1 Functions of Several Variables

Functions of Two Variables

Functions of Two Variables

Definition A function *f* of two variables is a rule that assigns to each ordered pair of real numbers (x, y) in a set D a unique real number denoted by $f(x, y)$. The set D is the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x,y) \mid (x,y) \in D\}$.

We often write $z = f(x, y)$ to make explicit the value taken on by f at the general point (x, y). The variables x and y are **independent variables** and z is the **dependent variable**. [Compare this with the notation $y = f(x)$ for functions of a single variable.]

Graphs

Graphs

Another way of visualizing the behavior of a function of two variables is to consider its graph.

Definition If f is a function of two variables with domain D, then the **graph** of f is the set of all points (x, y, z) in $\mathbb{R}^3$ such that $z = f(x, y)$ and (x, y) is in D.

The graph of a function f of two variables is a surface S with equation $z = f(x, y)$.

Graphs

We can visualize the graph *S* of *f* as lying directly above or below its domain *D* in the *xy*-plane (see Figure 5).

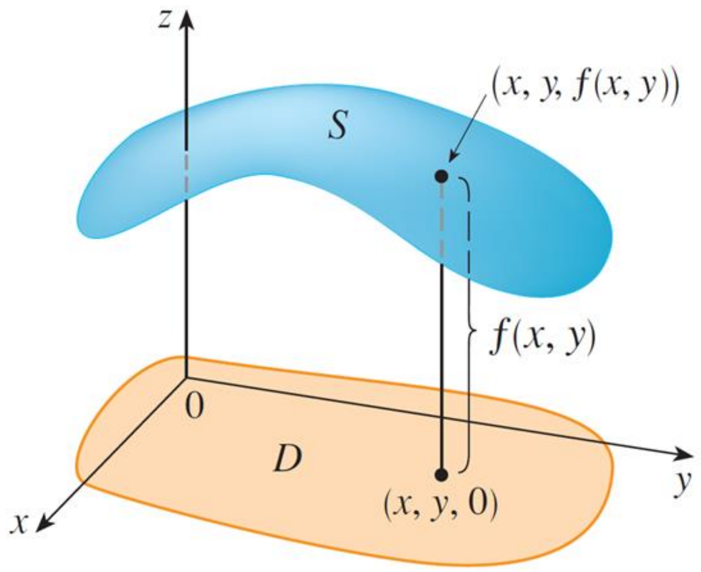


Figure 5

Example

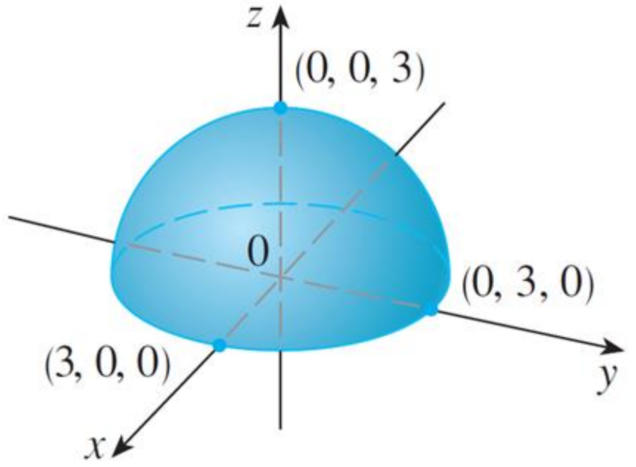
Sketch the graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$.

Solution:

In Example 2 we found that the domain of g is the disk with center (0, 0) and radius 3. The graph of t has equation $z = \sqrt{9 - x^2 - y^2}$.

We square both sides of this equation to obtain $z^2 = 9 - x^2 - y^2$, or $x^2 + y^2 + x^2 = 9$, which we recognize as an equation sphere with center the origin and radius 3.

But, since $z \geq 0$, the graph of g is just the top half of this sphere (see Figure 7).



Graph of $g(x,y) = \sqrt{9 - x^2 - y^2}$

Figure 7

Functions of Three or More Variables

Functions of Three or More Variables

A **function of three variables**, f, is a rule that assigns to each ordered triple (x, y, z) in a domain $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.

For instance, the temperature T at a point on the surface of the earth depends on the longitude x and latitude y of the point and on the time t, so we could write $T = f(x, y, t)$.

2 Limits and Continuity

Limits of Functions of Two Variables

Limits of Functions of Two Variables

In general, we use the notation

$$\lim_{(x,y)\to(a,b)} f(x,y) = L$$

to indicate that the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) (staying within the domain of f).

Limits of Functions of Two Variables

Other notations for the limit in Definition 1 are

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = L$$

and

$$f(x, y) \to L \text{ as } (x, y) \to (a, b)$$

Example

Evaluate $\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y)$.

Solution:

Let $f(x,y) = x^2y^3 - x^3y^2 + 3x + 2y$ is a polynomial, we can find the limit by direct substitution:

$$\lim_{(x,y)\to(1,2)}(x^2y^3 - x^3y^2 + 3x + 2y) = 1^2 \cdot 2^3 - 1^3 \cdot 2^2 + 3 \cdot 1 + 2 \cdot 2 = 11$$

Continuity

Continuity

We know that evaluating limits of *continuous* functions of a single variable is easy.

It can be accomplished by direct substitution because the defining property of a continuous function is $\lim_{x \to a} f(x) = f(a)$.

Continuous functions of two variables are also defined by the direct substitution property.

6 Definition A function *f* of two variables is called **continuous at** (a, b) if

$$\lim_{(x,y) \to (a,b)} f(x, y) = f(a, b)$$

We say *f* is **continuous on** *D* if *f* is continuous at every point (a, b) in *D*.

Functions of Three or More Variables

Functions of Three or More Variables

Everything that we have done in this section can be extended to functions of three or more variables.

The notation

$$\lim_{(x,y,z)\to(a,b,c)} f(x,y,z) = L$$

means that the values of $f(x, y, z)$ approach the number L as the point (x, y, z) approaches the point (a, b, c) (staying within the domain of f).

3 Partial Derivatives

Partial Derivatives of Functions of Two Variables

Partial Derivatives of Functions of Two Variables

In general, if f is a function of two variables x and y, suppose we let only x vary while keeping y fixed, say $y = b$, where b is a constant.

Then we are really considering a function of a single variable x, namely, $g(x) = f(x, b)$. If g has a derivative at a, then we call it the **partial derivative of *f* with respect to *x* at (*a*, *b*)** and denote it by $f_x(a, b)$. Thus

1 $$f_x(a,b) = g'(a) \quad \text{where} \quad g(x) = f(x,b)$$

Partial Derivatives of Functions of Two Variables

By the definition of a derivative, we have

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$$

and so Equation 1 becomes

$$\textbf{2} \qquad f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,\, b) - f(a,\, b)}{h}$$

Partial Derivatives of Functions of Two Variables (11 of 14)

Similarly, the **partial derivative of *f* with respect to *y* at (*a*, *b*)**, denoted by $f_y(a, b)$, is obtained by keeping x fixed ($x = a$) and finding the ordinary derivative at b of the function $G(y) = f(a, y)$:

3 $$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

Partial Derivatives of Functions of Two Variables (12 of 14)

If we now let the point (a, b) vary in Equations 2 and 3, f_x and f_y become functions of two variables.

4 Definition If f is a function of two variables, its **partial derivatives** are the functions f_x and f_y defined by

$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Partial Derivatives of Functions of Two Variables (13 of 14)

There are many alternative notations for partial derivatives. For instance, instead of f_x we can write f_1 or D_1f (to indicate differentiation with respect to the *first* variable) or $\partial f / \partial x$.

But here $\partial f / \partial x$ can't be interpreted as a ratio of differentials.

Notations for Partial Derivatives If $z = f(x, y)$, we write

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Partial Derivatives of Functions of Two Variables (14 of 14)

To compute partial derivatives, all we have to do is remember from Equation 1 that the partial derivative with respect to x is just the *ordinary* derivative of the function g of a single variable that we get by keeping y fixed.

Thus we have the following rule.

Rule for Finding Partial Derivatives of $z = f(x, y)$

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.

2. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

Example

If $f(x,y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution:

Holding y constant and differentiating with respect to x, we get

$$f_x(x, y) = 3x^2 + 2xy^3$$

and so $f_x(2, 1) = 3 \cdot 2^2 + 2 \cdot 2 \cdot 1^3 = 16$

Holding x constant and differentiating with respect to y, we get

$$f_y(x, y) = 3x^2y^2 - 4y$$

$$f_y(2, 1) = 3 \cdot 2^2 \cdot 1^2 - 4 \cdot 1 = 8$$

Interpretations of Partial Derivatives

Interpretations of Partial Derivatives (1 of 4)

To give a geometric interpretation of partial derivatives, we know that the equation $z = f(x, y)$ represents a surface S (the graph of f). If $f(a, b) = c$, then the point $P(a, b, c)$ lies on S.

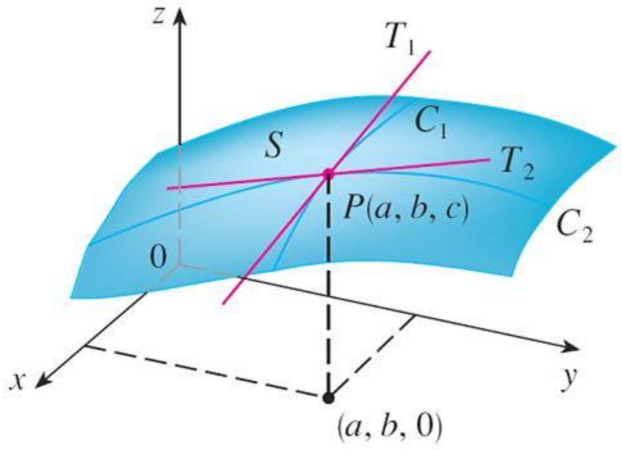
By fixing $y = b$, we are restricting our attention to the curve C_1 in which the vertical plane $y = b$ intersects S. (In other words, C_1 is the trace of S in the plane $y = b$.)

Interpretations of Partial Derivatives (2 of 4)

Likewise, the vertical plane $x = a$ intersects S in a curve C_2. Both of the curves C_1 and C_2 pass through the point P. (See Figure 1.)

Note that the curve C_1 is the graph of the function $g(x) = f(x, b)$, so the slope of its tangent T_1 at P is $g'(a) = f_x(a, b)$.

The curve C_2 is the graph of the function $G(y) = f(a, y)$, so the slope of its tangent T_2 at P is $G'(b) = f_y(a, b)$.



The partial derivatives of f at (a, b) are the slopes of the tangents to C_1 and C_2.

Figure 1

Interpretations of Partial Derivatives (3 of 4)

Thus the partial derivatives $f_x(a, b)$ and $f_y(a, b)$ can be interpreted geometrically as the slopes of the tangent lines at $P(a, b, c)$ to the traces C_1 and C_2 of S in the planes $y = b$ and $x = a$.

As we have seen in the case of the heat index function at the beginning of this section, partial derivatives can also be interpreted as *rates of change*.

If $z = f(x, y)$, then $\partial z/\partial x$ represents the rate of change of z with respect to x when y is fixed. Similarly, $\partial z/\partial y$ represents the rate of change of z with respect to y when x is fixed.

Example 3

If $f(x, y) = 4 - x^2 - 2y^2$, find $f_x(1, 1)$ and $f_y(1, 1)$ and interpret these numbers as slopes.

Solution:

We have

$$f_x(x, y) = -2x \qquad f_y(x, y) = -4y$$

$$f_x(1, 1) = -2 \qquad f_y(1, 1) = -4$$

Example 3 – Solution (1 of 2)

The graph of *f* is the paraboloid $z = 4 - x^2 - 2y^2$ and the vertical plane $y = 1$ intersects it in the parabola $z = 2 - x^2$, $y = 1$. (As in the preceding discussion, we label it C_1 in Figure 2.)

The slope of the tangent line to this parabola at the point (1, 1, 1) is $f_x(1, 1) = -2$.

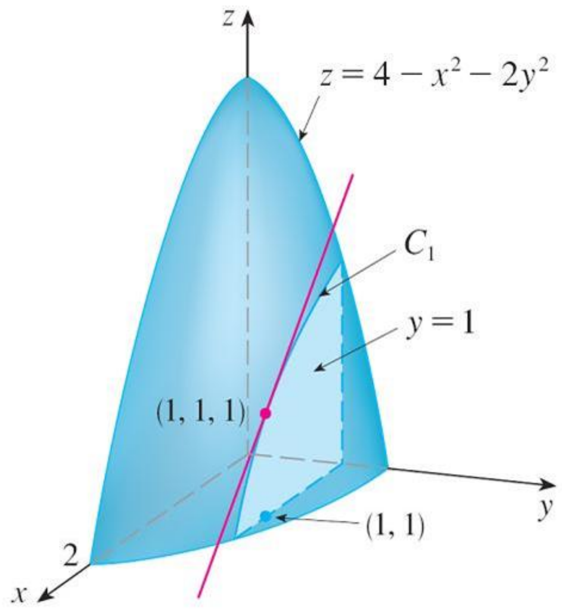


Figure 2

Example 3 – Solution (2 of 2)

Similarly, the curve C_2 in which the plane $x = 1$ intersects the paraboloid is the parabola $z = 3 - 2y^2$, $x = 1$, and the slope of the tangent line at (1, 1, 1) is $f_y(1, 1) = -4$. (See Figure 3.)

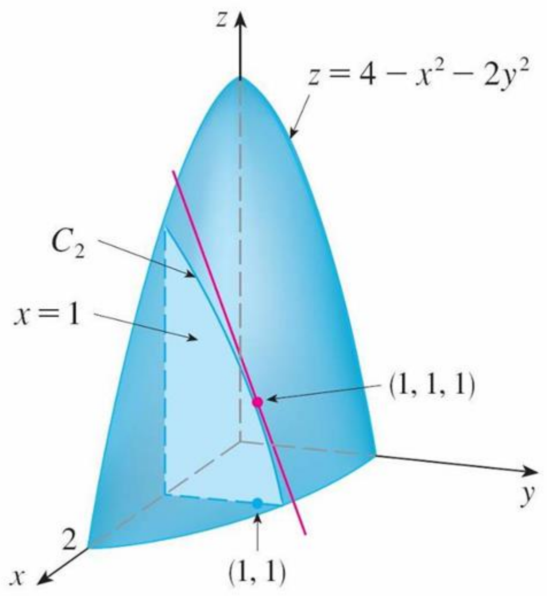


Figure 3

Functions of Three or More Variables

Functions of Three or More Variables (1 of 2)

Partial derivatives can also be defined for functions of three or more variables. For example, if *f* is a function of three variables *x*, *y*, and *z*, then its partial derivative with respect to *x* is defined as

$$f_x(x, y, z) = \lim_{h \to 0} \frac{f(x + h, y, z) - f(x, y, z)}{h}$$

and it is found by regarding *y* and *z* as constants and differentiating *f*(*x*, *y*, *z*) with respect to *x*.

Higher Derivatives

Higher Derivatives (1 of 4)

If f is a function of two variables, then its partial derivatives f_x and f_y are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of f.

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}$$

Higher Derivatives (2 of 4)

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation f_{xy} $\left(\text{or } \partial^2 f/\partial y \partial x\right)$ means that we first differentiate with respect to x and then with respect to y, whereas in computing f_{yx} the order is reversed.

Example 7

Find the second partial derivatives of

$$f(x,y) = x^3 + x^2y^3 - 2y^2$$

Solution:

In Example 1 we found that

$$f_x(x,y) = 3x^2 + 2xy^3 \qquad f_y(x,y) = 3x^2y^2 - 4y$$

Therefore

$$\begin{aligned} f_{xx} &= \frac{\partial}{\partial x}(3x^2 + 2xy^3) \\ &= 6x + 2y^3 \end{aligned}$$

Example 7 – Solution

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3)$$

$$= 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y)$$

$$= 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y)$$

$$= 6x^2y - 4$$

Higher Derivatives (3 of 4)

Notice that $f_{xy} = f_{yx}$ in Example 7. This is not just a coincidence. It turns out that the mixed partial derivatives f_{xy} and f_{yx} are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713−1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

Clairaut's Theorem Suppose f is defined on a disk D that contains the point (a, b). If the functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

4 Tangent Planes and Linear Approximations

Tangent Planes

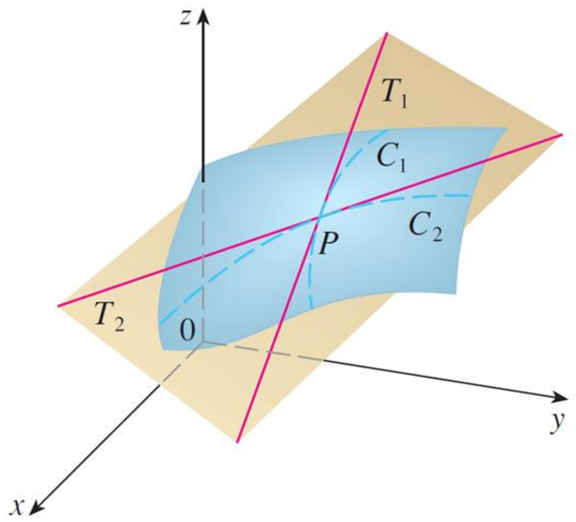
Tangent Planes

Suppose a surface S has equation $z = f(x, y)$, where f has continuous first partial derivatives, and let $P(x_0, y_0, z_0)$ be a point on S.

Let C_1 and C_2 be the curves obtained by intersecting the vertical planes $y = y_0$ and $x = x_0$ with the surface S. Then the point P lies on both C_1 and C_2. Let T_1 and T_2 be the tangent lines to the curves C_1 and C_2 at the point P.

Tangent Planes

Then the **tangent plane** to the surface S at the point P is defined to be the plane that contains both tangent lines T_1 and T_2. (See Figure 1.)



The tangent plane contains the tangent lines T_1 and T_2.

Figure 1

Tangent Planes

If C is any other curve that lies on the surface S and passes through P, then its tangent line at P also lies in the tangent plane.

Therefore you can think of the tangent plane to S at P as consisting of all possible tangent lines at P to curves that lie on S and pass through P. The tangent plane at P is the plane that most closely approximates the surface S near the point P.

We know that any plane passing through the point $P(x_0, y_0, z_0)$ has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$$

Tangent Planes

By dividing this equation by C and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$\textbf{1} \quad z - z_0 = a(x - x_0) + b(y - y_0)$$

If Equation 1 represents the tangent plane at P, then its intersection with the plane $y = y_0$ must be the tangent line T_1. Setting $y = y_0$ in Equation 1 gives

$$z - z_0 = a(x - x_0) \qquad \text{where } y = y_0$$

and we recognize this as the equation (in point-slope form) of a line with slope a.

Tangent Planes

But we know that the slope of the tangent T_1 is $f_x(x_0, y_0)$. Therefore $a = f_x(x_0, y_0)$.

Similarly, putting $x = x_0$ in Equation 1, we get $z - z_0 = b(y - y_0)$, which must represent the tangent line T_2, so $b = f_y(x_0, y_0)$.

2 Equation of a Tangent Plane Suppose f has continuous partial derivatives. An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example

Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).

Solution:

Let $f(x, y) = 2x^2 + y^2$.

Then

$$f_x(x, y) = 4x \qquad f_y(x, y) = 2y$$

$$f_x(1, 1) = 4 \qquad f_y(1, 1) = 2$$

Then (2) gives the equation of the tangent plane at (1, 1, 3) as

$$z - 3 = 4(x - 1) + 2(y - 1)$$

or

$$z = 4x + 2y - 3$$