Project 2

Comparison of Iterative Methods for Solving Nonlinear Equations

Govher Sapardurdyyeva

South Dakota State University

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Jung-Han Kimn

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Abstract

This paper explores three iterative methods—**Bisection Method**, **Newton's Method**, and **Secant Method**—to determine the most efficient approach for solving nonlinear equations. Each method is examined in terms of its convergence rate, computational efficiency, and suitability for various types of nonlinear functions. We apply these methods to specific nonlinear equations and analyze their performance by comparing the number of iterations required to reach the root, the accuracy of the solutions, and the reliability of each method. The results highlight the strengths and limitations of each approach, providing insight into the best practices for solving nonlinear equations in different scenarios.

Introduction

Numerous iterative techniques have been developed to approximate solutions for nonlinear equations, which are an essential task in numerical analysis. Since determining the root or roots of a nonlinear equation can be extremely challenging and perhaps impossible, we usually use repeating numerical techniques to do so. Nowadays, nonlinear equations and a variety of complicated and challenging problems can be resolved thanks to computer advancements and the expansion of applied software.

In order to identify at least one root , our goal in this project is to apply and evaluate the Bisection, Newton, and Secant methods to the following equations:

$$f(1) = x^2 - 4sin(x)$$

$$f(2) = x^2 - 1$$

$$f(3) = x^3 - 3x^2 + 3x - 1$$

Every one of these equations has distinct difficulties and provides a chance to investigate the efficacy of various root-finding strategies. We will examine each method's termination criteria, examine the rates of convergence, and compare the results of the three approaches.

Implementing the methods

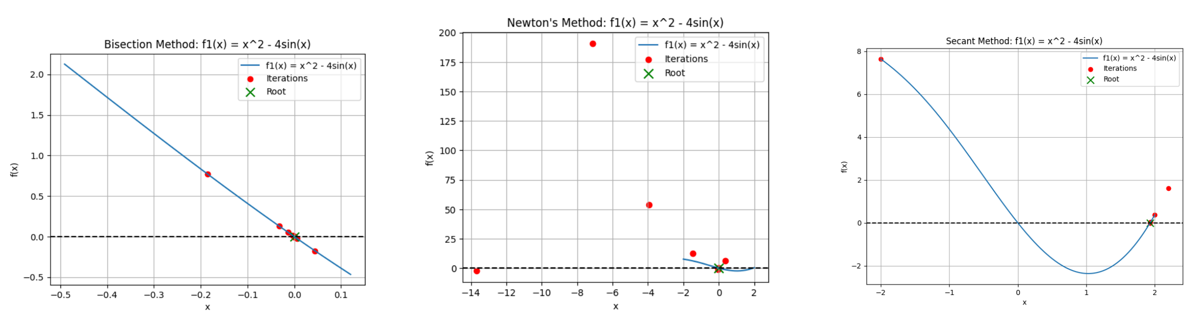
To find the roots of the functions using the **Bisection Method**, I initially considered the interval $[-2,2]$ for each function. However, to enhance the robustness and accuracy of the results, I implemented the **Monte Carlo method** to randomize the initial guesses within the interval $[-2,2]$. The idea of incorporating the **Monte Carlo method** into the **Bisection Method** was inspired by a research paper on root-finding techniques. The Monte Carlo method allowed for a more dynamic and flexible selection of the initial interval, increasing the likelihood of finding the root by exploring different regions of the function.

However, implementing the **Monte Carlo Method** for randomizing the initial values did not prove to be effective for the **Newton** and **Secant methods**. These methods rely on starting points that are close to the actual root to ensure fast and reliable convergence. Since the randomized values were not close to the root of the functions, the program often entered an endless loop or failed to converge. This is because the methods, particularly Newton's Method, can behave unpredictably when starting from a point too far from the root, leading to incorrect results or failure to find the root within the given iterations.

Instead of randomizing the values, we chose to use **1.0** as the initial guess for both methods. This value was selected because it is a reasonable starting point for the given functions, and it helps to strike a balance between computational efficiency and convergence reliability. By starting from a fixed, sensible initial value, we aimed to ensure the methods would converge more quickly to the correct root, especially for functions that have a single root or well-behaved derivatives near the root.

Analyzing the results

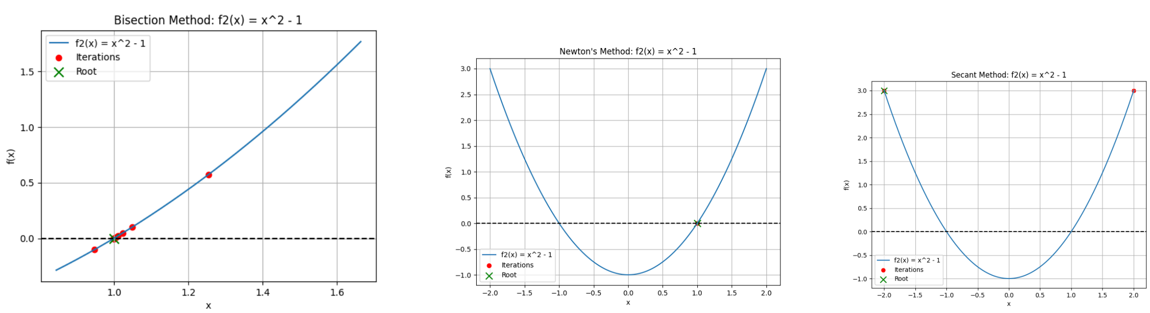
Now let's analyze the graphs. The graphs for **Function 1** show the behavior of the **Bisection**, **Newton**, and **Secant** methods in finding the root within the interval $[-2,2]$.



(Figure 1)

The **Bisection Method** progresses slowly, narrowing the interval with each iteration, and guarantees convergence, as indicated by the red points gradually approaching the root located at x=1.9337. It requires 19 iterations to reach the approximate accuracy. In contrast, **Newton's Method** exhibits much faster convergence, with the red dots quickly moving towards the root in just 9 iterations. However, the graph also reveals that the method starts far from the root in the initial steps, which can lead to large jumps if the initial guess is not ideal. Despite this, it converges quadratically, demonstrating the method's rapid efficiency when the initial guess is well-chosen. Similarly, the **Secant Method** converges in 7 iterations, providing a similar speed to Newton's method but without requiring the function's derivative. This method exhibits superlinear convergence, making it faster than the Bisection Method but slower than Newton's Method. In conclusion, the Bisection Method is slow but dependable, while Newton's and Secant methods converge more quickly, each with different advantages and challenges related to the starting points and the need for derivatives.

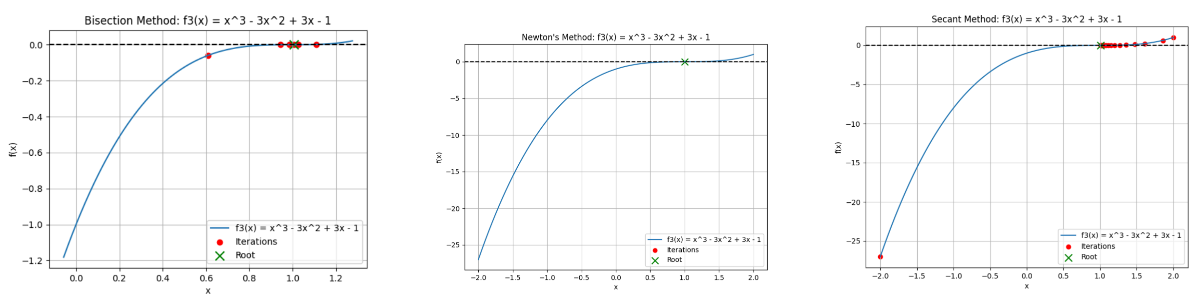
Now let's take a look at Function 2.



(Figure 2)

The **Bisection Method** starts by halving the interval [1,1.6] and moves gradually towards the root at x=1.0, taking several iterations to converge. While slow, the method is reliable and guarantees finding the root. In contrast, **Newton's Method** converges very quickly, reaching the root in just **1 iteration** due to its quadratic convergence rate, demonstrating its efficiency for this function. Similarly, the **Secant Method** also converges rapidly, finding the root in **2 iterations**, making it a fast alternative to the Bisection Method, though it does not require the derivative of the function. Overall, while the Bisection Method is slow but reliable, Newton's and Secant methods offer much faster convergence, with Newton's being the most efficient for this particular function.

Finally, let's take a look at **Function 3**. The **Bisection Method** converges to the root at x=1 after several iterations, gradually narrowing the interval in which the root lies. While reliable, it is slower compared to the other methods.



(Figure 3)

Newton's Method, on the other hand, demonstrates rapid convergence, reaching the root in just a few iterations, showcasing its quadratic convergence rate. However, this method can be sensitive to the initial guess. The **Secant Method** also finds the root relatively quickly, although it requires more iterations than Newton's Method. While it is slower than Newton's Method, the Secant Method is still faster than Bisection and does not require the function's derivative. In summary, Newton's Method is the fastest for this function, followed by the Secant Method, with Bisection being the most reliable but slowest option.

Table 1

	Number of Iterations		
Method used	Function 1	Function 2	Function 3
Bisection Method	**21**	**21**	**8**
Newton Method	**9**	**1**	**0**
Secant Method	**7**	**2**	**19**

Analysis

For iterative methods such as the **Bisection**, **Newton**, and **Secant methods**, a suitable termination criterion ensures the methods stop once an adequate approximation of the root is found. For the **Bisection Method**, the most common termination criterion is when the width of the interval $|b-a|$ is smaller than a specified tolerance δ, or when the function value at the midpoint $|f(c)|$ is smaller than a specified tolerance ϵ. For **Newton's Method** and **Secant Method**, the termination criteria are often based on the change in the approximation, the function value, or the change in the function value between iterations. Specifically, if the change in x or f(x) is smaller than a threshold, the methods will stop. For this analysis, the stopping criteria for all methods were based on function value tolerance and the change in the approximations.

The Bisection Method demonstrates linear convergence, which means that the error is cut in half with each iteration, according to the results. Even though convergence is assured, the rate of convergence is somewhat slow, particularly in comparison with other techniques. Newton's method achieves quadratic convergence, which means that as it gets closer to the root, the error drops exponentially. Once the procedure is close enough to the root, this leads to very quick convergence. However, it requires a good initial guess to avoid divergence. The Secant Method converges more slowly than Newton's Method but more quickly than the Bisection Method due to its superlinear convergence. The golden ratio, or its convergence rate, is roughly 1.618.

The Bisection Method took the longest, requiring 21 iterations for Functions 1 and 2, and 8 iterations for Function 3 (**Refer to table 1)**. It was reliable, but out of the three approaches, it converged the slowest. The fastest convergence, however, was attained by Newton's Method, which showed quadratic convergence by obtaining the root for Function 2 in just one iteration and Function 3 in zero. It took nine iterations for Function 1. The Secant Method required seven iterations for Function 1, two for Function 2, and nineteen for Function 3. It converged more quickly than the Bisection Method but more slowly than Newton's Method.

Conclusion

In this project, we developed and tested three important iterative methods for solving nonlinear equations: bisection, Newton, and secant. Each method was evaluated based on its ability to find the root of three distinct functions. The Bisection Method was the slowest and required a lot more iterations than the other two approaches, but it was dependable and guaranteed convergence. With quadratic convergence and the fewest iterations needed to discover the root, Newton's Method demonstrated the quickest convergence, particularly for functions with well-behaved derivatives. However, the original guess's selection had a significant impact on its performance. Even while the Secant Method was slower than Newton's, it offered a good compromise, allowing for quick convergence without the need for the derivative of the function, making it a flexible substitute. In conclusion, the Secant Method provides a good trade-off between speed and complexity, the Newton Method is the most effective for quick convergence, and the Bisection Method is still the most dependable but slowest approach, particularly when derivative information is not available or the initial guess is not certain. The properties of the function and the computational resources at hand determine how effective each method is.

To achieve optimal results in real-world problems, one effective approach could be combining the **Bisection** and **Newton** methods. This combination would provide both speed and reliability, with the **Bisection Method** helping to narrow down the best initial guess before applying the faster **Newton Method** for rapid convergence.

References

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