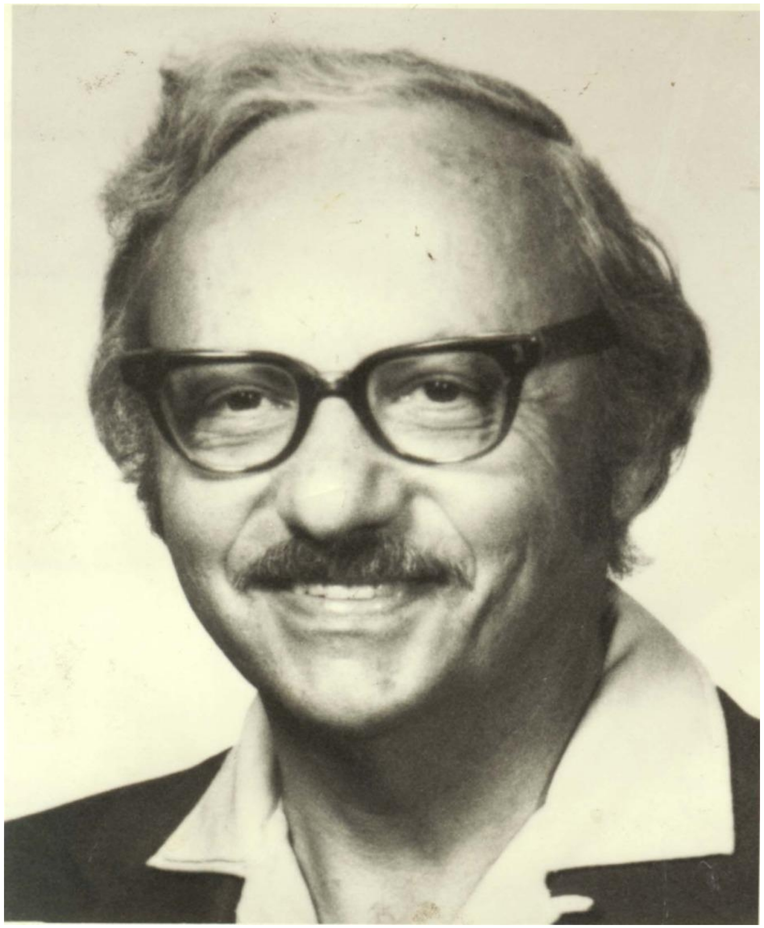
MDI210 : Linear Programming

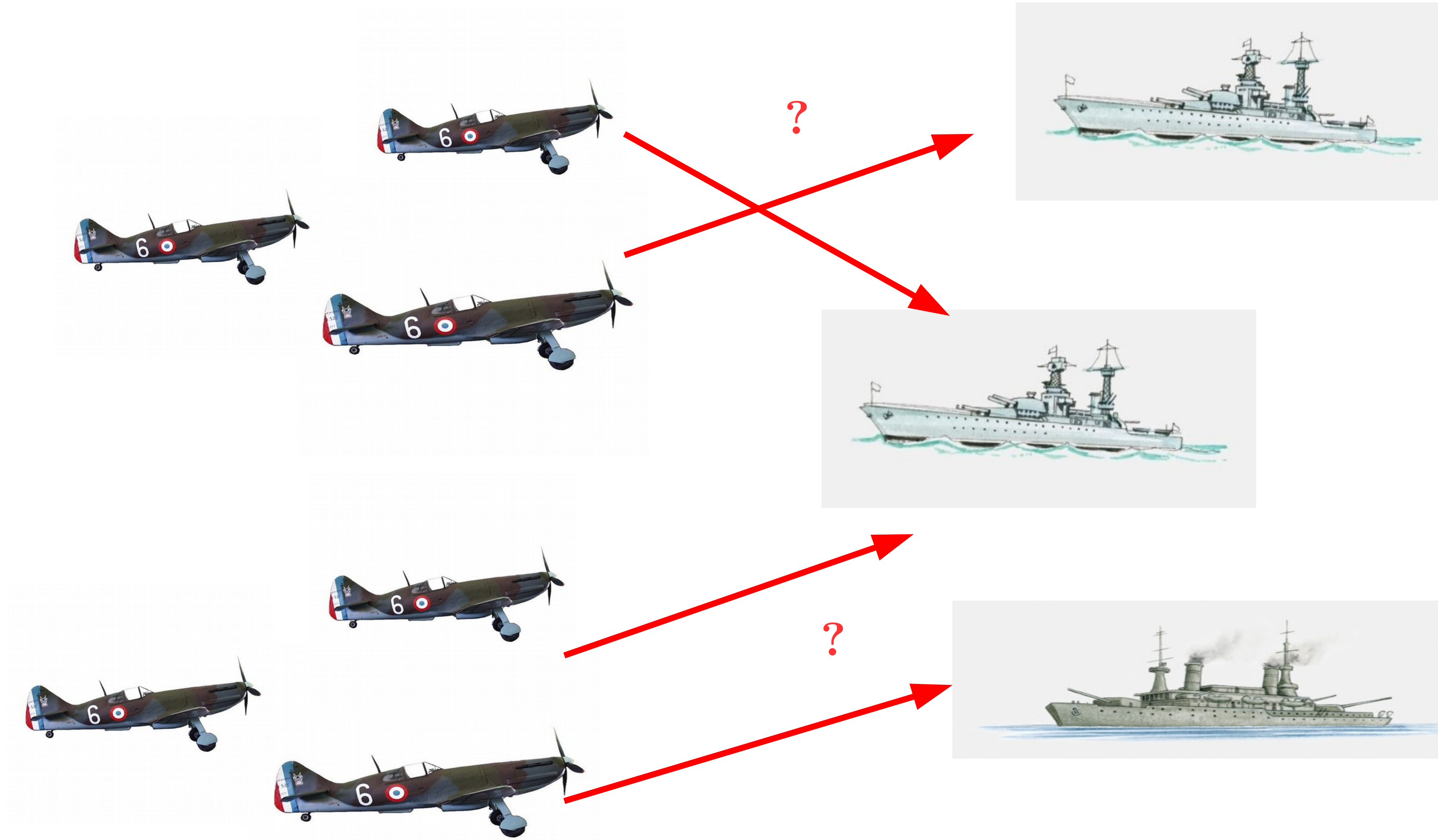
Robert M. Gower

Linear Programming History (1939)

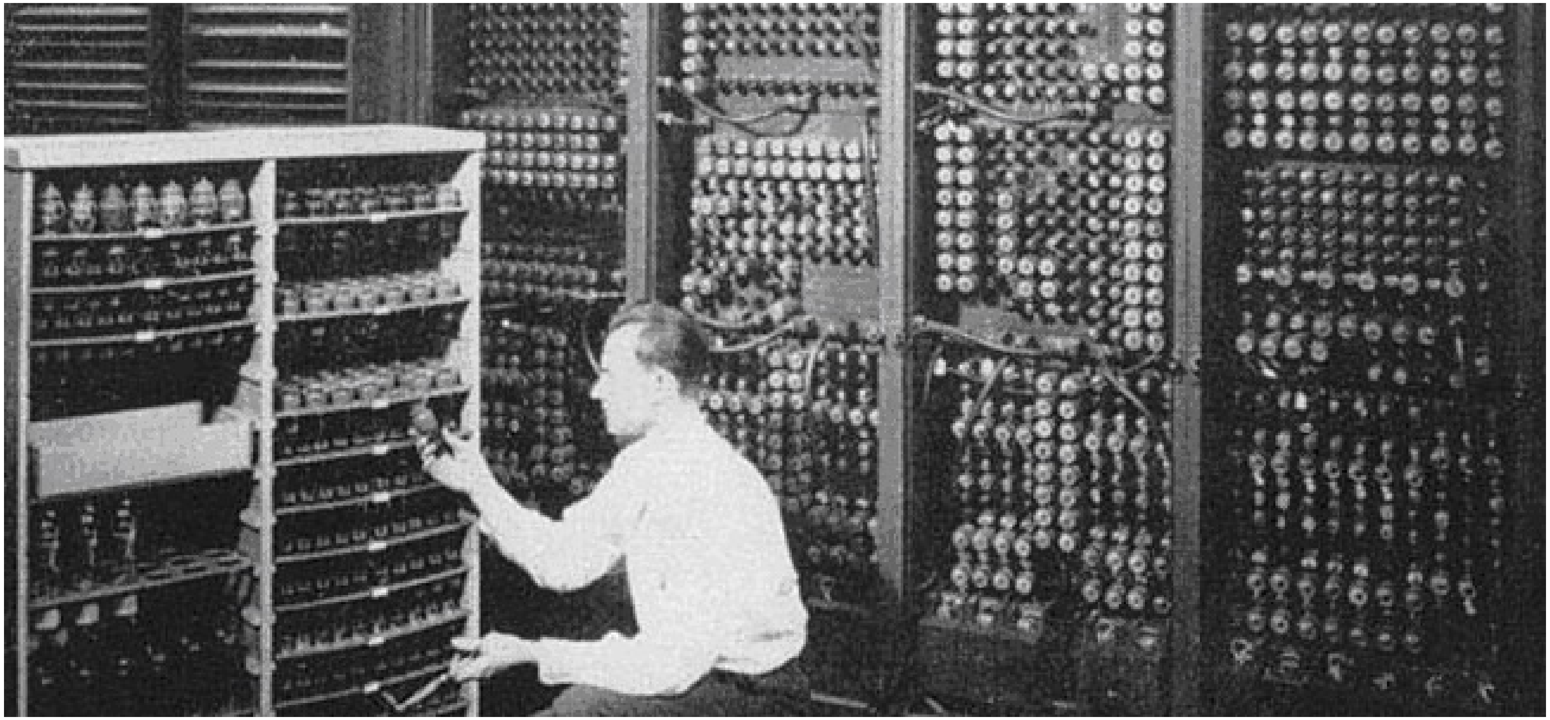


- 1947: George Dantzig, advising U.S. Air Force, invents Simplex.
- Assignment 70 people to 70 jobs (more possibilities than particles).

Linear Programming History (1941)



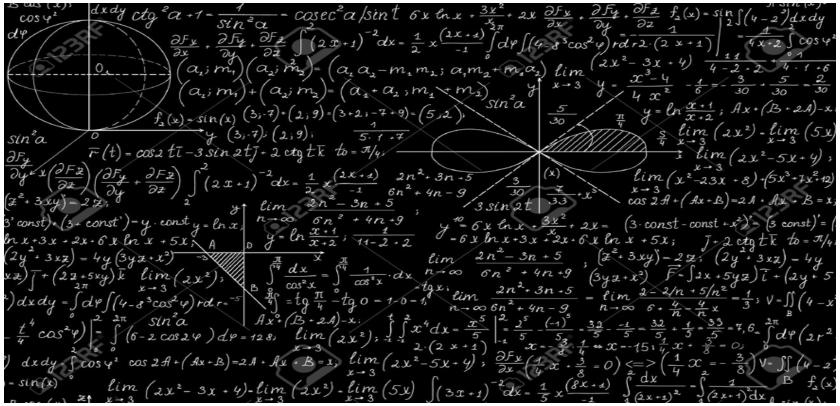
Army Builds Killing Machine (1949)



1949 SCOOP: Scientific Computation Of Optimal Programs

Mathematical Programming: Math used to figured out Flight and logistic programs/schedules

Dantzig the Urban Legend



Dantzig, George B. "On the Non-Existence of Tests of 'Student's' Hypothesis Having Power Functions Independent of Sigma." Annals of Mathematical Statistics. No. 11; 1940 (pp. 186-192).

Dantzig, George B. and Abraham Wald. "On the Fundamental Lemma of Neyman and Pearson." Annals of Mathematical Statistics. No. 22; 1951 (pp. 87-93).

Optimization and Numerical Analysis: Linear Programming

Robert Gower

September 17, 2019

Table of Contents

The Problem: Linear Programming

$$\begin{aligned} \max_{x} \, z &\overset{\text{def}}{=} c^\top x \\ \text{subject to } Ax &\leq b, \\ x &\geq 0, \end{aligned}$$

where $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, and $b \in \mathbb{R}^m$. Equivalently

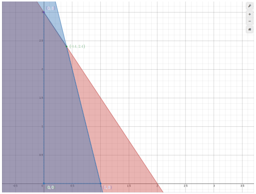
$$\begin{aligned} \max_{x} \, z &\overset{\text{def}}{=} \sum_{j=1}^{n} c_j x_j \\ \text{subject to } \sum_{j=1}^{n} a_{ij} x_j &\leq b_i, \qquad \text{for } i = 1, \ldots, m. \\ x &\geq 0. \end{aligned}$$

First example Simplex

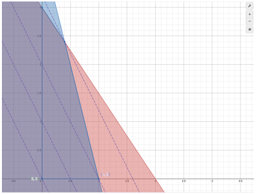
The problem

$$
\begin{aligned}
\max \quad & 4x_1 + 2x_2 & \\
& 3x_1 + 2x_2 & \leq 600 \\
& 4x_1 + 1x_2 & \leq 400 \\
& x_1 \geq 0, x_2 \geq 0. &
\end{aligned}
$$

We can solve this graphically:



With level sets $\Rightarrow$



How to do this systematically?

First example Simplex

The problem

$$\begin{aligned} \max \quad & 4x_1 + 2x_2 & \\ & 3x_1 + 2x_2 & \leq 600 \\ & 4x_1 + 1x_2 & \leq 400 \\ & x_1 \geq 0, x_2 \geq 0. & \end{aligned}$$

Can be transformed into

$$\begin{array}{rcccc} & \max & 4x_1 & + & 2x_2 \\ x_3 = 600 & - & 3x_1 & - & 2x_2 \\ x_4 = 400 & - & 4x_1 & - & x_2, \end{array}$$

where x_3 and x_4 are *slack variables*. This is known as the the *dictionary* format and is often written as:

$$\begin{array}{ccccccc} x_3 & = & 600 & - & 3x_1 & - & 2x_2 \\ x_4 & = & 400 & - & 4x_1 & - & x_2 \\ \hline z & = & & & 4x_1 & + & 2x_2 \end{array}$$

First example Simplex

The *dictionary* format

$$\begin{array}{rcrcrcr} x_3 & = & 600 & - & 3x_1 & - & 2x_2 \\ x_4 & = & 400 & - & 4x_1 & - & x_2 \\ \hline z & = & & & 4x_1 & + & 2x_2 \end{array}$$

admits obvious solution

$$(x_1^*, x_2^*, x_3^*, x_4^*) = (0, 0, 600, 400).$$

The objective z will improve if $x_1 > 0$. Increasing x_1 as much as possible

$$\begin{array}{lcl} x_3 \geq 0 & \Rightarrow & 600 - 3x_1 \geq 0 & \Rightarrow & x_1 \leq 200, \\ x_4 \geq 0 & \Rightarrow & 400 - 4x_1 \geq 0 & \Rightarrow & x_1 \leq 100. \end{array}$$

Thus $x_1 \leq 100$ to guarantee $x_4 \geq 0$. This means x_4 will leave the basis and x_1 will enter the basis. Using row operations $r_3 \leftarrow r_3 + r_2$ and $r_1 \leftarrow r_1 - \frac{3}{4} r_2$ to isolate x_1 on row$_2$.

$$\begin{array}{rcrcrcr} x_3 & = & 300 & + & \frac{3}{4}x_4 & - & \frac{5}{4}x_2 \\ x_1 & = & 100 & - & \frac{x_4}{4} & - & \frac{x_2}{4} \\ \hline z & = & 400 & - & x_4 & + & x_2 \end{array}$$

First example Simplex

From

$$\begin{array}{ccccccc} x_3 & = & 300 & + & \frac{3}{4}x_4 & - & \frac{5}{4}x_2 \\ x_1 & = & 100 & - & \frac{x_4}{4} & - & \frac{x_2}{4} \\ \hline z & = & 400 & - & x_4 & + & x_2 \end{array}$$

Now we are at the vertex $(x_1^*, x_2^*) = (100, 0)$. Next we see that increasing x_2 increases the objective value but

$$\begin{array}{ccccc} x_3 \geq 0 & \Rightarrow & 300 - \frac{5}{4}x_2 \geq 0 & \Rightarrow & 240 \geq x_2, \\ x_1 \geq 0 & \Rightarrow & 100 - \frac{x_2}{4} \geq 0 & \Rightarrow & 400 \geq x_2. \end{array}$$

Increase x_2 upto 240 while respecting the positivity constraints of x_3. Thus x_3 will *leave* the basis and x_2 will *enter* the basis. Performing a row elimination again via $r_3 \leftarrow r_3 + \frac{4}{5}r_1$ and $r_2 \leftarrow r_2 - \frac{1}{5}r_1$, we have that

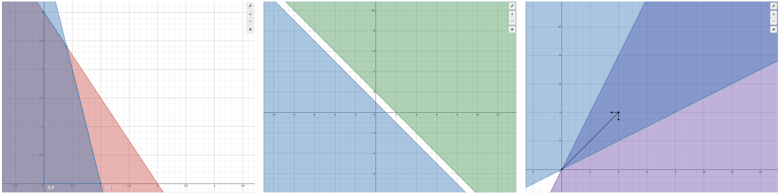
$$\begin{array}{ccccccc} x_2 & = & 240 & + & \frac{3}{5}x_4 & - & \frac{4}{5}x_3 \\ x_1 & = & 40 & - & \frac{2}{5}x_4 & - & \frac{1}{5}x_3 \\ \hline z & = & 640 & - & \frac{2}{5}x_4 & - & \frac{4}{5}x_3 \end{array}$$

Now $(x_1^*, x_2^*) = (40, 240)$. Increasing x_4 or x_3 will decrease z. THE END

Theorem (Fundamental Theorem of Linear Programming)

Let $P = \{x \mid Ax = b, x \geq 0\}$ *then either*

1. $P = \{\emptyset\}$
2. $P \neq \{\emptyset\}$ *and there exists a vertex* v *of* P *such that* $v \in \arg\min_{x \in P} c^\top x$
3. *There exists* $x, d \in \mathbb{R}^n$ *such that* $x + td \in P$ *for all* $t \geq 0$ *and* $\lim_{t \to \infty} c^\top (x + td) = \infty$.



Problem Notation

We will now formalize the definitions we introduced in the examples.

- There are n variables and m constraints
- The linear objective function $z = \sum_{j=1}^{n} c_j x_j$
- The m inequality constraints in standard form

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i, \text{ for } i \in \{1, \ldots, m\}.$$

- The n positivity constraints $x_j \geq 0$, for $j \in \{1, \ldots, n\}$.
- x_i^* denotes the value of ith variable.
- We call $(x_1^*, \ldots, x_n^*) \in \mathbb{R}^n$ a feasible solution if it satisfies the inequality and positivity constraints.

Dictionary Notation

- The slack variables $(x_{n+1}, \ldots, x_{n+m}) \in \mathbb{R}^m$ (*variables d'écart*)
- The initial dictionary

$$
\begin{array}{lcl}
x_{n+1} & = & b_1 - \sum_{j=1}^{n} a_{1j} x_j \\
\vdots & & \\
x_{n+i} & = & b_i - \sum_{j=1}^{n} a_{ij} x_j \\
\vdots & & \\
x_{n+m} & = & b_m - \sum_{j=1}^{n} a_{mj} x_j \\
\hline
z & = & \sum_{j=1}^{n} c_j x_j
\end{array}
$$

- Valid dictionary if m of the variables $(x_1, \ldots, x_{n+m})$ can be expressed as function of the remaining n variables.
- The m variables on the left-hand side are the basic variable (*variable de base*). The n variables on the right-hand side are the non-basic (*variable hors-base*).

Dictionary Notation

After row elimination operations we have a new basis.

- Basic variable set $I \subset \{1, \ldots, n+m\}$ and non-basic set $J = \{1, \ldots, n+m\} \setminus I$ with $|I| = m$ and $|J| = n$
- Current objective value $z^* = \sum_{j=1}^{n} c_j x_j^*$.
- For each basis set I there is a corresponding dictionary

$$\begin{array}{rcl} x_i & = & b'_i - \sum_{j \in J} a'_{ij} x_j, \text{ for } i \in I \\ \hline z & = & z^* + \sum_{j \in J} c'_j x_j, \end{array}$$

  where $a'_{ij}, b'_i, z^* \in \mathbb{R}$ are coefficients resulting from the row operations. For this to a feasible dictionary we require that $b'_i \geq 0$.
- A basic solution: $x_i^* = b'_i$ for $i \in I$ and $x_j^* = 0$ for $j \in J$.

Variable entering/leaving the basis

- If $j_0 \in J$ with $c'_{j_0} > 0$ then increasing x_{j_0} will improve the objective since
$$z = z^* + \sum_{j \in J} c'_j x_j.$$

Variable entering/leaving the basis

- If $j_0 \in J$ with $c'_{j_0} > 0$ then increasing x_{j_0} will improve the objective since
$$z = z^* + \sum_{j \in J} c'_j x_j.$$
- How much can we increase x_{j_0}? Until there is a $x_i = 0$ since
$$x_i^* \ = \ b'_i - a'_{ij_0} x_{j_0}^* \ \geq \ 0$$

Variable entering/leaving the basis

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$$x_i^* \ = \ b'_i - a'_{ij_0} x_{j_0}^* \ \geq \ 0 \quad \Rightarrow \quad a'_{ij_0} x_{j_0}^* \ \leq \ b'_i, \quad \forall i \in I.$$

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- If $a'_{ij_0} \leq 0$, then increasing $x_{j_0}^*$ will increase x_i^*

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- If $a'_{ij_0} \leq 0$, then increasing $x_{j_0}^*$ will increase x_i^*
- If $a'_{ij} > 0$, then $x_{j_0}^* \leq b'_i / \, a'_{ij_0}$

Variable entering/leaving the basis

- If $j_0 \in J$ with $c'_{j_0} > 0$ then increasing x_{j_0} will improve the objective since
$$z = z^* + \sum_{j \in J} c'_j x_j.$$
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- If $a'_{ij_0} \leq 0$, then increasing $x_{j_0}^*$ will increase x_i^*
- If $a'_{ij} > 0$, then $x_{j_0}^* \leq b'_i / \, a'_{ij_0}$
- Thus
$$x_{j_0}^* \;=\; \min_{i \in I,\; a'_{ij_0} > 0} \frac{b'_i}{a'_{ij_0}}$$

Variable entering/leaving the basis

- If $j_0 \in J$ with $c'_{j_0} > 0$ then increasing x_{j_0} will improve the objective since

$$z = z^* + \sum_{j \in J} c'_j x_j.$$

- How much can we increase x_{j_0}? Until there is a $x_i = 0$ since

$$x_i^* \;=\; b'_i - a'_{ij_0} x^*_{j_0} \;\geq\; 0 \quad \Rightarrow \quad a'_{ij_0} x^*_{j_0} \;\leq\; b'_i, \quad \forall i \in I.$$

- If $a'_{ij_0} \leq 0$, then increasing $x^*_{j_0}$ will increase x^*_i
- If $a'_{ij} > 0$, then $x^*_{j_0} \leq b'_i / a'_{ij_0}$
- Thus

$$x^*_{j_0} \;=\; \min_{i \in I,\, a'_{ij_0} > 0} \frac{b'_i}{a'_{ij_0}}$$

- In this case, which $x^*_i = 0$ (which i leaves the basis?)

A Step of the Simplex Method

Input: $I = \{n+1, \ldots, n+m\}$, $J = \{1, \ldots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

A Step of the Simplex Method

Input: $I = \{n+1, \dots, n+m\}$, $J = \{1, \dots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

---

if $c'_i \leq 0$ for all $i \in J$ **then**
    STOP;   # Optimal point found.

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---

if $c'_i \leq 0$ for all $i \in J$ **then**
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Choose a variable j_0 to **enter the basis** from the set $j_0 \in \{j \in J \,:\, c'_j > 0\}$.
if $a'_{ij_0} \leq 0$ for all $i \in J$ **then**
    STOP;   # The problem is unbounded.

A Step of the Simplex Method

Input: $I = \{n+1, \ldots, n+m\}$, $J = \{1, \ldots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

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Choose a variable i_0 to **leave the basis** from the set $i_0 \in \arg \min\limits_{i \in I, a'_{ij_0} > 0} \left\{ \dfrac{b'_i}{a'_{ij_0}} \right\}$.

A Step of the Simplex Method

Input: $I = \{n+1, \ldots, n+m\}$, $J = \{1, \ldots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

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$I \leftarrow (I \setminus \{i_0\})$ and $J \leftarrow J \cup \{i_0\}$ ▷ Move i_0 from basic to non-basic
for $i \in I$ **do**

$$a'_{i:} \leftarrow a'_{i:} - \frac{a'_{ij_0}}{a'_{i_0 j_0}} a'_{i_0:}$$

▷ Row elimination on pivot (i_0, j_0).

A Step of the Simplex Method

Input: $I = \{n+1, \ldots, n+m\}$, $J = \{1, \ldots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

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for $i \in I$ **do**

$$a'_{i:} \leftarrow a'_{i:} - \frac{a'_{ij_0}}{a'_{i_0 j_0}} a'_{i_0:}$$ ▷ Row elimination on pivot (i_0, j_0).

$a'_{i_0:} \leftarrow \frac{1}{a'_{i_0 j_0}} a'_{i_0:}$ and $a'_{i_0 j_0} \leftarrow \frac{1}{a'_{i_0 j_0}}$ ▷ Normalize the coefficient of $a'_{i_0 j_0}$

A Step of the Simplex Method

Input: $I = \{n+1, \dots, n+m\}$, $J = \{1, \dots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

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if $c'_i \leq 0$ for all $i \in J$ **then**
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▷ Row elimination on pivot (i_0, j_0).

$$a'_{i_0:} \leftarrow \frac{1}{a'_{i_0 j_0}} a'_{i_0:} \quad \text{and} \quad a'_{i_0 j_0} \leftarrow \frac{1}{a'_{i_0 j_0}}$$
▷ Normalize the coefficient of $a'_{i_0 j_0}$

$$c' \leftarrow c' - \frac{c'_{j_0}}{a'_{i_0 j_0}} a'_{i_0:}$$
▷ Update the cost coefficients.

A Step of the Simplex Method

Input: $I = \{n+1, \ldots, n+m\}$, $J = \{1, \ldots, n\}$, $a'_{ij} \in \mathbb{R}$, $b'_i \geq 0$, $c'_i \in \mathbb{R}$.

---

if $c'_i \leq 0$ for all $i \in J$ **then**
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$I \leftarrow (I \setminus \{i_0\})$ and $J \leftarrow J \cup \{i_0\}$ ▷ Move i_0 from basic to non-basic

for $i \in I$ **do**

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$a'_{i_0:} \leftarrow \frac{1}{a'_{i_0 j_0}} a'_{i_0:}$ and $a'_{i_0 j_0} \leftarrow \frac{1}{a'_{i_0 j_0}}$ ▷ Normalize the coefficient of $a'_{i_0 j_0}$

$c' \leftarrow c' - \frac{c'_{j_0}}{a'_{i_0 j_0}} a'_{i_0:}$ ▷ Update the cost coefficients.

$I \leftarrow I \cup \{j_0\}$ and $J \leftarrow (J \setminus \{j_0\})$ ▷ Move j_0 from non-basic to basic

---

How to choose who enters the basis?

$$\boxed{j_0 \in \{j \in J \,:\, c'_j > 0\}}$$

1. The mad hatter rule: Choose the first one you see costs: O(1)
2. Dantzig's 1st rule: $j_0 = \arg\max_{j \in J} c_j$ cost: O(n)
3. Dantzig's 2nd rule: Choose j_0 that maximizes the increase in z.

$$j_0 = \arg\max_{j \in J} \left\{ c_j \min_{i \in I, a_{ij} > 0} \left\{ \frac{b_i}{a_{ij}} \right\} \right\} \quad \mathit{costs} : O(n)$$

   Effective, but computationally expensive. costs: O(nm)
4. Bland's rule: Choose the smallest indices j_0 and i_0. That is, choose

$$j_0 = \arg\min\{j \in J \,:\, c_j > 0\} \quad \mathit{costs} : O(n)$$

$$i_0 = \min \left\{ \arg \min_{i \in I, a_{ij_0} > 0} \left\{ \frac{b_i}{a_{ij_0}} \right\} \right\}.$$

Degeneracy

If any of the basic variables are zero, then we say that the solution is degenerate.

Consider the initial dictionary:

$$
\begin{aligned}
x_4 &= 1 + 0 + 0 - 2x_3 \\
x_5 &= 3 - 2x_1 + 4x_2 - 6x_3 \\
x_6 &= 2 + x_1 - 3x_2 - 4x_3 \\
\hline
z &= 0 + 2x_1 - x_2 + 8x_3
\end{aligned}
$$

If x_3 enters then who leaves?

Degeneracy

If any of the basic variables are zero, then we say that the solution is degenerate.

Consider the initial dictionary:

$$
\begin{array}{rl}
x_4 = & 1 + 0 \;+\; 0 \;-2x_3 \\
x_5 = & 3 - 2x_1 + 4x_2 - 6x_3 \\
x_6 = & 2 + x_1 - 3x_2 - 4x_3 \\
\hline
z = & 0 + 2x_1 - x_2 + 8x_3
\end{array}
$$

If x_3 enters then who leaves? Both x_5 and x_6 are set to zero, so either one. Choosing x_4 and pivoting on a'_{13} we have.

$$
\begin{array}{rl}
x_3 = & 0.5 + 0 \;+ 0 \;- 0.5x_4 \\
x_5 = & 0 - 2x_1 + 4x_2 + 3x_4 \\
x_6 = & 0 + x_1 - 3x_2 + 2x_4 \\
\hline
z = & 4 + 2x_1 - x_2 - 4x_4
\end{array}
$$

Only x_1 can enter the basis, but it doesn't increase in value :(

Example in lecture notes.

Bland's rule for degeneracy

Bland's rule

Choose the smallest indices j_0 and i_0. That is, choose

$$j_0 = \arg\min\{j \in J \,:\, c_j > 0\}.$$

$$i_0 = \min\left\{\arg\min_{i \in I, a_{ij_0} > 0}\left\{\frac{b_i}{a_{ij_0}}\right\}\right\}.$$

Theorem

If Bland's rule is used on all degenerate dictionaries, then the simplex algorithm will not cycle.

The zero is not alway feasible

$$
\begin{array}{lrrrl}
\max & x_1 & -x_2 & +x_3 & \\
 & 2x_1 & -x_2 & +2x_3 & \leq 4 \\
 & 2x_1 & -3x_2 & +x_3 & \leq -5 \\
 & -x_1 & +x_2 & -2x_3 & \leq -1 \\
 & x_1, & x_2, & x_3, & \geq 0.
\end{array}
$$

The point $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ is not feasible.

The zero is not alway feasible

$$\begin{array}{lllll}
\max & x_1 & -x_2 & +x_3 & \\
 & 2x_1 & -x_2 & +2x_3 & \leq 4 \\
 & 2x_1 & -3x_2 & +x_3 & \leq -5 \\
 & -x_1 & +x_2 & -2x_3 & \leq -1 \\
 & x_1, & x_2, & x_3, & \geq 0.
\end{array}$$

The point $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$ is not feasible.
Setup an auxiliary problem

$$\begin{array}{llllr}
\max & -x_0 & & & \\
 & 2x_1 & -x_2 & +2x_3 & -x_0 \leq 4 \\
 & 2x_1 & -3x_2 & +x_3 & -x_0 \leq -5 \\
 & -x_1 & +x_2 & -2x_3 & -x_0 \leq -1 \\
 & x_1, & x_2, & x_3, & x_0 \geq 0.
\end{array}$$

Setup initial dictionary

Initial phase one dictionary:

$$\begin{array}{rrrrr} x_4 = 4 & -2x_1 & +x_2 & -2x_3 & +x_0 \\ x_5 = -5 & -2x_1 & +3x_2 & -x_3 & +x_0 \\ x_6 = -1 & +x_1 & -x_2 & +2x_3 & +x_0 \\ w = & & & & -x_0. \end{array}$$

Pivot on "most infeasible" basis. Thus x_5 leaves the basis and x_0 enters the basis. Gaussian elimination :

$r_1 \leftarrow r_1 - r_2$.

$r_3 \leftarrow r_3 - r_2$.

$w \leftarrow w + r_2$.

Initial phase one dictionary:

$$\begin{array}{rrrrr}
x_4 = 4 & -2x_1 & +x_2 & -2x_3 & +x_0 \\
x_5 = -5 & -2x_1 & +3x_2 & -x_3 & +x_0 \\
x_6 = -1 & +x_1 & -x_2 & +2x_3 & +x_0 \\
w = & & & & -x_0.
\end{array}$$

Pivot on "most infeasible" basis. Thus x_5 leaves the basis and x_0 enters the basis. Gaussian elimination :

$r_1 \leftarrow r_1 - r_2$.

$r_3 \leftarrow r_3 - r_2$.

$w \leftarrow w + r_2$.

$$\begin{array}{rrrrr}
x_4 = 9 & +0 & -2x_2 & -x_3 & +x_5 \\
x_0 = 5 & 2x_1 & -3x_2 & +x_3 & +x_5 \\
x_6 = 4 & +3x_1 & -4x_2 & +3x_3 & +x_5 \\
w = -5 & -2x_1 & +3x_2 & -x_3 & -x_5.
\end{array}$$

Now x_2 enters and who leaves?

Initial phase one dictionary:

$$\begin{array}{rrrrr} x_4 = 4 & -2x_1 & +x_2 & -2x_3 & +x_0 \\ x_5 = -5 & -2x_1 & +3x_2 & -x_3 & +x_0 \\ x_6 = -1 & +x_1 & -x_2 & +2x_3 & +x_0 \\ w = & & & & -x_0. \end{array}$$

Pivot on "most infeasible" basis. Thus x_5 leaves the basis and x_0 enters the basis. Gaussian elimination :

$r_1 \leftarrow r_1 - r_2$.

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$$\begin{array}{rrrrr} x_4 = 9 & +0 & -2x_2 & -x_3 & +x_5 \\ x_0 = 5 & 2x_1 & -3x_2 & +x_3 & +x_5 \\ x_6 = 4 & +3x_1 & -4x_2 & +3x_3 & +x_5 \\ w = -5 & -2x_1 & +3x_2 & -x_3 & -x_5. \end{array}$$

Now x_2 enters and who leaves? x_6 leaves the basis

Upper Bounds Using Duality

The LP in standard form

$$\max_x z \stackrel{\text{def}}{=} c^\top x$$
$$\text{subject to } Ax \leq b,$$
$$x \geq 0, \tag{LP}$$

We want to find $w \in \mathbb{R}$ so that $z = c^\top x \leq w$ for all $x \in \mathbb{R}^n$.
Combine rows of constraints?
Look for $y \geq 0 \in \mathbb{R}^m$ so that $y^\top A \approx c^\top$, consequently

$$c^\top x \approx (y^\top A)x \leq y^\top b = w.$$

Precisely, let $y \geq 0 \in \mathbb{R}^m$ be such that $y^\top A \geq c^\top$ or equivalently $A^\top y \geq c$. Then

$$c^\top x \leq (y^\top A)x \leq y^\top b.$$

Can we make this upper bound as tight as possible? Yes, by minimizing $y^\top b$. That is, we need to the *dual* linear program.

Dual definition

The LP in standard form

$$\max_x z \stackrel{\text{def}}{=} c^\top x$$
$$\text{subject to } Ax \le b,$$
$$x \ge 0, \tag{LP}$$

The dual LP:

$$\max_x w \stackrel{\text{def}}{=} y^\top b$$
$$\text{subject to } A^\top y \ge c,$$
$$y \ge 0. \tag{DP}$$

Lemma (Weak Duality)

If $x \in \mathbb{R}^n$ is a feasible point for (LP) and $y \in \mathbb{R}^m$ is a feasible point for (DP) then

$$c^\top x \le y^\top Ax \le y^\top b. \tag{1}$$

Weak Duality

Lemma (Weak Duality)

If $x \in \mathbb{R}^n$ is a feasible point for (LP) *and $y \in \mathbb{R}^m$ is a feasible point for* (DP) *then*

$$c^\top x \leq y^\top A x \leq y^\top b. \tag{2}$$

Consequently

- *If* (LP) *has an unbounded solution, that is $c^\top x \to \infty$, then there exists no feasible point y for* (DP)
- *If* (DP) *has an unbounded solution, that is $y^\top b \to -\infty$, then there exists no feasible point x for* (LP)
- *If x and y are primal and dual feasible, respectively, and $c^\top x = y^\top b$, then x and y are the primal and dual optimal points, respectively.*

Strong Duality

Theorem (Strong Duality)

If (LP) *or* (DP) *is feasible, then* $z^* = w^*$. *Moreover, if* c^* *is the cost vector of the optimal dictionary of the primal problem* (LP), *that is, if*

$$z = z^* + \sum_{i=1}^{n+m} c_i^* x_i, \tag{3}$$

then $y_i^* = -c_{n+i}^*$ *for* $i = 1, \ldots, m$.

First $c_i^* \leq 0$ for $i = 1, \ldots, m+n$ because dictionary is optimal.
Consequently $y_i^* = -c_{n+i}^* \geq 0$ for $i = 1, \ldots, m$.

Strong duality: Proof I

By the definition of the slack variables we have that

$$x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j, \quad \text{for } i = 1, \dots, m. \tag{4}$$

Consequently, setting $y_i^* = -c_{n+i}^*$, we have that

$$\begin{aligned}
z &\overset{(3)}{=} z^* + \sum_{j=1}^{n} c_j^* x_j + \sum_{i=n+1}^{n+m} c_i^* x_i \\
&\overset{(4)}{=} z^* + \sum_{j=1}^{n} c_j^* x_j - \sum_{i=1}^{m} y_i^* (b_i - \sum_{j=1}^{n} a_{ij} x_j) \\
&= z^* - \sum_{i=1}^{m} y_i^* b_i + \sum_{j=1}^{n} \left(c_j^* + \sum_{i=1}^{m} y_i^* a_{ij} \right) x_j \\
&= \sum_{j=1}^{n} c_j x_j, \qquad \forall x_1, \dots, x_n. \qquad (5)
\end{aligned}$$

Last line followed by definition $z = \sum_{j=1}^{n} c_j x_j$. Since the above holds for all $x \in \mathbb{R}^n$, we can match the coefficients.

Strong duality: Proof II

Matching coefficients on x_j's we have

$$
\begin{aligned}
z^* &= \sum_{i=1}^{m} y_i^* b_i && (6)\\
c_j &= c_j^* + \sum_{i=1}^{m} y_i^* a_{ij}, \qquad \text{for } j = 1, \ldots, n. && (7)
\end{aligned}
$$

Since $c_j^* \leq 0$ for $j = 1, \ldots, n$, the above is equivalent to

$$
\begin{aligned}
z^* &= \sum_{i=1}^{m} y_i^* b_i && (8)\\
\sum_{i=1}^{m} y_i^* a_{ij} &\leq c_j, \qquad \text{for } j = 1, \ldots, n. && (9)
\end{aligned}
$$

The inequalities (9) prove that y_i^*'s satisfies the constraints in (DP), and thus is feasible. The equality (8) shows that $z^* = \sum_{i=1}^{m} y_i^* b_i = w$, a consequently by week duality the y_i^*'s are dual optimal. $\square$

G.,R & P Richtárik, Randomized Iterative Methods for Linear Systems arXiv:1506.03296