

# Circuits and Transforms

Gowri Govindaraj

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**Abstract**—This manual provides a simple introduction to Transforms

## 1 DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of  $g(t)$  is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

## 2 LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes  $q_1 \mu C$ . Then S is switched to position Q. After a long time, the charge on the capacitor is  $q_2 \mu C$ .

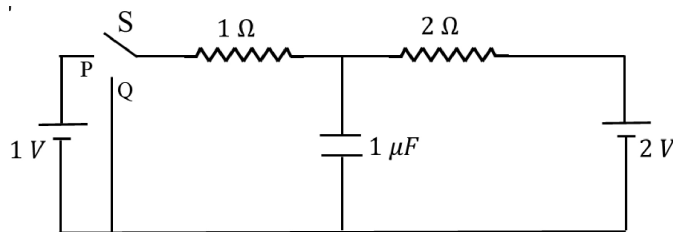


Fig. 2.1

2. Find  $q_1$ .

**Solution:** The equivalent circuit at steady-state when the switch is at P is shown alongside.

Assuming the circuit to be grounded at G and the relative potential at point X to be  $V$ , we use KCL at X and get

$$\frac{V-1}{1} + \frac{V-2}{2} = 0 \quad (2.1)$$

$$\Rightarrow V = \frac{4}{3} \text{ V} \quad (2.2)$$

Hence,

$$q_1 = CV = \frac{4}{3} \mu C \quad (2.3)$$

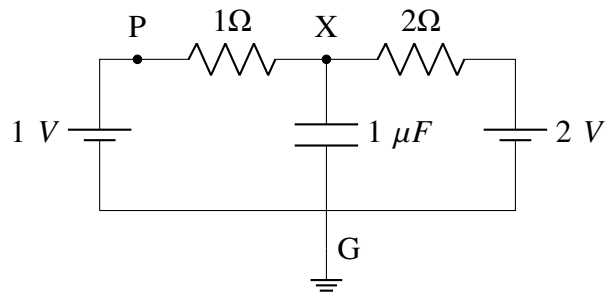


Fig. 2.2

3. Show that the Laplace transform of  $u(t)$  is  $\frac{1}{s}$  and find the ROC.

**Solution:** We have,

$$u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-st} dt \quad (2.4)$$

$$= \int_0^0 \frac{1}{2} e^{-st} dt + \int_0^{\infty} e^{-st} dt \quad (2.5)$$

$$= \frac{1}{s}, \quad \Re(s) > 0 \quad (2.6)$$

4. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.7)$$

and find the ROC.

**Solution:** Note that by substituting  $s := s + a$  in (2.6), and considering  $a \in \mathbb{R}$ ,

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.8)$$

$$= \frac{1}{s+a}, \quad \Re(s) > -a \quad (2.9)$$

5. Now consider the following resistive circuit transformed from Fig. 2.1 where

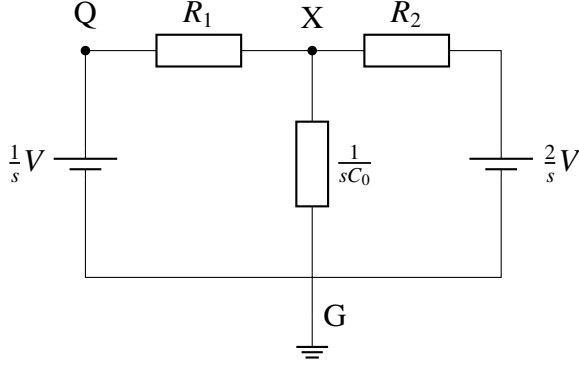


Fig. 2.3

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.10)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.11)$$

Find the voltage across the capacitor  $V_{C_0}(s)$ .

**Solution:** We see that

$$V_1(s) = \frac{1}{s} V_2(s) = \frac{2}{s} \quad (2.12)$$

Now, labelling points G and X as in Fig. 2.2, we use KCL at X.

$$\frac{V - \frac{1}{s}}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 V = 0 \quad (2.13)$$

$$V \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{s} \left( \frac{1}{R_1} + \frac{2}{R_2} \right) \quad (2.14)$$

$$V(s) = \frac{\frac{1}{R_1} + \frac{2}{R_2}}{s \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right)} \quad (2.15)$$

$$= \frac{\frac{1}{R_1} + \frac{2}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2}} \left( \frac{1}{s} - \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (2.16)$$

6. Find  $v_{C_0}(t)$ . Plot using python.

**Solution:** Taking the inverse Laplace transform

in (2.16),

$$V(s) \xleftrightarrow{\mathcal{L}} \frac{2R_1 + R_2}{R_1 + R_2} u(t) \left( 1 - e^{-\left( \frac{1}{R_1} + \frac{1}{R_2} \right) \frac{t}{C_0}} \right) \quad (2.17)$$

$$= \frac{4}{3} \left( 1 - e^{-(1.5 \times 10^6)t} \right) u(t) \quad (2.18)$$

The python code codes/2\_6.py plots the graph below.

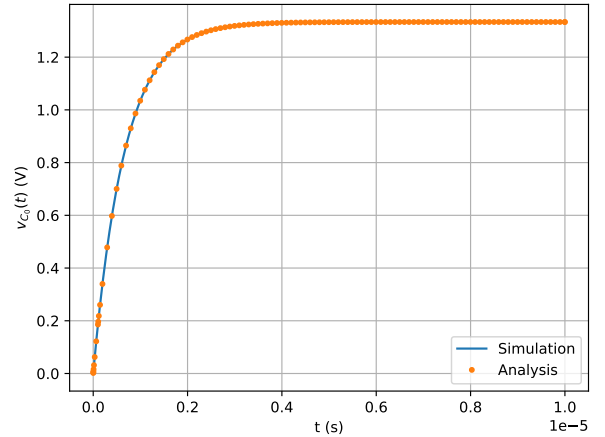


Fig. 2.4:  $v_{C_0}(t)$  before the switch is flipped

7. Verify your result using ngspice.

**Solution:** The ngspice script codes/2\_7.cir simulates the given circuit and the generated output is depicted in Fig. (2.4)

### 3 INITIAL CONDITIONS

1. Find  $q_2$  in Fig. 2.1.

**Solution:** The equivalent circuit at steady state when the switch is at Q is shown below.

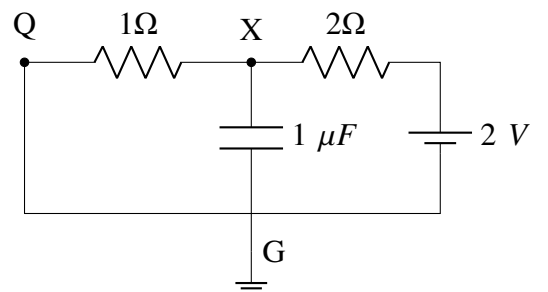


Fig. 3.1

Since capacitor behaves as an open circuit, we use KCL at X.

$$\frac{V-0}{1} + \frac{V-2}{2} = 0 \implies V = \frac{2}{3} V \quad (3.1)$$

and hence,  $q_2 = \frac{2}{3} \mu C$ .

2. Draw the equivalent  $s$ -domain resistive circuit when S is switched to position Q. Use variables  $R_1, R_2, C_0$  for the passive elements.

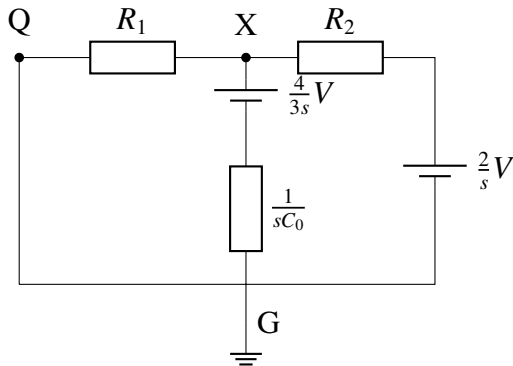


Fig. 3.2

3.  $V_{C_0}(s) = ?$

**Solution:** Using KCL at node X in Fig. 3.2

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0 \left( V - \frac{4}{3s} \right) = 0 \quad (3.2)$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.3)$$

4.  $v_{C_0}(t) = ?$  Plot using python.

**Solution:** From (3.3),

$$V_{C_0}(s) = \frac{4}{3} \left( \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( \frac{1}{s} - \frac{1}{\frac{1}{C_0} \left( \frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.4)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{2}{R_2 \left( \frac{1}{R_1} + \frac{1}{R_2} \right)} \left( 1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \quad (3.5)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left( 1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.6)$$

The Python code `codes/3_4.py` plots the

graph below.

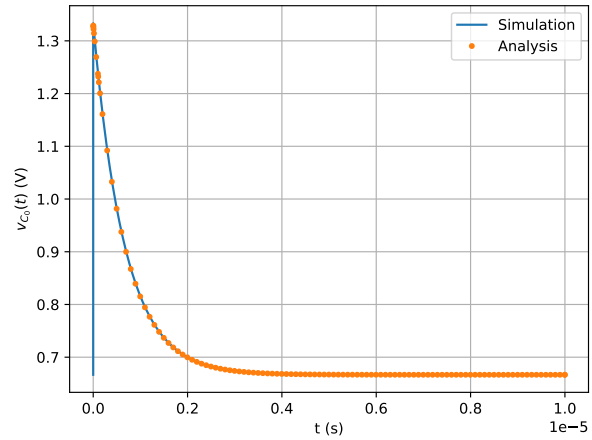


Fig. 3.3:  $v_{C_0}(t)$  after the switch is flipped

5. Verify your result using ngspice.

**Solution:** The ngspice script `codes/3_5.cir` simulates the given circuit and the generated output is depicted in Fig. (3.3).

6. Find  $v_{C_0}(0-)$ ,  $v_{C_0}(0+)$  and  $v_{C_0}(\infty)$ .

**Solution:** From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3} V \quad (3.7)$$

Using (3.6),

$$v_{C_0}(0+) = \lim_{t \rightarrow 0+} v_{C_0}(t) = \frac{4}{3} V \quad (3.8)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3} V \quad (3.9)$$

7. Obtain the Fig. in problem 3.2 using the equivalent differential equations.

**Solution:** The equivalent circuit in the  $t$ -domain is shown below.

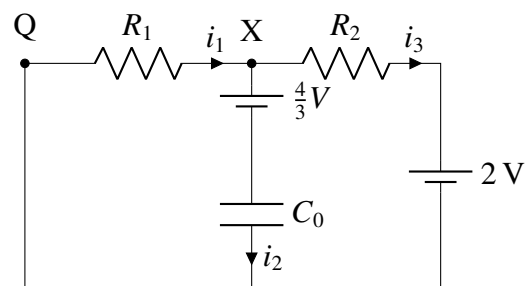


Fig. 3.4

From KCL and KVL,

$$i_1 = i_2 + i_3 \quad (3.10)$$

$$i_1 R_1 + \frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (3.11)$$

$$\frac{4}{3} + \frac{1}{C_0} \int_0^t i_2 dt - i_3 R_2 - 2 = 0 \quad (3.12)$$

$$(3.13)$$

Taking Laplace Transforms on both sides and using the properties of Laplace Transforms,

$$I_1 = I_2 + I_3 \quad (3.14)$$

$$I_1 R_1 + \frac{4}{3} + \frac{1}{sC_0} I_2 = 0 \quad (3.15)$$

$$\frac{4}{3} + \frac{1}{sC_0} I_2 - I_3 R_2 - 2 = 0 \quad (3.16)$$

$$(3.17)$$

where  $i(t) \xleftrightarrow{\mathcal{L}} I(s)$ . Note that the capacitor is equivalent to a resistive element of resistance  $R_C = \frac{1}{sC_0}$  in the  $s$ -domain. Equations (3.14) - (3.16) precisely describe Fig. 3.2.

#### 4 BILINEAR TRANSFORM

1. In Fig. 2.1, consider the case when  $S$  is switched to  $Q$  right in the beginning. Formulate the differential equation. **Solution:** The equivalent circuit in the  $t$ -domain is shown below.

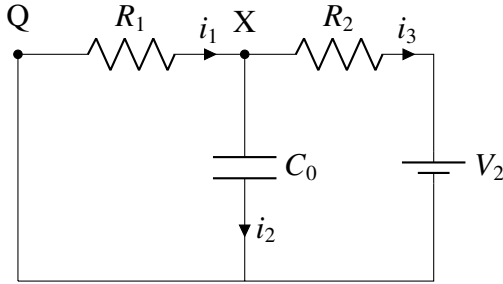


Fig. 4.1

Applying KCL and KVL,

$$i_1 = i_2 + i_3 \quad (4.1)$$

$$i_1 R_1 + \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (4.2)$$

$$i_3 R_2 + 2 - \frac{1}{C_0} \int_0^t i_2 dt = 0 \quad (4.3)$$

Differentiating the above equations,

$$\frac{di_1}{dt} = \frac{di_2}{dt} + \frac{di_3}{dt} \quad (4.4)$$

$$R_1 \frac{di_1}{dt} + \frac{i_2}{C_0} = 0 \quad (4.5)$$

$$R_2 \frac{di_3}{dt} - \frac{i_2}{C_0} = 0 \quad (4.6)$$

Using (4.4) and (4.6) in (4.5),

$$R_1 \left( \frac{di_2}{dt} + \frac{di_3}{dt} \right) + \frac{i_2}{C_0} = 0 \quad (4.7)$$

$$R_1 \frac{di_2}{dt} + \left( 1 + \frac{R_1}{R_2} \right) \frac{i_2}{C_0} = 0 \quad (4.8)$$

$$\frac{di_2}{dt} + \left( \frac{1}{R_1} + \frac{1}{R_2} \right) \frac{i_2}{C_0} = 0 \quad (4.9)$$

$$\frac{di_2}{dt} + \frac{i_2}{\tau} = 0 \quad (4.10)$$

where  $\tau = \frac{C_0 R_1 R_2}{R_1 + R_2}$  is the RC time constant of the circuit. Note that  $i_2(0) = \frac{V_2}{R_2}$  A and  $i_2 = C_0 \frac{dV}{dt}$ , where  $V$  is the voltage of the capacitor. Hence, integrating (4.10),

$$C_0 \frac{dV}{dt} - \frac{V_2}{R_2} + \frac{C_0 V}{\tau} = 0 \quad (4.11)$$

$$\Rightarrow \frac{dV}{dt} + \frac{V}{\tau} = \frac{V_2}{C_0 R_2} \quad (4.12)$$

2. Find  $H(s)$  considering the output voltage at the capacitor. **Solution:** Transforming Fig. 4.1 to the  $s$ -domain, Applying nodal analysis at X,

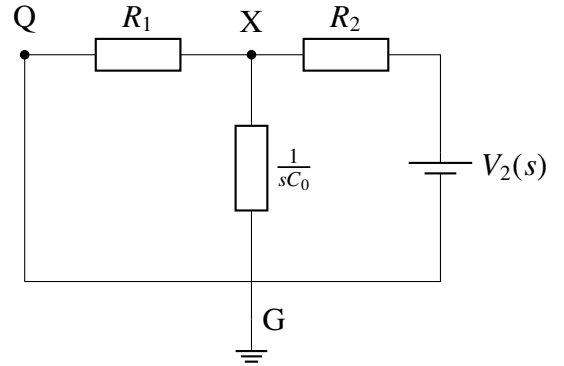


Fig. 4.2

and noting that  $H(s) = \frac{V(s)}{V_2(s)}$ ,

$$\frac{V}{R_1} + \frac{V}{\frac{1}{sC_0}} + \frac{V - V_2}{R_2} = 0 \quad (4.13)$$

$$H(s) \left( \frac{1}{R_1} + \frac{1}{R_2} + sC_0 \right) = \frac{1}{R_2} \quad (4.14)$$

$$H(s) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.15)$$

3. Plot  $H(s)$ . What kind of filter is it? **Solution:** The Python code `codes/4_3.py` plots  $H(s)$ . Clearly,  $H(s)$  is a low-pass filter.

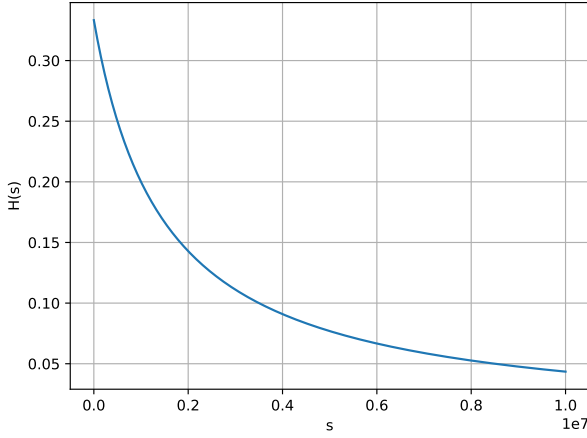


Fig. 4.3: Plot of  $H(s)$ .

4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.16)$$

**Solution:** Integrating (4.12) between limits  $n$  to  $n + 1$  and applying the trapezoidal formula,

$$v(n+1) - v(n) + \frac{v(n) + v(n+1)}{2\tau} = \frac{V_2(u(n) + u(n+1))}{C_0 R_2} \quad (4.17)$$

$$v(n)(2\tau + 1) + v(n-1)(2\tau - 1) = \frac{V_2\tau(u(n) + u(n-1))}{C_0 R_2} \quad (4.18)$$

for  $n > 0$ , where  $v(0) = 0$ .

5. Find  $H(z)$ . **Solution:** Note that for the input voltage,  $v_i(n) = 2u(n)$  and so,  $V_i(z) = \frac{2}{1-z^{-1}}$ . Applying the Z-transform on both sides of

(4.18),

$$\begin{aligned} V(z) \left[ (2\tau + 1) - z^{-1}(2\tau - 1) \right] \\ = \frac{\tau(1 + z^{-1}) V_i(z)}{C_0 R_2} \end{aligned} \quad (4.19)$$

Hence,

$$H(z) = \frac{\tau(1 + z^{-1})}{C_0 R_2 ((2\tau + 1) - (2\tau - 1)z^{-1})} \quad (4.20)$$

since  $\left| \frac{2\tau-1}{2\tau+1} \right| < 1$ , the ROC is  $|z| > 1$ .

6. How can you obtain  $H(z)$  from  $H(s)$ ? **Solution:** We use the bilinear transformation. Setting

$$s := \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (4.21)$$

we get

$$H(z) = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + \frac{2C_0}{T} \frac{1-z^{-1}}{1+z^{-1}}} \quad (4.22)$$

$$= \frac{T\tau(1 + z^{-1})}{C_0 R_2 ((2\tau + T) - (2\tau - T)z^{-1})} \quad (4.23)$$

Setting  $T = 1$  gives (4.20).

7. Find  $v(n)$ . Verify using ngspice and the differential equation.

**Solution:** We have,

$$V(z) = H(z)V_i(z) \quad (4.24)$$

$$\begin{aligned} &= \frac{TV_2\tau(1 + z^{-1})}{C_0 R_2 (1 - z^{-1})((2\tau + T) - (2\tau - T)z^{-1})} \\ &= \frac{V_2\tau(z + 1)}{2C_0 R_2} \sum_{k=-\infty}^{\infty} (1 - p^k) u(k) z^{-k} \end{aligned} \quad (4.25)$$

$$= \frac{V_2\tau(z + 1)}{2C_0 R_2} \sum_{k=-\infty}^{\infty} (1 - p^k) u(k) z^{-k} \quad (4.26)$$

where  $p := \frac{2\tau-T}{2\tau+T}$ . Thus,

$$v(n) = \frac{V_2\tau}{C_0 R_2} \left[ u(n)(1 - p^n) + u(n+1)(1 - p^{n+1}) \right] \quad (4.27)$$

where  $p := \frac{2\tau-1}{2\tau+1}$ . We take  $T = 10^{-7}$  as the sampling interval. The python code `codes/4_7.py` verifies these equalities.

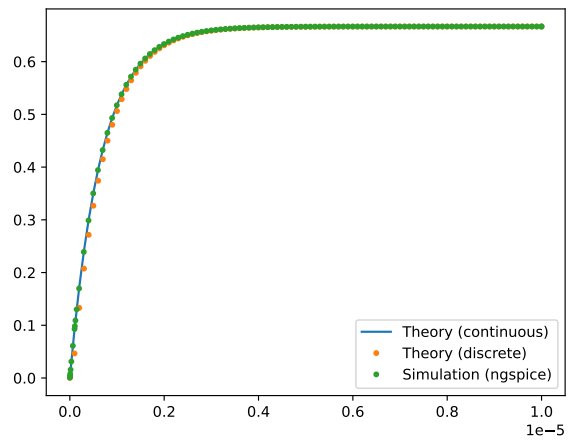


Fig. 4.4: Representation of output across  $C_0$ .