PMA 9 : Functions of Several Variables

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Contents

9 Linear Transformations

9.1 Definition

(d) dim $X = r$ if V contains an independent set of r vectors but contains no independent set of $r+1$ vectors.

(e) An independent subset of a vector space X which spans X is called a *basis* of X.

9.2 Theorem

Let r be a positive integer. If a vector space X is spanned by a set of r vectors, then dim $X \leq r$.

Proof. If this is false, there is a vector space X which cotains an independent set $Q = \{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{r+1}\}$ and which is spanned by a set $S_0 = \{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r\}$.

$[i = 0]$: Since S_0 spans X, $\boldsymbol{y}_1$ is in the span of S_0. Hence,

$$\boldsymbol{y}_1 + b_1\boldsymbol{x}_1 + \cdots + b_r\boldsymbol{x}_r = \mathbf{0}$$

Since $\boldsymbol{y}_1 \neq \mathbf{0}$, there exists k_0 such that $b_{k_0} \neq 0$ and $\boldsymbol{x}_{k_0}$ can be expressed as a linear combination of $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r, \boldsymbol{y}_1\} \setminus \{\boldsymbol{x}_{k_0}\}$. It follows that X is spanned by $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_r, \boldsymbol{y}_1\} \setminus \{\boldsymbol{x}_{k_0}\}$, which now we call S_1 and rename the elements by writting $S_1 = \{\boldsymbol{y}_1, \boldsymbol{x}_2, \cdots, \boldsymbol{x}_r\}$.

$[i = 1]$: Since S_1 spans X, $\boldsymbol{y}_2$ is in the span of S_1. Hence,

$$a_1\boldsymbol{y}_1 + \boldsymbol{y}_2 + b_2\boldsymbol{x}_2 + \cdots + b_r\boldsymbol{x}_r = \mathbf{0}$$

Since $\{\boldsymbol{y}_1, \boldsymbol{y}_2\}$ is independent, there exists k_1 such that $\boldsymbol{x}_{k_1} \neq 0$ and $\boldsymbol{x}_{k_1}$ can be expressed as a linear combination of $\{\boldsymbol{x}_2, \cdots, \boldsymbol{x}_r, \boldsymbol{y}_1, \boldsymbol{y}_2\} \setminus \{\boldsymbol{x}_{k_1}\}$. It follows that X is spanned by $\{\boldsymbol{x}_2, \cdots, \boldsymbol{x}_r, \boldsymbol{y}_1, \boldsymbol{y}_2\} \setminus \{\boldsymbol{x}_{k_1}\}$, which now we call S_2 and rename the elements by wrtting $S_2 = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_3, \cdots, \boldsymbol{x}_r\}$.

$[i = r-1]$: After the step $[i = r-2]$, we have $S_{r-1} = \{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{r-1}, \boldsymbol{x}_r\}$ which spans X. Since S_{r-1} spans X, $\boldsymbol{y}_r$ is in the span of S_{r-1}. Hence,

$$a_1\boldsymbol{y}_1 + \cdots + a_{r-1}\boldsymbol{y}_{r-1} + \boldsymbol{y}_r + b_r\boldsymbol{x}_r = \mathbf{0}.$$

Since $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\}$ is independent, $b_r \neq 0$ and $\boldsymbol{x}_r$ can be expressed as a linear combination of $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\}$. It follows that X is spanned by $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\}$, which we call S_r.

Since the set $Q = \{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_{r+1}\}$ was independent, $\boldsymbol{y}_{r+1}$ is outside of the span of S_r or the set X. This contradiction proves the theorem. $\square$

Corollary : dim $R^n = n$.

Proof. R^n is spanned by the set $E = \{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_r\}$. The previous theorem says that dim $R^n \leq n$. Moreover, E is independent set of n vectors, for which dim $R^n \geq n$. $\square$

9.3 Theorem

Suppose X is a vectors space, and dim $X = n$.

(a) A set E of n vectors in X spans X if and only if E is independent.

(b) X has a basis, and every basis consists of n vectors.

(c) If $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\}$ is an independent set in X such that $1 \leq r \leq n$, then X has a basis containing $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\}$.

Proof. (a) Suppose that $E = \{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ is independent and that $\boldsymbol{y} \in X$. The set $E \cup \{\boldsymbol{y}\}$ is dependent since dim $X = n$. That is, $a_1\boldsymbol{x}_1 + \cdots + a_n\boldsymbol{x}_n + b\boldsymbol{y} = \mathbf{0}$ for a set of coefficients a_i's and b, all of which is not zero. Then, $b \neq 0$ owing to the independence of E. Thus, $\boldsymbol{y}$ is in the span of E. Conversely, suppose that E is dependent. Then, one of its members can be removed (to constitute a set E_0 of $n-1$ vectors) without changing the span of E. If E spans X, then E_0 spans X too. Then dim $X \leq n-1$ by the previous theorem, which is a contradiction. Thus E spans X.

(b) Since dim $X = n$, X contains an independent set of n vectors. By (a), this set spans X and is a basis of X.

(c) Note first that any subset A of X, consting of more than n elements, is dependent. For, if A were independent, then dim $X > n$ by **9.1.(d)**.

Now, let $\{\boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ be a basis for X. The set $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_n\}$ is a dependent set which spans X. We can remove an element $\boldsymbol{x}_{k_i}$ for $1 \leq i \leq n-r$ from the set, without changing the span of X, to construct $\{\boldsymbol{y}_1, \cdots, \boldsymbol{y}_r\} \setminus \{x_{k_1}, \cdots, x_{k_{n-r}}\}$.

□