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The Lipschitz metric for real-valued continuous functions

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ABSTRACT

For a continuous real function f defined on a metric space X, let $\alpha(f)$ denote its minimal Lipschitz constant if f is Lipschitz and put $\alpha(f) = \infty$ otherwise. We study the extended real-valued metric on the continuous real functions defined by $d(f, g) = \max\{|f(x_0) - g(x_0)|, \alpha(f - g)\}$. When $X = [a, b]$ this metric provides new insight into a classical result regarding the derivative of a limit of a sequence of real-valued functions defined on the interval.

1. Introduction

Let $d : X \times X \to [0, \infty)$ be a metric. A function $f : X \to \mathbb{R}$ is called *λ-Lipschitz* for a nonnegative real number λ if $|f(x) - f(w)| \leq \lambda d(x, w)$ whenever x and w are in X. We denote the class of λ-Lipschitz functions by $\mathrm{Lip}_\lambda(X)$ and we write $\mathrm{Lip}(X)$ for the class of (continuous) real-valued functions on X that are λ-Lipschitz for some nonnegative λ. While $\mathrm{Lip}_\lambda(X)$ is stable under pointwise convergence, $\mathrm{Lip}(X)$ is not even stable under uniform convergence, as by the Weierstrass approximation theorem, each continuous non-Lipschitz function on $[a, b]$ is the uniform limit of a sequence of polynomials [15].

Now for each Lipschitz function f on X, there is a smallest nonnegative λ for which f is λ-Lipschitz, often called *the Lipschitz constant* for f, which we denote by $\alpha(f)$ in the sequel. Evidently,

$$\alpha(f) = \sup\left\{\frac{|f(x) - f(w)|}{d(x, w)} : x \neq w\right\}.$$

Of course, $\mathrm{Lip}(X)$ is a vector space and it is easy to check that $f \mapsto \alpha(f)$ is a seminorm on $\mathrm{Lip}(X)$ ($\alpha(f) = 0$ if and only f is a constant function). We put, consistently with the formula above, $\alpha(f) = \infty$ whenever f fails to be Lipschitz.

Of particular interest are the bounded Lipschitz functions which we denote by $\mathrm{Lip}^b(X)$; they are stable under pointwise products and thus form an algebra of functions. There are two favored norms with which we can equip $\mathrm{Lip}^b(X)$. The first $\|\cdot\|_S$, studied in the seminal papers of Sherbert [16,17] (see also [4,8,14]), makes $\mathrm{Lip}^b(X)$ a commutative Banach algebra with identity:

$$\|f\|_S := \alpha(f) + \|f\|_\infty \quad \text{for } f \text{ in } \mathrm{Lip}^b(X).$$

The second, given primacy in the monograph of Weaver [19], is evidently equivalent to the first:

$$\|f\|_W := \max\{\alpha(f), \|f\|_\infty\} \quad \text{for } f \text{ in } \mathrm{Lip}^b(X).$$

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While Weaver's norm fails to be submultiplicative, it perhaps has better properties overall, and has equal status in the literature (see, e.g., [6,7,9,13]).

When $\langle X, d\rangle$ is an unbounded metric space, there will be unbounded Lipschitz functions, and neither of the formulas above gives a norm on $\mathrm{Lip}(X)$. Instead, fixing $x_0 \in X$, one may use the following formula to define a norm:

$$\|f\|_L := \max\{|f(x_0)|, \alpha(f)\} \quad \text{for } f \text{ in } \mathrm{Lip}(X).$$

Replacing x_0 in the definition by a different point of X yields an equivalent norm, as does replacing the maximum by a sum in the defining formula. It can be shown that $\langle \mathrm{Lip}(X), \|\cdot\|_L\rangle$ is a Banach space [10, Prop. 2.5], which in particular means that if $\langle f_n\rangle$ is a sequence of Lipschitz functions that converges in $\|\cdot\|_L$-distance to a real-valued function g, then g must be Lipschitz as well, which is not the case for uniform convergence. Whether we choose to use a maximum or a sum to define our norm, a remarkable fact about either space is that the map $x \to \widehat{x}$ where $\widehat{x}$ is the evaluation functional on $\mathrm{Lip}(X)$ is an isometric embedding of the metric space $\langle X, d\rangle$ into the continuous dual space $(\mathrm{Lip}(X))^*$ of the normed space $\mathrm{Lip}(X)$ [10, Lemma 3.4]. Of course, since dual spaces are complete, this leads to an attractive construction of the completion of $\langle X, d\rangle$ that does not seem to be so well-known. Curiously, the same mapping is an isometry with respect to the Weaver norm on $\mathrm{Lip}^b(X)$ if and only if $\mathrm{Diam}(X) \le 2$, as shown by Vasavada [18] (see also [19, p. 25]).

When X itself is a real normed linear space and x_0 is the origin of the space, then of course $X^* \subseteq \mathrm{Lip}(X)$ and $\|\cdot\|_L$ coincides with the operator norm on X^* [12, p. 62]. Thus convergence with respect to $\|\cdot\|_L$ in X^* means uniform convergence on bounded subsets of X. For general real-valued Lipschitz functions, $\|\cdot\|_L$-convergence is considerably stronger than uniform convergence on bounded subsets.

Our norm $\|\cdot\|_L$ on $\mathrm{Lip}(X)$ gives rise to the metric

$$d_L(f, g) := \max\{|f(x_0) - g(x_0)|, \alpha(f - g)\}.$$

Replacing $\mathrm{Lip}(X)$ by the continuous real-valued functions $C(X)$ on X, this formula defines an extended real-valued metric that can assume values of ∞ which still turns out to be complete. We will call this the *Lipschitz extended metric* on $C(X)$. Evidently, this extended metric falls out of an extended norm on $C(X)$.

We will show that this space provides the ideal framework in which to address the following classical question from analysis: given a sequence $\langle f_n\rangle$ in $C([a, b])$, each function differentiable on (a, b) and a prospective limit function f, what kind of convergence notion will ensure that f is differentiable on (a, b) and for each x, $f'(x) = \lim_{n\to\infty} f_n'(x)$ as well? We give necessary and sufficient conditions on the structure of $\langle X, d\rangle$ for our Lipschitz extended metric topology to reduce to the topology of uniform convergence on bounded subsets and reconcile it with other extended real-valued metrics that arise from other Lipschitz norms. Finally, we will characterize those subsets of the function space that are compact in terms of a perhaps unexpected definition of boundedness in such a function space.

While we define our Lipschitz extended metric only for scalar-valued functions, it could be defined more generally for functions taking values in n-dimensional space or even in a Banach space, and many of our results go through. We have chosen our framework as we have for two reasons: (1) to coincide with that of the monograph of Weaver [19] on normed linear spaces of Lipschitz functions and the majority of the literature in the field, and (2) to show how such spaces illuminate a basic result from elementary analysis that everyone knows.

2. Preliminaries

An extended real-valued metric d on a set X is a function $d : X \times X \to [0, \infty]$ that satisfies the three standard properties that an ordinary finite-valued metric enjoys. In the sequel, we shall call such a function an *extended metric*, and we call a set equipped with such a metric an *extended metric space*, adopting the notation $\langle X, d\rangle$ for such an object. When we wish the extended metric to be real-valued, we shall simply call it a metric.

If $x_0 \in X$ and $\mu \in (0, \infty)$, we put $B_d(x_0; \mu)$ for the open ball in X with center x_0 and radius μ, that is, $B_d(x_0; \mu) := \{x \in X : d(x, x_0) < \mu\}$. The topology on X determined by d is defined in the expected way: a subset V of X is declared open provided it contains an open ball about each of its points. If we replace the extended metric d with the bounded metric $\min\{d, 1\}$, we obtain the same topology. Two extended metrics will of course be called *topologically equivalent* if they determine the same topology. Defining the Cauchy sequence in the usual way, we see that $\langle x_n\rangle$ is Cauchy with respect to d if and only if it is Cauchy with respect to $\min\{d, 1\}$, and so completeness of d is equivalent to completeness of the bounded equivalent metric.

Why not pass immediately to the bounded topologically equivalent metric? For one thing, the particular geometric meaning of distance may be obscured, as is the case when we define *Hausdorff distance* H_d [2] between nonempty closed sets as determined by a metric d on X:

$$H_d(E, F) := \max\ \{\sup_{x\in F} d(x, E), \sup_{x\in E} d(x, F)\}.$$

Moreover, passing to this bounded metric obliterates the large structure of $\langle X, d\rangle$ which is crucial to our analysis in the final section of this paper.

We call a nonempty subset B of $\langle X, d\rangle$ *bounded* provided it is contained in a finite union of open balls. Equivalently, B is bounded if B is a finite union of subsets of finite diameter, where for $A \neq \emptyset$, $\mathrm{Diam}(A) := \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$.

If d is real-valued, then a bounded subset will be contained in a single open ball. But this is not the case for a general extended metric. For example, if X is a finite set equipped with the extended metric

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ \infty & \text{otherwise,} \end{cases}$$

then X will be bounded as we have defined it, yet open balls in X contain exactly one point. As we should insist that our definition of bounded set be consistent with the properties that (1) compactness implies boundedness, and (2) the terms of a convergent sequence form a bounded set, the usual definition that a bounded set be one that has finite diameter or, equivalently, that it be contained in a single ball, is simply not appropriate in this context.

We leave the proof of the following fact as an easy exercise.

Proposition 2.1. *Let K be a compact subset of an extended metric space $\langle X, d\rangle$. Then K is bounded.*

The family of nonempty d-bounded subsets of $\langle X, d\rangle$ forms a *bornology* on X, that is, a hereditary family of subsets of X that forms a cover of X and that is stable under finite unions [3,11]. We denote this by $\mathscr{B}_d(X)$ in the sequel. Other bornologies of note are the family of nonempty subsets with compact closure and the family of nonempty totally bounded subsets (defined in the expected way). The largest bornology on a general nonempty set X is the family of its nonempty subsets $\mathscr{P}_0(X)$ and the smallest is the family of all nonempty finite subsets $\mathscr{F}_0(X)$.

Given a bornology $\mathscr{B}$ on $\langle X, d\rangle$, we define the *topology of $\mathscr{B}$-uniform convergence* on $C(X)$ [3] to be the topology $\tau_{\mathscr{B}}$ induced by the uniformity $\mathcal{U}_{\mathscr{B}}$ on $C(X)$ having as a base all sets of the form $[B, \varepsilon]$, where

$$[B, \varepsilon] := \{(f, g) \in C(X) : \sup_{x \in B} |f(x) - g(x)| < \varepsilon\} \quad \text{for } B \in \mathscr{B},\ \varepsilon > 0.$$

The larger $\mathscr{B}$, the finer the function space topology. The coarsest such topology occurs when the bornology consists of the finite subsets, and we obtain the *topology of pointwise convergence*. As the uniformity is separated even in this case, all such topologies must be Hausdorff. When $\mathscr{B}$ is the bornology of subsets with compact closure, we get the *topology of uniform convergence on compact subsets* which we denote by τ_c.

If $\mathscr{B}$ has a countable cofinal subset with respect to inclusion, then the uniformity has a countable base, and the function space topology is metrizable [20, p. 257]. When d is a metric, then this is always the case for $\mathscr{B}_d(X)$. For a general extended metric, this is the case when the equivalence relation $x \sim y$ provided $d(x, y) < \infty$ has countably many (clopen) equivalence classes.

Let $\langle X, d\rangle$ be a metric space. We call a subfamily Δ of $\mathrm{Lip}(X)$ *uniformly Lipschitz* if $\sup\{\alpha(f) : f \in \Delta\} < \infty$. We will need the following result in the final section.

Proposition 2.2. *Let d be a metric on a set X. Then for each $g \in C(X)$ and each $\beta > 0$, $\{f - g : f \in B_{d_L}(g; \beta)\}$ is pointwise bounded and uniformly Lipschitz.*

Proof. For each f in the ball, $\alpha(f - g) < d_L(f, g) < \beta$, so the family is uniformly Lipschitz. Now let $x \in X$ be arbitrary. We compute $|(f - g)(x) - (f - g)(x_0)| \le \beta d(x, x_0)$, so

$$|f(x) - g(x)| \le |(f - g)(x_0)| + \beta d(x, x_0) \le (1 + d(x, x_0))\beta,$$

and this establishes pointwise boundedness. □

3. Basic properties of the extended Lipschitz metric on $C(X)$

In the sequel $\langle X, d\rangle$ is a metric space. We will assume X to contain at least two points.

We begin with a simple yet stunning example that shows the strength of our extended metric d_L, even restricted to Lipschitz functions.

Example 3.1. Suppose that $\langle x_n\rangle$ is a sequence in a metric space $\langle X, d\rangle$ convergent to a point p in X where for each n, $x_n \neq p$. Let $f_n = d(\cdot, x_n)$ for $n = 1, 2, 3, \ldots$ and let $f = d(\cdot, p)$. Each of these functions belongs to $\mathrm{Lip}_1(X)$ and evidently $\langle f_n\rangle$ converges uniformly to f on X. But for each $n \in \mathbb{N}$,

$$\alpha(f_n - f) \ge \left| \frac{(f_n - f)(p) - (f_n - f)(x_n)}{d(p, x_n)} \right| = 2,$$

so $d_L(f_n, f) = \|f_n - f\|_L \nrightarrow 0$.

Let d be a metric on a set X. In the case where X is a finite set, each real-valued function on X is bounded and Lipschitz, and the d_L-topology agrees with all of the other topologies mentioned in this article. Suppose now that X is an infinite set. While the Lipschitz functions on X equipped with d_L form a topological vector space, as the metric is inherited from the norm $\|\cdot\|_L$, this is not the case for $C(X)$! With X infinite, $C(X)$ will contain some non-Lipschitz function g. From this fact, $(\beta, f) \to \beta f$ cannot be jointly continuous on $\mathbb{R} \times C(X)$, as $\langle \frac{1}{n} g \rangle$ fails to d_L-converge to the zero function.

We leave the proof of the following proposition as an easy exercise.

Proposition 3.2. *Let $\langle X, d\rangle$ be a metric space. Suppose that $\langle f_n\rangle$ and $\langle g_n\rangle$ are sequences in $C(X)$ that are d_L-convergent to f and g respectively. Then whenever β and μ are scalars, we have $\lim_{n\to\infty} d_L(\beta f_n + \mu g_n, \beta f + \mu g) = 0$.*

Proposition 3.3. *Let d be a metric on a set X. Then $\langle C(X), d_L\rangle$ is a complete extended metric space.*

Proof. Let $\langle f_n\rangle$ be a d_L-Cauchy sequence. Then $\langle f_n(x_0)\rangle$ is convergent to some real number γ_{x_0}, and since

$$\lim_{n,m\to\infty} \alpha(f_n - f_m) \leq \lim_{n,m\to\infty} d_L(f_n - f_m) = 0,$$

and for $x \neq x_0$,

$$|f_n(x) - f_m(x)| \leq |f_n(x_0) - f_m(x_0)| + \alpha(f_n - f_m)d(x, x_0),$$

it follows that the sequence $\langle f_n(x)\rangle$ is Cauchy in $\mathbb{R}$ for each x in X and so converges to some point in $\mathbb{R}$ which we may take as the value, $f(x)$, of our limit function.

To show that $\lim_{n\to\infty}\alpha(f_n - f) = 0$, let $\varepsilon > 0$ be arbitrary. Choose $k \in \mathbb{N}$ so large that $d_L(f_n, f_m) \leq \varepsilon$ provided $m > n \geq k$. Fix $n \geq k$; since $\forall m \geq k$, $f_n - f_m \in \mathrm{Lip}_\varepsilon(X)$, and $\forall x \in X$, $\lim_{m\to\infty}(f_n - f_m)(x) = (f_n - f)(x)$, we conclude that $f_n - f \in \mathrm{Lip}_\varepsilon(X)$.

The easy verification of continuity of f is left to the reader. $\square$

Example 3.4. As we noted in the Introduction, $\langle \mathrm{Lip}(X), \|\cdot\|_L\rangle$ is a Banach space, which is to say that $\mathrm{Lip}(X)$ is d_L-closed in $C(X)$. The bounded Lipschitz functions need not be closed in $\langle \mathrm{Lip}(X), \|\cdot\|_L\rangle$, i.e., this vector subspace need not be a Banach space. To see this, let $X = [1, \infty)$ with distinguished point $x_0 = 1$ and for each positive integer n, define $f_n \in \mathrm{Lip}^b(X)$ by

$$f_n(x) = \begin{cases} \sqrt{x} & \text{if } 1 \leq x \leq n \\ \sqrt{n} & \text{if } x > n. \end{cases}$$

Clearly $\langle f_n\rangle$ is norm convergent to $f(x) = \sqrt{x}$ on $[1, \infty)$.

Proposition 3.5. *Let $\langle X, d\rangle$ be a metric space. Let $f \in C(X)$, and suppose that $\langle f_n\rangle$ is a sequence in $C(X)$ such that $\lim_{n\to\infty} d_L(f_n, f) = 0$. Then $\langle f_n\rangle$ converges uniformly on bounded subsets of X to f.*

Proof. Let $\rho > 0$. We want to show that $\langle f_n\rangle$ converges uniformly to f on $B(x_0; \rho)$. Let $\varepsilon > 0$ be arbitrary, and choose $k \in \mathbb{N}$ so large that for all $n \geq k$ and $x \in X$, we have both

$$\begin{aligned} &\text{(i)} \quad |f_n(x_0) - f(x_0)| < \tfrac{\varepsilon}{2} \\ \text{and} \quad &\text{(ii)} \quad |(f_n - f)(x) - (f_n - f)(x_0)| \leq \tfrac{\varepsilon}{2\rho} d(x, x_0). \end{aligned}$$

It follows from (ii) that if $n \geq k$ and $d(x, x_0) \leq \rho$ we have

$$|f_n(x) - f(x) - (f_n(x_0) - f(x_0))| \leq \frac{\varepsilon}{2\rho} \cdot \rho,$$

so by (i), $|f_n(x) - f(x)| < (\varepsilon/2) + (\varepsilon/2) = \varepsilon$. $\square$

Note that the proof of Proposition 3.5 does not require that the functions be continuous. As continuity is stable under uniform convergence on bounded sets, this fact can be used to complete the proof of Proposition 3.3.

Corollary 3.6. *Let $\langle X, d\rangle$ be a metric space. Then on $C(X)$, τ_{d_L} is finer than $\tau_{\mathscr{B}_d}$ and so τ_{d_L} is finer than τ_c.*

Proof. As $\tau_{\mathscr{B}_d}$ is metrizable, sequences determine both τ_{d_L} and $\tau_{\mathscr{B}_d}$, and as each relatively compact set is bounded, $\tau_{\mathscr{B}_d} \supseteq \tau_c$. $\square$

Our next result gives necessary and sufficient conditions on the structure of X for our topology to reduce to $\tau_{\mathscr{B}_d}$, in either the environment $C(X)$ or $\mathrm{Lip}(X)$.

Theorem 3.7. *Let $\langle X, d\rangle$ be a metric space. The following conditions are equivalent.*

(1) *X is bounded and there exists $\delta > 0$ such that whenever $x_1 \neq x_2$ in X, we have $d(x_1, x_2) > \delta$;*
(2) *$\tau_{d_L} = \tau_{\mathscr{B}_d}$ on $C(X)$;*
(3) *$\tau_{d_L} = \tau_{\mathscr{B}_d}$ on $\mathrm{Lip}(X)$.*

Proof. (1) $\Rightarrow$ (2). Suppose that X is bounded and uniformly discrete as described. Without loss of generality we may assume that $\delta < 1$. Let $\langle f_n \rangle$ be $\tau_{\mathscr{B}_d}$-convergent to f in $C(X)$. Let $\varepsilon > 0$ be arbitrary and choose $k \in \mathbb{N}$ such that whenever $n \geq k$, $\sup_{x \in X} |f_n(x) - f(x)| < \delta\varepsilon/2$. If $x_1 \neq x_2$ and $n \geq k$, then

$$\left| \frac{(f_n - f)(x_1) - (f_n - f)(x_2)}{d(x_1, x_2)} \right| < \frac{\delta\varepsilon}{d(x_1, x_2)} < \varepsilon.$$

As $\delta < 1$, we have $|f_n(x_0) - f(x_0)| < \varepsilon/2 < \varepsilon$, and we get $d_L(f_n, f) \leq \varepsilon$.

(2) $\Rightarrow$ (3). This is trivial.

(3) $\Rightarrow$ (1). Assume that condition (1) fails. We distinguish two exhaustive cases for the structure of X: (i) X is unbounded, and (ii) for each $n \in \mathbb{N}$, we can find x_n and y_n such that $0 < d(x_n, y_n) < 1/n$.

In case (i), with x_0 the usual distinguished point of X, consider the following three functions in $\mathrm{Lip}_2(X)$: $h_1(x) = 0$, $h_2(x) = d(x, x_0)$, and $h_3(x) = 2d(x, x_0) - 2n$. As $\mathrm{Lip}_2(X)$ is a lattice, $f_n := \max\{h_1, \min\{h_2, h_3\}\}$ is again 2-Lipschitz. Explicitly, the values of each f_n are given by

$$f_n(x) = \begin{cases} 0 & \text{if } d(x, x_0) < n \\ 2d(x, x_0) - 2n & \text{if } n \leq d(x, x_0) < 2n \\ d(x, x_0) & \text{if } 2n \leq d(x, x_0). \end{cases}$$

With g being the zero function, we have uniform convergence of $\langle f_n \rangle$ on bounded subsets to g. But choosing for each n a point y_n with $d(y_n, x_0) > 2n$, we have

$$d_L(f_n, g) \geq \left| \frac{(f_n - g)(y_n) - (f_n - g)(x_0)}{d(y_n, x_0)} \right| = 1.$$

In case (2), for each $n \in \mathbb{N}$, pick x_n, y_n with $0 < d(x_n, y_n) < 1/n$, and define $f_n \in \mathrm{Lip}_1(X)$ by

$$f_n(x) = \begin{cases} d(x_n, x) & \text{if } d(x_n, x) < d(x_n, y_n) \\ d(x_n, y_n) & \text{otherwise.} \end{cases}$$

Here, $\langle f_n \rangle$ converges uniformly to the zero function g, while

$$d_L(f_n, g) \geq \left| \frac{(f_n - g)(y_n) - (f_n - g)(x_n)}{d(y_n, x_n)} \right| = 1. \quad \square$$

The remainder of the section is concerned with the equivalence of function space extended metrics that have as their provenance norms on Lipschitz or on bounded Lipschitz functions. The proof of our first result in this direction is routine and is left to the reader.

Proposition 3.8. *Let d be a metric on X and let $x_0, x_1 \in X$ (not necessarily distinct). Then* $d_L(f, g) := \max\{|f(x_0) - g(x_0)|, \alpha(f - g)\}$ *and* $d_{L'}(f, g) := |f(x_1) - g(x_1)| + \alpha(f - g)$ *are equivalent extended metrics on* $C(X)$.

Starting with Sherbert norm, we are motivated to look at this extended metric d_S on $C(X)$:

$$d_S(f, g) := \sup_{x \in X} |f(x) - g(x)| + \alpha(f - g).$$

Replacing the sum by a maximum to get a "Weaver extended metric" of course results in an equivalent extended metric on $C(X)$. When $\mathrm{Diam}(X) = \infty$, the extended metrics d_L and d_S not only fail to be equivalent on $C(X)$; they also fail to be equivalent on $\mathrm{Lip}^b(X)$ (where they are induced by the norms $\|\cdot\|_L$ and $\|\cdot\|_S$).

Example 3.9. If d is an unbounded metric on X with distinguished point x_0, the norm $\|\cdot\|_S$ on $\mathrm{Lip}^b(X)$ is always properly stronger than $\|\cdot\|_L$. To see this, define $f_n : X \to \mathbb{R}$ by $f_n(x) = \arctan(d(x, x_0)/n)$ for $n \in \mathbb{N}$. Note that $f_n(x_0) = 0$, $\|f_n\|_\infty = \pi/2$, and $\alpha(f_n) \leq 1/n$. Thus, $\langle f_n \rangle$ is convergent to the zero function with respect to $\|\cdot\|_L$ while for each n in $\mathbb{N}$, we have $\alpha(f_n) + \|f_n\|_\infty \geq \pi/2$.

However, all is well when d is bounded.

Proposition 3.10. *Let d be a bounded metric on X. Then d_L and d_S are equivalent extended metrics on $C(X)$.*

Proof. Clearly, $d_L \leq d_S$. Conversely, fix $f \in C(X)$ and $\varepsilon > 0$. We must produce $\delta > 0$ such that $d_L(f, g) < \delta \Rightarrow d_S(f, g) < \varepsilon$. Choose δ so small that $\delta + \delta\mathrm{Diam}(X) < \varepsilon/2$ and fix $g \in B_{d_L}(f; \delta)$. For each $x \in X$, we compute

$$|(f - g)(x)| \leq |(f - g)(x_0)| + \alpha(f - g)d(x, x_0) < \delta + \delta\mathrm{Diam}(X),$$

and so $\sup_{x \in X} |f(x) - g(x)| < \varepsilon/2$. We conclude that $d_S(f, g) < \varepsilon$ as required. $\square$

4. On the limit of a sequence of differentiable functions

In this section, we intend to show that the space $\langle C(X), d_L\rangle$ provides an appropriate framework in which to appreciate properly the following classical result involving interchanging limits and derivatives for sequences of real-valued functions defined on a closed interval $[a, b]$ (see, e.g., [15, Theorem 7.17] or [1, Theorem 8.2.3]).

Theorem. *Let $\langle f_n\rangle$ be a sequence in $C([a, b])$ such that f_n is differentiable on (a, b) for each n in $\mathbb{N}$. Suppose that $\lim_{n\to\infty} f_n(x_0)$ exists for some x_0 in $[a, b]$, and $\langle f_n'\rangle$ is uniformly convergent on (a, b) to some function g. Then $\langle f_n\rangle$ is uniformly convergent to a function f on $[a, b]$ which is differentiable on (a, b) with $f'(x) = g(x)$.*

A key tool in our analysis is the following folk-theorem that we record as a lemma. Note that there is no assumption that the function be Lipschitz.

Lemma 4.1. *Suppose that $f \in C([a, b])$ and is differentiable on (a, b). Then $\alpha(f) = \sup\{|f'(x)| : a < x < b\}$.*

Proof. That $\alpha(f) \le \sup\{|f'(x)| : a < x < b\}$ is an immediate consequence of the Mean Value Theorem. For the reverse inequality, let $\mu < \sup\{|f'(x)| : a < x < b\}$ be arbitrary, and take x in (a, b) with $|f'(x)| > \mu$. By the definition of derivative, there is a w in $[a, b]$ different from x with

$$\left|\frac{f(w) - f(x)}{w - x}\right| > \mu.$$

It follows that $\alpha(f) > \mu$, and so the reverse inequality holds. □

We are led to consider this vector subspace of $C([a, b])$:

$$\Delta([a, b]) := \{f \in C([a, b]) : f'(x) \text{ exists } \forall x \in (a, b)\}.$$

Notice that if $f, g \in \Delta([a, b])$, then

$$d_L(f, g) = \max\{|f(x_0) - g(x_0)|,\ \sup_{a<x<b} |f'(x) - g'(x)|\}.$$

Our main result of this section now follows.

Theorem 4.2. *If $[a, b]$ is a closed bounded interval in $\mathbb{R}$, then $\Delta([a, b])$ is a closed subset of the extended metric space $\langle C([a, b]), d_L\rangle$. Moreover, if $\langle f_n\rangle$ is a sequence in $\Delta([a, b])$ with $d_L(f_n, f) \to 0$, then $\lim_{n\to\infty} f_n'(x) = f'(x)$ for each x in (a, b).*

Proof. Suppose that $f \in C([a, b])$ and $\langle f_n\rangle$ is a sequence in $\Delta([a, b])$ which is d_L-convergent to f. Since $[a, b]$ is bounded, the convergence is uniform. Let $\varepsilon > 0$. There is an index N_1 such that $d_L(f_n, f) < \varepsilon/3$ whenever $n \ge N_1$. In particular, $\alpha(f_n - f) < \varepsilon/3$. If $x \in (a, b)$ and n and m are at least as large as N_1, then, using Lemma 4.1 and the triangle inequality for d_L,

$$\begin{aligned}|f_n'(x) - f_m'(x)| &= |(f_n - f_m)'(x)| \le \alpha(f_n - f_m)\\ &\le d_L(f_n, f) + d_L(f, f_m) < \tfrac{2\varepsilon}{3}.\end{aligned}$$

Thus the sequence of derivatives is uniformly Cauchy on (a, b) and must converge uniformly on (a, b) to some function $g : (a, b) \to \mathbb{R}$. There is an index N_2 such that

$$\text{(i)}\quad \left|\tfrac{(f_n-f)(w)-(f_n-f)(x)}{w-x}\right| \le \alpha(f_n - f) < \tfrac{\varepsilon}{3}$$

$$\text{and}\quad \text{(ii)}\quad |f_n'(x) - g(x)| < \tfrac{\varepsilon}{3}$$

for all w and x in (a, b) with $w \ne x$ whenever $n \ge \max\{N_1, N_2\}$. For x in (a, b), fix such an n. There is a positive δ such that

$$\text{(iii)}\quad \left|\frac{f_n(w) - f_n(x)}{w - x} - f_n'(x)\right| < \frac{\varepsilon}{3} \quad \text{whenever } 0 < |w - x| < \delta.$$

Combining (i)–(iii), we get

$$0 < |w - x| < \delta \Longrightarrow \left|\frac{f(w) - f(x)}{w - x} - g(x)\right| < \varepsilon.$$

This shows that f is differentiable on (a, b), and so $f \in \Delta([a, b])$ and $f'(x) = g(x) = \lim_{n\to\infty} f_n'(x)$ as claimed. □

The classical theorem above is an immediate consequence of our last result.

Proof of the classical theorem. The uniform convergence of $\langle f_n'\rangle$ to g on (a, b) implies that $\langle f_n'\rangle$ is uniformly Cauchy on (a, b). Since $\langle f_n(x_0)\rangle$ is Cauchy as well, Lemma 4.1 guarantees that $\langle f_n\rangle$ is a d_L-Cauchy sequence in $\Delta([a, b])$. By the completeness of $\langle C([a, b]), d_L\rangle$, $\langle f_n\rangle$ is d_L-convergent (and thus uniformly convergent) to some continuous function f. Apply Theorem 4.2. □

5. Two topological theorems

We first characterize compactness of subsets of our function space $\langle C(X), d_L\rangle$. Our proof relies on the classical Arzelà–Ascoli Theorem [5, p. 267]. Recall that a family Ω of real-valued functions on a topological space X is called *equicontinuous* if $\forall x \in X,\ \forall \varepsilon > 0$, there exists a neighborhood U of x such that $f \in \Omega,\ w \in U \Rightarrow |f(x) - f(w)| < \varepsilon$.

The Arzelà–Ascoli Theorem. *Let X be a topological space and let Ω be an equicontinuous family of real-valued functions on X. Suppose that for each $x \in X$, $\{f(x) : f \in \Omega\}$ is a bounded set of reals. Then $\mathrm{cl}_{\tau_c}(\Omega)$ is τ_c-compact.*

Theorem 5.1. *Let d be a metric on X. Then $E \subseteq C(X)$ is τ_{d_L}-compact if and only if the following three conditions are satisfied:*

(1) *E is d_L-bounded;*
(2) *the relative τ_{d_L} and τ_c topologies on E agree;*
(3) *E is τ_c-closed.*

Proof. We begin with necessity of the conditions. Condition (1) follows from Proposition 2.1. Condition (2) follows from Corollary 3.6 and the fact that a set equipped with a compact topology can admit no properly coarser Hausdorff topology [5, p. 226]. Condition (3) follows from condition (2), because τ_c is a Hausdorff topology, and a compact subset of a Hausdorff space must be closed.

For sufficiency, choose by condition (1) $\{g_1, g_2, \ldots, g_n\} \subseteq C(X)$ and positive scalars $\{\mu_1, \mu_2, \ldots \mu_n\}$ such that $E \subseteq \cup_{j=1}^{n} B_{d_L}(g_j; \mu_j)$. It suffices to show that for each $j \le n$, the τ_c-closure of $E \cap B_{d_L}(g_j; \mu_j)$ is τ_L-compact, as by condition (3),

$$E = \mathrm{cl}_{\tau_c}(E) = \bigcup_{j=1}^{n} \mathrm{cl}_{\tau_c}(E \cap B_{d_L}(g_j; \mu_j)).$$

To this end, fix $j \le n$ and consider $A_j = \{f - g_j : f \in E \cap B_{d_L}(g_j; \mu_j)\}$. By Proposition 2.2, A_j satisfies the hypotheses of the Arzelà–Ascoli Theorem, and so $\mathrm{cl}_{\tau_c}(A_j)$ is τ_c-compact. We compute

$$\mathrm{cl}_{\tau_c}(E \cap B_{d_L}(g_j; \mu_j)) = \mathrm{cl}_{\tau_c}(A_j + g) = g + \mathrm{cl}_{\tau_c}(A_j).$$

Since a translate of a compact set in the locally convex space $\langle C(X), \tau_c\rangle$ is again compact, we get τ_c-compactness of $\mathrm{cl}_{\tau_c}(E \cap B_{d_L}(g_j; \mu_j))$. Apply condition (2). $\square$

The result fails if condition (3) is replaced by the weaker condition (3*): E is d_L-closed.

Example 5.2. On $C(\mathbb{R})$, the topology τ_c coincides with the topology of uniform convergence on bounded sets and is thus metrizable. Consider for each $n \in \mathbb{N}$ the function $f_n \in \mathrm{Lip}_1(\mathbb{R})$ defined by

$$f_n(x) = \begin{cases} x - (2n-1) & \text{if } 2n-1 \le x < 2n \\ (2n+1) - x & \text{if } 2n \le x < 2n+1 \\ 0 & \text{otherwise.} \end{cases}$$

Put $E = \{f_n : n \in \mathbb{N}\}$. Then E is uniformly discrete with respect to d_L and is thus d_L-closed and d_L-noncompact. Further, denoting the zero function by g we have $E \subseteq B_{d_L}(g; 2)$, whatever x_0 may be. Finally, for both τ_{d_L} and τ_c on E, the trace is the discrete topology.

Example 5.3. There is an equally plausible way to define boundedness in $\langle C(X), d_L\rangle$ with respect to the equivalence relation given by $f \sim g$ provided $d_L(f, g) < \infty$. Declare $E \subseteq C(X)$ to be *weakly bounded* provided the trace of E on each equivalence class is bounded. We will show that boundedness cannot be replaced by weak boundedness in condition (1).

Consider for each $n \in \mathbb{N}$, the function $f_n \in \mathrm{Lip}(\mathbb{R})$ defined by

$$f_n(x) = \begin{cases} n^2x + n & \text{if } -1/n \le x < 0 \\ n - n^2x & \text{if } 0 \le x < 1/n \\ 0 & \text{otherwise.} \end{cases}$$

Whenever $\mathbb{M} \subseteq \mathbb{N}$ is infinite and $g \in C(X)$, the family $\{f_n - g : n \in \mathbb{M}\}$ fails to be uniformly Lipschitz, and so the trace of $E = \{f_n : n \in \mathbb{N}\}$ on each equivalence class is finite. Thus, E is weakly bounded. The relative τ_{d_L} and τ_c topologies on E are both discrete, and E is τ_c-closed, as the sequence has no τ_c-convergent subsequence. Clearly, τ_{d_L}-compactness of E fails.

The last example shows that conditions (2) and (3) of Theorem 5.1 together do not ensure that (1) holds. Example 3.1 shows that conditions (1) and (3) do not guarantee that (2) holds with $E = \{f, f_1, f_2, f_3, \ldots\}$. Finally, in an arbitrary metric space $\langle X, d\rangle$, if $\{g, g_1, g_2, g_3, \ldots\} \subseteq C(X)$ where all functions are distinct and $g = d_L - \lim g_n$, then $E = \{g_n : n \in \mathbb{N}\}$ satisfies (1) and (2) but not (3).

Our final result characterizes second countability of the Lipschitz extended metric topology.

Theorem 5.4. *Let $\langle X, d\rangle$ be a metric space. Then $\langle C(X), d_L\rangle$ is second countable if and only if X is a finite set.*

Proof. Second countability in this setting is equivalent to separability. If X is finite, then the family of rational-valued functions is easily seen to be d_L-dense. Otherwise, we distinguish two exhaustive cases: (1) X has a sequence $\langle x_n\rangle$ with distinct terms but with no cluster point; (2) X has a convergent sequence $\langle x_n\rangle$ with distinct terms convergent to some point p where for each n, $x_n \neq p$.

The same proof works in both cases. In the first case, we may assume that the set of terms of the sequence excludes some point $p \in X$. For each subset E of $\{x_n : n \in \mathbb{N}\}$, use the Tietze extension theorem [5, p. 149] to choose a function $f_E \in C(X)$ satisfying $f_E(p) = 0$, $f_E(x_n) = d(p, x_n)$ if $x_n \in E$, and $f_E(x_n) = 0$ otherwise. If E_1 fails to be a subset of E_2, then choosing $x_n \in E_1 \setminus E_2$, we have

$$d_L(f_{E_1}, f_{E_2}) \geq \alpha(f_{E_1}, f_{E_2}) \geq \left|\frac{(f_{E_1} - f_{E_2})(x_n) - (f_{E_1} - f_{E_2})(p)}{d(x_n, p)}\right| = 1.$$

This shows that the function space has an uncountable uniformly discrete set, and so it must fail to be second countable. □

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