

# Arima Model - 5

강의 : 김성범 교수님

정리 : 김선중

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이것은 ARIMA 모델 개요 - Part 5 강의에 대한 노트이다.

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## 아리마 모델 (ARIMA Model) - Part 5

ARIMA  
Backward Shift operator function

$$\begin{aligned} B \cdot X_t &= X_{t-1} \\ B^2 \cdot X_t &= B \cdot (B \cdot X_t) = B \cdot X_{t-1} = X_{t-2} \\ B^3 \cdot X_t &= X_{t-3} \dots B^m \cdot X_t = X_{t-m} \end{aligned}$$

$$\begin{aligned} AR(1) \quad X_t &= \phi X_{t-1} + a_t \\ &= \phi B \cdot X_{t-1} + a_t \\ (1 - \phi B) X_t &= a_t \\ X_t &= \frac{a_t}{(1 - \phi B)} \\ &= (1 + \phi B + \phi^2 B^2 + \phi^3 B^3 + \dots) a_t \\ &= a_t + \phi B a_{t-1} + \phi^2 B^2 a_{t-2} + \phi^3 B^3 a_{t-3} + \dots \\ &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \phi^3 a_{t-3} + \dots \end{aligned}$$



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$$AR(2) \quad X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t$$

$$= \phi_1 B \cdot X_t + \phi_2 B^2 \cdot X_t + a_t$$

$$(1 - \phi_1 B - \phi_2 B^2) X_t = a_t$$

$$X_t = \frac{a_t}{1 - \phi_1 B - \phi_2 B^2} \cdot a_t$$

$$= \frac{1}{(1 - \phi_1 B)(1 - \phi_2 B^2)} \cdot a_t$$

$$= \frac{1}{(1 - \phi_1 B)} \cdot \frac{1}{(1 - \phi_2 B^2)} \cdot a_t$$

$$= (1 + \phi_1 B + \phi_1^2 B^2 + \dots) (1 + \phi_2 B^2 + \phi_2^2 B^4 + \dots) a_t$$

$$= a_t + (\phi_1 + \phi_1^2) B a_{t-1} + (\phi_1^2 + \phi_1 \phi_2 B^2 + \phi_2^2 B^4) B^2 a_{t-2} + \dots$$

\* Forward Shift operator

$$F \cdot X_t = X_{t+1} \quad \dots \quad F^K \cdot X_t = X_{t+K}$$

ARMA(1,1)

$$\begin{aligned} X_t &= \phi X_{t-1} + a_t + \theta a_{t-1} \quad \rightarrow (1 - \phi B) X_t = a_t + \theta a_{t-1} \\ X_t \cdot X_{t-1} &= \phi X_{t-1} \cdot X_{t-1} + a_t \cdot X_{t-1} + \theta a_{t-1} \cdot X_{t-1} \quad \rightarrow X_t = (1 - \phi B)^{-1} (a_t + \theta a_{t-1}) \\ E(X_t \cdot X_{t-1}) &= \phi E(X_{t-1} \cdot X_{t-1}) + E(a_t \cdot X_{t-1}) + \theta E(a_{t-1} \cdot X_{t-1}) \quad = (1 + \phi B + \phi^2 B^2 + \dots) (a_t + \theta a_{t-1}) \\ Cov(X_t, X_{t-1}) &= \phi Cov(X_{t-1}, X_{t-1}) + E(a_t \cdot X_{t-1}) + \theta E(a_{t-1} \cdot X_{t-1}) \quad = a_t + \theta a_{t-1} + \phi a_{t-1} + \phi \theta a_{t-2} + \dots \\ &= a_t + (\phi + \theta) a_{t-1} + (\phi^2 + \phi \theta + \theta^2) a_{t-2} + \dots \end{aligned}$$

$$h=0, \quad Y_X(0) = \phi Y_X(1) + E(a_t \cdot X_{t-1}) + \theta E(a_{t-1} \cdot X_{t-1}) \quad \boxed{1} \quad \boxed{2} \quad \boxed{3} \quad \boxed{4}$$

$$\begin{aligned} ① E(a_t \cdot X_{t-1}) &= E(a_t \cdot (a_{t-1} + \phi a_{t-2} + \dots)) \\ &= E(a_t \cdot a_{t-1}) + \phi E(a_t \cdot a_{t-2}) + \dots \\ &= Cov(a_t, a_{t-1}) \\ &= 6a^2 \end{aligned}$$

$$\begin{aligned} ② E(a_{t-1} \cdot (a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \dots)) &= E(a_{t-1} \cdot a_t) + \phi E(a_{t-1} \cdot a_{t-1}) \\ &= E(a_{t-1} \cdot a_t) + \phi^2 E(a_{t-1} \cdot a_{t-1}) \\ &= 4\phi^2 a^2 \end{aligned}$$

$$\therefore Y_X(0) = 6a^2 = \phi Y_X(1) + 6a^2 + \theta \phi^2 a^2$$

Auto covariance function of ARMA(1,1)

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$$h=1, \quad Cov(X_t, X_{t-1}) = \phi Cov(X_{t-1}, X_{t-1}) + E(a_t \cdot X_{t-1}) + \theta E(a_{t-1} \cdot X_{t-1})$$

$$Y_X(1) = \phi Y_X(0) + \theta a^2 \quad \boxed{1}$$

$$Y_X(0) = \phi Y_X(1) + 6a^2 + \theta(\phi + \theta)6a^2 \quad \boxed{2}$$

$$\boxed{1}, \boxed{2} \quad Y_X(1) = \frac{(1+2\phi+\theta^2) \cdot 6a^2}{1-\phi^2} \quad h=0$$

$$Y_X(1) = \frac{(1+\theta\phi)(\phi+\theta) \cdot 6a^2}{1-\phi^2} \quad h=L$$

$$h \geq 2, \quad Cov(X_t, X_{t-h}) = \phi Cov(X_{t-1}, X_{t-h}) + E(a_t \cdot X_{t-h}) + \theta E(a_{t-1} \cdot X_{t-h})$$

$$Y_X(h) = \phi Y_X(h-1)$$

$$Y_X(h) = \phi^{h-1} Y_X(1) \quad \text{leave}$$

$$= \phi^{h-1} \cdot \frac{(1+\theta\phi)(\phi+\theta) \cdot 6a^2}{1-\phi^2}$$

$$\begin{aligned} AR \text{ process } AR(p) &\xrightarrow{\text{WN}} \phi^{-1}(B) \rightarrow AR(p) \\ X_t &= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + a_t \\ X_t - \phi_1 X_{t-1} - \phi_2 X_{t-2} - \dots - \phi_p X_{t-p} &= a_t \\ X_t - \phi_1 B X_t - \phi_2 B^2 X_t - \dots - \phi_p B^p X_t &= a_t \\ (1 - \phi_1 B - \phi_2 B^2 - \dots - \phi_p B^p) X_t &= a_t \\ \phi^{-1}(B) X_t &= a_t \\ X_t &= \phi^{-1}(B) \cdot a_t \end{aligned}$$

⇒ AR process can be thought of as the output  $X_t$  from a linear filter with transfer function  $\phi^{-1}(B)$  when the input is white noise  $a_t$ .

MA( $\theta$ )

$$\begin{aligned} X_t &= a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \dots - \theta_q a_{t-q} \\ &= a_t - \theta_1 B \cdot a_t - \theta_2 B^2 \cdot a_t - \dots - \theta_q B^q \cdot a_t \\ &= (1 - \theta_1 B - \theta_2 B^2 - \dots - \theta_q B^q) a_t \\ &= \theta(B) \cdot a_t. \end{aligned}$$

$$\xrightarrow{\text{at}} \theta(B) \rightarrow MA(\theta)$$

ARMA(p,q)

$$X_t = \underbrace{\phi_1 X_{t-1} + \dots + \phi_p X_{t-p}}_{AR} + \underbrace{a_t - \theta_1 a_{t-1} - \dots - \theta_q a_{t-q}}_{MA}$$

$$\phi^{-1}(B) X_t = \theta(B) \cdot a_t$$

$$X_t = \underbrace{\phi^{-1}(B) \cdot \theta(B)}_{ARMA} a_t$$

$$\xrightarrow{\text{at}} \underbrace{X_t}_{ARMA} \xrightarrow{\text{at}} X_t^{(p,q)}$$

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# 1 Backward shift operator $B$

$B$  is called the *backward shift operator*. It is a function that maps a random variable which depends on time (i.e.  $X_t$  or  $a_t$ , backward by one step.

$$B(X_t) = X_{t-1} \quad (1)$$

We occasionally write  $BX_t$  instead of  $B(X_t)$  to stand for the same thing. Note that

$$\begin{aligned} BX_t &= X_{t-1} \\ B^2X_t &= X_{t-2} \\ &\vdots \\ B^mX_t &= X_{t-m} \end{aligned}$$

for any positive integer  $m$ .

## 1.1 AR(1) model described by $B$

Recall that AR(1) is called the *autoregressive model* of order 1, which predict  $X_t$  by means of  $X_{t-1}$  and  $a_t$ .

$$X_t = \phi X_{t-1} + a_t \quad (2)$$

$a_t$  is a white noise so that the expectation of  $a_t$  is 0

$$\mathbb{E}(a_t) = 0, \quad (3)$$

the variance of  $a_t$  is independent of  $t$

$$\mathbb{V}(a_t) = \sigma_a^2 \quad (4)$$

and  $a_t$ 's in distinct timesteps are independent

$$\text{Cov}(a_t, a_{t+h}) = 0 \quad \text{for } h \neq 0. \quad (5)$$

We can rewrite the above equation as

$$\begin{aligned} X_t &= \phi BX_t + a_t \\ (1 - \phi B)X_t &= a_t \end{aligned}$$

where  $1 - \phi B$  can be thought of as an operator, which maps  $X_t$  to  $X_t - \phi BX_t$ . But, I think the professor think of  $1 - \phi B$  as a ‘1 minus a real number  $\phi B$ . Moreover, he assumes that  $|\phi B| < 1$  in that

$$\begin{aligned} X_t &= \frac{1}{1 - \phi B} a_t \\ &= (1 + \phi B + \phi^2 B^2 + \dots) a_t \\ &= a_t + \phi a_{t-1} + \phi^2 a_{t-2} + \dots \end{aligned} \quad (6)$$

where he use the formula for geometric series.

As a result,  $X_t$  is expressed as a (infinite) linear combination of  $a_{t-h}$  for  $h = 0, 1, 2, \dots$

## 1.2 AR(2) model described by $B$

Recall that AR(2) is an autoregressive model of order 2, which predict  $X_t$  by means of  $X_{t-1}$ ,  $X_{t-2}$  and  $a_t$ .

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + a_t \quad (7)$$

Using the backward shift operator  $B$ , we get

$$\begin{aligned} X_t &= \phi_1 B X_t + \phi_2 B^2 X_t + a_t \\ (1 - \phi_1 B - \phi_2 B^2) X_t &= a_t \end{aligned}$$

Again, regarding  $B$  as a real number,  $1 - \phi_1 B - \phi_2 B^2$  can be thought of as a quadratic polynomial. So there exists numbers<sup>1</sup>  $\alpha_1$  and  $\alpha_2$  such that  $(\alpha_1 + \alpha_2 = \phi_1, \alpha_1 \alpha_2 = -\phi_2)$

$$1 - \phi_1 B - \phi_2 B^2 = (1 - \alpha_1 B)(1 - \alpha_2 B).$$

Thus,

$$\begin{aligned} X_t &= \frac{1}{1 - \phi_1 B - \phi_2 B^2} a_t \\ &= \frac{1}{(1 - \alpha_1 B)(1 - \alpha_2 B)} a_t \\ &= (1 + \alpha_1 B + \alpha_1^2 B^2 + \dots)(1 + \alpha_2 B + \alpha_2^2 B^2 + \dots) a_t \\ &= a_t + (\alpha_1 + \alpha_2) a_{t-1} + (\alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2^2) a_{t-2} + (\alpha_1^3 + \alpha_1^2 \alpha_2 + \alpha_1 \alpha_2^2 + \alpha_2^3) a_{t-3} + \dots \end{aligned}$$

## 1.3 Forward shift operator $F$

In contrast to the backward shift operator  $B$ , the *forward shift operator*  $F$  shift a random variable forward.

$$\begin{aligned} F X_t &= X_{t+1} \\ F^2 X_t &= X_{t+2} \\ &\vdots \\ F^k X_t &= X_{t+k} \end{aligned} \quad (8)$$

for any positive integer  $k$ .

## 2 ARMA(1,1) model

ARMA(1,1) model is a combination of AR(1) model and MA(1) model. We are assuming that  $X_t$  is obtained from  $X_{t-1}$ ,  $a_t$  and  $a_{t-1}$ .

$$X_t = \phi X_{t-1} + a_t + \theta a_{t-1}. \quad (9)$$

Here, we compute the covariance function of this model for  $h = 0, 1, 2, \dots$

To get the autocovariance function  $\gamma(h) = \text{Cov}(X_t, X_{t-h})$ , multiply (9) by  $X_{t-h}$  on both sides

$$X_t X_{t-h} = \phi X_{t-1} X_{t-h} + a_t X_{t-h} + \theta a_{t-1} X_{t-h}$$

---

<sup>1</sup>real numbers if  $D = b^2 - 4ac = (-\phi_1)^2 - 4 \cdot (-\phi_2) \cdot 1 > 0$ . but in general, they are complex numbers

and take the expectation on both sides to get<sup>2</sup>;

$$\begin{aligned}
\gamma(h) &= \mathbb{E}[X_t X_{t-h}] \\
&= \phi \mathbb{E}[X_{t-1} X_{t-h}] + \mathbb{E}[a_t X_{t-h}] + \theta \mathbb{E}[a_{t-1} X_{t-h}] \\
&= \phi \text{Cov}(X_{t-1}, X_{t-h}) + \mathbb{E}[a_t X_{t-h}] + \theta \mathbb{E}[a_{t-1} X_{t-h}]
\end{aligned} \tag{10}$$

## 2.1 autocovariance for $h = 0 : \gamma(0)$

Suppose  $h = 0$ . Before computation, we can convert the equation (9) using the operator  $B$  ;

$$\begin{aligned}
(1 - \phi B)X_t &= a_t + \theta a_{t-1} \\
X_t &= \frac{1}{1 - \phi B}(a_t + \theta a_{t-1}) \\
&= (1 + \phi B + \phi^2 B^2 + \dots)(a_t + \theta a_{t-1}) \\
&= a_t + \theta a_{t-1} + \phi a_{t-1} + \phi \theta a_{t-2} + \phi^2 a_{t-2} + \phi^2 \theta a_{t-3} + \dots \\
&= a_t + (\theta + \phi)a_{t-1} + \phi(\theta + \phi)a_{t-2} + \dots \\
&= a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots
\end{aligned} \tag{11}$$

By (11) the second term and the third term of (10) are evalauted as

$$\begin{aligned}
\mathbb{E}[a_t X_t] &= \mathbb{E}[a_t(a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots)] \\
&= \mathbb{E}[a_t^2] + \psi_1 \mathbb{E}[a_t a_{t-1}] + \psi_2 \mathbb{E}[a_t a_{t-2}] + \dots \\
&= \sigma_a^2 + 0 + 0 + \dots = \sigma_a^2,
\end{aligned}$$

and

$$\begin{aligned}
\theta \mathbb{E}[a_{t-1} X_t] &= \theta \mathbb{E}[a_{t-1}(a_t + \psi_1 a_{t-1} + \psi_2 a_{t-2} + \dots)] \\
&= \theta (\mathbb{E}[a_{t-1} a_t] + \psi_1 \mathbb{E}[a_{t-1}^2] + \psi_2 \mathbb{E}[a_{t-1} a_{t-2}] + \dots) \\
&= \theta (0 + \psi_1 \sigma_a^2 + 0 + \dots) \\
&= \theta \psi_1 \sigma_a^2;
\end{aligned}$$

where we used (3), (4) and (5) in the computations. Therefore, the autocovariance function of ARMA(1,1) when  $h = 0$  is

$$\begin{aligned}
\gamma(0) &= \phi \text{Cov}(X_{t-1}, X_t) + \sigma_a^2 + \theta \psi_1 \sigma_a^2 \\
&= \phi \gamma(1) + \sigma_a^2 + \theta(\theta + \phi) \sigma_a^2,
\end{aligned} \tag{12}$$

which is also the variance of the ARMA(1,1) model.

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<sup>2</sup>여기에서  $\mathbb{E}[X_t] = \mathbb{E}[X_{t-h}] = 0$ 임을 가정하고 있는 것일까? Covariance의 정의는  $\text{Cov}(X, Y) = \mathbb{E}[(X - [X])(Y - [Y])]$ 인데 마치  $\text{Cov}(X, Y) = \mathbb{E}[XY]$ 인 것처럼 쓰고 있다. Shumway의 책에서는  $x_t$ 의 평균이 0임을 가정하고 있다.  $x_t$ 의 평균이 0이 아닐 경우 평균  $\mu$ 를 뺀  $x_t - \mu$ 를 고려하라고 되어 있다. (p. 86)

## 2.2 autocovariance for $h = 1 : \gamma(1)$

Suppose  $h = 1$ . The equation (9) applied to  $t - 1$  yields

$$\begin{aligned} X_{t-1} &= \phi X_{t-2} + a_{t-1} + \theta a_{t-2} \\ X_{t-1} &= \phi B X_{t-1} + a_{t-1} \\ (1 - \phi B)X_{t-1} &= a_{t-1} \\ X_{t-1} &= (1 + \phi B + \phi^2 B^2 + \dots) a_{t-1}. \end{aligned}$$

Plugging  $h = 1$  and the above equation into (10) yield

$$\begin{aligned} \gamma(1) &= \phi \text{Cov}(X_{t-1}, X_{t-1}) + \mathbb{E}[a_t X_{t-1}] + \theta \mathbb{E}[a_{t-1} X_{t-1}] \\ &= \phi \gamma(0) + \mathbb{E}[a_t(a_{t-1} + \phi a_{t-2} + \phi^2 a_{t-3} + \dots)] + \theta \mathbb{E}[a_{t-1}(a_{t-1} + \phi a_{t-2} + \phi^2 a_{t-3} + \dots)] \\ &= \phi \gamma(0) + 0 + \theta \times \sigma_a^2 \\ &= \phi \gamma(0) + \theta \sigma_a^2. \end{aligned} \tag{13}$$

As a result, we get a system of linear equations (12) and (13) with unknowns  $\gamma(0)$  and  $\gamma(1)$ . Solving the system by substitution yields

$$\begin{aligned} \gamma(0) &= \phi \gamma(1) + \sigma_a^2 + \theta(\phi + \theta)\sigma_a^2 \\ &= \phi \gamma(0) + \phi \theta \sigma_a^2 + \sigma_a^2 + \theta(\phi + \theta)\sigma_a^2 \\ \gamma(0) &= \frac{1 + 2\theta\phi + \theta^2}{1 - \phi^2} \sigma_a^2 \end{aligned} \tag{14}$$

and

$$\begin{aligned} \gamma(1) &= \theta \gamma(0) + \theta \sigma_a^2 \\ &= \phi^2 \gamma(1) + \phi \sigma_a^2 + \theta \phi(\phi + \theta) \sigma_a^2 + \theta \sigma_a^2 \\ \gamma(1) &= \frac{\phi + \theta \phi^2 + \theta^2 \phi + \theta}{1 - \phi^2} \sigma_a^2 = \frac{(\phi + \theta)(1 + \phi\theta)}{1 - \phi^2} \sigma_a^2 \end{aligned} \tag{15}$$

## 2.3 autocovariance for $h \geq 2 : \gamma(h)$

By the similar reasoning as in 2.1 and 2.2, we have

$$X_{t-h} = (1 + \phi B + \phi^2 B^2 + \dots) a_{t-h}.$$

Substituting the above equation to (10) yields

$$\begin{aligned} \gamma(h) &= \phi \text{Cov}(X_{t-1}, X_{t-h}) + \mathbb{E}[a_t X_{t-h}] + \theta \mathbb{E}[a_{t-1} X_{t-h}] \\ &= \phi \gamma(h-1) + \mathbb{E}[a_t(a_{t-h} + \phi a_{t-h-1} + \phi^2 a_{t-h-2} + \dots)] + \theta \mathbb{E}[a_{t-1}(a_{t-h} + \phi a_{t-h-1} + \phi^2 a_{t-h-2} + \dots)] \\ &= \phi \gamma(h-1). \end{aligned} \tag{16}$$

Using induction on the above recurrence formula (16), we have  $\gamma(h) = \phi^{h-1}\gamma(1)$  for  $h \geq 2$ . Combined with (14) and (15), we get

$$\gamma(h) = \begin{cases} \frac{1+2\theta\phi+\theta^2}{1-\phi^2}\sigma_a^2 & (h=0) \\ \frac{(\phi+\theta)(1+\phi\theta)\phi^{h-1}}{1-\phi^2}\sigma_a^2 & (h \geq 1) \end{cases} \quad (17)$$

Since  $\gamma(-h) = \gamma(h)$ , equation (17) is the explicit formula of  $\gamma(h)$  for all integers  $h$ .

### 3 AR, MA and ARMA described by operators

The lecture ends by giving a summary for AR model, MA model and ARMA model. The moral is that those models can be expressed in terms of *polynomials* of  $B$ .

#### 3.1 AR model

Consider the AR( $p$ ) model, characterized by the equation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t \quad (18)$$

The above equation can be converted into

$$(1 - \phi B - \phi^2 B^2 - \cdots - \phi^p B^p)X_t = a_t \quad (19)$$

We denote

$$\phi(B) = 1 - \phi B - \phi^2 B^2 - \cdots - \phi^p B^p, \quad (20)$$

so that  $\phi(B)$  is an operator (a polynomial of operator  $B$ ) acting on a random variable, where 1 should be regarded as the identity operator. The equation (19) can be expressed as

$$\phi(B)X_t = a_t \quad (21)$$

If the inverse operator  $\phi^{-1}(B)$  of  $\phi(B)$  exists<sup>3</sup>, the above equation goes

$$X_t = \phi^{-1}(B)a_t.$$

Here is a comment by the professor, summarizing the above equation :

AR process can be thought of as making the output  $X_t$  from a linear filter with transfer function  $\phi^{-1}(B)$  when the input is a white noise  $a_t$ .

#### 3.2 MA model

The MA( $q$ ) model is governed by the equation<sup>4</sup>

$$X_t = a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}. \quad (22)$$

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<sup>3</sup>actually, it exists if  $|\phi| < 1$  whose explicit form is written in (??). The way of geometric series doesn't seem to be rigorous. Shumway explain an alternative way in section 3.2.

We have

$$X_t = (1 - \theta B - \theta^2 B^2 - \cdots - \theta^q B^q) a_t \quad (23)$$

$$\theta(B) = 1 - \theta B - \theta^2 B^2 - \cdots - \theta^q B^q \quad (24)$$

$$X_t = \theta(B) a_t \quad (25)$$

### 3.3 ARMA model

The ARMA( $p, q$ ) model has the equation

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \cdots + \phi_p X_{t-p} + a_t - \theta_1 a_{t-1} - \theta_2 a_{t-2} - \cdots - \theta_q a_{t-q}. \quad (26)$$

By the similar procedures that we did earlier,

$$\phi(B) X_t = \theta(B) a_t \quad (27)$$

holds for the same  $\phi(B)$  and  $\theta(B)$  defined in (20) and (24), respectively. Taking the inverse  $\phi^{-1}(B)$  of  $\phi(B)$  on both sides yields

$$X_t = \phi^{-1}(B) \theta(B) a_t. \quad (28)$$

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<sup>4</sup>Note that the professor wrote minus sign instead of plus sign. The choice of signs are irrelevant of the context. Confer to the second footnote at page 90, in Shumway's book