EE1390

Matrix Analysis Project

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Outline

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Geometry Problem

Q.No 55 from JEE Advanced 2013 Paper 2

Problem Statement

- Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a, a > 0$.
- Length of chord PQ is :

Options

- 7a
- 5a
- 2a
- 3a

Outline

Matrix Transformation

Equation of a general conic

$$X^T VX + 2u^T X + F = 0$$

Observe that the given parabola can be represented as:

$$x^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} x + 2 \begin{pmatrix} -2a & 0 \end{pmatrix} x = 0$$

and the given line can be represented as

$$\begin{pmatrix} 2 & -1 \end{pmatrix} x + a = 0$$

Matrix Transformation

Thus the question can be re formulated as

Question

Let PQ be a focal chord of the parabola $x^T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}x + 2\begin{pmatrix} -2a & 0 \end{pmatrix}x = 0$. The tangents at the parabola at P and Q meet at point lying on the line $\begin{pmatrix} 2 & -1 \end{pmatrix}x + a = 0, a > 0$. Length of the chord PQ is :

Outline

Finding the intersection point of tangents

Since points P and Q lie on the parabola,

$$P^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} P + 2 \begin{pmatrix} -2a & 0 \end{pmatrix} P = 0 \qquad (1)$$

$$Q^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Q + 2 \begin{pmatrix} -2a & 0 \end{pmatrix} Q = 0 \qquad (2)$$

Finding the intersection point of tangents

General tangent equation at p on a conic

$$(p^T V + u^T)x + p^T u + F = 0$$

Here the equations of tangents at P and Q respectively are:

$$\left[P^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2a & 0 \end{pmatrix}\right] x + P^T \begin{pmatrix} -2a \\ 0 \end{pmatrix} = 0 \quad (3)$$

$$\left[Q^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2a & 0 \end{pmatrix}\right] x + Q^T \begin{pmatrix} -2a \\ 0 \end{pmatrix} = 0 \quad (4)$$

Finding the intersection point of tangents

Since PQ is a focal chord, the line joining them passes through the focus.
ie.

$$\begin{pmatrix} a \\ 0 \end{pmatrix} = P + \lambda(Q - P)$$
$$\begin{pmatrix} a \\ 0 \end{pmatrix} = P(1 - \lambda) + \lambda Q \tag{5}$$

Finding the intersection point of tangents

Multiply (3) with $1-\lambda$ and (4) with λ and add,

$$\begin{aligned}((1-\lambda)P^T+\lambda Q^T)\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} &+ (1-\lambda+\lambda)\begin{pmatrix}-2a & 0\end{pmatrix}x \\ &+ ((1-\lambda)P^T+\lambda Q^T))\begin{pmatrix}-2a\\ 0\end{pmatrix} \\ &= 0\end{aligned}$$

Finding the intersection point of tangents

Using equation (5),

$$\Rightarrow \begin{pmatrix} a & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2a & 0 \end{pmatrix} x + \begin{pmatrix} a & 0 \end{pmatrix} \begin{pmatrix} -2a \\ 0 \end{pmatrix} = 0$$

On simplification this gives,

$$\begin{pmatrix} -2a & 0 \end{pmatrix} x = 2a^2$$
$$\begin{pmatrix} 1 & 0 \end{pmatrix} x = -a \tag{6}$$

Finding the intersection point of tangents

Also, given that this point lies on

$$\begin{aligned} \begin{pmatrix} 2 & -1 \end{pmatrix} x &= -a \\ \Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} x &= \begin{pmatrix} -a \\ -a \end{pmatrix} \\ \Rightarrow x &= \begin{pmatrix} -a \\ -a \end{pmatrix} \end{aligned} \tag{7}$$

Finding $P - Q$

Substitute (7) in (3),

$$\left[P^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2a & 0 \end{pmatrix}\right] \begin{pmatrix} -a \\ -a \end{pmatrix} + P^T \begin{pmatrix} -2a \\ 0 \end{pmatrix} = 0$$

$$P^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -a \\ -a \end{pmatrix} + \begin{pmatrix} -2a & 0 \end{pmatrix} \begin{pmatrix} -a \\ -a \end{pmatrix} + P^T \begin{pmatrix} -2a \\ 0 \end{pmatrix} = 0$$

On Simplifying and taking transpose on both sides,

$$\begin{pmatrix} 2 & 1 \end{pmatrix} P = 2a \tag{8}$$

Similarly on substituting (7) in (4),

$$\begin{pmatrix} 2 & 1 \end{pmatrix} Q = 2a \tag{9}$$

Finding $P - Q$

Subtract (9) from (8),

$$\begin{pmatrix} 2 & 1 \end{pmatrix}(P - Q) = 0 \tag{10}$$

Observe that $\begin{pmatrix} 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = 0$

Note

$$n^T a = n^T b = 0 \Rightarrow a = kb$$

$$\Rightarrow P - Q = k\begin{pmatrix} 1 \\ -2 \end{pmatrix} \tag{11}$$

Finding $P - Q$

Thus put $P = Q + \begin{pmatrix} k \\ -2k \end{pmatrix}$ in (1)

$$\left(Q + \begin{pmatrix} k \\ -2k \end{pmatrix}\right)^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \left(Q + \begin{pmatrix} k \\ -2k \end{pmatrix}\right) + 2\begin{pmatrix} -2a & 0 \end{pmatrix}\left(Q + \begin{pmatrix} k \\ -2k \end{pmatrix}\right) = 0$$

Finding $P - Q$

On simplifying,

$$Q^T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} Q + Q^T \begin{pmatrix} 0 \\ -2k \end{pmatrix} + \begin{pmatrix} 0 & -2k \end{pmatrix} Q + 4k^2 + 2 \begin{pmatrix} -2a & 0 \end{pmatrix} Q - 4ak = 0$$

Further simplification using (2) gives,

$$\begin{pmatrix} 0 & 1 \end{pmatrix} Q = k - a \tag{12}$$

Finding $P - Q$

From (9) and (12),

$$\begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} Q = \begin{pmatrix} k - a \\ 2a \end{pmatrix}$$
$$\implies Q = \begin{pmatrix} \frac{3a-k}{2} \\ k - a \end{pmatrix} \tag{13}$$

Finding $P - Q$

Substitute (13) in (2) with x = Q,

$$\left[\begin{pmatrix} \frac{3a-k}{2} & k-a \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -2a & 0 \end{pmatrix}\right]\begin{pmatrix} \frac{3a-k}{2} \\ k-a \end{pmatrix} + \begin{pmatrix} \frac{3a-k}{2} & k-a \end{pmatrix}\begin{pmatrix} -2a \\ 0 \end{pmatrix} = 0$$

Finding $P - Q$

On simplification finally yields,

$$k^2 = 5a^2$$
$$\Rightarrow k = \pm\sqrt{5}a$$
$$P - Q = \pm\sqrt{5}a\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$
$$\Rightarrow ||P - Q|| = 5a$$

Answer

Thus the length $PQ = 5a$

Outline

Python verification

The points and lines given in the question and those obtained in solution are plotted using pyplot library. The distance PQ is verified as $5a$

Python verification

Plot