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Homework - 1

(1)

(a) To prove $||Rq - Rp|| = ||q - p||$

$\forall q, p \in R^3$

Since R is orthogonal,
multiplying R^T on both sides

$$R^T ||Rq - Rp|| = R^T ||q - p||$$

$$||(R^T R)q - R^T (R) p|| = ||R^T (q - p)||$$

Now: $R^T R = I$ (property of orthogonal matrix)

and $||R^T|| = ||R||$

$$\therefore ||q - p|| = ||R(q-p)||$$

$$= ||Rq - Rp||$$

Hence proved

(b) Let the general form of the R matrix be

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Let the general format of V be
$V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$ and of W be
$W = W_1\hat{i} + W_2\hat{j} + W_3\hat{k}$

Calculating..
$R(V \times W)$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} V_2W_3 - W_2V_3 \\ -V_1W_3 + W_1V_3 \\ V_1W_2 - V_2W_1 \end{bmatrix}$$

Calculating the first term we get →

$$r_{11}V_2W_3 - r_{11}W_2V_3 - r_{12}V_1W_3 + r_{12}W_1V_3 + r_{13}V_1W_2$$
$$- r_{13}V_2W_1 \longrightarrow L \cdot H \cdot S$$

Now, Calculating $(RV) \times (RW)$ on the $R \cdot H \cdot S$

$$= \begin{bmatrix} r_{11}V_1 + r_{12}V_2 + r_{13}V_3 \\ r_{21}V_1 + r_{22}V_2 + r_{23}V_3 \\ r_{31}V_1 + r_{32}V_2 + r_{33}V_3 \end{bmatrix} \begin{bmatrix} r_{11}W_1 + r_{12}W_2 + r_{13}W_3 \\ r_{21}W_1 + r_{22}W_2 + r_{23}W_3 \\ r_{31}W_1 + r_{32}W_2 + r_{33}W_3 \end{bmatrix}$$

Since we are only comparing the first terms...

$$(r_{21}V_1 + r_{22}V_2 + r_{23}V_3)(r_{31}W_1 + r_{32}W_2 + r_{33}W_3)$$

$$-(r_{21}W_1 + r_{22}W_2 + r_{23}W_3)(r_{31}V_1 + r_{32}V_2 + r_{33}V_3)$$

$$= r_{21}V_1 r_{31}W_1 + r_{21}V_1 r_{32}W_2 + r_{21}V_1 r_{33}W_3$$

$$+ r_{22}V_2 r_{31}W_1 + r_{22}V_2 r_{32}W_2 + r_{22}V_2 r_{33}W_3$$

$$+ r_{23}V_3 r_{31}W_1 + r_{23}V_3 r_{32}W_2 + r_{23}V_3 r_{33}W_3$$

$$- r_{21}W_1 r_{31}V_1 - r_{21}W_1 r_{32}V_2 - r_{21}W_1 r_{33}V_3$$

$$- r_{22}W_2 r_{31}V_1 - r_{22}W_2 r_{32}V_2 - r_{22}W_2 r_{33}V_3$$

$$- r_{23}W_3 r_{31}V_1 - r_{23}W_3 r_{32}V_2 - r_{23}W_3 r_{33}V_3$$

After cancelling out the terms and accounting for the fact that each element of a real orthogonal matrix is equal to its cofactor we can write the remainder of the terms as follows:

$$= V_1 W_2 \underbrace{\left[r_{21}r_{32} - r_{22}r_{31}\right]}_{\text{cofactor of } r_{13}} + (-V_1 W_3)\underbrace{\left[r_{21}r_{33} - r_{23}r_{31}\right]}_{\text{cofactor of } r_{12}}$$

$$+ (-V_2 W_1)\underbrace{\left[r_{22}r_{31} - r_{21}r_{32}\right]}_{\text{cofactor of } r_{13}} + V_2 W_3 \underbrace{\left[r_{22}r_{33} - r_{23}r_{32}\right]}_{\text{cofactor of } r_{11}}$$

$$-(-w_1 v_3)\Big[\underbrace{r_{21} r_{33} - r_{23} r_{31}}_{\text{Cofactor of } r_{12}}\Big] - w_2 v_3 \Big[\underbrace{r_{22} r_{33} - r_{33} r_{32}}_{\text{Cofactor of } r_{11}}\Big]$$

$$= r_{11} v_2 w_3 - r_{11} w_2 v_3 - r_{12} v_1 w_3 + r_{12} v_3 w_1 + r_{13} v_1 w_2$$

$$- r_{13} v_2 w_1 = \underline{\underline{R\cdot H\cdot S}} = \underline{\underline{L\cdot H\cdot S}}$$

Hence proved that

$$\boxed{R(v \times w) = (Rv) \times (Rw)}$$

(2) The Rotation matrices can be expressed in terms of quarternion

$R_{x,\phi} = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}\,\hat{i}$

$R_{z,\theta} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\hat{k}$

$R_{y,\psi} = \cos\psi + \sin\frac{\psi}{2}\,\hat{j}$

Now, $(R_{x,\phi})(R_{z,\theta})(R_{y,\phi})$

$= \left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\,\hat{i}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\hat{k}\right)\left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\,\hat{j}\right)$

$= \left(\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \cos\frac{\phi}{2}\sin\frac{\theta}{2}\,\hat{k} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\,\hat{i} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\,(-\hat{j})\right)\left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\,\hat{j}\right)$

Now onwards writing $\cos\frac{\theta}{2}$, $\cos\frac{\phi}{2}$, $\cos\frac{\psi}{2}$, $\sin\frac{\theta}{2}$, $\sin\frac{\phi}{2}$ and $\sin\frac{\psi}{2}$ as $C\theta$, $C\phi$, $C\psi$, $S\theta$, $S\phi$, $S\psi$ respectively.

$C\theta C\phi C\psi + C\theta C\phi S\psi\,\hat{j} + C\phi S\theta C\psi\,\hat{k} + C\phi S\theta S\psi\,(-\hat{i})$
$+ C\theta S\phi C\psi\,\hat{i} + C\theta S\phi S\psi\,\hat{k} - S\theta S\phi C\psi\,\hat{j} + S\theta S\phi S\psi$

$= c\theta c\phi c\psi + s\theta s\phi s\psi + (c\theta s\phi c\psi - s\theta c\phi s\psi)\hat{i}$

$+ (c\theta c\phi s\psi - s\theta s\phi s\psi)\hat{j} + (s\theta c\phi c\psi + c\theta s\phi s\psi)\hat{k}$

$$A = \cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}$$

$$u_x = \cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2} - \sin\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2}$$

$$u_y = \cos\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2}$$

$$u_z = \sin\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}$$

As calculated $\sqrt{A^2 + u_x^2 + u_y^2 + u_z^2}$, should be given a value of 1 in order to signify a unit quaternion

To verify

$A^2 = c^2\theta c^2\phi c^2\psi + s^2\theta s^2\phi s^2\psi + 2c\theta c\phi c\psi s\theta s\phi s\psi$

$u_x^2 = c^2\theta s^2\phi c^2\psi + s^2\theta c^2\phi s^2\psi - 2c\theta c\phi c\psi s\theta s\phi s\psi$

$u_y^2 = c^2\theta c^2\phi s^2\psi + s^2\theta s^2\phi c^2\psi - 2c\theta c\phi c\psi s\theta s\phi s\psi$

$u_z^2 = s^2\theta c^2\phi c^2\psi + c^2\theta s^2\phi s^2\psi + 2c\theta c\phi c\psi s\theta s\phi s\psi$

Now adding A^2, u_x^2, u_y^2 and u_z^2 →

$= c^2\theta c^2\phi c^2\psi + c^2\theta c^2\phi s^2\psi + c^2\theta s^2\phi c^2\psi$

$+ c^2\theta s^2\phi s^2\psi + s^2\theta s^2\phi s^2\psi + s^2\theta s^2\phi c^2\psi$

$+ s^2\theta c^2\phi s^2\psi + s^2\theta c^2\phi c^2\psi$

$+ 4c\theta c\phi c\psi s\theta s\phi s\psi - 4c\theta c\phi c\psi s\theta s\phi s\psi$

$= c^2\theta c^2\phi (c^2\psi + s^2\psi) + c^2\theta s^2\phi (c^2\psi + s^2\psi)$

$+ s^2\theta s^2\phi (s^2\psi + c^2\psi) + s^2\theta c^2\phi (s^2\psi + c^2\psi)$

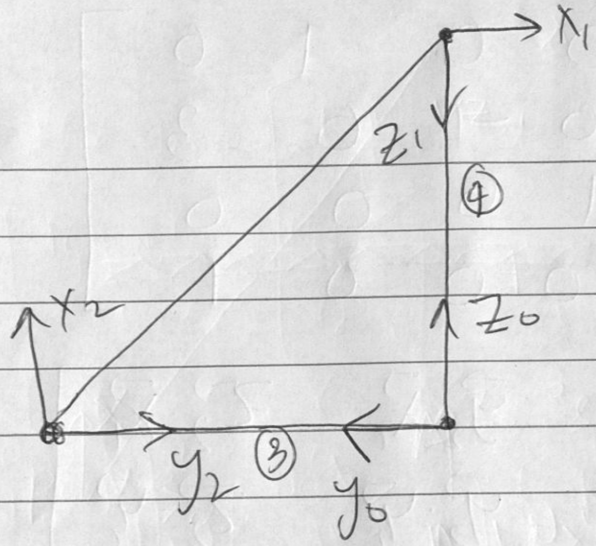
$= c^2\theta c^2\phi + c^2\theta s^2\phi + s^2\theta s^2\phi + s^2\theta c^2\phi$

$= c^2\theta (c^2\phi + s^2\phi) + s^2\theta (s^2\phi + c^2\phi)$

$= c^2\theta + s^2\theta = \underline{\underline{1}}$

Hence proved

(3)



The rotation matrix between Frame '0' and Frame '1' will be of the form:

$$R_1^0 = \begin{bmatrix} \vec{X_1}\cdot\vec{X_0} & \vec{y_1}\cdot\vec{x_0} & \vec{Z_1}\cdot\vec{X_0} \\ \vec{X_1}\cdot\vec{y_0} & \vec{y_1}\cdot\vec{y_0} & \vec{Z_1}\cdot\vec{y_0} \\ \vec{X_1}\cdot\vec{Z_0} & \vec{y_1}\cdot\vec{Z_0} & \vec{Z_1}\cdot\vec{Z_0} \end{bmatrix}$$

Hence this along with a translation of 4 along the 'Z' axis will be represented as:

$$H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly $R_2^0 = \begin{bmatrix} \vec{X_2}\cdot\vec{X_0} & \vec{y_2}\cdot\vec{X_0} & \vec{Z_2}\cdot\vec{X_0} \\ \vec{X_2}\cdot\vec{y_0} & \vec{y_2}\cdot\vec{y_0} & \vec{Z_2}\cdot\vec{y_0} \\ \vec{X_2}\cdot\vec{Z_0} & \vec{y_2}\cdot\vec{Z_0} & \vec{Z_2}\cdot\vec{Z_0} \end{bmatrix}$

Hence this along with a translation of 3 along the Y axis can be represented as

$$H_2^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$R_2^1 = \begin{bmatrix} \vec{x}_2\cdot\vec{x}_1 & \vec{y}_2\cdot\vec{x}_1 & \vec{z}_2\cdot\vec{x}_1 \\ \vec{x}_2\cdot\vec{y}_1 & \vec{y}_2\cdot\vec{y}_1 & \vec{z}_2\cdot\vec{y}_1 \\ \vec{x}_2\cdot\vec{z}_1 & \vec{y}_2\cdot\vec{z}_1 & \vec{z}_2\cdot\vec{z}_1 \end{bmatrix}$$

This alongwith an traslation of -3 along X and 4 units along Z gives us a homogeneous transformation represented as H_2^1

$$H_2^1 = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now: $H_1^0 \times H_2^1$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= H_2^0$. <u>Hence proved.</u> [Solution provided in MATLAB as well]