Workshop 10

March 31, 2014

Topics: Sequences and Series

1. **Explicit Sequences** Determine if the following sequences diverge or converge as $n \to \infty$. If they converge, give the limit (with proof!). If they diverge, prove that they diverge.
   (a) $a_n = \frac{3n^2-1}{10n+5n^2}$
   (b) $(-1)^n$
   (c) $\frac{(-1)^n}{n}$
   (d) $\frac{n^n}{n!}$
   (e) $\frac{2^n}{n!}$
   (f) $\frac{n+47}{\sqrt{n^2+3n}}$
   (g) $\sqrt{n+47} - \sqrt{n}$

Solution:

(a) Manipulate and use Limit Laws. Converges to 3/5

(b) Suppose there exists an $L = \lim a_n$ and there exists N such that $n > N$ we have that $|(-1)^n - L| < \epsilon$. Not true by choosing $\epsilon = 1/2$.

(c) Use the "absolute value " theorem and first show $|a_n| \to 0$. Converges to 0

(d) Comparison with the sequence $b_n = n$, for $\frac{n^{n-1}}{n!} \geq 1$ so

$$\frac{n^n}{n!} = \frac{n.\overbrace{n \dots n}^{n-1} 1}{\underbrace{n(n-1)(n-2)\dots 2}_{n-1} 1} \geq n$$

Tends to ∞

(e) Comparison with the sequence $b_n = 1/n$. Converges to 0

(f) $\frac{n+47}{\sqrt{n^2+3n}} = \frac{1+47/n}{\sqrt{1+3/n}}$ now use the limit laws.

(g) Multiply above and below:

$$\frac{\sqrt{n+47}+\sqrt{n}}{\sqrt{n+47}+\sqrt{n}}.$$

Converges to 0

2. **Recurrsive Sequences.** Use the "Bounded Monotic Theorem" to prove convergence. You will need to use mathematical induction to do this.

   (a) $a_1 = 1$ and $a_{n+1} = 3 - 1/a_n$ for $n \geq 1$. Prove that $1 \leq a_n < 3$ for all n and that the sequence is increasing. Find $\lim_{n\to\infty} a_n$.

   (b) $a_1 = \sqrt{2}$ and $a_n = \sqrt{2 + a_{n-1}}$. Prove that $a_n \leq 3$, that it is increasing and converges. Find $\lim_{n\to\infty} a_n$.

Solution:

(a)
- **I) Is it Bounded?** Induction Hypothesis is that $1 \leq a_n < 3$. For $n = 1$ it is true as $a_1 = 1$. Suppose that $1 \leq a_n < 3$ and let us try and prove that $1 \leq a_{n+1} < 3$.

$$a_{n+1} = 3 - 1/a_n < 3 - 1/3 < 3.$$
$$a_{n+1} = 3 - 1/a_n \geq 3 - 1/1 \geq 1.$$

  II) Is it monotic? Try a few $a_1 = 1$, $a_2 = 5/3$, $a_3 = 3 - 3/5 = 13/5$, seems to be growing. Induction $a_n \geq a_{n-1}$. Then

$$a_{n+1} = 3 - 1/a_n \geq 3 - 1/a_{n-1} = a_n.$$

  III) What is the limit ? $L = 3 - 1/L$ thus $L^2 - 3L + 1 = 0$ and $L = \frac{3\pm\sqrt{5}}{2}$, thus $L = \frac{3+\sqrt{5}}{2}$.

(b)
- **I) Is it Bounded?** Induction Hypothesis is that $a_n < 3$. For $n = 1$ it is true as $a_1 = \sqrt{2}$. Suppose that $a_n < 3$ and let us try and prove that $a_{n+1} < 3$.

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 3} < 3.$$

  II) Is it monotic? Try a few $a_1 = \sqrt{2}$, $a_2 = \sqrt{2 + \sqrt{2}}$, seems to be growing. Induction $a_n \geq a_{n-1}$. Then

$$a_{n+1} = \sqrt{2 + a_n} \geq \sqrt{2 + a_{n-1}} = a_n.$$

  III) What is the limit ? $L = \sqrt{2 + L}$ thus $L^2 - L - 2 = 0$ and $L = \frac{1\pm\sqrt{5}}{2}$, thus $L = \frac{1+\sqrt{5}}{2}$ as each a_n must be positive (Square roots!).

3. **Series.** Using basic properties, sum of geometric series and comparison test, see if the series converges of diverges.

$$\textbf{Geometric Series:} \quad \sum_{n=0}^{\infty} \frac{1}{r^n} = \frac{1}{1 - 1/r}.$$

   (a) $\sum_{n=1}^{\infty} 1/2$

   Calculate the following Geometric Series:

   (b) $\sum_{n=0}^{\infty} e^{1-2n}$. Calculate the limit sum.

   (c) $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{2n+1}}$. Calculate the limit sum.

   Use comparison

   (d) $\sum_{n=1}^{\infty} \frac{2+n}{n^3}$

   (e) $\sum_{n=1}^{\infty} \frac{2}{n^2+1}$

   (f) $\sum_{n=1}^{\infty} \frac{1+n}{n+n^{3/2}}$

   (g) $\sum_{n=1}^{\infty} \frac{1+n}{3^n}$.

(h) $\sum_{n=1}^{\infty} 1/\ln(2+2n)$

Solution:

(a) $s_m = \sum_{n=1}^{m} 1/2 = m/2$ thus $\lim_{m\to\infty} s_m = \infty$

(b) It is a Geometric Series: $\sum_{n=0}^{\infty} e^{1-2n} = e^1 \sum_{n=0}^{\infty} \left(e^{-2}\right)^n = e\frac{1}{1-e^{-2}}$.

(c) It is a Geometric Series: $\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{2n+1}} = \frac{2^{-1}}{3} \sum_{n=1}^{\infty} \left(\frac{2}{3^2}\right)^n = \frac{2^{-1}}{3}\frac{1}{1-2/3^2}$.

(d) Compare with $1/n^2$. $\frac{2+n}{n^3} \leq \frac{2}{n^2}$ for n big enough (in this case $n \geq 2$ but that's not important!) thus $\sum_{n=m}^{\infty} \frac{2+n}{n^3} \leq \sum_{n=m}^{\infty} \frac{2}{n^2}$ for m big enough.

(e) Compare with $1/n^2$. $\sum_{n=m}^{\infty} \frac{2}{n^2+1} \leq \sum_{n=m}^{\infty} \frac{2}{n^2}$ for m big enough (in this case $m \geq 1$!)

(f) Doesn't converge, compare with $1/\sqrt{n}$. We have

$$\frac{1+n}{n+n^{3/2}} \geq \frac{n}{n+n^{3/2}} = \frac{1}{1+\sqrt{n}} \geq \frac{1}{2\sqrt{n}}$$

for n big enough (In this case $n \geq 1$). Thus $\sum_{n=1}^{\infty} \frac{1+n}{n+n^{3/2}} \to \infty$.

(g) does converge, compare with $1/2^n$, for n big enough ($n \geq 3$ in this case).

(h) Compare with $1/n$ or integrate. We have that $\frac{1}{\ln(2+2n)} \geq \frac{1}{2\ln(2n)} \geq \frac{1}{2n}$ for n big enough ($n \geq 1$!), thus $\sum_{n=1}^{\infty} 1/\ln(2+2n) \geq \sum_{n=1}^{\infty} \frac{1}{n} \to \infty$.

4. **Series.** Using basic properties, alternating series, comparison, ratio tests, see if it converges or diverges

- **Alternating Test:** If $\sum_{n=1}^{\infty}(-1)^n a_n$ is such that $\lim_{n\to\infty} a_n = 0$ and $0 \leq a_{n+1} \leq a_n$ then $\sum_{n=1}^{\infty}(-1)^n a_n$ converges.
- **Ratio test:** If $\sum_{n=1}^{\infty} a_n$ is such that

$$\lim_{\infty} \frac{a_{n+1}}{a_n} = L$$

  The ratio test states that:
  if $L < 1$ then the series converges absolutely;
  if $L > 1$ then the series does not converge;
  if $L = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.
- **Root test:** If $\sum_{n=1}^{\infty} a_n$ is such that

$$\lim_{\infty} \sqrt[n]{a_n} = C$$

  The ratio test states that:
  if $C < 1$ then the series converges absolutely;
  if $C > 1$ then the series does not converge;
  if $C = 1$ or the limit fails to exist, then the test is inconclusive, because there exist both convergent and divergent series that satisfy this case.

(a) **Apply Alternating test:** $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(2n)}$

(b) **Apply Alternating test or Comparison?** $\sum_{n=1}^{\infty} \frac{(-1)^{2n+1}}{n}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^{1/10}}$

(d) **Apply the Ratio Test:**
$\sum_{n=1}^{\infty} \frac{2^{n-1}}{3^{2n+1}}$. (Was the Ratio test necessary here?)

(e) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

(f) $\sum_{n=1}^{\infty} \frac{n^4}{(2n)!}$

(g) $\sum_{n=1}^{\infty} \frac{3^{n^3}}{n!}$

(h) **Try Root Test:**
$\sum \frac{(-3n)^n}{(2n\sqrt{n+2})^n}$

(i) $\sum_{n=1}^{\infty}(1+1/n)^n$. **TIP:** $\lim_{n\to\infty}(1+1/n)^n = e$.

(j) $\sum_{n=1}^{\infty}(1+1/n)^{n^2}$

Solution:

(a) Alternating, decreasing and positive. $\frac{1}{\ln(2n)} \geq \frac{1}{\ln(2(n+1))} \geq 0$. Also, $\lim_{n\to\infty} \frac{1}{\ln(2n)} = 0$.

(b) $\frac{(-1)^{2n+1}}{n} = -\frac{1}{n}$ the harmonic series diverges.

(c) $\frac{(-1)^{n+1}}{n^{1/10}}$ is alternating and decreasing.

(d) Ratio test

$$\lim_{n\to\infty} \frac{2^n}{3^{2n+3}} \frac{3^{2n+1}}{2^{n-1}} = \lim_{n\to\infty} \frac{2}{3^2} < 1$$

.

(e) Ratio test $\lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2}{(1+n)} = 0$.

(f) Ratio test $\lim_{n\to\infty} \frac{(n+1)^4}{(2(n+2))!} \frac{2n!}{n^4} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^4 \frac{1}{2(n+2)(n+1)} = 0$

(g) Ratio test

$$\begin{aligned}
\lim_{n\to\infty} \frac{3^{(n+1)^3}}{(n+1)!} \frac{n!}{3^{n^3}} &= \lim_{n\to\infty} \frac{3^{(n+1)^3 - n^3}}{(n+1)} \\
&= \lim_{n\to\infty} \frac{3^{3n^2+3n+1}}{(n+1)} \\
&= \lim_{x\to\infty} \frac{3^{3x^2+3x+1}}{(x+1)} \\
&=^{\text{L'H}} \infty
\end{aligned}$$

(h) Using a root test $\sqrt[n]{|a_n|} = \sqrt[n]{\left(\left|\frac{(-1)3n}{(2n\sqrt{n+2})}\right|\right)^n} = \frac{3n}{(2n\sqrt{n+2})}$. Thus $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 0$.

(i) Does not trail off to zero! $\lim_{x\to\infty}(1+1/x)^x = e$!

(j) Root test $\lim_{n\to\infty} \left((1+1/n)^{n^2}\right)^{1/n} = \lim_{n\to\infty}(1+1/n)^n = e > 1$! Diverges!