# MDI210 : Numerical Analysis and Continuous Optimization

**Robert M. Gower**

TELECOM ParisTech

# Who am I?

**Robert M. Gower**

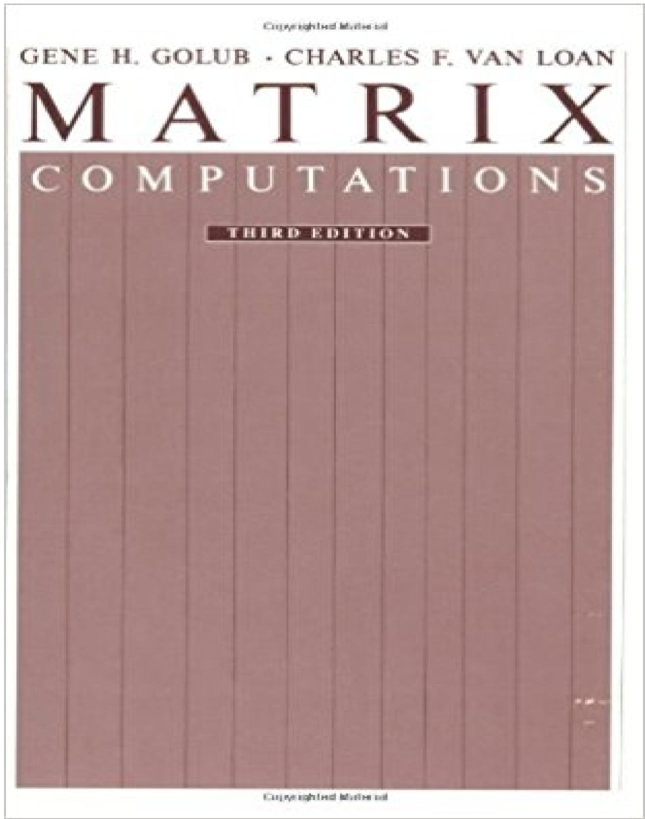
- **Assistant prof at Telecom**
- **robert.gower@telecom-paristech.fr**
- **www.ens.fr/~rgower**
- **Research topics: Stochastic algorithms for optimization, numerical linear algebra, quasi-Newton methods and automatic differentiation (backpropagation).**

# Core Info

- **Where** : Telecom ParisTech
- **Location** : C 48
- **Volume** : 28h
- **When** : 8 weeks
- **Exam: One exam** on 31st of October
- **Online:** Find lecture notes on my homepage

  http://www.di.ens.fr/~rgower/teaching.html
- **Exercices:** Do all exercises in the MDI210 lecture notes

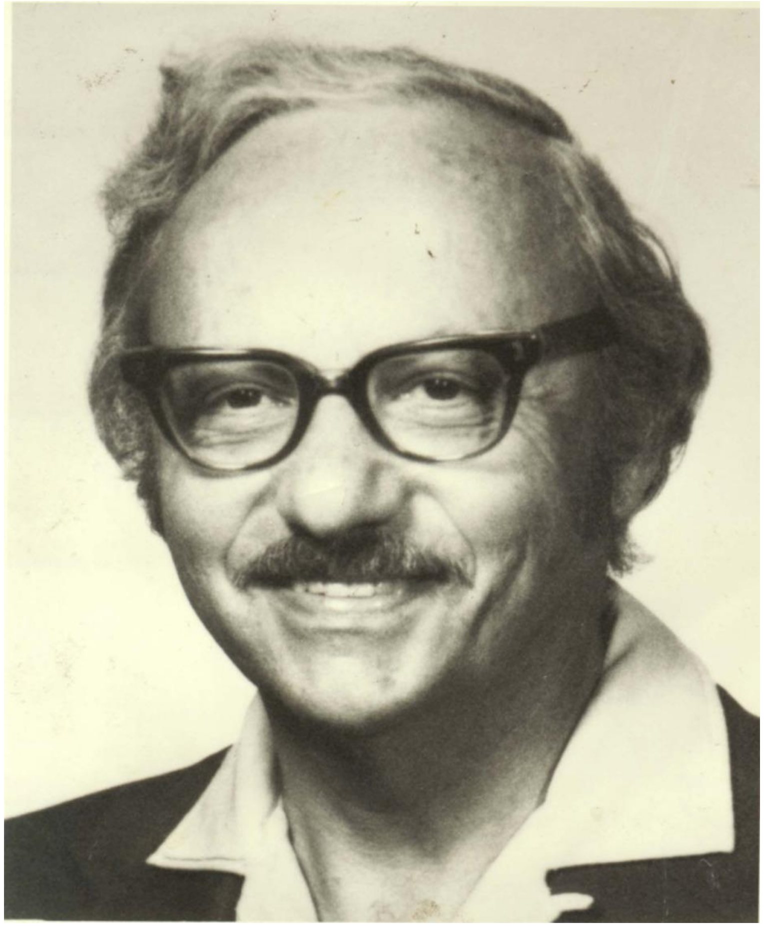
# Additional References for Numerical Analysis




**Matrix Computations:**
Gene H. Golub and
Charles F. Van Loan

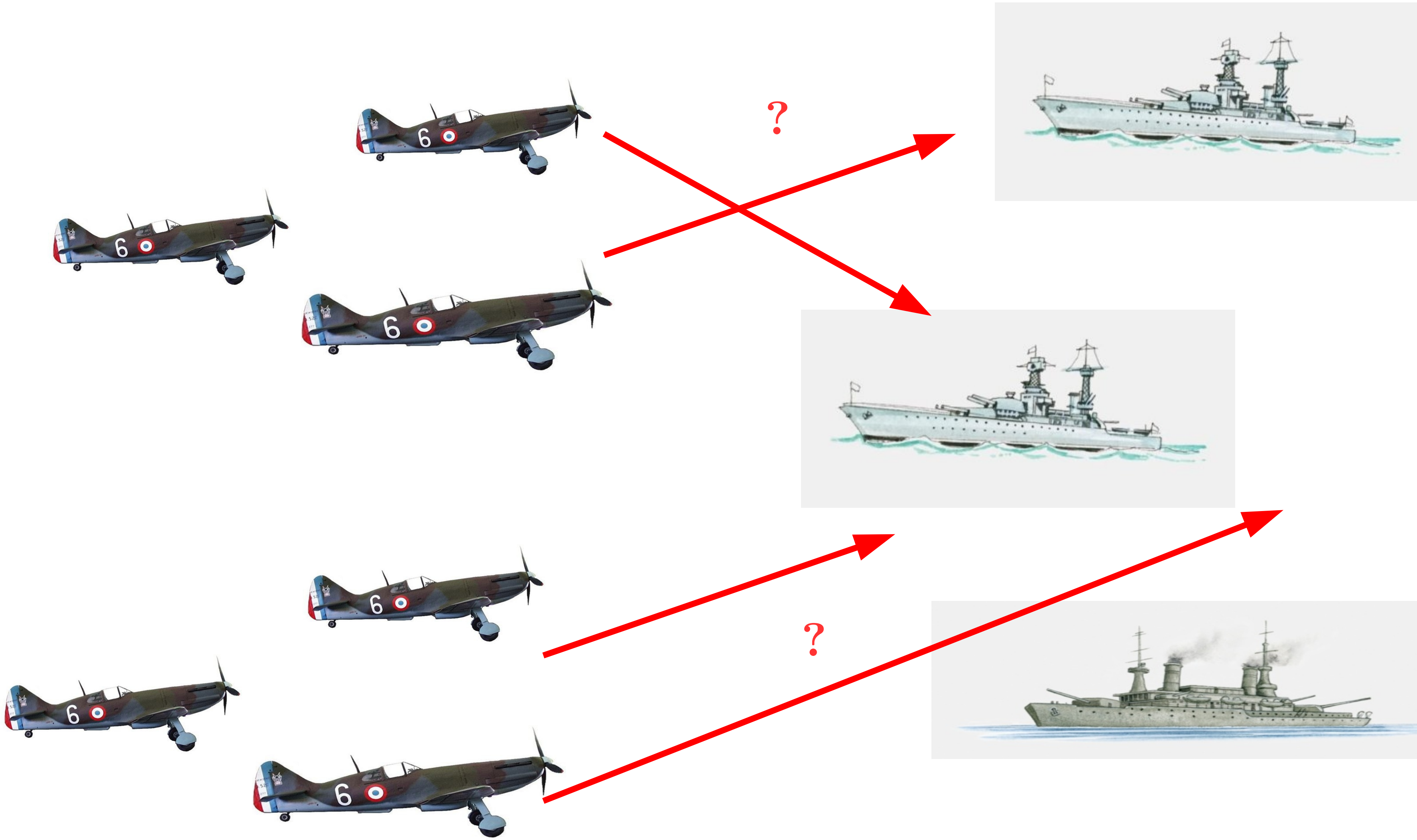
**Three copies in the library on the 8$^{th}$ floor!**

# Linear Programming History (1939)

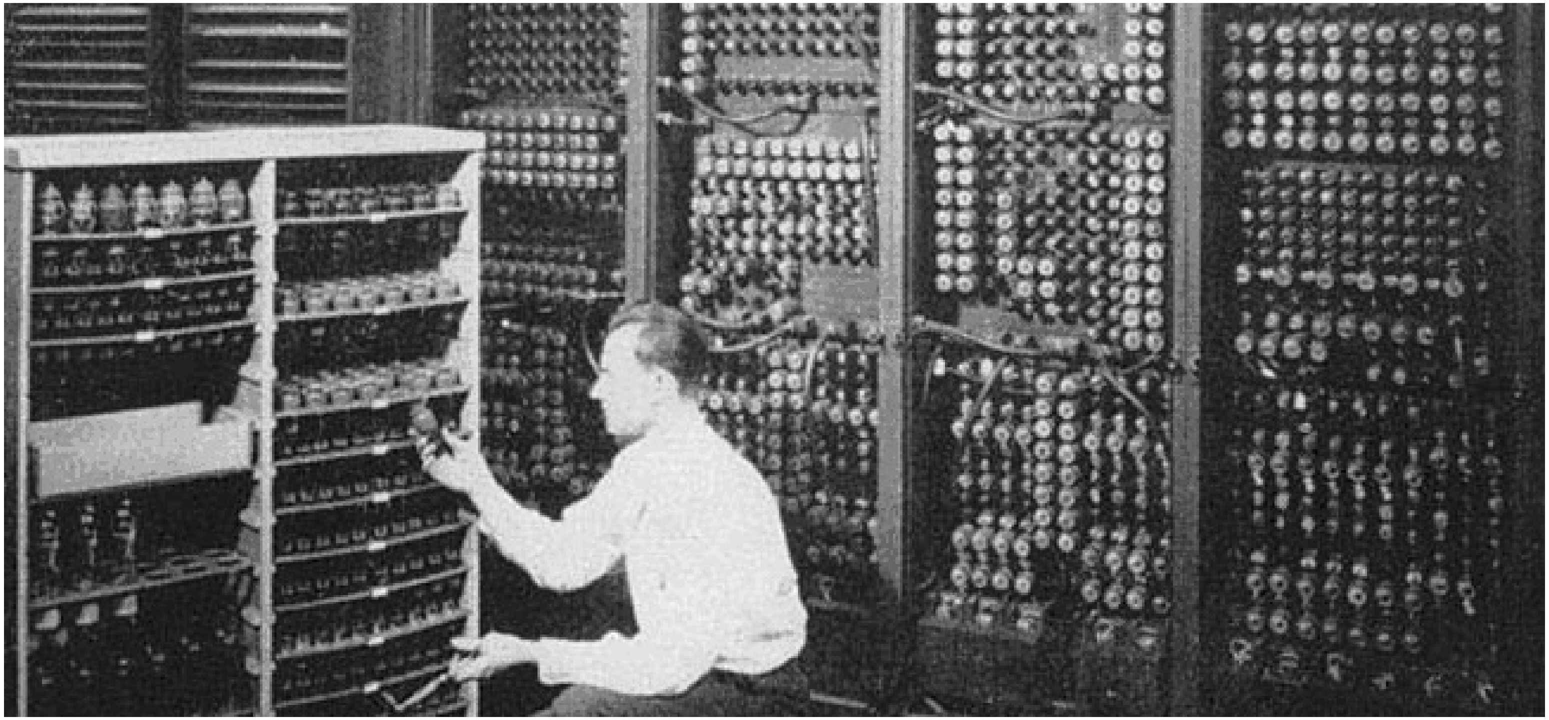



- 1947: George Dantzig, advising U.S. Air Force, invents Simplex.
- Assignment 70 people to 70 jobs (more possibilities than particles).

# Linear Programming History (1941)



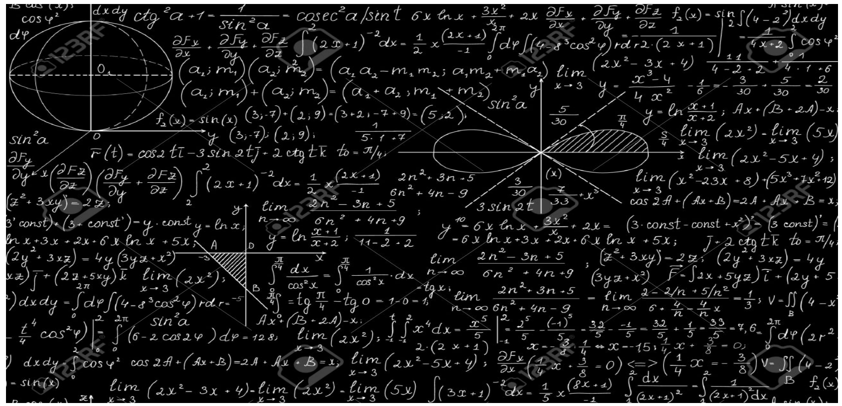
# Army Builds Killing Machine (1949)



1949 SCOOP: Scientific Programming Of Optimal Programs

Mathematical Programming: Math used to figured out Flight and logistic programs/schedules

# Dantzig the Urban Legend



Dantzig, George B. "On the Non-Existence of Tests of 'Student's' Hypothesis Having Power Functions Independent of Sigma." Annals of Mathematical Statistics. No. 11; 1940 (pp. 186-192).

Dantzig, George B. and Abraham Wald. "On the Fundamental Lemma of Neyman and Pearson." Annals of Mathematical Statistics. No. 22; 1951 (pp. 87-93).

# Optimization and Numerical Analysis: Linear Programming

Robert Gower

September 26, 2017

# Table of Contents

## The Problem: Linear Programming

$$\begin{aligned} \max_x z &\stackrel{\text{def}}{=} c^\top x \\ \text{subject to } Ax &\leq b, \\ x &\geq 0, \end{aligned}$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Equivalently

$$\begin{aligned} \max_x z &\stackrel{\text{def}}{=} \sum_{j=1}^n c_j x_j \\ \text{subject to } \sum_{j=1}^n a_{ij} x_j &\leq b_i, \qquad \text{for } i = 1, \ldots, m. \\ x &\geq 0. \end{aligned}$$

**Theorem (Fundamental Theorem of Linear Programming)**

*Let* $P = \{x \mid Ax = b, x \geq 0\}$ *then either*

1. $P = \{\emptyset\}$
2. $P \neq \{\emptyset\}$ *and there exists a vertex* $v$ *of* $P$ *such that* $v \in \arg\min_{x \in P} c^\top x$
3. *There exists* $x, d \in \mathbb{R}^n$ *such that* $x + td \in P$ *for all* $t \geq 0$ *and* $\lim_{t \to \infty} c^\top (x + td) = \infty$.

# First example Simplex

The problem

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 & \\ & 3x_1 + 2x_2 & \leq 600 \\ & 4x_1 + 1x_2 & \leq 400 \\ & x_1 \geq 0, x_2 \geq 0. & \end{aligned}$$

Can be transformed into

$$\begin{array}{rcccc} \max & 4x_1 & + & 2x_2 \\ x_3 = 600 & - & 3x_1 & - & 2x_2 \\ x_4 = 400 & - & 4x_1 & - & x_2, \end{array}$$

where $x_3$ and $x_4$ are *slack variables*. This is known as the the *dictionary* format and is often written as:

$$\begin{array}{ccccccc} x_3 & = & 600 & - & 3x_1 & - & 2x_2 \\ x_4 & = & 400 & - & 4x_1 & - & x_2 \\ \hline z & = & & & 4x_1 & + & 2x_2 \end{array}$$

# First example Simplex

The *dictionary* format

$$
\begin{array}{ccccccc}
x_3 & = & 600 & - & 3x_1 & - & 2x_2 \\
x_4 & = & 400 & - & 4x_1 & - & x_2 \\
\hline
z & = & & & 4x_1 & + & 2x_2
\end{array}
$$

admits obvious solution

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 600, 400).$$

The objective $z$ will improve if $x_1 > 0$. Increasing $x_1$ as much as possible

$$
\begin{array}{ccccc}
x_3 \geq 0 & \Rightarrow & 600 - 3x_1 \geq 0 & \Rightarrow & x_1 \leq 200, \\
x_4 \geq 0 & \Rightarrow & 400 - 4x_1 \geq 0 & \Rightarrow & x_1 \leq 100.
\end{array}
$$

Thus $x_1 \leq 100$ to guarantee $x_4 \geq 0$. This means $x_4$ will leave the basis and $x_1$ will enter the basis. Using row operations to isolate $x_1$ on row$_2$.

$$
\begin{array}{ccccccc}
x_3 & = & 300 & & 0 & - & \frac{5}{4}x_2 \\
x_1 & = & 100 & - & \frac{x_4}{4} & - & \frac{x_2}{4} \\
\hline
z & = & 400 & - & x_4 & + & x_2
\end{array}
$$

# First example Simplex

From

$$\begin{array}{ccccccc} x_3 & = & 300 & & 0 & - & \frac{5}{4}x_2 \\ x_1 & = & 100 & - & \frac{x_4}{4} & - & \frac{x_2}{4} \\ \hline z & = & 400 & - & x_4 & + & x_2 \end{array}$$

Now we are at the vertex $(x_1^*, x_2^*) = (100, 0)$. Next we see that increasing $x_2$ increases the objective value but

$$\begin{aligned} x_3 \geq 0 \quad &\Rightarrow \quad 240 \geq x_2, \\ x_1 \geq 0 \quad &\Rightarrow \quad 400 \geq x_4. \end{aligned}$$

Increase $x_2$ upto 240 while respecting the positivity constraints of $x_3$. Thus $x_3$ will leave the basis and $x_2$ will enter the basis. Performing a row elimination again, we have that

$$\begin{array}{ccccccc} x_2 & = & 240 & & 0 & - & \frac{4}{5}x_3 \\ x_1 & = & 40 & - & \frac{x_4}{4} & - & \frac{1}{5}x_3 \\ \hline z & = & 640 & - & x_4 & - & \frac{4}{5}x_3 \end{array}$$

Now $(x_1^*, x_2^*) = (40, 240)$. Increasing $x_4$ or $x_3$ will decrease $z$.

# Problem Notation

We will now formalize the definitions we introduced in the examples.

- There are $n$ variables and $m$ constraints
- The linear objective function $z = \sum_{j=1}^{n} c_j x_j$
- The $m$ inequality constraints in standard form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \text{ for } i \in \{1, \ldots, m\}.$$

- The $n$ positivity constraints $x_j \geq 0$, for $j \in \{1, \ldots, n\}$.
- $x_i^*$ denotes the value of $i$th variable.
- We call $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ a feasible solution if it satisfies the inequality and positivity constraints.

# Dictionary Notation

- The slack variables $(x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^m$ (*variables d'écart*)
- The initial dictionary

$$\begin{array}{lcl} x_{n+1} & = & b_1 - \sum_{j=1}^{n} a_{1j} x_j \\ \vdots & & \\ x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j \\ \vdots & & \\ x_{n+m} & = & b_m - \sum_{j=1}^{n} a_{mj} x_j \\ \hline z & = & \sum_{j=1}^{n} c_j x_j, \end{array}$$

- Valid dictionary if $m$ of the variables $(x_1, \ldots, x_{n+m})$ can be expressed as function of the remaining $n$ variables.
- The $m$ variables on the left-hand side are the basic variable (*variable de base*). The $n$ variables on the right-hand side are the non-basic (*variable hors-base*).

# Dictionary Notation

After row elimination operations we have a new basis.

- Basic variable set $I \subset \{1, \ldots, n+m\}$ and non-basic set $J = \{1, \ldots, n+m\} \setminus I$
- Current objective value $z^* = \sum_{j=1}^{n} c_j x_j^*$.
- For each basis set $I$ there is a corresponding dictionary

$$\begin{array}{rcl} x_i & = & b'_i - \sum_{j \in J} a'_{ij} x_j, \ \text{for } i \in I \\ \hline z & = & z^* + \sum_{j \in J} c'_j x_j, \end{array}$$

  where $a'_{ij}, b'_i, z^* \in \mathbb{R}$ are coefficients resulting from the row operations. For this to a feasible dictionary we require that $b'_i \geq 0$.
- A basic solution: $x_i^* = b'_i$ for $i \in I$ and $x_j^* = 0$ for $j \in J$.

# A Step of the Simplex Method

**Input:** A basic index set $I \subset \{1, \ldots, n+m\}$, $J = I \setminus \{1, \ldots, n+m\}$, constraint coefficients $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$ and $c'_i \in \mathbb{R}$.

---

**if** $c_i \leq 0$ for all $i \in J$ **then**

  **STOP**;  # Optimal point found.

Choose a variable $j_0$ to **enter the basis** from the set $j_0 \in \{j \in J \,:\, c'_j > 0\}$.

**if** $a'_{ij} \geq 0$ for all $i \in J$ **then**

  **STOP**;  # The problem is unbounded.

Choose a variable $i_0$ to **leave the basis** from the set $i_0 \in \arg \min\limits_{i \in I, a'_{ij_0} > 0} \left\{ \frac{b'_i}{a'_{ij_0}} \right\}$.

**for** $i \in I$ **do**

  $a'_{i:} \leftarrow a'_{i:} - \frac{a'_{ij_0}}{a'_{i_0 j_0}} a'_{i_0:}$  # Row elimination on pivot $(i_0, j_0)$.

$c' \leftarrow c' - \frac{c'_{j_0}}{a'_{i_0 j_0}} a'_{i_0:}$.  # Update the cost coefficients.

$I \leftarrow (I \setminus \{i_0\}) \cup \{j_0\}$.  # Update basis

---

**Output:** $I, a'_{ij}, b'_j, c'_j$.

# How to choose who enters the basis?

1. The mad hatter rule: Choose the first one you see.
2. Dantzig's 1st rule: $j_0 = \arg\max_{j \in J} c_j$.
3. Dantzig's 2nd rule: Choose $j_0 \in \{j \in J : c_j > 0\}$ that so that maximizes increase in $z$.

$$j_0 = \arg\max_{j \in J} \left\{ c_j \min_{i \in I, a_{ij} > 0} \left\{ \frac{b_i}{a_{ij}} \right\} \right\}.$$

   Effective, but computationally expensive.

4. Bland's rule: Choose the smallest indices $j_0$ and $i_0$. That is, choose

$$j_0 = \arg\min\{j \in J : c_j > 0\}.$$

$$i_0 = \min \left\{ \arg \min_{i \in I, a_{ij_0} > 0} \left\{ \frac{b_i}{a_{ij_0}} \right\} \right\}.$$

# Degeneracy

If any of the basic variables are zero, then we say that the solution is degenerate. Example on Board and in other lecture slides.

# Upper Bounds Using Duality

The LP in standard form

$$\max_x z \stackrel{\text{def}}{=} c^\top x$$
$$\text{subject to } Ax \le b,$$
$$x \ge 0, \tag{LP}$$

We want to find $w \in \mathbb{R}$ so that $z = c^\top x \le w$ for all $x \in \mathbb{R}^n$.
Combine rows of constraints?
Look for $y \ge 0 \in \mathbb{R}^m$ so that $y^\top A \approx c^\top$, consequently

$$c^\top x \approx (y^\top A)x \le y^\top b = w.$$

Precisely, let $y \ge 0 \in \mathbb{R}^m$ be such that $y^\top A \ge c^\top$ or equivalently $A^\top y \ge c$. Then

$$c^\top x \le (y^\top A)x \le y^\top b.$$

Can we make this upper bound as tight as possible? Yes, by minimizing $y^\top b$. That is, we need to the *dual* linear program.

# Dual definition

The LP in standard form

$$\begin{aligned} \max_x\ z &\overset{\text{def}}{=} c^\top x \\ \text{subject to } Ax &\le b, \\ x &\ge 0, \end{aligned} \tag{LP}$$

The dual LP:

$$\begin{aligned} \max_x\ w &\overset{\text{def}}{=} y^\top b \\ \text{subject to } A^\top y &\ge c, \\ y &\ge 0. \end{aligned} \tag{DP}$$

## Lemma (Weak Duality)

*If $x \in \mathbb{R}^n$ is a feasible point for* (LP) *and $y \in \mathbb{R}^m$ is a feasible point for* (DP) *then*

$$c^\top x \le y^\top A x \le y^\top b. \tag{1}$$

# Weak Duality

## Lemma (Weak Duality)

*If $x \in \mathbb{R}^n$ is a feasible point for* (LP) *and $y \in \mathbb{R}^m$ is a feasible point for* (DP) *then*

$$c^\top x \leq y^\top Ax \leq y^\top b. \tag{2}$$

*Consequently*

- *If* (LP) *has an unbounded solution, that is $c^\top x \to \infty$, then there exists no feasible point $y$ for* (DP)
- *If* (DP) *has an unbounded solution, that is $y^\top b \to -\infty$, then there exists no feasible point $x$ for* (LP)
- *If $x$ and $y$ are primal and dual feasible, respectively, and $c^\top x = y^\top b$, then $x$ and $y$ are the primal and dual optimal points, respectively.*

# Strong Duality

## Theorem (Strong Duality)

*If* (LP) *or* (DP) *is feasible, then* $z^* = w^*$. *Moreover, if* $c^*$ *is the cost vector of the optimal dictionary of the primal problem* (LP), *that is, if*

$$z = z^* + \sum_{i=1}^{n+m} c_i^* x_i, \tag{3}$$

*then* $y_i^* = -c_{n+i}^*$ *for* $i = 1, \ldots, m$.

First $c_i^* \leq 0$ for $i = 1, \ldots, m+n$ because dictionary is optimal.
Consequently $y_i^* = -c_{n+i}^* \geq 0$ for $i = 1, \ldots, m$.

# Strong duality: Proof I

By the definition of the slack variables we have that

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad \text{for } i = 1, \ldots, m. \tag{4}$$

Consequently, setting $y_i^* = -c_{n+i}^*$, we have that

$$\begin{aligned}
z &\overset{(3)}{=} z^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=n+1}^{n+m} c_i^* x_i \\
&\overset{(4)}{=} z^* + \sum_{j=1}^{n} c_j^* x_j - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j) \\
&= z^* - \sum_{i=1}^{m} y_i^* b_i + \sum_{j=1}^{n} \left( c_j^* + \sum_{i=1}^{m} y_i^* a_{ij} \right) x_j \\
&= \sum_{j=1}^{n} c_j x_j, \qquad \forall x_1, \ldots, x_n. \qquad (5)
\end{aligned}$$

Last line followed by definition $z = \sum_{j=1}^{n} c_j x_j$. Since the above holds for all $x \in \mathbb{R}^n$, we can match the coefficients.

# Strong duality: Proof II

Matching coefficients on $x_j$'s we have

$$\begin{aligned} z^* &= \sum_{i=1}^{m} y_i^* b_i && (6) \\ c_j &= c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}, \qquad \text{for } j = 1, \ldots, n. && (7) \end{aligned}$$

Since $c_j^* \leq 0$ for $j = 1, \ldots, n$, the above is equivalent to

$$\begin{aligned} z^* &= \sum_{i=1}^{m} y_i^* b_i && (8) \\ \sum_{i=1}^{m} y_i^* a_{ij} &\leq c_j, \qquad \text{for } j = 1, \ldots, n. && (9) \end{aligned}$$

The inequalities (9) prove that $y_i^*$'s satisfies the constraints in (DP), and thus is feasible. The equality (8) shows that $z^* = \sum_{i=1}^{m} y_i^* b_i = w$, a consequently by week duality the $y_i^*$'s are dual optimal. $\square$

G.,R & P Richtárik, Randomized Iterative Methods for Linear Systems arXiv:1506.03296