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ENPM 662 - 2019
Homework - 1

(1)

(a) To prove $||Rq - Rp|| = ||q - p|| \quad \forall q, p \in R^3$

Since R is orthogonal, multiplying RT on both sides

$$R^T ||Rq - Rp|| = R^T ||q - p||$$

$$||(R^T R)q - R^T(R)p|| = ||R^T(q - p)||$$

Now: $R^T R = 1$ (property of orthogonal matrix)

and $||R^T|| = ||R||$

$$\therefore ||q - p|| = ||R(q - p)||$$
$$= ||Rq - Rp||$$

Hence proved

(2) The Rotation matrices can be expressed in terms of quaternions.

$R_{x,\phi} = \cos\frac{\phi}{2} + \sin\frac{\phi}{2}\,\hat{i}$

$R_{z,\theta} = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}\,\hat{k}$

$R_{y,\psi} = \cos\frac{\psi}{2} + \sin\left(\frac{\psi}{2}\right)\hat{j}$

Now, $R_{x,\phi} * R_{z,\theta} * R_{y,\psi}$

$= \left(\cos\frac{\phi}{2} + \sin\frac{\phi}{2}\hat{i}\right)\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{k}\right)\left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\hat{j}\right)$

$= \left(\cos\frac{\theta}{2}\cos\frac{\phi}{2} + \cos\frac{\phi}{2}\sin\frac{\theta}{2}\hat{k} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\hat{i}\right.$

$\left. - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\hat{j}\right)\left(\cos\frac{\psi}{2} + \sin\frac{\psi}{2}\hat{j}\right)$

$= \cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \sin\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2}\hat{k} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2}\hat{i}$

$- \sin\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2}\hat{j} + \cos\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2}\hat{j} - \sin\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2}\hat{i}$

$+ \cos\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}\hat{k} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}$

$= \cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2} + \left[\cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2}\right.$

$\left.- \sin\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2}\right)\hat{i} + \left(\cos\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2}\right)\hat{j}$

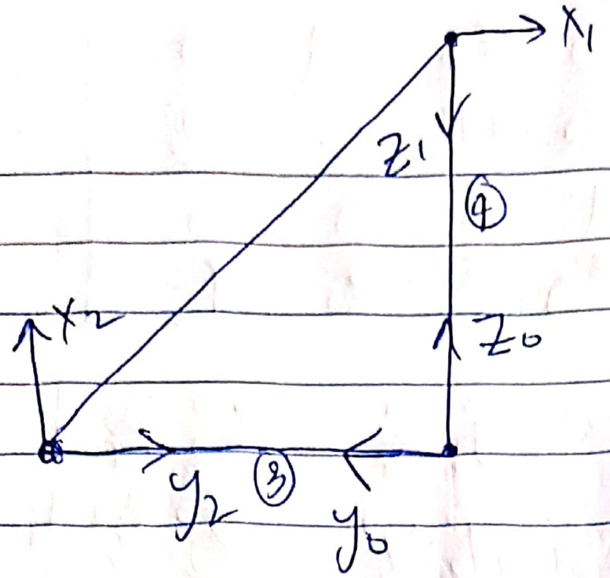
$+ \left(\sin\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}\right)\hat{k}$

Hence, of the format is $A + u_x \hat{i} + u_y \hat{j} + u_z \hat{k}$

- $A = \cos\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \sin\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}$
- $u_x = \cos\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2} - \sin\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2}$
- $u_y = \cos\frac{\theta}{2}\cos\frac{\phi}{2}\sin\frac{\psi}{2} - \sin\frac{\theta}{2}\sin\frac{\phi}{2}\cos\frac{\psi}{2}$
- $u_z = \sin\frac{\theta}{2}\cos\frac{\phi}{2}\cos\frac{\psi}{2} + \cos\frac{\theta}{2}\sin\frac{\phi}{2}\sin\frac{\psi}{2}$

When calculated $\sqrt{A^2 + u_x^2 + u_y^2 + u_z^2}$ should give a value of 1 inorder to signify unit quaternion.

(3)



The rotation matrix between Frame '0' and Frame '1' will be of the form:

$$R_1^0 = \begin{bmatrix} \vec{X_1}\cdot\vec{X_0} & \vec{Y_1}\cdot\vec{X_0} & \vec{Z_1}\cdot\vec{X_0} \\ \vec{X_1}\cdot\vec{Y_0} & \vec{Y_1}\cdot\vec{Y_0} & \vec{Z_1}\cdot\vec{Y_0} \\ \vec{X_1}\cdot\vec{Z_0} & \vec{Y_1}\cdot\vec{Z_0} & \vec{Z_1}\cdot\vec{Z_0} \end{bmatrix}$$

Hence this along with a translation of 4 along the 'Z' axis will be represented as:

$$H_1^0 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Similarly $R_2^0 = \begin{bmatrix} \vec{X_2}\cdot\vec{X_0} & \vec{Y_2}\cdot\vec{X_0} & \vec{Z_2}\cdot\vec{X_0} \\ \vec{X_2}\cdot\vec{Y_0} & \vec{Y_2}\cdot\vec{Y_0} & \vec{Z_2}\cdot\vec{Y_0} \\ \vec{X_2}\cdot\vec{Z_0} & \vec{Y_2}\cdot\vec{Z_0} & \vec{Z_2}\cdot\vec{Z_0} \end{bmatrix}$

Hence this along with a translation of 3 along the Y axis can be represented as

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$$H_2^0 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now,

$$R_2^1 = \begin{bmatrix} \vec{x_2}\cdot\vec{x_1} & \vec{y_2}\cdot\vec{x_1} & \vec{z_2}\cdot\vec{x_1} \\ \vec{x_2}\cdot\vec{y_1} & \vec{y_2}\cdot\vec{y_1} & \vec{z_2}\cdot\vec{y_1} \\ \vec{x_2}\cdot\vec{z_1} & \vec{y_2}\cdot\vec{z_1} & \vec{z_2}\cdot\vec{z_1} \end{bmatrix}$$

This alongwith an translation of -3 along X and 4 units along Z gives us a homogeneous transformation represented as H_2^1

$$H_2^1 = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now: $H_1^0 \times H_2^1$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= H_2^0$. Hence proved. [Solution provided in MATLAB as well]