보충 Approximate Integration

Approximate Integration

There are two situations in which it is impossible to find the exact value of a definite integral.

The first situation arises from the fact that in order to evaluate $\int_a^b f(x)\,dx$ using the Fundamental Theorem of Calculus we need to know an antiderivative of *f*.

Sometimes, however, it is difficult, or even impossible, to find an antiderivative. For example, it is impossible to evaluate the following integrals exactly:

$$\int_0^1 e^{x^2}\,dx \quad \int_{-1}^1 \sqrt{1+x^3}\,dx$$

Approximate Integration

The second situation arises when the function is determined from a scientific experiment through instrument readings or collected data. There may be no formula for the function.

In both cases we need to find approximate values of definite integrals. We already know one such method.

Approximate Integration

We know that the definite integral is defined as a limit of Riemann sums, so any Riemann sum could be used as an approximation to the integral: If we divide $[a, b]$ into n subintervals of equal length $\Delta x = \dfrac{(b-a)}{n}$, then we have

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where x_i^* is any point in the ith subinterval $[x_{i-1}, x_i]$. If x_i^* is chosen to be the left endpoint of the interval, then $x_i^* = x_{i-1}$ and we have

$$\int_a^b f(x)\,dx \approx L_n = \sum_{i=1}^{n} f(x_{i-1})\,\Delta x \tag{1}$$

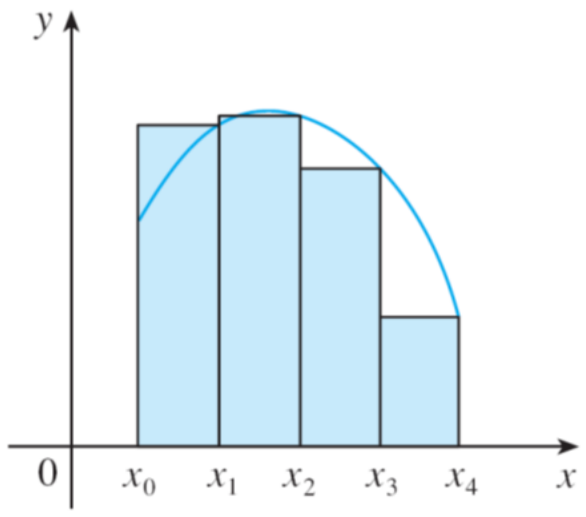
Approximate Integration

If we choose x_i^* to be the right endpoint, then $x_i^* = x_i$ and we have

2 $$\int_a^b f(x)\,dx \approx R_n = \sum_{i=1}^{n} f(x_i)\,\Delta x$$

[See Figure 1(b).]

The approximations L_n and R_n defined by Equations 1 and 2 are called the **left endpoint approximation** and **right endpoint approximation**, respectively.



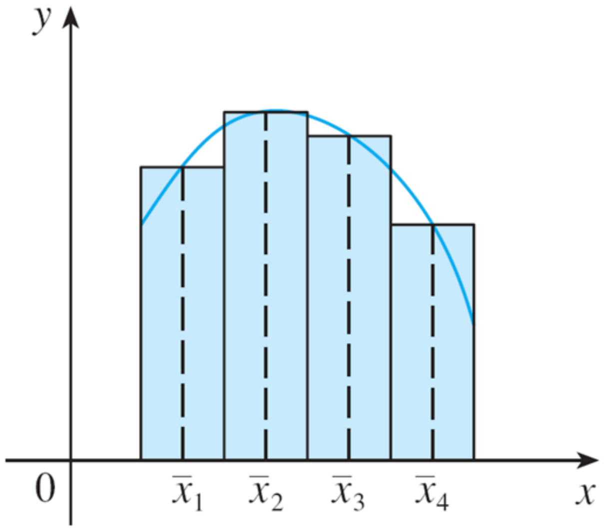
Right endpoint approximation

Figure 1(b)

The Midpoint and Trapezoidal Rules

The Midpoint and Trapezoidal Rules

We also considered the case where x_i^* is chosen to be the midpoint $\bar{x}_i$ of the subinterval $[x_{i-1}, x_i]$. Figure 2 shows the midpoint approximation M_n, which appears to be better than either L_n or R_n.



Midpoint approximation

Figure 2

The Midpoint and Trapezoidal Rules

Midpoint Rule

$$\int_a^b f(x)\,dx \approx M_n = \Delta x\left[f(\bar{x}_1) + f(\bar{x}_2) + \cdots + f(\bar{x}_n)\right]$$

where

$$\Delta x = \frac{b-a}{n}$$

and

$$\bar{x}_i = \frac{1}{2}(x_{i-1} + x_i) = \text{midpoint of } [x_{i-1}, x_i]$$

The Midpoint and Trapezoidal Rules

Another approximation, called the Trapezoidal Rule, results from averaging the approximations in Equations 1 and 2:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{1}{2}\left[\sum_{i=1}^{n} f(x_{i-1})\Delta x + \sum_{i=1}^{n} f(x_i)\Delta x\right] = \frac{\Delta x}{2}\left[\sum_{i=1}^{n}\left(f(x_{i-1}) + f(x_i)\right)\right] \\ &= \frac{\Delta x}{2}\left[\left(f(x_0)+f(x_1)\right)+\left(f(x_1)+f(x_2)\right)+\cdots+\left(f(x_{n-1})+f(x_n)\right)\right] \\ &= \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]\end{aligned}$$

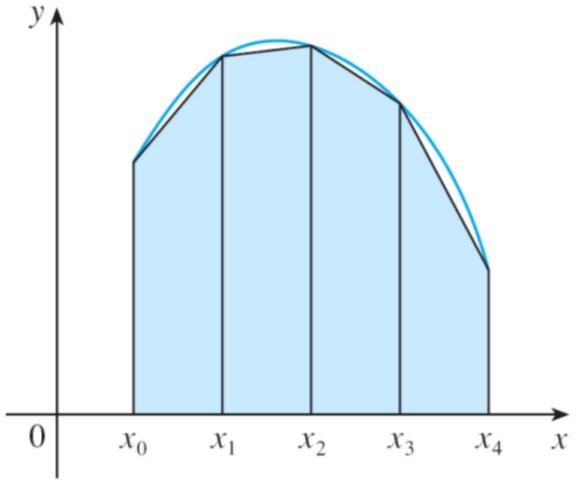
Trapezoidal Rule

$$\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+2f(x_{n-1})+f(x_n)\right]$$

where $\Delta x = \dfrac{(b-a)}{n}$ and $x_i = a + i\Delta x$.

The Midpoint and Trapezoidal Rules

The reason for the name Trapezoidal Rule can be seen from Figure 3, which illustrates the case with $f(x) \geq 0$ and $n = 4$.



Trapezoidal approximation

Figure 3

The Midpoint and Trapezoidal Rules

The area of the trapezoid that lies above the *i*th subinterval is

$$\Delta x\left(\frac{f\left(x_{i-1}\right)+f\left(x_i\right)}{2}\right)=\frac{\Delta x}{2}\left[f\left(x_{i-1}\right)+f\left(x_i\right)\right]$$

and if we add the areas of all these trapezoids, we get the right side of the Trapezoidal Rule.

Error Bounds for the Midpoint and Trapezoidal Rules

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3 Error Bounds Suppose $|f''(x)| \le K$ for $a \le x \le b$. If E_T and E_M are the errors in the Trapezoidal and Midpoint Rules, then

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \quad \text{and} \quad |E_M| \le \frac{K(b-a)^3}{24n^2}$$

Let's apply this error estimate to the Trapezoidal Rule approximation in Example 1.

$$\text{If } f(x) = \frac{1}{x}, \text{ then } f'(x) = \frac{-1}{x^2} \text{ and } f''(x) = \frac{2}{x^3}.$$

Simpson's Rule

Simpson's Rule

Another rule for approximate integration results from using parabolas instead of straight line segments to approximate a curve.

As before, we divide [*a*, *b*] into *n* subintervals of equal length $h = \Delta x = \frac{(b-a)}{n}$, but this time we assume that *n* is an *even* number.

Simpson’s Rule

Then on each consecutive pair of intervals we approximate the curve $y = f(x) \geq 0$ by a parabola as shown in Figure 8.

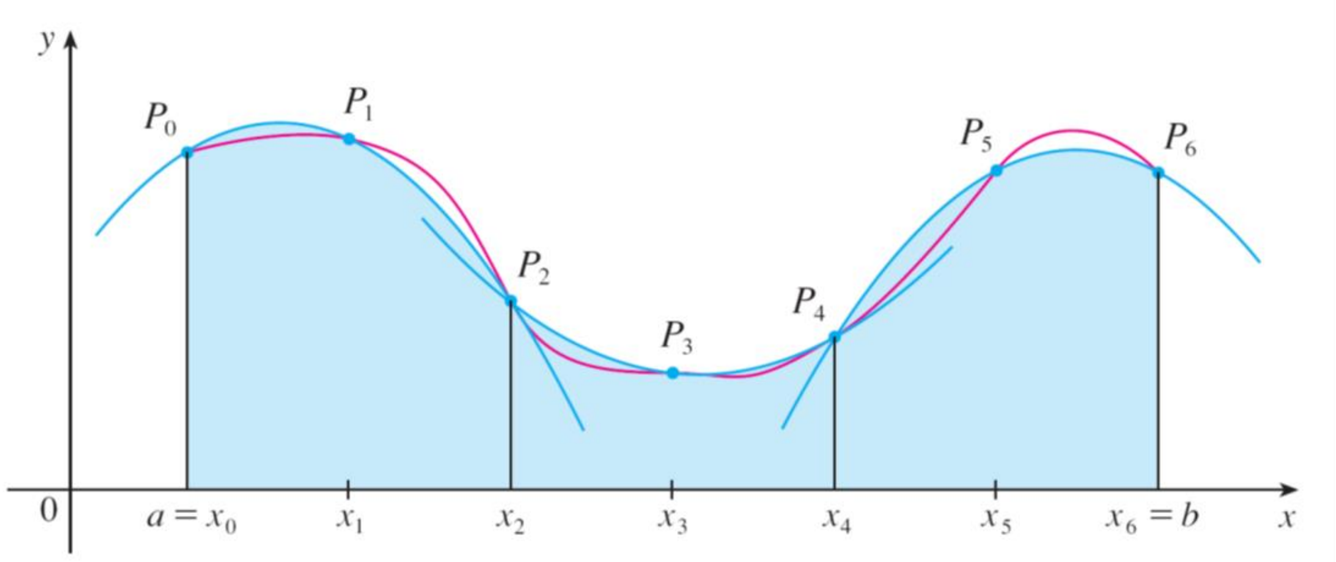

Figure 8

If $y_i = f(x_i)$, then P_i (x_i, y_i) is the point on the curve lying above x_i.

Simpson's Rule

A typical parabola passes through three consecutive points P_i, P_{i+1}, and P_{i+2}.

To simplify our calculations, we first consider the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. (See Figure 9.)

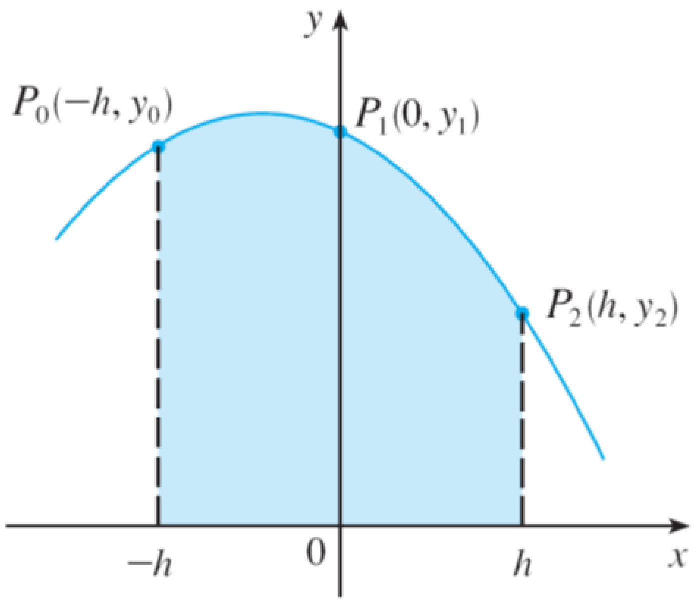

Figure 9

Simpson's Rule

We know that the equation of the parabola through P_0, P_1, and P_2 is of the form $y = Ax^2 + Bx + C$ and so the area under the parabola from $x = -h$ to $x = h$ is

$$\begin{aligned}
\int_{-h}^{h} \left(Ax^2 + Bx + C\right) dx &= 2\int_{0}^{h} \left(Ax^2 + C\right) dx \\
&= 2\left[A\frac{x^3}{3} + C_x\right]_0^h \\
&= 2\left(A\frac{h^3}{3} + Ch\right) = \frac{h}{3}\left(2Ah^2 + 6C\right)
\end{aligned}$$

Simpson's Rule

But, since the parabola passes through $P_0(-h, y_0)$, $P_1(0, y_1)$, and $P_2(h, y_2)$, we have

$$
\begin{aligned}
y_0 &= A(-h)^2 + B(-h) + C = Ah^2 - Bh + C \\
y_1 &= C \\
y_2 &= Ah^2 + Bh + C
\end{aligned}
$$

and therefore $$y_0 + 4y_1 + y_2 = 2Ah^2 + 6C$$

Thus we can rewrite the area under the parabola as

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

Simpson's Rule

Now by shifting this parabola horizontally we do not change the area under it.

This means that the area under the parabola through P_0, P_1, and P_2 from $x = x_0$ to $x = x_2$ in Figure 8 is still

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right)$$

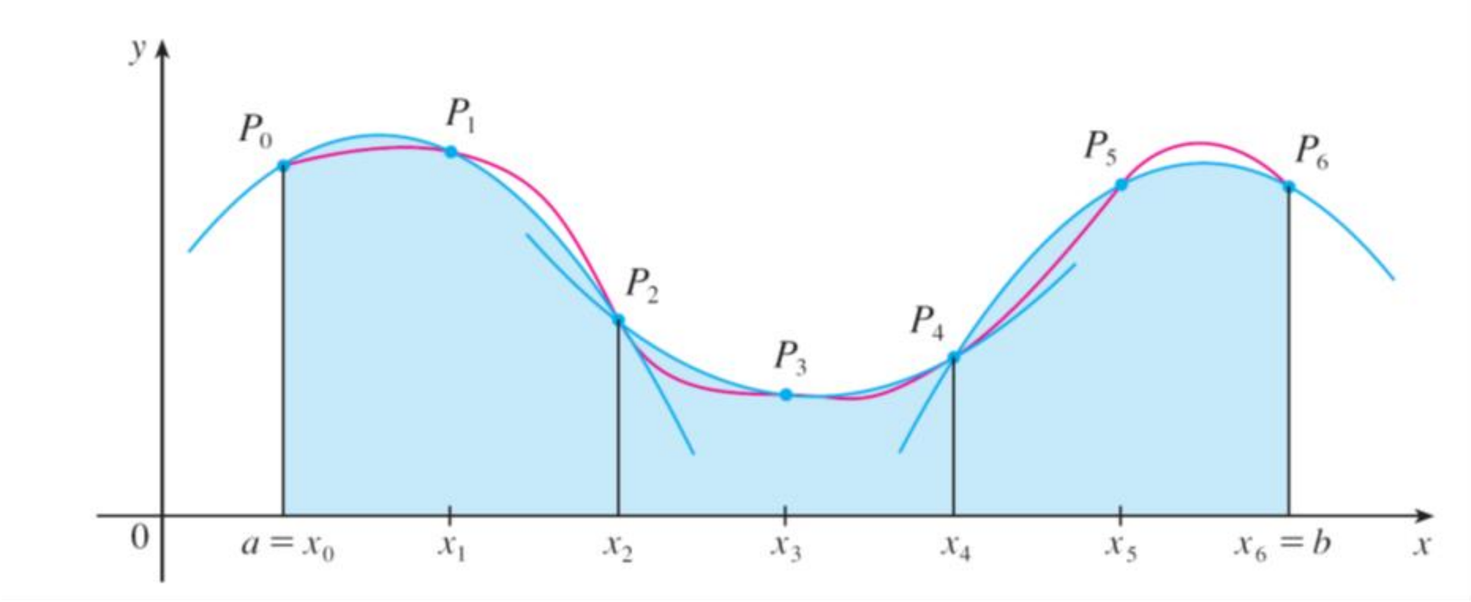

Figure 8

Simpson's Rule

Similarly, the area under the parabola through P_2, P_3, and P_4 from $x = x_2$ to $x = x_4$ is

$$\frac{h}{3}\left(y_2 + 4y_3 + y_4\right)$$

If we compute the areas under all the parabolas in this manner and add the results, we get

$$\begin{aligned}\int_a^b f(x)\,dx &\approx \frac{h}{3}\left(y_0 + 4y_1 + y_2\right) + \frac{h}{3}\left(y_2 + 4y_3 + y_4\right) + \cdots + \frac{h}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right)\end{aligned}$$

Simpson's Rule

Although we have derived this approximation for the case in which $f(x) \geq 0$, it is a reasonable approximation for any continuous function f and is called Simpson's Rule after the English mathematician Thomas Simpson (1710–1761).

Note the pattern of coefficients:

1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 4, 1.

Simpson's Rule

$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

where n is even and $\Delta x = \dfrac{(b-a)}{n}$.

Error Bound for Simpson’s Rule

Error Bound for Simpson's Rule

That is consistent with the appearance of n^4 in the denominator of the following error estimate for Simpson's Rule.

It is similar to the estimates given in (3) for the Trapezoidal and Midpoint Rules, but it uses the fourth derivative of *f*.

4 Error Bound for Simpson's Rule Suppose that $\left|f^{(4)}(x)\right| \le K$ for $a \le x \le b$. If

E_S is the error involved in using Simpson's Rule, then

$$\left|E_s\right| \le \frac{K(b-a)^5}{180n^4}$$