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Chapter 4.3

3. Derive the approximation formula

$$f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$$

Show that its error term is of the form $\frac{1}{3}h^2 f'''(\xi)$.

Taylor expunsion at $x+h$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4)$$

at $x+2h$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{6}f'''(x) + O(h^4) =$$

$$= f(x) + 2hf'(x)\ 2h^2 f''(x) + \frac{4h^3}{3}f'''(x) + O(h^4)$$

Subsitute:

$$\frac{1}{2}\Big[4\Big(f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{6}f'''(x) + O(h^4)\Big) - 3f(x) -$$

$$- f(x) + 2h^2 f'(x) + 2h^2 f''(x) + \frac{4h^3}{3}f'''(x) + O(h^4)\Big]$$

$$\Rightarrow f'(x) = \frac{1}{2h}\Big[4hf'(x) + 2h^2 f''(x) + \frac{2}{3}h^3 - 2hf'(x) -$$

$$- \frac{4}{3}h^3 f'''(x)\Big]$$

$$\Rightarrow f'(x) = \frac{1}{\cancel{2h}}\Big[\cancel{2}hf'(x) - \frac{\cancel{2}}{3}h^{\cancel{3}}f'''(x)\Big]$$

$$\Rightarrow f'(x) = f'(x) - \frac{h^2}{3}f'''(x)$$

$\therefore$ error term is

$$\frac{h^2}{3}f'''(x)$$

5. Averaging the forward-difference formula $f'(x) \approx [f(x+h) - f(x)]/h$ and the backward-difference formula $f'(x) \approx [f(x) - f(x-h)]/h$, each with error term $\mathcal{O}(h)$, results in the central-difference formula $f'(x) \approx [f(x+h) - f(x-h)]/(2h)$ with error $\mathcal{O}(h^2)$. Show why.
Hint: Determine at least the first term in the error series for each formula.

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \dots$$

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \dots$$

①

$$hf'(x) = f(x+h) - f(x) - \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \dots$$

$$\Rightarrow \frac{f(x+h - f(x)}{h} - \frac{h}{2} f''(x) - \frac{h^2}{6} f'''(x) + \dots$$

$$\Rightarrow \frac{f(x+h) - f(x)}{h} + O(h)$$

②

$$hf'(x) = f(x) - f(x-h) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \dots$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2} f''(x) - \frac{h^2}{6} f'''(x) + \dots$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$

Now 1−2

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^3}{6} f'''(x) + \dots$$

$$\Rightarrow 2hf'(x) = f(x+h) - f(x-h) - \frac{h^3}{6} f'''(x) - \dots$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{h^2}{6} f'''(x) - \dots$$

$$\Rightarrow f'(x) = \underbrace{\frac{f(x+h) - f(x-h)}{2h}}_{\text{central difference}} + \underbrace{O(h^2)}_{\text{error term}}$$

8. Derive the two formulas

~~**a.**~~ $f'(x) \approx \frac{1}{4h}[f(x+2h) - f(x-2h)]$

b. $f''(x) \approx \frac{1}{4h^2}[f(x+2h) - 2f(x) + f(x-2h)]$

Establish formulas for the errors in using them.

$$f''(x) \approx \frac{1}{4h^2}\left[f(x+2h) - 2f(x) + f(x-2h)\right]$$

$$f(x+2h) = f(x) + 2hf'(x) + \frac{4h^2}{2}f''(x) + \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^4(x) + O(h^5)$$

$$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^4(x) + O(h^5)$$

Substitute in given formula:

$$f''(x) \approx \frac{1}{4h^2}\Big[f(x) + 2hf'(x) + 2h^2f''(x) + \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^4(x) - 2f(x)$$

$$+ f(x) - 2hf'(x) + 2h^2f''(x) - \frac{8h^3}{6}f'''(x) + \frac{16h^4}{24}f^4(x)\Big]$$

$$\Rightarrow \frac{1}{\cancel{4h^2}}\left[\cancel{4h^2}f''(x) + \frac{2h^4}{3}f^4(x) + O(h^6)\right]$$

$$\Rightarrow f''(x) \approx \underbrace{\frac{h^2}{6}f^4(x)}_{\text{error term.}} + O(h^6)$$