_1 b_0 + a_0 b_1$

There for both of them satisfy fo $n=1$

Therefore

$a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n \, a_n b_{n-1}$ for all n

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15. (Continuation) Can $a_0 = a_1 = a_2 = \cdots$ happen?

16. (Continuation) Let $c_n = (a_n + b_n)/2$. Show that

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$$

15)

$a_0 = a_1 = a_2 = \ldots$ can happen

if and only if

$f(a_0) = 0$, so iff a_0 is the root

16)

from previous

$b_n - a_n = \frac{b_0 - a_0}{2^n}$

interval decreases as $n \to \infty$,

so a_n and b_n get closer

to each othe which

makes them converge to the

same limit. So,

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ and

Suppose $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = L$

exists for L

then

$c_n = \frac{a_n + b_n}{2}$ can be

$\lim_{n\to\infty} c_n = \frac{L}{2} = L$

$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n$

22. If the bisection method is applied with starting interval $[2^m, 2^{m+1}]$, where m is a positive or negative integer, how many steps should be taken to compute the root to full machine precision on a 32-bit word-length computer?

To compute number of iterations required for bisection method:

$$\frac{b-a}{2^n} \leq \varepsilon$$

$$n = \log_2\left(\frac{b-a}{\varepsilon}\right)$$

$$\varepsilon = 1.19 \cdot 10^{-7} \approx 2^{-23}$$

interval is $[2^m, 2^{m+1}]$

$$n = \log_2\left(\frac{2^{m+1} - 2^m}{2^{-23}}\right)$$

$$= \log_2\left(\frac{2^m}{2^{-23}}\right)$$

$$= \log_2\left(2^{m+23}\right)$$

$$= m + 23$$

$\therefore$ number of steps required is

$$n = m + 23$$