October 1 - 2019

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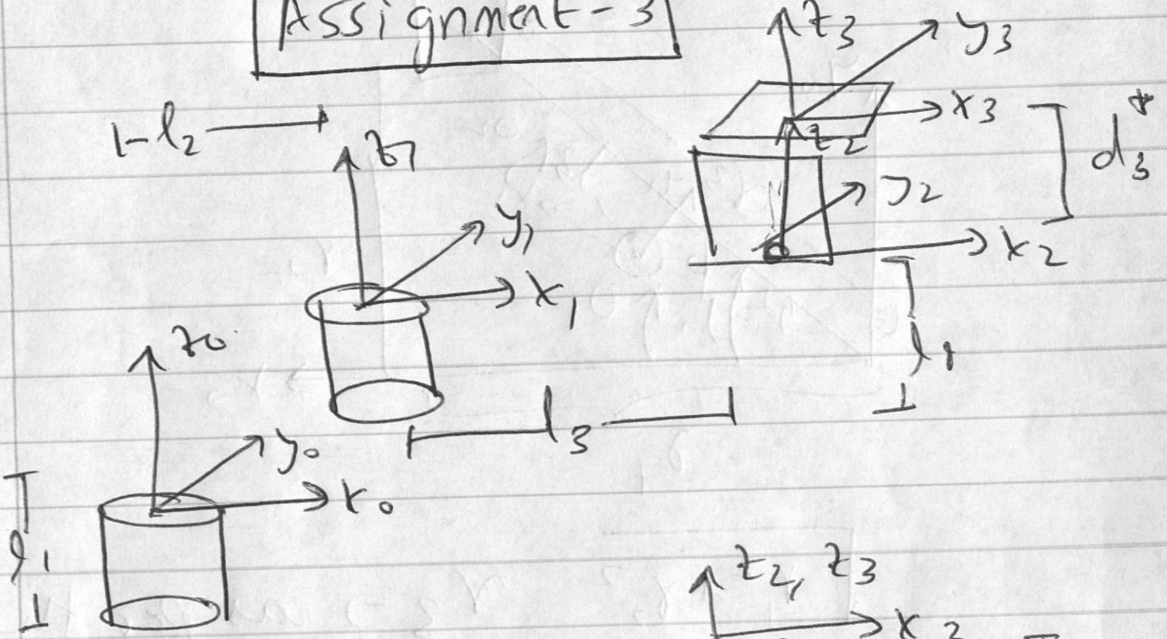
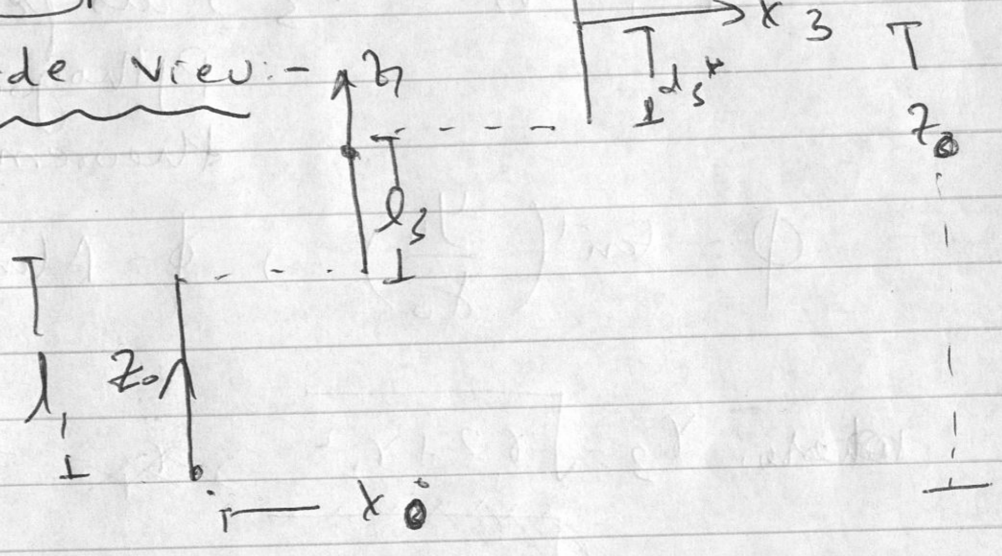
ENPM662

Assignment-3

(1)



Side view:-



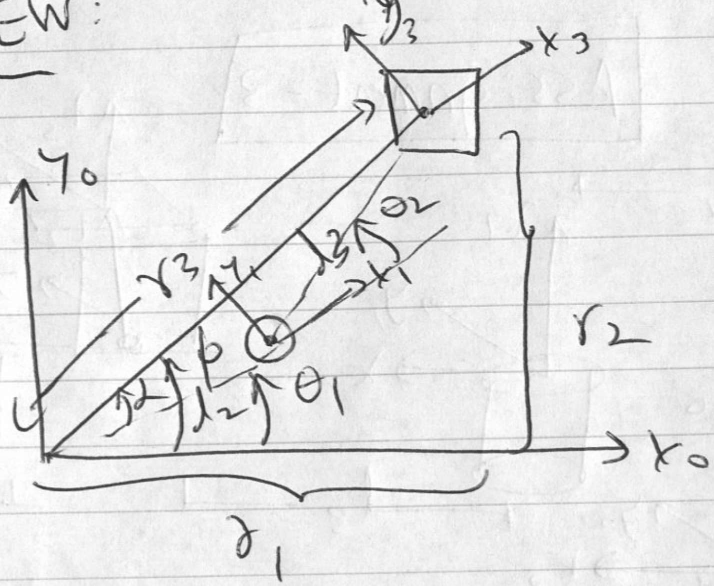
Here:

$$l_1 + l_4 + d_3^* = Z_0$$

$$\Rightarrow \boxed{d_3^* = Z_0 - l_1 - l_4}$$

Z_0 here is the height along the Z-axis

TOP VIEW:



$\sqrt{r_1^2 + r_2^2} = r_3 \Rightarrow$ as per the Pythogorean theorem.

$\phi = \tan^{-1}\left(\frac{y_0}{x_0}\right) \Rightarrow \phi = \mathrm{Atan2}(x_0, y_0)$

~~where~~ $r_3 = \sqrt{r_1^2 + r_2^2}$ $= \sqrt{x_0^2 + y_0^2}$

$\theta_1 = \phi - \alpha$

Here $\alpha = \cos^{-1}\left(\frac{l_2^2 + r_3^2 - l_3^2}{2\, l_2\, r_3}\right)$

$$r_3 = \sqrt{x_0^2 + y_0^2}$$

$$\theta_1 = \phi - \alpha$$

$$\theta_1 = \phi - \cos^{-1}\left(\frac{l_2^2 + r_3^2 - l_3^2}{2 l_2 r_3}\right)$$

$$\Rightarrow \boxed{\theta_1 = \tan^{-1}\left(\frac{y_0}{x_0}\right) - \cos^{-1}\left(\frac{l_2^2 + r_3^2 - l_3^2}{2 l_2 r_3}\right)}$$

where $r_3 = \sqrt{x_0^2 + y_0^2}$

Now, $180 - \theta_2 = \cos^{-1}\left(\frac{l_2^2 + l_3^2 - r_3^2}{2 l_2 l_3}\right)$

$$\Rightarrow \boxed{\theta_2 = 180 - \cos^{-1}\left(\frac{l_2^2 + l_3^2 - r_3^2}{2 l_2 l_3}\right)}$$

(2) ⓐ To compute the desired coordinates, O_c^0:

$$\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix} = \begin{bmatrix} O_x - d_6 r_{13} \\ O_y - d_6 r_{23} \\ O_z - d_6 r_{33} \end{bmatrix}$$

because: $O_c^0 = O - d_6 R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Now from the forward kinematic calculation we know that:-

$r_{13} = C_1(C_2C_4S_5 + S_2C_5) - S_1S_4S_5$

$r_{23} = S_1(C_2C_4C_5 + S_2C_5) + C_1S_4S_5$

$r_{33} = -S_2C_4S_5 + C_2C_5$

Now:-

$x_c = d_x - d_6 r_{13} = C_1S_2d_3 - S_1d_2 +$

$d_6(C_1C_2C_4S_5 + S_2C_1C_5 - S_1S_4S_5)$

$- d_6(C_1((C_4C_2S_5) + (S_2C_5)) - S_1S_4S_5)$

$= \underline{\underline{C_1S_2d_3 - S_1d_2 = x_c}}$

Because:-

$$d_x = C_1 S_2 d_3 - S_1 d_2 + d_6 (C_1 C_2 C_4 S_5$$
$$+ C_1 C_5 S_2 - S_1 S_4 S_5)$$

Similarly,

$$d_y = S_1 S_2 d_3 + C_1 d_2 + d_6 (C_1 S_4 S_5 +$$
$$C_2 C_4 S_1 S_5 + C_5 S_1 S_2)$$

and

$$d_z = C_2 d_3 + d_6 (C_2 C_5 - C_4 S_2 S_5)$$

Hence:

$$y_c = d_y - d_6 r_{23}$$
$$= S_1 S_2 d_3 + C_1 d_2 + d_6 C_1 S_4 S_5 + d_6 C_2 C_4 S_1 S_5$$
$$+ d_6 C_5 S_1 S_2 - d_6 S_1 C_2 C_4 S_5 - d_6 S_1 S_2 C_5$$
$$- d_6 C_1 S_4 S_5$$

$$\Rightarrow \underline{\underline{y_c = S_1 S_2 d_3 + C_1 d_2}}$$

$z_c = d_z - d_6 r_{33}$

$= C_2 d_3 + d_6 C_2 C_5 - d_6 C_4 S_2 S_5$

$+ d_6 S_2 C_4 S_5 - d_6 C_5 C_2$

$\Rightarrow \underline{\underline{z_c = C_2 d_3}}$

(b) $A^0_3 = A^0_1 \times A_2 \times A_3$ $\boxed{\text{frames taken as per Spong Example 3.5}}$

Taking inverse on both the sides:

$$\boxed{(A^0_1)^{-1} A^0_3 = A_2 \times A_3} \quad \text{———} \; (1)$$

Now, from the most general form of the homogeneous transformation matrix, we know that

A^0_3 can be written as:

$$A^0_3 = \begin{bmatrix} n_x & s_x & a_x & d_x \\ n_y & s_y & a_y & d_y \\ n_z & s_z & a_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} n & s & a & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

also, from the properties of rotation matrices, it is clear that $(A^0_i)^{-1} = (A^0_i)^T$

$$(A_1^0) = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (A_1^0)^{-1} = \begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

plugging these values into equation (1) we get.

$$\begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_3 & s_3 & a_3 & p_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C_2 & 0 & S_2 & 0 \\ S_2 & 0 & -C_2 & 0 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Here $d_2 = l_2$, $d_1 = 0$ [as per the frames]

Since we just want the distances,

We can ignore the n, s and a columns

in the A_3^0 matrix.

Hence:

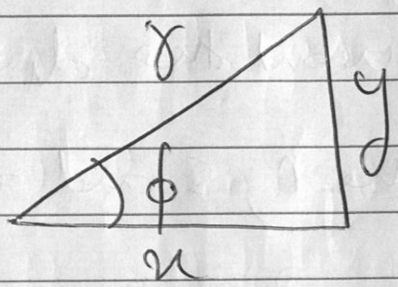
$$\begin{bmatrix} C_1 & S_1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -S_1 & C_1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cdots & p_x \\ \cdots & p_y \\ \cdots & p_z \\ \cdots & 1 \end{bmatrix} = \begin{bmatrix} C_2 & 0 & S_2 & d_3 S_2 \\ S_2 & 0 & -C_2 & -C_2 d_3 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \underbrace{\begin{bmatrix} C_1 P_x + S_1 P_y \\ -P_z \\ -S_1 P_x + C_1 P_y \end{bmatrix} = \begin{bmatrix} d_3 S_2 \\ -C_2 d_3 \\ d_2 \end{bmatrix}}_{\text{Comparing L.H.S \& R.H.S}}$$

Now, $C_1 P_x + S_1 P_y = d_3 S_2$ —— (1)

$-P_z = -C_2 d_3$ —— (2)

$-S_1 P_x + C_1 P_y = d_2$ —— (3)



From the figure:

$$\tan\phi = \left(\frac{y}{x}\right)$$

$$\Rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$r = \sqrt{x^2+y^2}$$

from equation 3

$$d_2 = -\sin\theta_1 P_x + \cos\theta_1 P_y \quad (4)$$

from the figure $\rightarrow$ $P_x = r\cos\phi$
$P_y = r\sin\phi$

Substituting that in equation 4

$$d_2 = r(-\cos\phi\sin\theta_1 + \cos\theta_1\sin\phi)$$

$$\Rightarrow d_2 = r\sin(\theta_1 - \phi)$$ {from the basic trignometric properties}

$$\Rightarrow \sin(\theta_1 - \phi) = \frac{d_2}{r}$$

$$\Rightarrow \cos(\theta_1 - \phi) = \sqrt{1 - \sin^2(\theta - \phi)}$$

$$= \sqrt{1 - \frac{d_2^2}{r^2}}$$

$$= \frac{\sqrt{r^2 - d_2^2}}{r}$$

Now, $\tan(\theta_1 - \phi) = \dfrac{\sin(\theta - \phi)}{\cos(\theta - \phi)} = \dfrac{d_2}{\sqrt{r^2 - d_2^2}}$

$$\Rightarrow \theta_1 - \phi = \pm \tan^{-1}\left(\frac{d_2}{\sqrt{r^2 - d_2^2}}\right)$$

$$\Rightarrow \theta_1 = \phi \pm \tan^{-1}\left(\frac{d_2}{\sqrt{r^2 - d_2^2}}\right)$$

But $\phi = \tan^{-1}\left(\dfrac{P_y}{P_x}\right)$

$$\Rightarrow \boxed{\theta_1 = \tan^{-1}\left(\frac{P_y}{P_x}\right) \pm \tan^{-1}\left(\frac{d_2}{\sqrt{r^2 - d_2^2}}\right)}$$

where $r = \sqrt{P_x^2 + P_y^2}$, $d_2 = l_2$

Hence θ_1 has 2 solutions

Now, from eq ①

$$\cos\theta_1 P_x + \sin\theta_1 P_y = d_3 \sin\theta_2$$

$$\Rightarrow \cos\theta_1 r\cos\phi + \sin\theta_1 r\sin\phi = d_3 \sin\theta_2$$

$$\Rightarrow r(\cos(\theta_1 - \phi)) = d_3 \sin\theta_2 \quad — ⑤$$

from equation ②

$$+P_z = +\cos\theta_2 d_3$$

$$\Rightarrow \theta_2 = \cos^{-1}\left(\frac{P_z}{d_3}\right)$$

$$\Rightarrow \sin\theta_2 = \sqrt{1 - \left(\frac{P_z}{d_3}\right)^2}$$

$$\boxed{\theta_2 = \sin^{-1}\left(\frac{\sqrt{d_3^2 - P_z^2}}{d_3}\right)}$$

from ⑤

$$r\cos(\theta_1 - \phi) = \cancel{d_3}\frac{\sqrt{d_3^2 - P_z^2}}{\cancel{d_3}}$$

$$\boxed{d_3 = \pm\sqrt{P_z^2 - r^2\cos^2(\theta_1 - \phi)}}$$

(c)

$$R_1^0 = \begin{bmatrix} C_1 & 0 & -S_1 \\ S_1 & 0 & C_1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$R_2^1 = \begin{bmatrix} C_2 & 0 & S_2 \\ S_2 & 0 & -C_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$R_3^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{R_3^0 = R_1^0 R_2^1 R_3^2}$$

$$\Rightarrow R_3^0 = \begin{bmatrix} C_1 & 0 & -S_1 \\ S_1 & 0 & C_1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} C_2 & 0 & S_2 \\ S_2 & 0 & -C_2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow R_3^0 = \begin{bmatrix} C_1 C_2 & -S_1 & C_1 S_2 \\ C_2 S_1 & C_1 & S_1 S_2 \\ -S_2 & 0 & C_2 \end{bmatrix}$$

Now, we know that R^3_6 is given as:

$$R^3_6 = \begin{bmatrix} C_4 C_5 S_6 - S_4 S_6 & -C_4 C_5 S_6 - S_4 C_6 & C_4 S_5 \\ S_4 C_5 C_6 + C_4 S_6 & -S_4 C_5 S_6 + C_4 S_6 & S_4 S_5 \\ -S_5 C_6 & S_5 S_6 & C_5 \end{bmatrix}$$

$$\boxed{R^0_6 = R^0_3 R^3_6}$$

Taking the general format of $\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$

for R^0_6 and taking $(R^0_3)^{-1}$ on both the sides.

$$R^3_6 = (R^0_3)^{-1} R$$

$$\Rightarrow (R^3_6)^{-1} = \begin{bmatrix} C_4 C_5 C_6 - S_4 S_6 & S_4 C_5 C_6 + C_4 S_6 & -S_5 C_6 \\ -C_4 S_6 C_5 - S_4 S_6 & -S_4 C_5 S_6 + C_4 C_6 & S_5 S_6 \\ C_4 S_5 & S_4 S_5 & C_5 \end{bmatrix}$$

and $R^{0^{-1}}_3 = \begin{bmatrix} C_1 C_2 & -S_1 & C_1 S_2 \\ C_2 S_1 & C_1 & S_1 S_2 \\ -S_2 & 0 & C_2 \end{bmatrix}^T$