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ENPM662 - HW5

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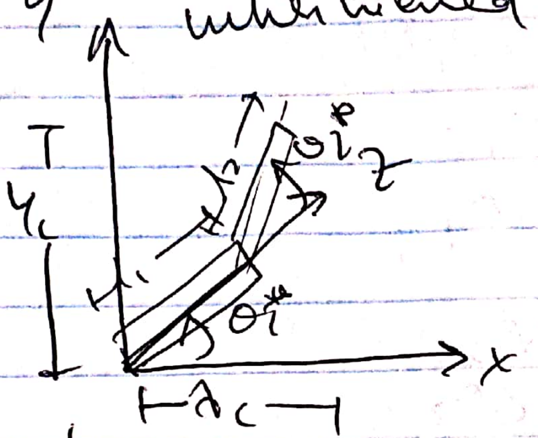
(1)

given: $l_1 = 2 l_2$

link 1 $\longrightarrow$ massless

link 2 $\longrightarrow$ massless

when viewed from the top



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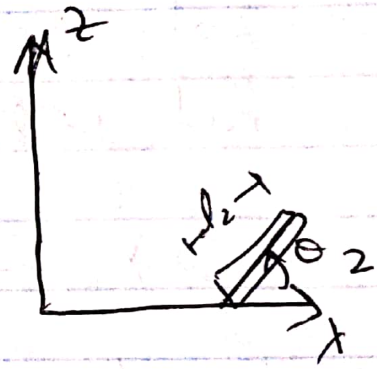

$$x_c = (l_1 + l_2 \cos\theta_2)\cos\theta_1$$

$$y_c = (l_1 + l_2 \cos\theta_2)\sin\theta_1$$

$$z_c = l_2 \sin\theta_2$$

We know the formula of kinetic Energy as:

$$k.E = \frac{m}{2}\left[\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2\right]$$

Now,

$$\frac{d}{dt} x_c = -l_1 \sin\theta_1 \dot{\theta}_1 + (-l_2 \cos\theta_1 \dot{\theta}_1) \cos\theta_2 - l_2 \cos\theta_2 (\dot{\theta}_2) \cos\theta_1)$$

$$= l_2 [-\sin\theta_2 \cos\theta_1 \dot{\theta}_2 - \cos\theta_2 \sin\theta_1 \dot{\theta}_1] - l_1 (\sin\theta_1 \dot{\theta}_1]$$

Let $\sin\theta_1 = S\theta_1$;
Let $\cos\theta_1 = C\theta_1$

Let $\sin\theta_2 = S\theta_2$
Let $\cos\theta_2 = C\theta_2$

$$\Rightarrow \dot{x}_c = -l_2 [-S_2 C_1 \dot{\theta}_2 - C_2 S_1 \dot{\theta}_1] - l_1 S_1 \dot{\theta}_1 \quad \text{— (a)}$$

Now, $\dot{y}_c = l_1 C\theta_1 \dot{\theta}_1 + l_2 C\theta_2 C\theta_1 \dot{\theta}_1 + (-l_2 S\theta_1 S\theta_2 \dot{\theta}_2)$

$$\dot{y}_c = l_2 [C_2 C_1 \dot{\theta}_1 - S_2 S_1 \dot{\theta}_2] + l_1 C_1 \dot{\theta}_1 \quad \text{— (b)}$$

Now,

$$\dot{z}_c = l_2 C\theta_2 \dot{\theta}_2 \quad \text{— (c)}$$

$$K.E = \frac{1}{2} m (\dot{x}_c^2 + \dot{y}_c^2 + \dot{z}_c^2)$$

$$= \frac{1}{2} m \Big[\big(-l_2[-S_2 C_1 \dot{\theta}_2 - C_2 S_1 \dot{\theta}_1] - l_1 S_1 \dot{\theta}_1\big)^2 + \big(l_1 C_1 \dot{\theta}_1 + l_2 (C_2 C_1 \dot{\theta}_1 - S_2 S_1 \dot{\theta}_2)\big)^2 + \big(l_2 C\theta_2 \dot{\theta}_2\big)^2 \Big]$$

In order to simplify these calculations we leverage the information given to us:-

$l_1 = 2l_2$

and taking l_2 common, Now we get:-

$\therefore K.E =$

$\frac{Ml_2^2}{2}\Big[(\dot{\theta_2})^2 + (S_2C_1)^2 + (\dot{\theta_1})^2(2S_1+S_1C_2)^2$

$+2\dot{\theta_1}\dot{\theta_2}\big(S_2C_1(S_1C_2+2S_1)\big) + \dot{\theta_1}^2\big[2C_1+C_2C_1\big]^2$

$+(\dot{\theta_2})^2(S_2^2S_1^2) + (\dot{\theta_2})^2C_2^2 + 2\dot{\theta_1}\dot{\theta_2}\big[-S_2S_1(C_2C_1$

$+2C_1)\big]$

$\Rightarrow \frac{l_2^2M}{2}\Big[(\dot{\theta_1})^2\Big[4C_2S_1^2 + C_2^2S_1^2 + 4S_1^2 + 4C_1^2 + C_2^2C_1^2$

$+4C_2C_1^2\Big]$ ①

$+(\dot{\theta_2})^2\big(S_2^2(C_1^2+S_1^2)+C_2^2\big] + 2\dot{\theta_1}\dot{\theta_2}$

$\big[S_1S_2\cancel{C_1C_2} + \cancel{2S_1S_2C_1} - S_1S_2\cancel{C_1C_2} - \cancel{2S_1S_2C_1}\big]$

We know: $\sin^2\theta_1 + \cos^2\theta_1 = 1$

$\Rightarrow K.E = \frac{Ml_2^2}{2}\Big[(\dot{\theta_1})^2\big(4(S_1^2+C_1^2)+C_2^2(C_1^2+S_1^2)$

$+4C_2(S_1^2+C_1^2)\big)$

$+(\dot{\theta_2})^2\big[S_2^2+C_2^2\big]$

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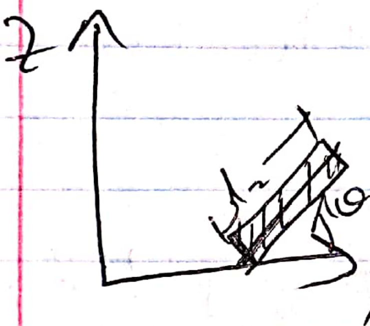
again utilising the same formula:-

$$\rightarrow K.E = \frac{M l_2^2}{2}\left[(\dot{\theta}_1)^2(4+C_2^2+4C_2)+(\dot{\theta}_2)^2\right]$$

$$\boxed{K.E = \frac{M l_2^2}{2}\left[(\dot{\theta}_1)^2((C_2+2)^2+(\dot{\theta}_2)^2\right]}$$

Now we have a ball of mass M.
This is bound to have a potential energy.
$P.E = mgh$.

Here height is in the z-direction



total height $= l_2 \sin\theta_2$

$\} l_2 S\theta_2$; Total potential energy

$= mg\, l_2 \sin\theta_2 = mg\, l_2 S_2$

Calculating the Lagrangian:-

$$L = K - P \quad \left\{\begin{array}{l} K = \text{kinetic Energy} \\ P = \text{potential Energy}\end{array}\right\}$$

$$\Rightarrow L = \frac{m l_2^2}{2}\left[(\dot{\theta}_2)^2 + ((_2+2)^2 (\dot{\theta}_1)^2 - mg l_2 S_2 \right]$$

$$\Rightarrow L = \frac{m l_2}{2}\left[l_2\left((\dot{\theta}_2)^2 - ((_2+2)^2 (\dot{\theta}_1)^2\right) - mg S_2 \right]$$

Now we know that:

$$\boxed{\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_1} - \frac{\partial T}{\partial q_1} = Q_1}$$

$$* \frac{\partial L}{\partial \dot{\theta}_1} = 0 - \frac{m l_2^2}{2}(2)((_2+2)^2 \dot{\theta}_1 \underset{-0}{} = \underline{\underline{m l_2^2 \dot{\theta}_1 ((_2+2)^2}}$$

$$* \frac{\partial L}{\partial \dot{\theta}_2} = \underline{\underline{m l_2^2 \dot{\theta}_2}}$$

$$* \frac{\partial L}{\partial \theta_1} = \underline{\underline{0}}$$ $\rightarrow$ Since θ_1 is not present at all.

$$* \frac{\partial L}{\partial \theta_2} = -m l_2^2 \left[((_2+2) S_2 \dot{\theta}_1)^2 + \frac{C_2}{l_2} g \right]$$

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$$Q_1 = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1}$$

$$Q_1 = \frac{d}{dt}\left(m l_2^2 \dot{\theta}_1 (c_2+2)^2\right) - 0$$

$$Q_1 = m l_2^2 \left[\frac{d}{dt}\left(\dot{\theta}_1\left(\cos^2\theta_2 + 4 + 4\cos\theta_2\right)\right)\right]$$

$$Q_1 = m l_2^2 \frac{d}{dt}\left[\dot{\theta}_1 \cos^2\theta_2 + \dot{\theta}_1(4) + 4\dot{\theta}_1 \cos\theta_2\right]$$

$$\cancel{= m l_2^2 \left[\dot{\theta}_1 \dot{\theta}_2 \cos^2\theta_2\right.}$$

$$Q_1 = m l_2^2 \left[\dot{\theta}_1\left(2\cos\theta_2(-\sin\theta_2)\dot{\theta}_2\right) + \ddot{\theta}_1 \cos^2\theta_2 + 4\ddot{\theta}_1 + (-4\dot{\theta}_1 \sin\theta_2)\dot{\theta}_2 + 4\ddot{\theta}_1 \cos\theta_2\right]$$

$$Q_1 = m l_2^2 \left(\ddot{\theta}_1\left[\cos^2\theta_2 + 4 + 4\cos\theta_2\right] + (\dot{\theta}_1 \dot{\theta}_2)\left[\left[\sin 2\theta_2\right] + \left[-4\sin\theta_2\right]\right]\right)$$

$$\Rightarrow Q_1 = m l_2^2 \left[\ddot{\theta}_1 \left((c_2+2)^2 - \dot{\theta}_1\dot{\theta}_2 (\sin 2\theta_2) + 4\sin\theta_2\right)\right]$$

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Now, calculating θ_2

$$Q_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2}$$

$$= \frac{d}{dt}\left(m l_2^2 \dot{\theta}_2\right) - \left(-m l_2^2\left[(c_2+2) s_2 (\dot{\theta}_1)^2 + \frac{c_2}{l_2} g\right]\right)$$

$$= m l_2^2 \ddot{\theta}_2 + m l_2^2 \left[(c_2+2) s_2 (\dot{\theta}_1)^2 + \frac{c_2}{l_2} \cdot g\right]$$

$$\Rightarrow Q_2 = m l_2^2 \left[\ddot{\theta}_2 + \left[(c_2+2)(s_2)(\dot{\theta}_1)^2 + \frac{c_2}{l_2} \cdot g\right]\right]$$

(2) $l_1 = 1.5$

$l_2 = 1$

$l_3 = 0.5$

$\theta_1 = 150^\circ$ $\quad$ $l_4^* = 2$

$\theta_2 = 45^\circ$

$\theta_3 = 30^\circ$

Refer to attached MATLAB Code for Jacobian Calculation :-

We use the following D.H. Table.

	θ	d	a	α
$0 \to 1$	$\theta_1 - 90$	l_1	0	-30
$1 \to 2$	$\theta_2 + 90$	l_2	0	-90
$2 \to 3'$	-30	0	0	0
$3' \to 4$	$\theta_3 - 90$	0	0	-90
$4 \to 5$	0	$l_3 + l_4^*$	0	0

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The following matrices are obtained.

$$z_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$z_1 = \begin{pmatrix} -0.4330 \\ 0.2500 \\ 0.8660 \end{pmatrix}$$

$$z_2 = \begin{pmatrix} 0.1768 \\ -0.9186 \\ 0.3536 \end{pmatrix}$$

$$z_3 = \begin{pmatrix} -0.8839 \\ -0.3062 \\ -0.3536 \end{pmatrix}$$

$$\text{Jacobian} = \begin{bmatrix} 0.5155 & 0.9919 & 1.0825 & -0.8839 \\ -2.692 & -2.296 & -0.625 & -0.3062 \\ 0 & 0.883 & -2.1651 & -0.3536 \\ 0 & -0.433 & 0.1768 & 0 \\ 0 & 0.2500 & -0.9186 & 0 \\ 1.000 & 0.866 & 0.3536 & 0 \end{bmatrix}$$

$$\text{Wrench} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = [1\ 0\ 0\ 0\ 0\ 0]^T$$

$$\text{Torque} \Rightarrow \begin{bmatrix} 0.5155 \\ 0.9919 \\ 1.0825 \\ -0.8839 \end{bmatrix} \rightarrow \text{check MATLAB}$$

```
clc
clear all
%%code for
%%answer 2 part a HW 5
%%rewriting the equations derived in the last question
%%every row has one alpha value

alpha1 = -30;
alpha2 = -90;
alpha3 = 0;
alpha4 = -90;
alpha5 = 0;

%%link lengths
l1 = 1.5;
l2 = 1;
l3 = 0.5;

%%configuration q
theta1 = 150;
theta2 = 45;
theta3 = 30;
l4= 2;

%%matrix 1
Rz_theta1=[cosd(theta1-90) -sind(theta1-90) 0 0;sind(theta1-90)
 cosd(theta1-90) 0 0;0 0 1 0;0 0 0 1];
Rx_alpha1=[1 0 0 0;0 cosd(alpha1) -sind(alpha1) 0;0 sind(alpha1)
 cosd(alpha1) 0;0 0 0 1];
Tz_d1=[1 0 0 0;0 1 0 0;0 0 1 l1;0 0 0 1];
Tx_a1=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%%matrix 2
Rz_theta2=[cosd(theta2+90) -sind(theta2+90) 0 0;sind(theta2+90)
 cosd(theta2+90) 0 0;0 0 1 0;0 0 0 1];
Rx_alpha2=[1 0 0 0;0 cosd(alpha2) -sind(alpha2) 0;0 sind(alpha2)
 cosd(alpha2) 0;0 0 0 1];
Tz_d2=[1 0 0 0;0 1 0 0;0 0 1 l2;0 0 0 1];
Tx_a2=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%matrix 3
Rz_theta3=[cosd(-30) -sind(-30) 0 0;sind(-30) cosd(-30) 0 0;0 0 1 0;0
 0 0 1];
Rx_alpha3=[1 0 0 0;0 cosd(alpha3) -sind(alpha3) 0;0 sind(alpha3)
 cosd(alpha3) 0;0 0 0 1];
Tz_d3=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
Tx_a3=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%%matrix 4
Rz_theta4=[cosd(theta3-90) -sind(theta3-90) 0 0;sind(theta3-90)
 cosd(theta3-90) 0 0;0 0 1 0;0 0 0 1];
```

```
Rx_alpha4=[1 0 0 0;0 cosd(alpha4) -sind(alpha4) 0;0 sind(alpha4)
 cosd(alpha4) 0;0 0 0 1];
Tz_d4=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
Tx_a4=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%%matrix 5
Rz_theta5=[cosd(0) -sind(0) 0 0;sind(0) cosd(0) 0 0;0 0 1 0;0 0 0 1];
Rx_alpha5=[1 0 0 0;0 cosd(alpha5) -sind(alpha5) 0;0 sind(alpha5)
 cosd(alpha5) 0;0 0 0 1];
Tz_d5=[1 0 0 0;0 1 0 0;0 0 1 (l3+l4);0 0 0 1];
Tx_a5=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%%multiplying to get homogeneous matrices
A_1=Rz_theta1*Tz_d1*Tx_a1*Rx_alpha1
A_2=Rz_theta2*Tz_d2*Tx_a2*Rx_alpha2
A_3=Rz_theta3*Tz_d3*Tx_a3*Rx_alpha3
A_4=Rz_theta4*Tz_d4*Tx_a4*Rx_alpha4
A_5=Rz_theta5*Tz_d5*Tx_a5*Rx_alpha5


A_1 =

    0.5000   -0.7500   -0.4330         0
    0.8660    0.4330    0.2500         0
         0   -0.5000    0.8660    1.5000
         0         0         0    1.0000


A_2 =

   -0.7071         0   -0.7071         0
    0.7071         0   -0.7071         0
         0   -1.0000         0    1.0000
         0         0         0    1.0000


A_3 =

    0.8660    0.5000         0         0
   -0.5000    0.8660         0         0
         0         0    1.0000         0
         0         0         0    1.0000


A_4 =

    0.5000         0    0.8660         0
   -0.8660         0    0.5000         0
         0   -1.0000         0         0
         0         0         0    1.0000


A_5 =
```

```
    1.0000         0         0         0
         0    1.0000         0         0
         0         0    1.0000    2.5000
         0         0         0    1.0000
```

```
%%FINDING HOMOGENEOUS, ROTATION MATRICES AND ORIGINS
%%FOR THE JACOBIAN CALCULATION

%%JOINT 1 - REVOLUTE JOINT
O_0_0=[0 0 0]'

H_1_0 = A_1
R_1_0 = H_1_0(1:3,1:3)
O_1_0 = H_1_0(1:3,4)

%%JOINT 2 - REVOLUTE JOINT
H_2_0 = A_1*A_2
R_2_0 = H_2_0(1:3,1:3)
O_2_0 = H_2_0(1:3,4)

%%JOINT 3 - REVOLUTE JOINT
H_3_0 = A_1*A_2*A_3*A_4
R_3_0 = H_3_0(1:3,1:3)
O_3_0 = H_3_0(1:3,4)

%%JOINT 4 - PRISMATIC JOINT
H_4_0 = A_1*A_2*A_3*A_4*A_5
R_4_0 = H_4_0(1:3,1:3)
O_4_0 = H_4_0(1:3,4)

fprintf("Answer B : ")
fprintf("The O matrices are given as follows : ")

O_1_0 = H_1_0(1:3,4)
O_2_0 = H_2_0(1:3,4)
O_3_0 = H_3_0(1:3,4)
O_4_0 = H_4_0(1:3,4)

fprintf("The Z matrices are given as follows : ")
z0 = eye(3)*[0 0 1]'
z1 = R_1_0*[0 0 1]'
z2 = R_2_0*[0 0 1]'
z3 = R_3_0*[0 0 1]'

Jacobian = [cross(z0,O_4_0-O_0_0) cross(z1,O_4_0-O_1_0)
 cross(z2,O_4_0-O_2_0) z3;z0 z1 z2 [0 0 0]']
wrench = [1 0 0 0 0 0]'
Torque = (Jacobian)'*(wrench)
```

```
O_0_0 =

     0
```

```
     0
     0


H_1_0 =

    0.5000   -0.7500   -0.4330         0
    0.8660    0.4330    0.2500         0
         0   -0.5000    0.8660    1.5000
         0         0         0    1.0000


R_1_0 =

    0.5000   -0.7500   -0.4330
    0.8660    0.4330    0.2500
         0   -0.5000    0.8660


O_1_0 =

         0
         0
    1.5000


H_2_0 =

   -0.8839    0.4330    0.1768   -0.4330
   -0.3062   -0.2500   -0.9186    0.2500
   -0.3536   -0.8660    0.3536    2.3660
         0         0         0    1.0000


R_2_0 =

   -0.8839    0.4330    0.1768
   -0.3062   -0.2500   -0.9186
   -0.3536   -0.8660    0.3536


O_2_0 =

   -0.4330
    0.2500
    2.3660


H_3_0 =

   -0.4330   -0.1768   -0.8839   -0.4330
    0.2500    0.9186   -0.3062    0.2500
    0.8660   -0.3536   -0.3536    2.3660
         0         0         0    1.0000
```

```
R_3_0 =

   -0.4330   -0.1768   -0.8839
    0.2500    0.9186   -0.3062
    0.8660   -0.3536   -0.3536


O_3_0 =

   -0.4330
    0.2500
    2.3660


H_4_0 =

   -0.4330   -0.1768   -0.8839   -2.6427
    0.2500    0.9186   -0.3062   -0.5155
    0.8660   -0.3536   -0.3536    1.4821
         0         0         0    1.0000


R_4_0 =

   -0.4330   -0.1768   -0.8839
    0.2500    0.9186   -0.3062
    0.8660   -0.3536   -0.3536


O_4_0 =

   -2.6427
   -0.5155
    1.4821
```

Answer B : The O matrices are given as follows :

```
O_1_0 =

         0
         0
    1.5000


O_2_0 =

   -0.4330
    0.2500
    2.3660


O_3_0 =
```

```
   -0.4330
    0.2500
    2.3660


O_4_0 =

   -2.6427
   -0.5155
    1.4821

The Z matrices are given as follows :
z0 =

     0
     0
     1


z1 =

   -0.4330
    0.2500
    0.8660


z2 =

    0.1768
   -0.9186
    0.3536


z3 =

   -0.8839
   -0.3062
   -0.3536


Jacobian =

    0.5155    0.4419    1.0825   -0.8839
   -2.6427   -2.2964   -0.6250   -0.3062
         0    0.8839   -2.1651   -0.3536
         0   -0.4330    0.1768         0
         0    0.2500   -0.9186         0
    1.0000    0.8660    0.3536         0


wrench =

     1
     0
```

```
     0
     0
     0
     0


Torque =

    0.5155
    0.4419
    1.0825
   -0.8839
```

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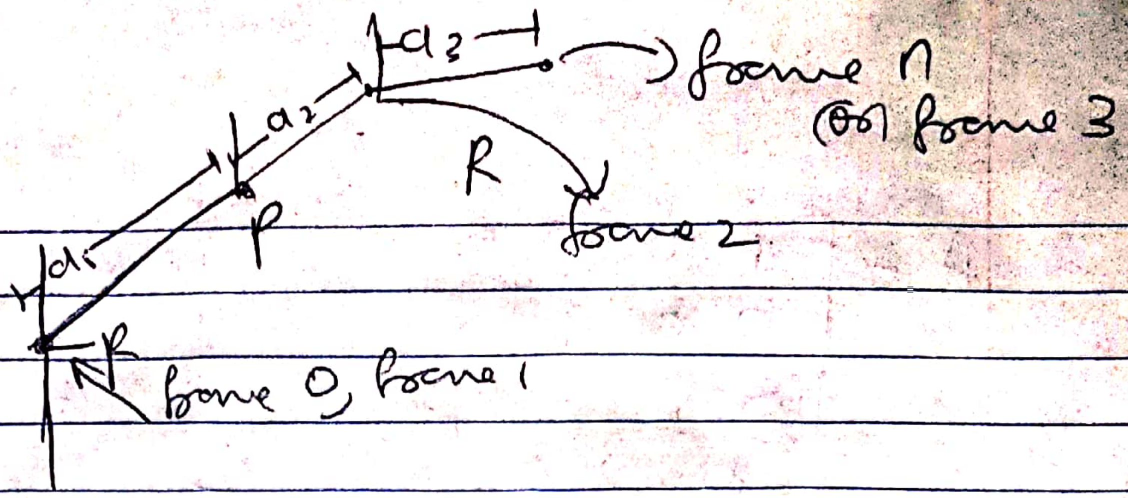
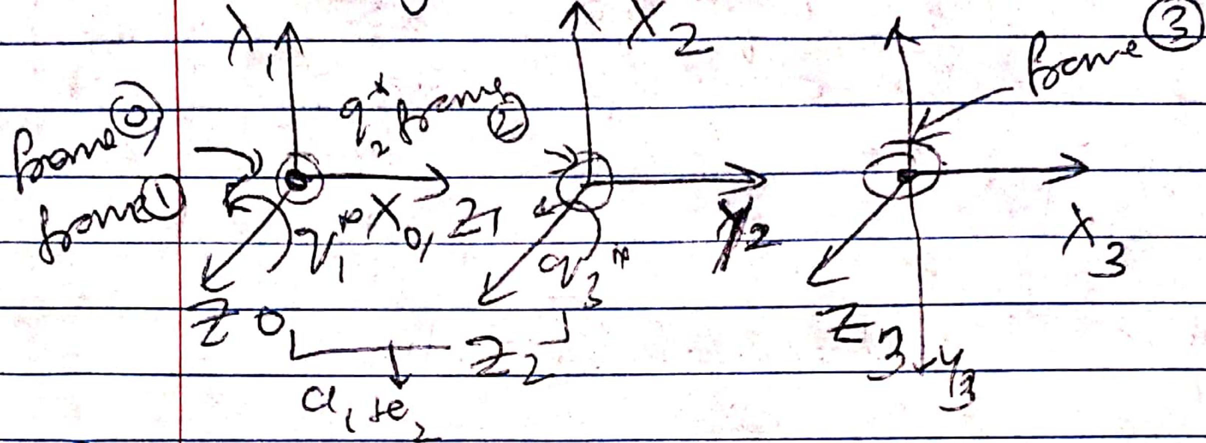


Drawing the coordinate frames:



→ Writing the D.H Table:

	θ_{i-1}	d_{i-1}	a_i	α_i
$0 \to 1$	$q_1^* + 90°$	0	0	$90°$
$1 \to 2$	0	$a_1 + a_2 + q_2^*$	0	-90
$2 \to 3$	$q_3^* - 90$	0	a_3	0

Now, writing the A matrices:

$$A_1 = \begin{bmatrix} -Sq_1^* & 0 & Cq_1^* & 0 \\ Cq_1^* & 0 & Sq_1^* & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T_1^0 is automatically A_1.

$$A_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & a_1 + a_2 + q_{v_2}^* \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} Sq_{v_3}^* & Cq_{v_3}^* & 0 & a_3 Sq_{v_3}^* \\ -Cq_{v_3}^* & Sq_{v_3}^* & 0 & -a_3 Cq_{v_3}^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

We already have $T_1^0 = A_1$

Now, $T_2^0 = A_1 \times A_2$ (performed on MATLAB)

$$= \begin{bmatrix} -Sq_{v_1}^* & -Cq_{v_1}^* & 0 & Cq_{v_1}^*(a_1 + a_2 + q_{v_2}^*) \\ Cq_{v_1}^* & -Sq_{v_1}^* & 0 & Sq_{v_1}^*(a_1 + a_2 + q_{v_2}^* \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$T_3^0 = A_1 \times A_2 \times A_3$ (performed on MATLAB

$$= \begin{bmatrix} C(q_{v_1}^* + q_{v_3}^*) & -S(q_{v_1}^* + q_{v_3}^*) & 0 & (a_1 + a_2 + q_{v_2}^*)\cos q_{v_1}^* + a_3 C(q_{v_1}^* + q_{v_3}^*) \\ S(q_{v_1}^* + q_{v_3}^*) & C(q_{v_1}^* + q_{v_3}^* & 0 & a_1 + a_2 + q_{v_3}^*)\sin q_{v_1}^* + a_3 S(q_{v_1}^* + q_{v_3}^*) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now, calculating the jacobian :-

We know :-

$$Z_0 = [0\ 0\ 1]^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Z_1 = [cq_1^* \ \ sq_1^* \ \ 0]^T = \begin{bmatrix} cq_1^* \\ sq_1^* \\ 0 \end{bmatrix}$$

$$Z_2 = [0\ 0\ 1]^T = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

also;

$$O_0 = [0\ 0\ 0]^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$O_1 = [0\ 0\ 0]^T = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$O_2 = \begin{bmatrix} cq_1^*(a_1 + a_2 + q_2^*) \\ sq_1^*(a_1 + a_2 + q_2^*) \\ 0 \end{bmatrix}$$

$$O_3 = \begin{bmatrix} (a_1 + a_2 + q_2^*)\, cq_1^* + a_3\, c(q_1^* + q_3^*) \\ (a_1 + a_2 + q_2^*)\, sq_1^* + a_3\, s(q_1^* + q_3^*) \\ 0 \end{bmatrix}$$

To calculate the Jacobian for linear velocity, we know that:

$$J_{v_1} = [Z_0(O_1 - O_0) \quad 0 \quad 0] \rightarrow$$ Not including the components for angular velocity

$$= [0 \quad 0 \quad 0]$$

$$J_{v_2} = [Z_0(O_2 - O_0) \quad Z_1 \quad 0]$$

$$= \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} (a_1 + a_2 + q_2^*) c q_1^* \\ (a_1 + a_2 + q_2^*) s q_1^* \\ 0 \end{bmatrix} \quad \begin{matrix} c q_1^* \\ s q_1^* \\ 0 \end{matrix} \quad \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \right]$$

$$J_{v_3} = [Z_0(O_3 - O_0) \quad Z_1 \quad Z_2(O_3 - O_2)]$$

$$= \begin{bmatrix} (-a_1 + a_2 + q_2^*) s q_1^* + q_3 (s(q_1^* + q_3^*)) & c_1 & -q_3 s(q_1^* + q_3^*) \\ (a_1 + q_2^* + a_2) c q_1^* + q_3 (c(q_1^* + q_3^*)) & s_1 & q_3 (c(q_1^* + q_3^*)) \\ 0 & 0 & 0 \end{bmatrix}$$

Also calculating the Jacobian for angular velocities we get:

$$J_{\omega 1} = [(0 0 1)^T \;\; 0 \;\; 0] = [z_0 \;\; 0 \;\; 0]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$J_{\omega 2} = [(0 0 1)^T \;\; 0 \;\; 0] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$
$$= [z_0 \;\; 0 \;\; 0]$$

$$J_{\omega 3} = [(0 0 1)^T \;\; 0 \;\; (0 0 1)^T] = [z_0 \;\; 0 \;\; z_2]$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now, we have three links.
Let them all have masses m_1, m_2 and m_3 respectively

When undergoing angular motion they will experience a moment of inertia.
Let link 1, link 2 and link 3 have M.O.I of I_1, I_2 and I_3 respectively.

The general formula for total kinetic energy is $\frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$

$$K.E_i = \frac{1}{2} M_i \dot{q}_i^T J_{V_i}^T J_{V_i} \dot{q} + \frac{1}{2} \dot{q}^T J_{\omega_i}^T I J_{\omega_1} \dot{q}$$

* Calculating for the Joint -1

$$K.E_1 = \underbrace{\frac{1}{2} M_1 \dot{q}^T J_{V_1}^T J_{V_1} \dot{q}}_{0} + \frac{1}{2} \dot{q}^T J_{\omega_1}^T I J_{\omega_1} \dot{q}$$

We know $\omega_1 = J_{\omega_1} \dot{q}$

$$\Rightarrow \omega_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

Now, $\therefore K.E_1$

$$= \frac{1}{2} M_1 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}^T \cdot 0 \cdot 0 \cdot (\dot{q}_1, \dot{q}_2, \dot{q}_3)$$

$$+ \frac{1}{2} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \cdot I_1 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$= \frac{I_1}{2} \dot{q}_1^2$$

Hence, total kinetic energy of (1) is $\frac{I_1}{2}\dot{q}_{v_1}^2$.

* **Calculating for Joint-2**

$$K.E_2 = \frac{1}{2} M_2 \dot{q}^T J_{v_2}^T J_{v_2} \dot{q} + \frac{1}{2} \dot{q}^T J_{w_1}^T \frac{I_2}{} J_{w_1} \dot{q}$$

Calculating KE for linear velocity

$$= \frac{1}{2} M_2 \begin{bmatrix} \dot{q}_{v_1} \\ \dot{q}_{v_2} \\ \dot{q}_{v_3} \end{bmatrix}^T \begin{bmatrix} -(a_1 + a_{c_2} + q_{v_2}^*) S_1 & C_1 & 0 \\ (a_1 + a_{c_2} + q_{v_2}^*) C_1 & S_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} -(a_1 + a_{c_2} + q_{v_2}^*) S_1 & (a_1 + a_{c_2} + q_{v_2}^*) C_1 & 0 \\ C_1 & S_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_{v_1} \\ \dot{q}_{v_2} \\ \dot{q}_{v_3} \end{bmatrix}$$

Here: $\begin{Bmatrix} S q_{v_1}^* = S_1 \\ C q_{v_1}^* = C_1 \end{Bmatrix}$

$$= \frac{1}{2} M_2 \begin{bmatrix} -\dot{q}_{v_1}(a_1 + a_{c_2} + q_{v_2}^*) S_1 + \dot{q}_{v_2} C_1 & \dot{q}_{v_1}(a_1 + a_{c_2} + q_{v_2}^*) C_1 + \dot{q}_{v_2} S_1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\dot{q}_{v_1}(a_1 + a_{c_2} + q_{v_2}^*) S_1 + \dot{q}_{v_2} C_1 \\ \dot{q}_{v_1}(a_1 + a_{c_2} + q_{v_2}^*) C_1 + \dot{q}_{v_2} S_1 \end{bmatrix}$$

$$= \frac{1}{2} m_2 \Big[\dot{q}_1^2 (a_1 + a_{c_2} + q_2^*)^2 s_1^2 + \dot{q}_2^2 c_1^2$$

$$+ \dot{q}_1^2 (a_1 + a_{c_2} + q_2^*)^2 c_1^2 + \dot{q}_2^2 s_1^2$$

$$\cancel{- 2 \dot{q}_1 \dot{q}_2 c_1 s_1 (a_1 + a_{c_2} + q_2^*)}$$

$$\cancel{+ 2 \dot{q}_1 \dot{q}_2 c_1 s_1 (a_1 + a_{c_2} + q_2^*)} \Big]$$

$$\Rightarrow k \cdot E_v = \frac{1}{2} m_2 \left[\dot{q}_1^2 (a_1 + a_{c_2} + q_2^*)^2 + \dot{q}_2^2 \right]$$

Now calculating the contribution ^{of k·E} by angular velocity:

$$\omega_2 = J_{\omega_2} \dot{q} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \dot{q}_1 \end{bmatrix}$$

$$\therefore k \cdot E_\omega = \frac{1}{2} I_2 \omega_2^2 = \frac{I_2 \dot{q}_1^2}{2}$$

$\therefore$ Total $k \cdot E_2 =$

$$\frac{m_2}{2} \left[(a_1 + a_2 + q_2^*)^2 \dot{q}_1^2 + \dot{q}_2^2 \right] + \frac{I_2}{2} \dot{q}_1^2$$

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* Calculating for Joint-3

$$K.E_3 = \frac{1}{2} m_3 \dot{q}^T J_{V_3}^T J_{V_3} \dot{q} + \frac{1}{2} \dot{q}^T J_{\omega_3}^T I_3 J_{\omega_3} \dot{q}$$

Calculating the contribution to K.E by V:

here $S_{13} = S(q_1^* + q_3^*)$

$C_{13} = C(q_1^* + q_3^*)$

$\Rightarrow$

$$K.E_3 = \frac{1}{2} m_3 \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}^T J_{V_3}^T J_{V_3} [\dot{q}_1 \; \dot{q}_2 \; \dot{q}_3]^T$$

$$= \frac{m_3}{2} \Big[[(a_1 + a_2 + q_2^*) S_1 + a_{c_3} S_{13}]^2 + C_1 \dot{q}_2^2 + a_{c_3}^2 S_{13}^2 \dot{q}_3^2 - 2 a_{c_3} C_1 S_{13} \dot{q}_2 \dot{q}_3$$

$$- 2 ((a_1 + a_2 + q_2^*) S_1 + a_{c_3} S_{13}) C_1 \dot{q}_1 \dot{q}_2$$

$$+ 2 a_{c_3} S_{13} [(a_1 + a_2 + q_2^*) S_1 + a_{c_3} S_{13}) \dot{q}_1 \dot{q}_3$$

$$+ ([(a_1 + a_2 + q_2^*) C_1 + a_{c_3} C_{13}] \dot{q}_1^2 + S_1^2 \dot{q}_2^2$$

$$+ a_{c_3}^2 C_{13}^2 \dot{q}_3^2 + 2 S_1 [(a_1 + a_2 + q_2^*) C_1 + a_{c_3} C_{13}) \dot{q}_1 \dot{q}_2$$

$$+ 2 a_{c_3} S_1 C_{13} \dot{q}_2 \dot{q}_3 + 2 a_{c_3} C_{13} ((a_1 + a_2 + q_2) C_1 + a_{c_3} C_{13}] \dot{q}_1 \dot{q}_3 \Big]$$

$$= \frac{m_3}{2}\Big[\dot q_1^2 (a_1 + a_2 + q_2^*)^2 + 2(a_1 + a_2 + q_2)\, a_{3c} c_3 \dot q_1^2$$

$$+ a_{3c}^2 \dot q_1^2 - 2 a_{3c} s_3 \dot q_1 \dot q_2 - 2 a_3 s_1 \dot q_2 \dot q_3$$

$$+ (2 a_{3c}^2 + 2 a_1 a_{3c} c_3 + 2 a_2 a_{3c} +$$

$$2 q_2^* a_{3c} c_3)\, \dot q_1 \dot q_3 + \dot q_2^2 + a_{3c}^2 \dot q_3^2\Big]$$

$$\Rightarrow K.E_{v_3} = \frac{m_3}{2}\Big[(a_1 + a_2 + q_2^*)^2 + a_{c_3}^2 + 2(a_1 + a_2 + q_2^*)\, a_{c_3} c_3\Big]\dot q_1^2$$

$$+ \dot q_2^2 + a_{c_3}^2 \dot q_3^2 - 2 a_{c_3} s_3 \dot q_1 \dot q_2$$

$$- 2 a_{c_3} s_3 \dot q_2 \dot q_3 + 2 a_{c_3}\Big[a_{c_3} +$$

$$(a_1 + a_2 + q_2^*)(c_3)\Big]\dot q_1 \dot q_3\Big]$$

Calculating the contribution by Angular velocity

$$\omega_3 = J_{\omega_3} \dot q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot q_1 \\ \dot q_2 \\ \dot q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot q_1 + \dot q_3 \end{bmatrix}$$

$$\therefore K.E_{\omega_3} = \frac{1}{2}\omega_3^T I_3 \omega_3 = \frac{I_3}{2}\left(\dot q_1^2 + 2\dot q_1 \dot q_3 + \dot q_3^2\right)$$

Adding them we get:

$K.E_3 =$

$$\frac{m_3}{2}\Big[(a_1+a_2+q_2^*)^2 + a_{c_3}^{*} + 2(a_1+a_2+q_2^*)a_{c_3} C_3]\dot{q}_1^2$$
$$+\dot{q}_2^2 + a_{c_3}^2\dot{q}_3^2 - 2a_{c_3}S_3\dot{q}_1\dot{q}_2$$
$$-2a_{c_3}S_3\dot{q}_2\dot{q}_3 + 2a_{c_3}[a_{c_3} + (a_1+a_2+q_2^*)C_3)\dot{q}_1\dot{q}_3]$$
$$+\frac{I_3}{2}(\dot{q}_1^2 + 2\dot{q}_1\dot{q}_3 + \dot{q}_3^2)$$

Now, finding potential energies:

$P.E_i = m_i g h_i$ (g = constant = acceleration due to gravity)

$P.E_1 = m_1 g h_1 = m_1 g (0) = 0$

$P.E_2 = m_2 g h_2 = m_2 g (a_1+a_2+q_2^*) S q_1^*$

$P.E_3 = m_3 g h_3 = m_3 g [(a_1+a_2+q_2^*)S_1 + a_3 S_{13}]$

$\boxed{\text{In all the above terms:}}$

$S_1 = Sq_1^*$ $S_2 = Sq_2^*$ $S_3 = Sq_3^*$ $S_{13} = S(q_1^* + q_3^*)$

$C_1 = Cq_1^*$ $C_2 = Cq_2^*$ $C_3 = Cq_3^*$ $C_{13} = C(q_1^* + q_3^*)$

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Now, as per the Lagrange equations.

$L = L_1 + L_2 + L_3$

$L_i = k_i - P_i$

* Calculating for link 1 ⟶

$L_1 = k_1 - P_1$

$= 0 + \frac{I_1}{2}\dot{q}_1^2 - 0 = \frac{I_1 \dot{q}_1^2}{2}$

* Calculating for link 2 ⟶

$L_2 = k_2 - P_2$

$= k_2 - P_2$

$= \frac{M_2}{2}\left[\dot{q}_1^2(a_1 + a_{c_2} + q_2^*)^2 + \dot{q}_2^2\right]$

$+ \frac{1}{2} I_2 \dot{q}_1^2 - \left[M_2 g S_1 (a_1 + a_{c_2} + q_2^*)\right]$

* Calculating for link 3 ⟶

$L_3 = k_3 - P_3$

$= \frac{1}{2} m_3 \{ [(a_1 + a_2 + q_2^*)^2 + a_{c_3}^2 + 2(a_1 + a_2 + q_2^*) a_{c_3} C_3] \dot{q}_1^2$

$+ \dot{q}_2^2 + a_{c_3} \dot{q}_3^2 - 2 a_{c_3} S_3 \dot{q}_1 \dot{q}_2 - 2 a_{c_3} S_3 \dot{q}_2 \dot{q}_3$

$+ 2 a_{c_3} [a_{c_3} + (a_1 + a_2 + q_2^*) C_3] \dot{q}_1 \dot{q}_3 \}$

$- m_3 g [(a_1 + a_2 + q_2^*) S_1 + a_{c_3} S_{13}] + \frac{I_3}{2}(\dot{q}_1^2 + \dot{q}_3^2 + 2\dot{q}_1\dot{q}_3)$

Adding L_1, L_2 and L_3:

$$L = I_1 \dot{q}_1^2 + \frac{m_2}{2}\left[(a_1 + a_{2c} + q_2^*)^2 \dot{q}_1^2 + \dot{q}_2^2\right]$$

$$+ \frac{I_2}{2}\dot{q}_1^2 + \frac{I_3}{2}(\dot{q}_1 + \dot{q}_3)^2 + \frac{m_3}{2}\left[(a_1 + a_2 + q_2^*)^2 + a_{3c}^2\right.$$

$$+ 2a_{3c}c_3(\dot{q}_1\dot{q}_3 + \dot{q}_2\dot{q}_3) + 2a_{3c}(a_{3c} +$$

$$\left.(a_1 + a_2 + q_2^*)c_3]\dot{q}_1\dot{q}_3 + \dot{q}_2^2 + a_{3c}^2\dot{q}_3^2\right]$$

$$- m_2 g(a_1 + a_{2c} + q_2^*)s_1 - m_3 g[(a_1 + a_2 + q_2^*)s_1$$

$$+ a_{3c}s_{13}]$$

Now, we know that

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \tau_i}$$

→ Calculating $\frac{\partial L}{\partial q_1}$

$$\frac{\partial L}{\partial q_1} = -m_2 g c_1(a_1 + a_{c2} + q_2) - m_3 g[(a_1 + a_2 + q_2)$$

$$c_1 + a_3 c_{13}]$$

→ Calculating $\frac{\partial L}{\partial q_i}$

$$\frac{\partial L}{\partial \dot{q}_1} = I_1 \dot{q}_1 + m_2 \dot{q}_1 (a_1 + a_{c_2} + q_2)^2 + I_2 \dot{q}_1$$

$$+ m_3 \left[(a_1 + a_2 + q_2)^2 + a_{c_3}^2 + 2\dot{q}_1 (a_1 + a_2 + q_2) a_{c_3} c_2 \right]$$

$$- m_3 a_{c_3} s_3 \dot{q}_2 + m_3 a_{c_3} \left[a_{c_3} (a_1 + a_2 + q_2) c_3 \right] \dot{q}_3$$

$$+ I_3 \dot{q}_1 + I_3 \dot{q}_3$$

→ Calculating $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)$

$$= I_1 \ddot{q}_1 + m_2 (a_1 + a_{c_2} + q_2)^2 + 2 m_2 \dot{q}_1 \dot{q}_2 (a_1 + a_{c_2} + q_2)$$

$$+ I_2 \ddot{q}_1 + m_3 \Big(2(a_1 + a_2 + q_2)\dot{q}_2 + 2 a_{c_3} c_3 \dot{q}_2$$

$$+ 2 a_{c_3} (a_1 + a_2 + q_2) \dot{q}_2 + m_3 \big[(a_1 + a_2 + q_2)^2$$

$$+ a_{c_3}^2 + 2(a_1 + a_2 + q_2) a_{c_3} c_3 \ddot{q}_1 \Big)$$

$$- m_3 a_{c_3} (s_3 \ddot{q}_2 + \dot{q}_2 \dot{q}_3 c_3) + m_3 a_{c_3} \ddot{q}_3$$

$$\left[a_{c_3} + (a_1 + a_2 + q_2) c_3 \right] + m_3 a_{c_3} \dot{q}_3 \Big[a_3 \dot{q}_2$$

$$+ (a_1 + a_2 + q_2) \dot{q}_3 (-s_3) \Big] + I_3 (\ddot{q}_1 + \ddot{q}_3)$$

$$= (I_1 + I_2 + I_3)\ddot{q}_1 + M_2(a_1 + a_{2c} + q_2^*)\ddot{q}_1$$

$$+ I_2 \ddot{q}_3 + M_2 \dot{q}_1 \dot{q}_2 + 2M_3(a_1 + a_2 + q_2 + a_{3c}C_3)$$

$$\dot{q}_1 \dot{q}_2 - M_3 a_{3c} C_3 \ddot{q}_2$$

$$+ M_3 a_{3c} C_3 \dot{q}_2 \dot{q}_2 + M_3 a_{3c}\{a_{3c} + (a_1 + a_2 + q_2) S_3\}\ddot{q}_3$$

$$+ M_3 a_{3c} S_3 \dot{q}_2 \dot{q}_3 - 2M_3 a_{3c}(a_1 + a_2 + q_2) S_3 \dot{q}_1 \dot{q}_3$$

$$- M_3 a_{3c}(a_1 + a_2 + q_2) S_3 \dot{q}_3$$

→ Calculating $\frac{\partial L}{\partial q_2}$

$$\frac{\partial L}{\partial q_2} = M_2(a_1 + a_2 + q_2^*)\dot{q}_1^2 + M_3(a_1 + a_2 + q_2$$

$$+ a_{3c}C_3)\dot{q}_1^2 + M_3 a_{3c} C_3 \dot{q}_1 \dot{q}_3$$

$$- (M_2 + M_3) g_s$$

→ Calculating $\frac{\partial L}{\partial \dot{q}_2}$

$$\frac{\partial L}{\partial \dot{q}_2} = M_2 \dot{q}_2 - M_3 a_{3c} C_3 (\dot{q}_1 + \dot{q}_3) + M_3 \dot{q}_2$$

→ Calculating $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \frac{d}{dt}\left(M_2 \dot{q}_2 - M_3 a_{3c} C_3 (\dot{q}_1 + \dot{q}_3) + M_3 \dot{q}_2\right)$$

$$= (M_2 + M_3)\ddot{q}_2 + M_3 a_{3c} S_3 (\dot{q}_1 + \dot{q}_3)\dot{q}_3 - M_3 a_{3c} C_3 (\ddot{q}_1 + \ddot{q}_3)$$

→ Calculating $\frac{\partial L}{\partial q_3} =$

$$m_3 a_{c_3}(a_1 + a_2 + q_2)(-s_3)\dot{q}_1^2 - m_3 a_{c_3} c_3 \dot{q}_1 \dot{q}_2 - m_3 a_{c_3} c_3 \dot{q}_2 \dot{q}_3 + m_3 a_{c_3}(-s_3)(a_1 + a_2 + q_2)\dot{q}_1 \dot{q}_2 - m_3 g a_{c_3} c_{13}$$

→ Calculating $\frac{\partial L}{\partial \dot{q}_3} = m_3 a_{c_3}^2 \dot{q}_3 - m_3 a_{c_3} s_3 \dot{q}_2 + m_3 a_{c_3}\left[a_{c_3} + (a_1 + a_2 + q_2) c_3\right]\dot{q}_1 + I_3(\dot{q}_1 + \dot{q}_3)$

$$= I_3(\dot{q}_1 + \dot{q}_3) - m_3 a_{3c} s_3 \dot{q}_2 + m_3 a_{3c} c_3 (a_1 + a_2 + q_2)\dot{q}_1 + m_3 a_{3c}^2 \dot{q}_3$$

→ Calculating $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_3}\right) = I_3(\ddot{q}_1 + \ddot{q}_3) + m_3 a_{3c} s_3 \dot{q}_2 \dot{q}_3 + m_3 a_{3c} s_3 (a_1 + a_2 + q_2)\dot{q}_1 \dot{q}_3 + m_3 a_{3c} s_3 \dot{q}_1 \dot{q}_2 + m_3 a_{3c} c_3 (a_1 + a_2 + q_2)\ddot{q}_1 + m_3 a_{3c}^2 \ddot{q}_3$$

But we already know that

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = \tau_1$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \tau_2$$

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_3}\right) - \frac{\partial L}{\partial q_3} = \tau_3$$

- Using all the pre-calculated values we get:

$\tau_1 =$

$$I_1 + m_2(a_1 + a_{c2} + q_2)^2 + I_2 + m_3[(a_1 + a_2 + q_2)^2 + I_3 + a_{c_3}^2 + 2(a_1 + a_2 + q_2)a_{c_3}c_3]\,\ddot{q}_1 +$$

$$m_2 g c_1(a_1 + a_{c2} + q_2) + m_3 g[(a_1 + a_2 + q_2)c_1 + a_{c_3}c_{13}] + 2m_2(a_1 + a_{c_2} + q_2)\dot{q}_1\dot{q}_2$$

$$+ (-m_3 a_{c_3} s_3 \dot{q}_2) + m_3[2(a_1 + a_2 + q_2)\dot{q}_2\dot{q}_1]$$

$$+ 2m_3 a_{c_3} c_3 \dot{q}_1\dot{q}_2 - 2m_3 a_{c_3}(a_1 + a_2 + q_2)$$

$$s_2\dot{q}_1\dot{q}_3 + \{m_3 a_{c_3}(a_{c_3} + (a_1 + a_2 + q_2)c_3]$$

$$+ I_2\}\ddot{q}_2 - m_3 a_{c_3}(a_1 + a_2 + a_2)s_3\dot{q}_3^2$$

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Now,

$$\tau_2 \Rightarrow -m_3 a_{c_3} S_3 \ddot{q}_1 + (m_3 + m_2)\ddot{q}_2 - m_3 a_{c_3} S_3 \ddot{q}_3$$

$$-2 m_3 a_{c_3} C_3 (\dot{q}_1)\dot{q}_3 - \dot{q}_1^2 [(a_1 + a_{c_2} + q_2) m_2$$

$$+ m_3 (a_1 + a_2 + q_2) + m_3 (a_{c_3} C_3)] - m_3 a_{c_3} C_3 \dot{q}_3^2$$

$$+ (m_2 + m_3) g S_1 = \tau_2$$

and lastly,

$$\tau_3 \Rightarrow m_3 a_{c_3} [a_{c_3} + (a_1 + a_2 + q_2) C_3] \ddot{q}_1$$

$$+ I_3 \ddot{q}_1 - m_3 a_{c_3} S_3 \ddot{q}_2 + (m_3 a_{c_3}^2 + I_3) \ddot{q}_3$$

$$+ 2 m_3 a_{c_3} C_3 \dot{q}_1 \dot{q}_2 + m_3 a_{c_3} (a_1 + a_2 + q_2) S_3 \dot{q}_1^2$$

$$+ m_3 g\, a_{c_3} C_{13} = \tau_3$$

Comparing with the general equation

$$\boxed{M(q)\ddot{q} + C(q, \dot{q})\dot{q} + g(q) = \tau}$$

Hence, we get the M, C and g as :—>

$$M = \begin{bmatrix} I_1+I_2+I_3+ m_2(a_1+a_{c_2}+q_2)^2 + m_3((a_1+a_2+q_2)^2 + a_{c_3}^2 + 2a_{c_3}(a_1+a_2+q_2)c_3 & -m_3 a_{c_3} s_3 & m_3 a_{c_3}[a_{c_3} + c_3(a_1+a_2+q_2) + I_3] \\ -m_3 a_{c_3} s_3 & m_2+m_3 & -m_3 a_{c_3} s_3 \\ -m_3 a_{c_3}[a_{c_3}+(a_1+a_2+q_2)c_3 + I_3] & -m_3 a_{c_3} s_3 & m_3 a_{c_3}^2 + I_3 \end{bmatrix}$$

and

$$C = \begin{bmatrix} 2m_2(a_1+a_{c_2}+q_2)\dot{q}_2 & 2m_3[(a_1+a_2+q_2)+a_3 c_3]\dot{q}_1 & (2\dot{q}_1+\dot{q}_3)m_3 a_{c_3}(a_1+a_2+q_2)s_3 \\ -(m_2(a_1+a_2+q_2)+m_3(a_1+a_2+q_2)+a_3 c_3)\dot{q}_1 & 0 & m_3 a_{3c} c_3 \dot{q}_1 + m_3 a_{3c} s_3(\dot{q}_1+\dot{q}_2) \\ -m_3 a_{3c} s_3(a_1+a_2+\dot{q}_2)\dot{q}_1 & m_3 a_{3c}(s_3+c_3)\dot{q}_1 & 0 \end{bmatrix}$$

and

$$g = \begin{bmatrix} -m_3 a_{3c} c_{13} g - (m_3 + m_2)(a_1 + a_2 + a/2) c_1 g \\ -(m_2 + m_3) s_1 g \\ -m_3 a_{3c} c_{13} g \end{bmatrix}$$

```
clc
clear all
%%code for
%%answer 3 HW 5
%%rewriting the equations derived in the last question
syms q1 q2 q3 a1 a2 a3

alpha1 = 90;
alpha2 = -90;
alpha3 = 0;

%%matrix 1
Rz_theta1=[cosd(q1+90) -sind(q1+90) 0 0;sind(q1+90) cosd(q1+90) 0 0;0
 0 1 0;0 0 0 1];
Rx_alpha1=[1 0 0 0;0 cosd(alpha1) -sind(alpha1) 0;0 sind(alpha1)
 cosd(alpha1) 0;0 0 0 1];
Tz_d1=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
Tx_a1=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%%matrix 2
Rz_theta2=[cosd(0) -sind(0) 0 0;sind(0) cosd(0) 0 0;0 0 1 0;0 0 0 1];
Rx_alpha2=[1 0 0 0;0 cosd(alpha2) -sind(alpha2) 0;0 sind(alpha2)
 cosd(alpha2) 0;0 0 0 1];
Tz_d2=[1 0 0 0;0 1 0 0;0 0 1 a1+a2+q2;0 0 0 1];
Tx_a2=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];

%matrix 3
Rz_theta3=[cosd(q3-90) -sind(q3-90) 0 0;sind(q3-90) cosd(q3-90) 0 0;0
 0 1 0;0 0 0 1];
Rx_alpha3=[1 0 0 0;0 cosd(alpha3) -sind(alpha3) 0;0 sind(alpha3)
 cosd(alpha3) 0;0 0 0 1];
Tz_d3=[1 0 0 0;0 1 0 0;0 0 1 0;0 0 0 1];
Tx_a3=[1 0 0 a3;0 1 0 0;0 0 1 0;0 0 0 1];


%%multiplying to get homogeneous matrices
A_1=Rz_theta1*Tz_d1*Tx_a1*Rx_alpha1
A_2=Rz_theta2*Tz_d2*Tx_a2*Rx_alpha2
A_3=Rz_theta3*Tz_d3*Tx_a3*Rx_alpha3

% T0_1 = A_1
% T0_2 = A_1*A_2
% T0_3 = A_1*A_2*A_3

% %%
% %%FINDING HOMOGENEOUS, ROTATION MATRICES AND ORIGINS
% %%FOR THE JACOBIAN CALCULATION
%
% %%JOINT 1 - REVOLUTE JOINT

%
H_1_0 = A_1;
```

```
R_1_0 = H_1_0(1:3,1:3)
O_1_0 = H_1_0(1:3,4);
%
% %%JOINT 2 - REVOLUTE JOINT
H_2_0 = A_1*A_2;
R_2_0 = H_2_0(1:3,1:3);
O_2_0 = H_2_0(1:3,4);
%
% %%JOINT 3 - REVOLUTE JOINT
H_3_0 = A_1*A_2*A_3;
R_3_0 = H_3_0(1:3,1:3);
O_3_0 = H_3_0(1:3,4);
%
fprintf("The Transformation matrices are given as follows : ")

H_1_0
H_2_0
H_3_0

fprintf("The O matrices are given as follows : ")

O_0_0=[0 0 0]'

O_1_0 = H_1_0(1:3,4)
O_2_0 = H_2_0(1:3,4)
O_3_0 = H_3_0(1:3,4)


fprintf("The Z matrices are given as follows : ")
z0 = eye(3)*[0 0 1]'
z1 = R_1_0*[0 0 1]'
z2 = R_2_0*[0 0 1]'


fprintf("The velocity jacobian matrices are given as follows : ")

J_v_1 = [cross(z0,(O_1_0-O_0_0)) [0 0 0]' [0 0 0]']
J_v_2 = [cross(z0,(O_2_0-O_0_0)) z1 [0 0 0]']
J_v_3 = [cross(z0,(O_3_0-O_0_0)) z1 cross(z2,(O_3_0-O_2_0))]


fprintf("The angular velocity (w) jacobian matrices are given as
 follows : ")

J_w_1 = [[0 0 1]' [0 0 0]' [0 0 0]']
J_w_2 = [[0 0 1]' [0 0 0]' [0 0 0]']
J_w_3 = [[0 0 1]' [0 0 0]' [0 0 1]']


A_1 =

[ cos((pi*(q1 + 90))/180), 0,  sin((pi*(q1 + 90))/180), 0]
[ sin((pi*(q1 + 90))/180), 0, -cos((pi*(q1 + 90))/180), 0]
[                       0, 1,                        0, 0]
```

```
[                       0, 0,                          0, 1]


A_2 =

[ 1,  0, 0,            0]
[ 0,  0, 1,            0]
[ 0, -1, 0, a1 + a2 + q2]
[ 0,  0, 0,            1]


A_3 =

[ cos((pi*(q3 - 90))/180), -sin((pi*(q3 - 90))/180), 0, a3*cos((pi*(q3
 - 90))/180)]
[ sin((pi*(q3 - 90))/180),  cos((pi*(q3 - 90))/180), 0, a3*sin((pi*(q3
 - 90))/180)]
[                       0,                        0, 1,
           0]
[                       0,                        0, 0,
           1]


R_1_0 =

[ cos((pi*(q1 + 90))/180), 0,  sin((pi*(q1 + 90))/180)]
[ sin((pi*(q1 + 90))/180), 0, -cos((pi*(q1 + 90))/180)]
[                       0, 1,                        0]

The Transformation matrices are given as follows :
H_1_0 =

[ cos((pi*(q1 + 90))/180), 0,  sin((pi*(q1 + 90))/180), 0]
[ sin((pi*(q1 + 90))/180), 0, -cos((pi*(q1 + 90))/180), 0]
[                       0, 1,                        0, 0]
[                       0, 0,                        0, 1]


H_2_0 =

[ cos((pi*(q1 + 90))/180), -sin((pi*(q1 + 90))/180), 0,  sin((pi*(q1 +
 90))/180)*(a1 + a2 + q2)]
[ sin((pi*(q1 + 90))/180),  cos((pi*(q1 + 90))/180), 0, -cos((pi*(q1 +
 90))/180)*(a1 + a2 + q2)]
[                       0,                        0, 1,
                        0]
[                       0,                        0, 0,
                        1]


H_3_0 =

[ cos((pi*(q1 + 90))/180)*cos((pi*(q3 - 90))/180) -
 sin((pi*(q1 + 90))/180)*sin((pi*(q3 - 90))/180), - cos((pi*(q1
```

```
 + 90))/180)*sin((pi*(q3 - 90))/180) - cos((pi*(q3 -
 90))/180)*sin((pi*(q1 + 90))/180), 0, sin((pi*(q1 + 90))/180)*(a1
 + a2 + q2) + a3*cos((pi*(q1 + 90))/180)*cos((pi*(q3 - 90))/180) -
 a3*sin((pi*(q1 + 90))/180)*sin((pi*(q3 - 90))/180)]
[ cos((pi*(q1 + 90))/180)*sin((pi*(q3 - 90))/180) +
 cos((pi*(q3 - 90))/180)*sin((pi*(q1 + 90))/180),   cos((pi*(q1
 + 90))/180)*cos((pi*(q3 - 90))/180) - sin((pi*(q1 +
 90))/180)*sin((pi*(q3 - 90))/180), 0, a3*cos((pi*(q1 +
 90))/180)*sin((pi*(q3 - 90))/180) - cos((pi*(q1 + 90))/180)*(a1 + a2
 + q2) + a3*cos((pi*(q3 - 90))/180)*sin((pi*(q1 + 90))/180)]
[
                            0,
                                                               0, 1,

                                                                        0]
[
                            0,
                                                               0, 0,

                                                                        1]

The O matrices are given as follows :
O_0_0 =

     0
     0
     0


O_1_0 =

 0
 0
 0


O_2_0 =

  sin((pi*(q1 + 90))/180)*(a1 + a2 + q2)
 -cos((pi*(q1 + 90))/180)*(a1 + a2 + q2)
                                       0


O_3_0 =

 sin((pi*(q1 + 90))/180)*(a1 + a2 + q2) + a3*cos((pi*(q1
 + 90))/180)*cos((pi*(q3 - 90))/180) - a3*sin((pi*(q1 +
 90))/180)*sin((pi*(q3 - 90))/180)
 a3*cos((pi*(q1 + 90))/180)*sin((pi*(q3 - 90))/180) - cos((pi*(q1 +
 90))/180)*(a1 + a2 + q2) + a3*cos((pi*(q3 - 90))/180)*sin((pi*(q1 +
 90))/180)


      0
```

```
The Z matrices are given as follows :
z0 =

     0
     0
     1


z1 =

  sin((pi*(q1 + 90))/180)
 -cos((pi*(q1 + 90))/180)
                        0


z2 =

 0
 0
 1

The velocity jacobian matrices are given as follows :
J_v_1 =

[ 0, 0, 0]
[ 0, 0, 0]
[ 0, 0, 0]


J_v_2 =

[ cos((pi*(conj(q1) + 90))/180)*(conj(a1) + conj(a2) + conj(q2)),
 sin((pi*(q1 + 90))/180), 0]
[ sin((pi*(conj(q1) + 90))/180)*(conj(a1) + conj(a2) + conj(q2)), -
cos((pi*(q1 + 90))/180), 0]
[                                                               0,
                   0, 0]


J_v_3 =

[ cos((pi*(conj(q1) + 90))/180)*(conj(a1) + conj(a2) +
 conj(q2)) - cos((pi*(conj(q1) + 90))/180)*sin((pi*(conj(q3) -
 90))/180)*conj(a3) - cos((pi*(conj(q3) - 90))/180)*sin((pi*(conj(q1)
 + 90))/180)*conj(a3),  sin((pi*(q1 + 90))/180), - cos((pi*(conj(q1) +
 90))/180)*sin((pi*(conj(q3) - 90))/180)*conj(a3) - cos((pi*(conj(q3)
 - 90))/180)*sin((pi*(conj(q1) + 90))/180)*conj(a3)]
[ sin((pi*(conj(q1) + 90))/180)*(conj(a1) + conj(a2) +
 conj(q2)) + cos((pi*(conj(q1) + 90))/180)*cos((pi*(conj(q3) -
 90))/180)*conj(a3) - sin((pi*(conj(q1) + 90))/180)*sin((pi*(conj(q3)
 - 90))/180)*conj(a3), -cos((pi*(q1 + 90))/180),   cos((pi*(conj(q1) +
 90))/180)*cos((pi*(conj(q3) - 90))/180)*conj(a3) - sin((pi*(conj(q1)
 + 90))/180)*sin((pi*(conj(q3) - 90))/180)*conj(a3)]
```

```
[
                                                                                    0,
                     0,

                        0]

The angular velocity (w) jacobian matrices are given as follows :
J_w_1 =

     0     0     0
     0     0     0
     1     0     0


J_w_2 =

     0     0     0
     0     0     0
     1     0     0


J_w_3 =

     0     0     0
     0     0     0
     1     0     1
```

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