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1. Verify that when Newton's method is used to compute $\sqrt{R}$ (by solving the equation $x^2 = R$), the sequence of iterates is defined by

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

2. (Continuation) Show that if the sequence $\{x_n\}$ is defined as in the preceding exercise, then

$$x_{n+1}^2 - R = \left[\frac{x_n^2 - R}{2x_n}\right]^2$$

(1) $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

$f(x) = x^2 - R$

$f'(x) = 2x$

$$x_{n+1} = x_n - \frac{x_n^2 - R}{2x_n}$$

$$x_{n+1} = x_n - \frac{x_n^{\cancel{2}}}{2\cancel{x_n}} + \frac{R}{x_n}$$

$$x_{n+1} = x_n - \frac{x_n}{2} + \frac{R}{x_n}$$

$$x_{n+1} = \frac{2x_n - x_n}{2} + \frac{R}{2x_n}$$

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

(2) We start with the formula

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

then for $x_{n+1}^2 - R$:

$$x_{n+1}^2 - R = \left(\frac{1}{2}\left(x_n + \frac{R}{x_n}\right)\right)^2 - R$$

$$= \frac{1}{4}\left(x_n + \frac{R}{x_n}\right)^2 - R$$

$$= \frac{1}{4}\left(x_n^2 + 2\cancel{x_n} \cdot \frac{R}{\cancel{x_n}} + \frac{R^2}{x_n^2}\right) - R$$

$$= \frac{1}{4}\left(x_n^2 + 2R + \frac{R^2}{x_n^2}\right) - R$$

$$= \frac{x_n^2}{4} + \frac{2R}{4} + \frac{2R^2}{4x_n^2} - R$$

$$= \frac{x_n^2}{4} + \frac{R}{2} + \frac{R^2}{2x_n^2} - \frac{2R}{2}$$

$$x_{n+1}^2 - R = \frac{x_n^2}{4} + \frac{R^2}{4x_n^2} - \frac{R}{2}$$

Now we consider the right handside

$$x_{n+1}^2 - R = \frac{x_n^4 + R^2 - 2Rx_n^2}{4x_n^2}$$

$$= \frac{(x^2)^2 + R^2 - 2Rx_n^2}{(2x_n^2)^2}$$

$$= \frac{(x_n^2 - R)^2}{(2x_n^2)^2}$$

$$x_{n+1}^2 - R = \left(\frac{x_n^2 - R}{2x_n}\right)^2$$

$\therefore$ Squared error satisfies the expression

13. Each of the following functions has $\sqrt[3]{R}$ as a zero for any positive real number R. Determine the formulas for Newton's method for each and any necessary restrictions on the choice for x_0.

[a]**a.** $a(x) = x^3 - R$ **b.** $b(x) = 1/x^3 - 1/R$

[a]**c.** $c(x) = x^2 - R/x$ **d.** $d(x) = x - R/x^2$

b) $b(x) = \frac{1}{x^3} - \frac{1}{R}$

$b'(x) = -\frac{3}{x^4}$

$x_{n+1} = x_n - \dfrac{\frac{1}{x^3} - \frac{1}{R}}{-\frac{3}{x^4}}$

$= x_n + \frac{x_n}{3}\left(1 - \frac{x_n^3}{R}\right)$

restriction on x_0 is $x_0 \neq 0$ $x_0 > 0$

$d(x) = x - \frac{R}{x^2}$

$d'(x) = 1 + \frac{2R}{x^3}$

$x_{n+1} = x_n - \dfrac{x_n - \frac{R}{x_n^2}}{1 + \frac{2R}{x_n^3}}$

$x_{n+1} = x_n - \dfrac{x_n^4 - R}{x_n^3 + 2R}$

$x_0 \neq 0$

14. Determine the formulas for Newton's method for finding a root of the function $f(x) = x - e/x$. What is the behavior of the iterates?

$f(x) = x - \frac{e}{x}$

$f'(x) = 1 + \frac{e}{x^2}$

$x_{n+1} = x_n - \dfrac{x_n - \frac{e}{x_n}}{1 + \frac{e}{x_n^2}}$

$= \dfrac{x_n\left(1 + \frac{e}{x_n^2}\right) - \left(x_n - \frac{e}{x_n}\right)}{1 + \frac{e}{x_n^2}}$

$= \dfrac{\cancel{x_n} + \frac{e}{x_n} - \cancel{x_n} + \frac{e}{x_n}}{1 + \frac{e}{x_n^2}}$

$= \dfrac{2e}{x_n + \frac{e}{x_n}}$

Iterates x_n converge to root of $f(x) = 0$, which is $x = \sqrt{e}$

$\therefore$ number of correct digit doubles each iteration

18. Determine Newton's iteration formula for computing the cube root of N/M for nonzero integers N and M.

$$x = \sqrt[3]{\frac{N}{M}}$$

$$(x)^3 = \left(\sqrt[3]{\frac{N}{M}}\right)^3$$

$$x^3 = \frac{N}{M}$$

$$f(x) = x^3 - \frac{N}{M}$$

$$f'(x) = 3x^2 - 0$$

$$x_{n+1} = x_n - \frac{x^3 - \frac{N}{M}}{3x^2}$$

$$x_{n+1} = x_n - \frac{x^3}{3x^2} - \frac{N/M}{3x^2}$$

$$x_{n+1} = x_n - \frac{x}{3} + \frac{N/M}{3x^2}$$

$$= \frac{3x - x}{3} + \left(\frac{N}{M} : \frac{1}{3x^2}\right)$$

$$= \frac{2x}{3} + \frac{3x^2 N}{M}$$

25. Newton's method for finding $\sqrt{R}$ is

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

Perform three iterations of this scheme for computing $\sqrt{2}$, starting with $x_0 = 1$, and of the bisection method for $\sqrt{2}$, starting with interval $[1, 2]$. How many iterations are needed for each method in order to obtain 10^{-6} accuracy?

Newton

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right) \quad \text{for } R = 2 \text{ and } x_0 = 1$$

$$x_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right) \qquad x_2 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right)$$

$$x_1 = \frac{3}{2} \qquad = 1.4167$$

$$x_1 = 1.5$$

$$x_3 = \frac{1}{2}\left(1.4167 + \frac{2}{1.4167}\right)$$

$$= 1.4142$$

Converges quadratically ∴ would require 5 iterations

Bisection $[1,2]$

$$x_n = \frac{a+b}{2}$$

$$x_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5$$

$$x_2 = \frac{1+1.5}{2} = 1.25$$

$$x_3 = \frac{1.25 + 1.5}{2} = 1.375$$

$$\frac{b-a}{2^n} \leq \varepsilon$$

$$n \geq \log_2\left(\frac{b-a}{\varepsilon}\right)$$

$$n \geq \frac{\log(1) - \log(10^{-6})}{\log 2}$$

$$n \geq \frac{6}{0.301}$$

$n \geq 20$ Bisection method would need 20 iterations