Analyzing the Performances of a Compliant 3R Planar Robot using the ESP Control

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1 Introduction

The safe and controlled interaction of a robotic system with its environment plays an ever-increasing role. For this reason, the introduction of compliant actuators improves mechanical robustness against rigid impacts as well as unknown contact forces. The elastic elements acting between motor and link inertia lead to a dynamical behavior of a low-pass filter on external loads and the price to pay is in term of complexity of the control action. In fact the plant becomes under-actuated and this means that the dimension of the configuration space is doubled w.r.t. the control input space.

Introducing elastic elements will also bring intrinsic oscillatory behaviours that must be handled correctly in order to obtain performances as much similar as the one of a rigid manipulator. Moreover, situations where a robotic system is subjected to hard and/or fast impacts may arise, and in these scenarios, the implementation of a damping term directly on the link side can be challenging since the actuators easily run into torque saturation. Motivated by this observation, we analyzed the Elastic Structure Preserving (ESP and $ESP+$) control approaches presented in [1], that simultaneously achieve motion tracking and assign damping for the link configuration variables of robots with compliant elements with nonlinear elastic transmissions. In fact the previous problem is theoretically well founded, it has been analytically proven the existence of a controller which provides global and uniform asymptotic stability, is practically feasible and achieves impressive performances. The idea behind this kind of control approaches relies on the fundamental concept of changing the original dynamics only to a minimal extent and, in particular, to preserve the (nonlinear) elastic structure. The dynamic gravity cancellation for flexible robot, as described in [2], is also present in the analyzed control laws.

The authors of [1], on the basis of a theoretical analysis made over a robotic system with a single joint, show that this approach changes the plant dynamics significantly less than linearization-based full state feedback control and, hence, suffers less from unmodeled dynamics, parameter uncertainties, actuators bandwidth and amplitude limitations.

By introducing new coordinates that reflect these damping and feedforward terms, through means of feedback control, we have obtained a feedback equivalent dynamics. In the new variables a link damping has been added whereas the gravity term has been removed from the picture. In addition, for the desired tracking and disturbance rejection behaviours of the link coordinates, a pure PD regulation control is implemented in the new motor coordinates.

The ESP+ is the extension of this "minimalistic" feedback control that solves the same problems for the link configuration variables, avoiding the scaling of the motor inertia to constant values in the new coordinates. To analyze this aspect, we have considered linear transmissions with the goal of shaping the motor inertia by assigning a new desired stiffness constant to the motor-side.

This project is structured as follow *Section 2* describes the dynamic model used for the simulations, *Section 3* reports the analytical computations on how the control strategy are derived using the model of the robotic system, *Section 4* shows the simulation made using MATLAB and finally *Section 5* reports the summary of what have been analyzed through this work.

2 Dynamic model

In our simulations we have considered a 3R planar robot with nonlinear elastic transmissions, i.e. a robot manipulator composed of three revolute joints which moves on a vertical plane and so it is subjected to the gravitational force. The dynamic model has been obtained considering the Euler-Lagrange formalism [3]: given the generalized coordinates $z = [q, \theta]^T$, where $q \in \mathbb{R}^N$ are the link coordinates and $\theta \in \mathbb{R}^N$ are the motor variables. The Euler-Lagrange equations that describe the dynamic evolution of the system are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{z}_i} - \frac{\partial L}{\partial z_i} = u_i$$

The above relation can be rewritten in such a way to obtain the dynamic model in a more compact form

$$M(z)\ddot{z} + c(z,\dot{z}) + \frac{\partial U(z)}{\partial z} = u$$

where $M(z)$ is the inertia matrix, $c(z,\dot{z})$ is the centrifugal-coriolis term and the last one is the derivative of the total potential energy of the system. First of all we define the position of the centers of mass and the motor positions.

$$p_{c1} = \begin{bmatrix} d_1c_1 \\ d_1s_1 \end{bmatrix} \qquad p_{c2} = \begin{bmatrix} \ell_1c_1 + d_2c_{12} \\ \ell_1s_1 + d_2s_{12} \end{bmatrix} \qquad p_{c3} = \begin{bmatrix} \ell_1c_1 + \ell_2c_{12} + d_3c_{123} \\ \ell_1s_1 + \ell_2s_{12} + d_3s_{123} \end{bmatrix}$$

$$p_1^m = 0 \qquad p_2^m = \begin{bmatrix} \ell_1c_1 \\ \ell_1s_1 \end{bmatrix} \qquad p_3^m = \begin{bmatrix} \ell_1c_1 + \ell_2c_{12} \\ \ell_1s_1 + \ell_2s_{12} \end{bmatrix}$$

Then, in order to build the inertia matrix, we obtained the total kinetic energy given by the sum of all the kinetic contributions of both the links and the motors.

$$\begin{aligned} T_{l1} &= \frac{1}{2}m_1v_{c1}^Tv_{c1} + \frac{1}{2}I_1\dot{q_1}^2 & T_{m1} &= \frac{1}{2}I_{m1}\dot{\theta_1}^2k_r^2 \\ T_{l2} &= \frac{1}{2}m_2v_{c2}^Tv_{c2} + \frac{1}{2}I_2(\dot{q_1}+\dot{q_2})^2 & T_{m_2} &= \frac{1}{2}I_{m_2}\dot{\theta_2}^2k_r^2 + \frac{1}{2}m_{m_2}v_2^Tv_2 \\ T_{l3} &= \frac{1}{2}m_3v_{c3}^Tv_{c3} + \frac{1}{2}I_3(\dot{q_1}+\dot{q_2}+\dot{q_3})^2 & T_{m_3} &= \frac{1}{2}I_{m_3}\dot{\theta_3}^2k_r^2 + \frac{1}{2}m_{m_3}v_3^Tv_3 \end{aligned}$$

In addition we have considered that, in the term derived by the kinetic energy due to the motors, the angular speed is related just to a rotation around their spinning axis. This reasoning is equivalent to consider high values for the stiffness of the springs so to avoid the energetic contributions due to the inertial coupling between motors and links.
We can write the total kinetic energy as follows:

$$\begin{aligned} T(q,\dot{q},\dot{\theta}) &= \frac{1}{2}\begin{bmatrix}\dot{q}^T & \dot{\theta}^T\end{bmatrix}\psi(q)\begin{bmatrix}\dot{q} \\ \dot{\theta}\end{bmatrix} = \\ &= \frac{1}{2}\begin{bmatrix}\dot{q}^T & \dot{\theta}^T\end{bmatrix}\begin{bmatrix}M(q) & 0 \\ 0 & B\end{bmatrix}\begin{bmatrix}\dot{q} \\ \dot{\theta}\end{bmatrix} \end{aligned}$$

The inertia matrix of the link coordinates is given in the parametrized form

$$M(q) = \begin{bmatrix} a_6 + 2a_3c_{23} + 2a_5c_2 + 2a_2c_3 & a_4 + a_3c_{23} + 2a_2c_3 + a_5c_2 & a_1 + a_2c_3 + a_3c_{23} \\ & a_4 + 2a_2c_3 & a_1 + a_2c_3 \\ symm & & a_1 \end{bmatrix} \tag{1}$$

while the motor inertia matrix is given by

$$B = \begin{bmatrix} I_{m1}k_r^2 & 0 & 0 \\ 0 & I_{m_2}k_r^2 & 0 \\ 0 & 0 & I_{m_3}k_r^2 \end{bmatrix}$$

where k_r is the reduction ratio. Once built the inertia matrix $M(q)$ we can obtain the coriolis and centrifugal term accordingly to the following expression

$$c_k(q, \dot{q}) = \dot{q}^T C_k \dot{q} \quad \text{with} \quad C_k = \frac{1}{2}\left(\frac{\partial m_k}{\partial q} + \left(\frac{\partial m_k}{\partial q}\right)^T - \frac{\partial M}{\partial q_k}\right) \qquad k = 1, .., N \tag{2}$$

where m_k represent the *k-th* column of the inertia matrix $M(q)$. The Coriolis term in *eq.(2)* can be factorized in the form

$$c(q, \dot{q}) = S(q, \dot{q})\dot{q}$$

where the term $S(q, \dot{q})$ satisfies the property that $\dot{M}(q) - 2S(q, \dot{q})$ is a skew symmetric matrix for all $(q,\ \dot{q}) \in \mathbb{R}^N \times \mathbb{R}^N$.
Finally the potential contribution is given both by the elastic potential energy, due to the compliant joints, and by the gravitational potential energy, since the robot is moving on a vertical plane. The elastic term is defined by

$$\psi(\theta - q) = \left(\frac{\partial U_{el}(\phi)}{\partial \phi}\right)\Bigg|_{\phi=\theta-q} \in \mathbb{R}^N$$

while the gravity term is expressed as

$$g(q) = \left(\frac{\partial U_g(q)}{\partial q}\right) \in \mathbb{R}^N \tag{3}$$

In conclusion the dynamic model of the robot is

$$\begin{aligned} M(q)\ddot{q} + S(q, \dot{q})\dot{q} + g(q) - \psi(\theta - q) &= \tau_{ext} \\ B\ddot{\theta} + \psi(\theta - q) &= u \end{aligned} \tag{4}$$

The main idea for designing the control input u is to inject a damping and a pseudo-feedforward term in the link side dynamics. This will ensure tracking performances and the resulting dynamics will evolves following the desired one, defined as

$$M(t, \tilde{q})\ddot{\tilde{q}} + S(t, \tilde{q}, \dot{\tilde{q}})\dot{\tilde{q}} + D\dot{\tilde{q}} - \psi(\eta - \tilde{q}) = \tau_{ext} \tag{5}$$

where $\tilde{q} = q - q_d(t) \in \mathbb{R}^N$ represents the link-side tracking error and for emphasizing the explicit time dependency of the closed-loop link dynamics (*eq.(5)*), the short forms have been used

$$\begin{aligned} M(\tilde{q}, t) &= M(q) = M(\tilde{q} + q_d(t)) \\ S(\tilde{q}, \dot{\tilde{q}}, t) &= S(q, \dot{q}) = S(\tilde{q} + q_d(t), \dot{\tilde{q}} + \dot{q}_d(t)) \\ g(\tilde{q}, t) &= g(q) = g(\tilde{q} + q_d(t)) \end{aligned}$$

The explicit form of all the parameters of *eq.(1)*, *eq.(2)* and *eq.(3)* can be found in *Appendix B*, *Appendix C* and *Appendix D* respectively.

3 Elastic Structure Preserving (ESP and ESP+)

We need to perform dynamics transformation before designing the ESP or ESP+ controller.
Introduction of new motor coordinates $\eta \in \mathbb{R}^N$ to reflect the link side of the new system behaviours is needed. First of all we impose equivalence of the new dynamic model (*eq.(5)*) and the original one (*eq.(4)*) to find an implicit relation.

$$\psi(\theta - q) = \psi(\eta - \tilde{q}) + n(t, \tilde{q}, \dot{\tilde{q}}) \tag{6}$$

Where the term $n(t, \tilde{q}, \dot{\tilde{q}})$ is defined as

$$n(t, \tilde{q}, \dot{\tilde{q}}) = g(t, \tilde{q}) - D\dot{\tilde{q}} + M(t, \tilde{q})\ddot{q}_d(t) + S(t, \tilde{q}, \dot{\tilde{q}})\dot{q}_d(t) \tag{7}$$

with time derivative

$$\dot{n}(t, \tilde{q}, \dot{\tilde{q}}) = \dot{g}(t, \tilde{q}) - D\ddot{\tilde{q}} + M(t, \tilde{q})\dddot{q}_d(t) + \dot{M}(t, \tilde{q})\ddot{q}_d(t) + \dot{S}(t, \tilde{q}, \dot{\tilde{q}})\dot{q}_d(t) + S(t, \tilde{q}, \dot{\tilde{q}})\ddot{q}_d(t)$$

Differentiating *eq.(6)* with respect to time gives us a differential relation between the old and new motor coordinates.

$$\kappa(\theta - q)(\dot{\theta} - \dot{q}) = \kappa(\eta - \tilde{q})(\dot{\eta} - \dot{\tilde{q}}) + \dot{n}(t, \tilde{q}, \dot{\tilde{q}}) \tag{8}$$

The local stiffness κ, i.e. the Hessian of the elastic potential energy U_{el}, is computed as follows

$$\kappa(\phi_0) = \left(\frac{\partial^2 U_{el}(\phi)}{\partial \phi^2}\right)\bigg|_{\phi=\phi_0} = \left(\frac{\partial \psi(\phi)}{\partial \phi}\right)\bigg|_{\phi=\phi_0} \in \mathbb{R}^{n\times n}$$

Solving *eq.(8)* for $\dot{\theta}$ is

$$\dot{\theta} = k^{-1}(\theta - q)k(\eta - \tilde{q})\dot{\eta} + \dot{\tilde{q}} + \dot{q}_d(t) + k^{-1}(\theta - q)\gamma(t, \eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) \tag{9}$$

where

$$\gamma(t, \eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = -\kappa(\eta - \tilde{q})\dot{\tilde{q}} + \dot{n}(t, \tilde{q}, \dot{\tilde{q}}) \tag{10}$$

With *eq.(6)* we can rewrite $\dot{\theta}$ as a function of the new states only

$$\dot{\theta} = A(t, \eta, \tilde{q}, \dot{\tilde{q}})\dot{\eta} + a(t, \eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) \tag{11}$$

See *Appendix A* for the computations of A and a. Differentiating $\dot{\theta}$ with respect to time yields

$$\ddot{\theta} = A(t, \eta, \tilde{q}, \dot{\tilde{q}})\ddot{\eta} + \dot{A}(t, \eta, \tilde{q}, \dot{\tilde{q}})\dot{\eta} + \dot{a}(t, \eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) \tag{12}$$

The relations *eq.(6)* and *eq.(12)* allow us to perform a coordinate transformation $[\theta, q] \rightarrow [\eta, \tilde{q}]$ for the robot dynamics *eq.(4)*. For the transformed motor dynamics we get

$$BA\ddot{\eta} + B\dot{A}\dot{\eta} + B\dot{a} + \psi(\theta - q) = u \tag{13}$$

3.1 ESP

ESP control focuses on preserving the inertial properties of motor dynamics.
We design the controller in three steps, such that the resulting controller $u = u_{ESP}$ is composed of three components

$$u_{ESP} = \check{u}_{ESP} + \hat{u}_{ESP} + \bar{u}_{ESP}$$

First, we pre-compensate some nonlinear terms by

$$\check{u}_{ESP} = B(\dot{A}\dot{\eta} + \dot{a})$$

Resulting in the following intermediary dynamics

$$BA\ddot{\eta} + \psi(\theta - q) = \hat{u}_{ESP} + \bar{u}_{ESP}$$

Second, we shape the motor inertia such that the original, constant motor inertia B results. In addition, we transform the spring torques into the new coordinates.

$$\hat{u}_{ESP} = \psi(\theta - q) - BAB^{-1}\psi(\eta - \tilde{q})$$

yields the intermediary dynamics

$$B\ddot{\eta} + \psi(\eta - \tilde{q}) = BA^{-1}B^{-1}\bar{u}_{ESP}$$

At last, we choose PD control in the new motor coordinates to achieve link-side motion tracking.

$$\bar{u}_{ESP} = -BAB^{-1}(K_D\dot{\eta} + K_P\eta)$$

In the end, we yield the following closed-loop motor dynamics

$$B\ddot{\eta} + \psi(\eta - \tilde{q}) = -K_D\dot{\eta} - K_P\eta \tag{14}$$

Fig.(1a) presents the block diagram that corresponds to the ESP control law.

3.2 ESP+

ESP+ control aims at minimizing the dynamic shaping of the motor inertia.
Pre-multiplying *eq.(13)* by A^T yields the following transformed motor dynamics

$$B_\eta\ddot{\eta} + S_\eta\dot{\eta} + A^T\psi(\theta - q) + A^TB\dot{a} = A^Tu$$

where naturally the arising inertia matrix is expressed as

$$B_\eta(t,\eta,\tilde{q},\dot{\tilde{q}}) = A^T(t,\eta,\tilde{q},\dot{\tilde{q}})BA(t,\eta,\tilde{q},\dot{\tilde{q}})$$

and coriolis and centrifugal matrix is expressed as

$$S_\eta(t,\eta,\dot{\eta},\tilde{q},\dot{\tilde{q}},\ddot{\tilde{q}}) = A^T(t,\eta,\tilde{q},\dot{\tilde{q}})B\dot{A}(t,\eta,\tilde{q},\dot{\tilde{q}})$$

Again, we derive the control law in three steps, such that the resulting control input $u = u_{ESP+}$ is composed of three components

$$u_{ESP+} = \check{u}_{ESP+} + \hat{u}_{ESP+} + \bar{u}_{ESP+}$$

We start by canceling some nonlinear terms by

$$\check{u}_{ESP+} = B\dot{a}$$

We proceed with transforming the spring torques into the new coordinates

$$\hat{u}_{ESP+} = \psi(\theta - q) - A^{-T}\psi(\eta - \tilde{q})$$

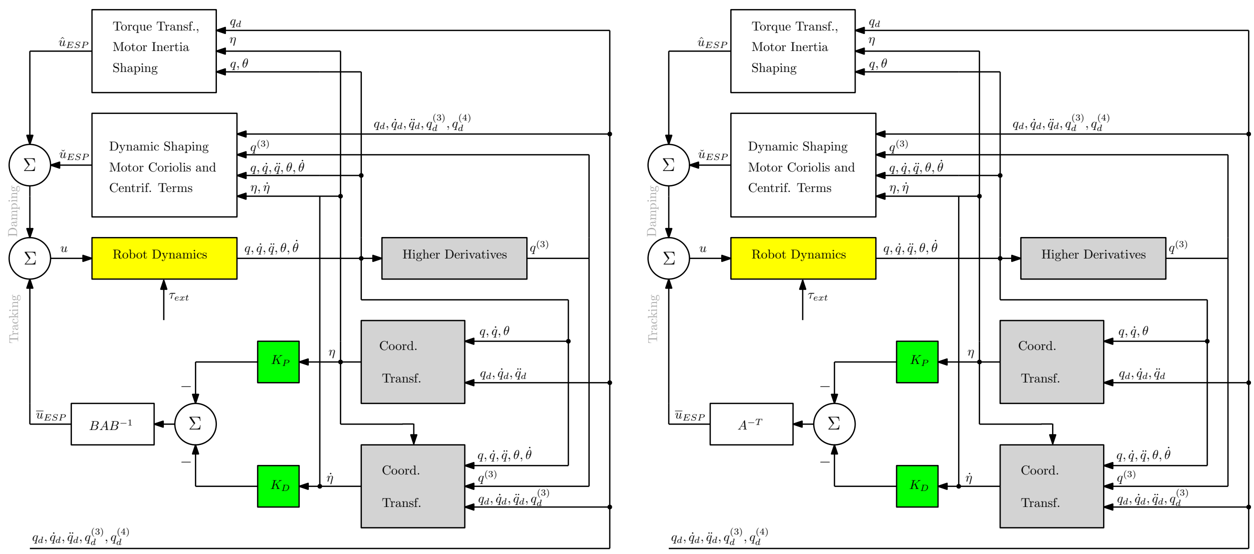
Finally, we choose PD control in the new coordinates to achieve link-side tracking behavior.

$$\bar{u}_{ESP+} = -A^{-T}(K_D\dot{\eta} + K_P\eta)$$

In the end, the following closed-loop motor dynamics results in

$$B_\eta\ddot{\eta} + S_\eta\dot{\eta} + \psi(\eta - \tilde{q}) = -K_D\dot{\eta} - K_P\eta$$

Fig.(1b) depicts the block diagram that corresponds to the ESP+ control law.



(a) *ESP block scheme* (b) *ESP+ control scheme*

Fig. 1: *Block schemes*

4 Computational Results

In this section we are going to show the experimental results obtained through the simulations done in MATLAB/Simulink following the block schemes depicted in *Fig.(1)*.
In order to exploit the *ESP* and *ESP+* controls we have considered the situation in which the robot has to remain still at the desired configuration, i.e with higher derivatives of $q_d(t)$ that are all zero, and then it is pushed by a force and so it has to recover its previous configuration. Differently from the work done in [1] we have applied the control framework to a planar 3R robot which lies on a vertical plane with nonlinear elastic transmissions of the form

$$\psi(\theta - q) = k[(\theta - q) + (\theta - q)^3]$$

where k is a stiffness constant listed in *Tab.(1)*.
We have analysed both frameworks by comparing them with a simple *PD* control law with gravity compensation [4] in order to see the effect of the damped behaviour imposed by the former controls. According to [4] the control law that globally stabilizes the system (*eq.(4)*) is

$$u = -K_P(\theta - \theta_d) - K_D\dot{\theta} + g(q_d) \tag{15}$$

where the desired value of the motor variables θ_d is related to the one of the link variables q_d through the relation

$$\theta_d = q_d + \psi(\theta - q)^{-1} g(q_d)$$

We have focused on the case in which the robot is pushed by a force on the *CoM* of the third link because it is where the effect of the force is higher. We have tested our control scheme in different scenarios by putting the robot in various configurations.
Moreover we have exploited the shape of the motor inertia that can be done using *ESP+* comparing the results with the *ESP* scenario. In this context linear elastic transmission have been used in order to act directly on the motor inertia through the control

4.1 Simulation parameters

Here we have listed all the physical parameters used for defining the dynamic model of the 3R planar robot and the control parameters and the elements of the diagonal damping matrix D.

The control terms define the overall closed loop dynamics as presented in *eq.(14)*, and are the parameters of K_P and K_D, whereas the damping term reflects the link-side damped behaviour we want to achieve following *eq.(5)*.

Parameter	Description	Value
m	link mass	1 [kg]
mm	motor mass	1 [kg]
d	position of the CoM of the link	0.5 [m]
l	link length	1 [m]
I	link inertia	3.33 [kg $\cdot$ m^2]
I_m	motor inertia	0.01 [kg $\cdot$ m^2]
k_r	reduction ratio	18
k	stiffness constant	500 [N/m]
d_i	D element	500
k_{pi}	K_P element	800
k_{di}	K_D element	500
T	simulation time	10 [s]
t_1	starting instant of the force	2 [s]
t_2	ending instant of the force	3 [s]
F	module of the force	1000 [N]

Tab. 1: *Simulation parameters*

In order to evaluate *eq.(6)*, we need to compute numerically η, but this requires to invert the nonlinear relation $\psi(\eta - \tilde{q})$ which is not an easy task in general. For solving this problem, we have used the numerical nonlinear solver provided by MATLAB, `fsolve`.

4.2 Generic configuration

In our first simulation the starting configuration of the robot is a generic pose (showed in *Fig.(2)*) and is selected as

$$q_0 = q_d = \begin{bmatrix} \frac{\pi}{3} \\ -\frac{\pi}{7} \\ \frac{\pi}{4} \end{bmatrix}$$

At $t \in [t_1,\ t_2]$ the robot feels a force acting on the third link and we can see from *Fig.(3)* that both ESP and $ESP+$ control give the robot a damped behaviour since we can see no overshoot in the evolution of the error $\tilde{q}$. Differently, when the robot is controlled through the simple PD action (*eq.(15)*), it oscillates around the desired configuration until it stabilizes at steady state. Moreover, we can see a common behaviour among all the control actions which is the fact that, since the force acts on the third link and because of the internal coupling of the robot dynamics, all the three link variables feel a torque due to the external force (*Fig.(3b)*).

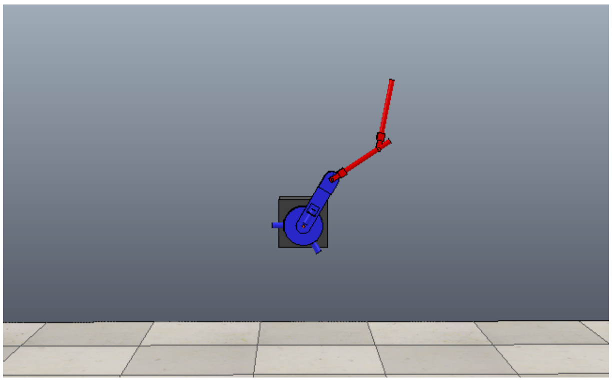
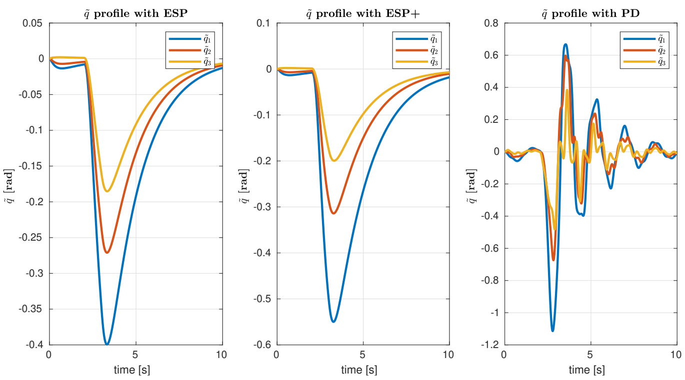
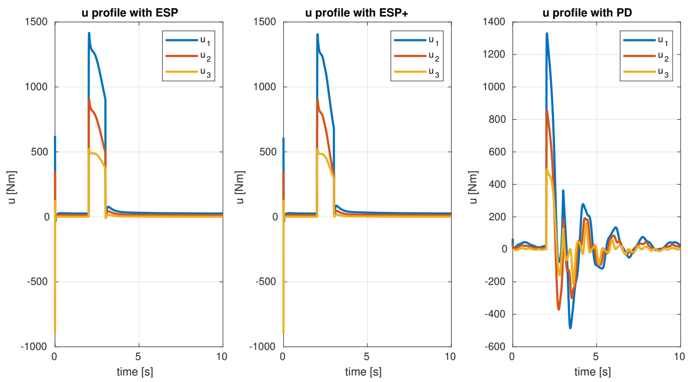


Fig. 2: Initial position in a generic pose



(a) *$\tilde{q}$ profile*



(b) *control profile*

Fig. 3: Force acting on the CoM of the third link: generic configuration

4.3 Stretched up arm configuration

For the second simulation the starting point of the robot have been set to the stretched up arm configuration (shown in *Fig.(4)*) that is

$$q_0 = q_d = \begin{bmatrix} \frac{\pi}{2} \\ 0 \\ 0 \end{bmatrix}$$

We have considered this situation to depict the case in which we need to apply the highest control effort to recover the force applied on the third link of the robot. The result of the simulation is presented in *Fig.(5)* where we can notice that the tracking error in this case is greater w.r.t. the previous case, i.e. by comparing *Fig.(3a)* with *Fig.(5a)*, as well as the control action that reach an higher peak (*Fig.(3b)* and *Fig.(5)*)

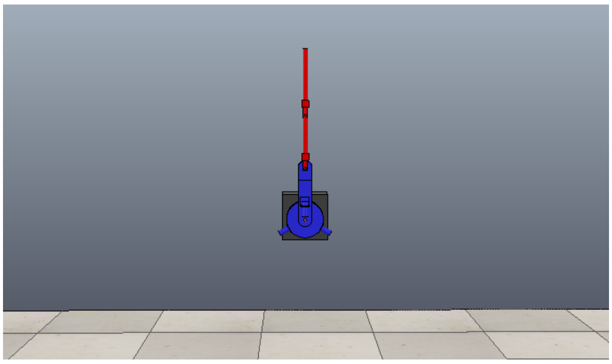
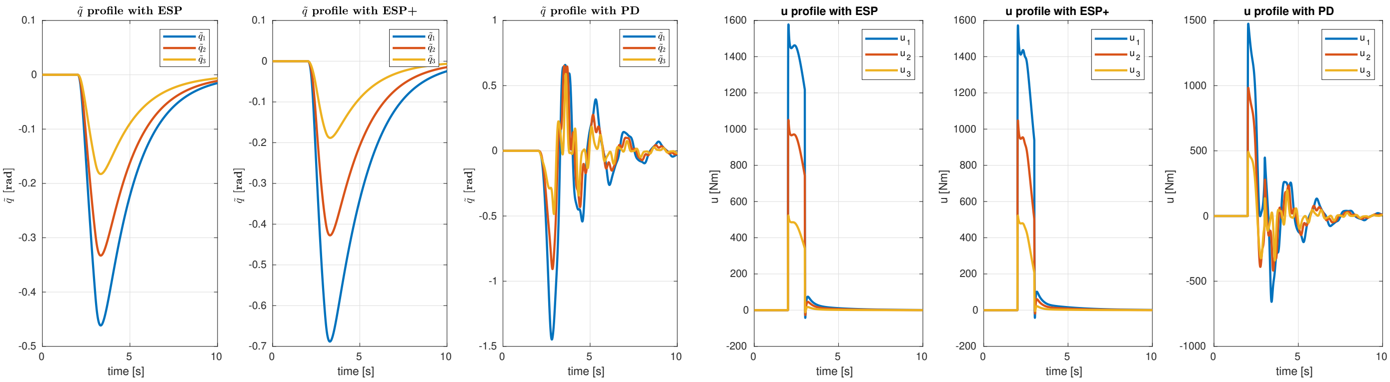


Fig. 4: Initial position in the unstable equilibrium



(a) $\tilde{q}$ *profile* (b) *control profile*

Fig. 5: Force acting on the CoM of the third link: stretched arm configuration

4.4 Linear elastic transmissions

In this section we propose a method for shaping the apparent inertia in the η variables by considering the case in which we have linear elastic transmission and we would like to impose a different elasticity. First of all let us start by redefining properly the change of coordinates related to the introduction of the new motor variable η. Let us consider *eq.(6)* using $\psi(\phi) = k_i \cdot \phi$, where ϕ is the angular displacement, obtaining that

$$k_1(\theta - q) = k_2(\eta - q) - D\dot{q} + g(q)$$

where k_1 is the original stiffness while k_2 is the desired one. From the above equation we can isolate θ that assumes the form

$$\theta = k_1^{-1}k_2\eta + k_1^{-1}(g - D\dot{q} - k_2 q) + k_1 q$$

By differentiating w.r.t time we obtain

$$\dot{\theta} = k_1^{-1}k_2\dot{\eta} + k_1^{-1}(\dot{g} - D\ddot{q} - k_2\dot{q}) + k_1\dot{q}$$

which can be rearranged as follows

$$\dot{\theta} = A\dot{\eta} + a$$

where $A = k_1^{-1}k_2$ is a constant matrix in this case. Differentiating again w.r.t. time we obtain

$$\ddot{\theta} = A\ddot{\eta} + \dot{a}$$

Thanks to the above relations we can now perform a coordinate transformation $[\theta, q] \rightarrow [\eta, \tilde{q}]$ for the robot dynamics *eq.(4)* that leads to the relation

$$BA\ddot{\eta} + B\dot{a} + k_1(\theta - q) = u \tag{16}$$

4.4.1 ESP

Since in the linear transmission case we have that the matrix A is constant, i.e. is not state dependent, its derivative is identically zero ($\dot{A} = 0$) and we can rewrite the expression of the ESP control considering the new desired elasticity k_2 in the variables η.

$$\begin{aligned}
\check{u}_{ESP} &= B\dot{a} \\
\hat{u}_{ESP} &= k_1(\theta - q) - BAB^{-1}k_2(\eta - \tilde{q}) \\
\bar{u}_{ESP} &= -BAB^{-1}(K_D\dot{\eta} + K_P\eta)
\end{aligned}$$

and using these equations for the control, the closed-loop dynamics assumes the form

$$B\ddot{\eta} + k_2(\eta - \tilde{q}) = -K_D\dot{\eta} - K_P\eta \tag{17}$$

4.4.2 ESP+

Also in this case the same reasoning done with nonlinear elastic transmissions has been applied, hence we pre-multiply the initial dynamics in *eq.(16)* by A^T obtaining

$$A^TBA\ddot{\eta} + A^TB\dot{a} + A^Tk_1(\theta - q) = A^Tu \tag{18}$$

and the equations of the control assume the following form

$$\begin{aligned}
\check{u}_{ESP+} &= B\dot{a} \\
\hat{u}_{ESP+} &= k_1(\theta - q) - A^Tk_2(\eta - \tilde{q}) \\
\bar{u}_{ESP+} &= -A^T(K_D\dot{\eta} + K_P\eta)
\end{aligned}$$

leading to a closed-loop dynamics

$$A^TBA\ddot{\eta} - k_2(\eta - \tilde{q}) = -K_D\dot{\eta} - K_P\eta \tag{19}$$

In this case we are able to act directly on the apparent motor inertia B_η through the matrix $A = k_1^{-1}k_2$ by choosing the desired stiffness to be applied to the motor-side k_2.

4.4.3 Simulations analysis

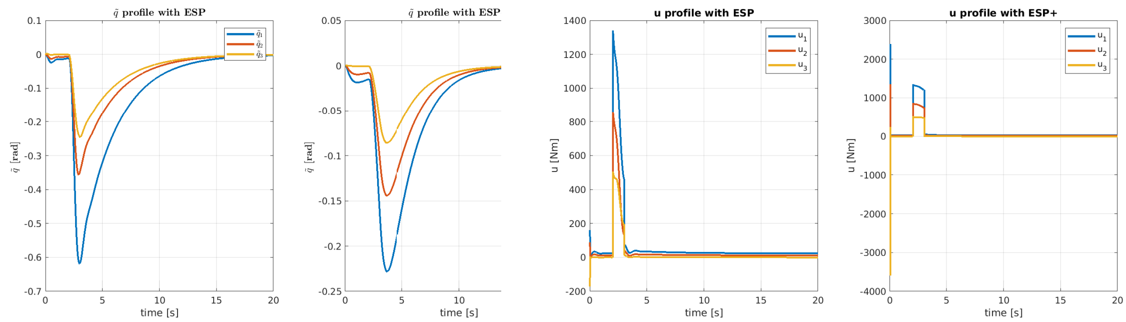
The simulations carried over in this section have been made considering that the arm is in the same generic configuration as in *Section 4.2*.
Let now consider the linear elastic transmissions as simple springs, hence their frequency of oscillation is described by the relation

$$\omega = Bk^{-1}$$

where B is the inertia of the spring and k is its stiffness.
Looking at *eq.(19)* could be notice that B is shaped through the matrix A, which appears in a quadratic form, in the new motor coordinates since $B_\eta = A^TBA$. Thus the desired stiffness chosen in the η variables affects the rigidity of the motors, i.e. their frequency of oscillation. Indeed the equation of the natural frequency in the η coordinates becomes

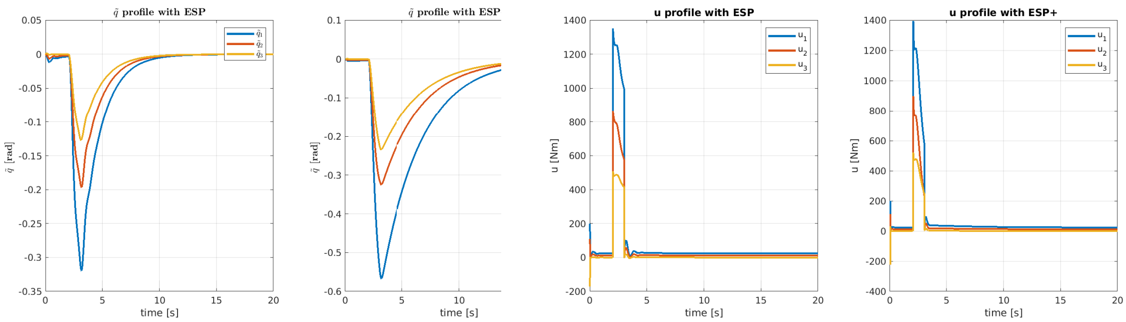
$$\omega_\eta = B_\eta k_2^{-1} = \frac{Bk_2^2}{k_1^2k_2} = \frac{Bk_2}{k_1^2} = \frac{B}{k_1}\frac{k_2}{k_1} = \omega\frac{k_2}{k_1} \tag{20}$$

where the direct relation between ω and the the desired stiffness k_2 is rendered explicit.
In conclusion we can state that by choosing a desired stiffness k_2 lower than the original one k_1, we are going to have a lower apparent inertia B_η and, even though we have a more elastic transmission, we obtain a more rigid robot since $\omega_\eta < \omega$. On the other hand the choice of k_2 bigger than k_1 results in the dual situation, i.e. a less rigid robot.



(a) $\tilde{q}$ *profile* (b) *control profile*

Fig. 6: k_2*=250*



(a) $\tilde{q}$ *profile* (b) *control profile*

Fig. 7: k_2*=1000*

Fig.(6) and *Fig.(7)* are experiments conducted by setting the desired stiffness as the double and the half of the original one, respectively, in order to see if the aforementioned reasoning is true in simulation. In *Fig.(6a)* we can notice that the choice of a lower stiffness k_2 has brought the robot to have an apparent inertia B_η which is lower but at the same time we have obtained more rigid transmissions. This is reflected in the position error, in fact $ESP+$ has a lower overshoot than ESP w.r.t the same external force applied to the robot. Conversely in *Fig.(7a)*, we have imposed a greater k_2 which reflects in an higher B_η and less rigid transmissions, in fact we have obtained the dual situation w.r.t *Fig.(6)* and this confirms the theoretical intuition that comes from *eq.(20)*.
Moreover we have that the control profiles in the 2 cases (*Fig.(6b)* and *Fig.(7b)*) do not have much differences.

5 Conclusion

Throughout this work we have analyzed the control scheme denoted as Elastic Structure Preserving (ESP) that has the goal of controlling a robotic system with nonlinear elastic elements in a free-gravity framework, assigning a desired damping at the link-side dynamics. This is achieved since the two systems are equivalent through feedback (as shown in [1] on a simple 1 D.O.F. case).

The case study is a 3R planar robot that lies on a vertical plane with nonlinear transmissions. After introducing analytically the model of the robot and the control strategies that have been applied, we have simulated the problem in two different configurations and we have compared the obtained results with the same system which implements a simple PD controller.

Moreover we have exploited the case of having a linear relation for the stiffness of the transmission in order to understand in an immediate way how the motor inertia could be shaped in the case of the $ESP+$ control scheme. This has been done by assigning a desired stiffness for the compliant element of the robot since the inertia is weighted using the matrix A that depends directly on it.

By considering the elastic transmissions as a linear spring we have been able to analyze the natural frequency of these components in the new motor coordinates, discovering that it depends on the ratio between the imposed stiffness constant and the original one. As a result, by increasing k_2 w.r.t k_1 we will obtain that the natural frequency grows linearly with it and this implies that the robot is less rigid while performing its motion, even though it has become less elastic. Conversely by decreasing it w.r.t k_1 the transmissions are more elastic, but this results in a more rigid motion.

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A Define A and a terms

Applying *eq.(6)* we can rewrite $\kappa(\theta - q)$ in new coordinates

$$\kappa(\theta - q) = \kappa(\psi^{-1}(\psi(\eta - \tilde{q}) + n(t, \tilde{q}, \dot{\tilde{q}}))) =: \chi(t, \eta, \tilde{q}, \dot{\tilde{q}}) \tag{21}$$

eq.(21) together with *eq.(9)*, allows us to define A as

$$A(\eta, \tilde{q}, \dot{\tilde{q}}) = \kappa^{-1}(\theta - q)\kappa(\eta - \tilde{q})$$

Its derivative takes the form

$$\dot{A}(\eta, \tilde{q}, \dot{\tilde{q}}) = -\kappa^{-1}(\theta - q)\dot{\kappa}(\theta - q)\kappa^{-1}(\theta - q)\kappa(\eta - \tilde{q}) + \kappa^{-1}(\theta - q)\dot{\kappa}(\eta - \tilde{q})(\dot{\eta} - \dot{\tilde{q}})$$

The equations *eq.(6)*, *eq.(10)* and *eq.(21)* yield a as follow

$$a(\eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = \dot{\tilde{q}} + \dot{q}_d + \kappa^{-1}(\theta - q)\gamma(\eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}})$$

Its derivative takes the form

$$\dot{a}(\eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = \ddot{\tilde{q}} + \ddot{q}_d - \kappa^{-1}(\theta - q)\dot{\kappa}(\theta - q)\kappa^{-1}(\theta - q)\gamma(\eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) + \kappa^{-1}(\theta - q)\dot{\gamma}(\eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}})$$

where

$$\dot{\gamma}(\eta, \tilde{q}, \dot{\tilde{q}}, \ddot{\tilde{q}}) = -\dot{\kappa}(\eta - \tilde{q})\dot{\tilde{q}} - \kappa(\eta - \tilde{q})\ddot{\tilde{q}} + \ddot{n}(\tilde{q}, \dot{\tilde{q}})$$

B Parameters for the inertia matrix M(q)

The dynamical parameters used in the parameterized expression of the inertia matrix *eq.(1)* are:

$$\begin{aligned}
a_1 &= m_3 d_3^2 + I_3 \\
a_2 &= \ell_2 m_3 d_3 \\
a_3 &= d_3 \ell_1 m_3 \\
a_4 &= I_2 + I_3 + d_2^2 m_2 + d_3^2 m_3 + \ell_2^2 m_3 + \ell_2^2 m_{3m} \\
a_5 &= m_2 \ell_1 d_2 + m_{3m} \ell_1 \ell_2 + \ell_1 \ell_2 m_3 \\
a_6 &= I_1 + I_2 + I_3 + d_1^2 m_1 + d_2^2 m_2 + d_3^2 m_3 + \ell_1^2 m_2 + \ell_1^2 m_3 + \ell_2^2 m_3 + \ell_1^2 m_{2m} + \ell_1^2 m_{3m} + \ell_2^2 m_{3m}
\end{aligned}$$

C Coriolis and Centrifugal term

Taking into account the parameterized inertia matrix the factorized expression of the coriolis and centrifugal term is:

$$S(q, \dot{q}) = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$

where

$$
\begin{aligned}
S_{11} &= -\dot{q}_3(a_3 s_{13} + a_2 s_3) - \dot{q}_2(a_3 s_{13} + a_5 s_2)\\
S_{12} &= -\dot{q}_1(a_3 s_{13} + a_5 s_2) - \dot{q}_3(a_3 s_{13} + a_2 s_3) - \dot{q}_2(a_3 s_{13} + a_5 s_2)\\
S_{12} &= -\dot{q}_1(a_3 s_{13} + a_2 s_3) - \dot{q}_2(a_3 s_{13} + a_2 s_3) - \dot{q}_3(a_3 s_{13} + a_2 s_3)\\
S_{21} &= \dot{q}_1(a_3 s_{13} + a_5 s_2) - a_2 \dot{q}_3 s_3\\
S_{22} &= -a_2 \dot{q}_3 s_3\\
S_{23} &= -a_2 \dot{q}_1 s_3 - a_2 \dot{q}_2 s_3 - a_2 \dot{q}_3 s_3\\
S_{31} &= \dot{q}_1(a_3 s_{13} + a_2 s_3) + a_2 \dot{q}_2 s_3\\
S_{32} &= a_2 \dot{q}_1 s_3 + a_2 \dot{q}_2 s_3\\
S_{33} &= 0
\end{aligned}
$$

D Gravity term

The gravity term is obtained accordingly to *eq.(3)* where the potential energy $U_g(q)$ is the sum of all the contributions given by the links, $U_i = -m_i[0 - g_0]p_{ci}$ with $i = 1, ..., N$, and its final expression is

$$
g(q) = \begin{bmatrix} g_0 m_2(d_2 c_{12} + \ell_1 c_1) + g_0 m_3(\ell_2 c_{12} + d_3 c_{123} + \ell_1 c_1) + d_1 g_0 m_1 c_1 \\ g_0 m_3(\ell_2 c_{12} + d_3 c_{123}) + d_2 g_0 m_2 c_{12} \\ d_3 g_0 m_3 c_{123} \end{bmatrix}
$$