Fourier Series

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Contents

Abstract—**This manual provides a simple introduction to Fourier Series.**

1 Periodic Function

Let

$$x(t) = A_0 \left| \sin\left(2\pi f_0 t\right) \right| \tag{1.1}$$

1.1 Plot $x(t)$.

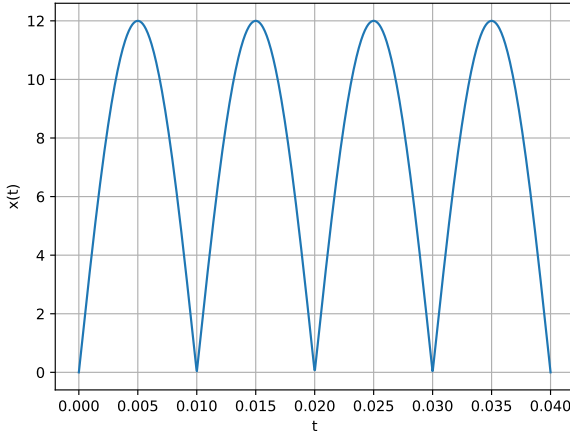**Solution:** The Python code `codes/1_1.py` plots $x(t)$ below.

Fig. 1.1: $x(t)$

1.2 Show that $x(t)$ is periodic and find its period.

Solution: From Fig. (1.1), we see that $x(t)$ is periodic. Further,

$$x\left(t + \frac{1}{2f_0}\right) = A_0 \left| \sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right) \right| \tag{1.2}$$

$$= A_0 \left| \sin\left(2\pi f_0 t + \pi\right) \right| \tag{1.3}$$

$$= A_0 \left| \sin\left(2\pi f_0 t\right) \right| \tag{1.4}$$

Hence the period of $x(t)$ is $\frac{1}{2f_0}$.

2 Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t} \, dt \tag{2.2}$$

Solution: We have for some $n \in \mathbb{Z}$,

$$x(t) e^{-j2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-n)f_0 t} \tag{2.3}$$

But we know from the periodicity of $e^{j2\pi k f_0 t}$,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi k f_0 t} \, dt = \frac{1}{f_0}\delta\left(k\right) \tag{2.4}$$

Thus,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} \, dt = \frac{c_n}{f_0} \tag{2.5}$$

$$\implies c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi n f_0 t} \, dt \tag{2.6}$$

2.2 Find c_k for (1.1)

Solution: Using (2.2),

$$c_n = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin \left(2\pi f_0 t \right) \right| e^{-\jmath 2\pi n f_0 t} \, dt \quad (2.7)$$

$$= f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin \left(2\pi f_0 t \right) \right| \cos \left(2\pi n f_0 t \right) \, dt$$

$$+ \jmath f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left| \sin \left(2\pi f_0 t \right) \right| \sin \left(2\pi n f_0 t \right) \, dt \tag{2.8}$$

$$= 2 f_0 \int_0^{\frac{1}{2f_0}} A_0 \sin \left(2\pi f_0 t \right) \cos \left(2\pi n f_0 t \right) \, dt \tag{2.9}$$

$$= f_0 A_0 \int_0^{\frac{1}{2f_0}} \left(\sin \left(2\pi \left(n + 1 \right) f_0 t \right) \right) \, dt$$

$$- f_0 A_0 \int_0^{\frac{1}{2f_0}} \left(\sin \left(2\pi \left(n - 1 \right) f_0 t \right) \right) \, dt \quad (2.10)$$

$$= A_0 \frac{1 + (-1)^n}{2\pi} \left(\frac{1}{n+1} - \frac{1}{n-1} \right) \tag{2.11}$$

$$= \begin{cases} \frac{2A_0}{\pi(1-n^2)} & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \tag{2.12}$$

2.3 Verify (2.1) using python.
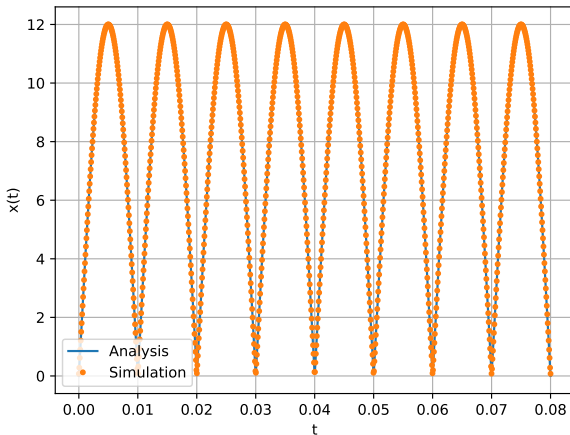   **Solution:** The Python code `codes/2_3.py` verifies (2.13).

Fig. 2.3: Verification of (2.1).

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} \left(a_k \cos \jmath 2\pi k f_0 t + b_k \sin \jmath 2\pi k f_0 t \right) \tag{2.13}$$

and obtain the formulae for a_k and b_k.
Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.14}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.15}$$

$$= c_0 + \sum_{k=1}^{\infty} \left(c_k + c_{-k} \right) \cos \left(2\pi k f_0 t \right)$$

$$+ \sum_{k=0}^{\infty} \left(c_k - c_{-k} \right) \sin \left(2\pi k f_0 t \right) \tag{2.16}$$

Hence, for $k \geq 0$,

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.17}$$

$$b_k = c_k - c_{-k} \tag{2.18}$$

2.5 Find a_k and b_k for (1.1)
   Solution: From (2.1), we see that since $x(t)$ is even,

$$x(-t) = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi k f_0 t} \tag{2.19}$$

$$= \sum_{k=-\infty}^{\infty} c_{-k} e^{\jmath 2\pi k f_0 t} \tag{2.20}$$

$$= \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.21}$$

where we substitute $k := -k$ in (2.20). Hence, we see that $c_k = c_{-k}$. So, from (2.17) and (2.18), for $k \geq 0$,

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k > 0, \ k \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

$$b_k = 0 \tag{2.23}$$

2.6 Verify (2.13) using python.
   Solution: The Python code `codes/2_6.py` verifies (2.13).

3 FOURIER TRANSFORM

3.1

$$\delta(t) = 0, \quad t \neq 0 \tag{3.1}$$

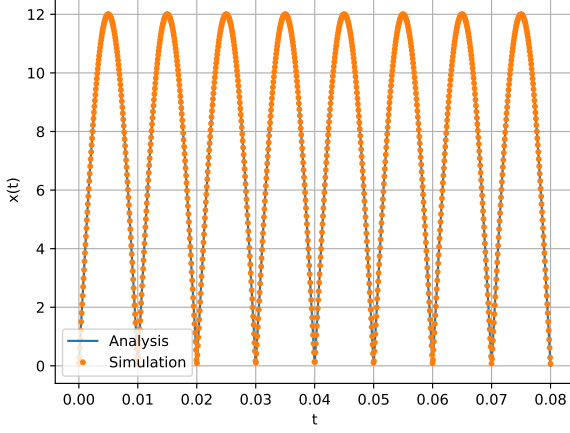$$\int_{-\infty}^{\infty} \delta(t) \, dt = 1 \tag{3.2}$$

Fig. 2.6: Verification of (2.13).

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t)e^{-j2\pi ft}\, dt \qquad (3.3)$$

3.3 Show that

$$g(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} G(f)e^{-j2\pi ft_0} \qquad (3.4)$$

Solution: We write, substituting $u := t - t_0$,

$$g(t - t_0) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t - t_0)e^{-j2\pi ft}\, dt \qquad (3.5)$$

$$= \int_{-\infty}^{\infty} g(u)e^{-j2\pi f(u+t_0)}\, du \qquad (3.6)$$

$$= G(f)e^{-j2\pi ft_0} \qquad (3.7)$$

where the last equality follows from (3.3).

3.4 Show that

$$G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \qquad (3.8)$$

Solution: Using the definition of the Inverse Fourier Transform,

$$g(t) = \int_{-\infty}^{\infty} G(f)e^{j2\pi ft}\, df \qquad (3.9)$$

Hence, setting $t := -f$ and $f := t$, which implies $df = dt$,

$$g(-f) = \int_{-\infty}^{\infty} G(t)e^{-j2\pi ft}\, dt \qquad (3.10)$$

$$\implies G(t) \overset{\mathcal{F}}{\longleftrightarrow} g(-f) \qquad (3.11)$$

3.5 $\delta(t) \overset{\mathcal{F}}{\longleftrightarrow}?$

Solution: We have, from the definition of $\delta(t)$,

$$\delta(t) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \delta(t)e^{-j2\pi ft}\, dt \qquad (3.12)$$

$$= \int_{-\infty}^{\infty} \delta(0)\, dt \qquad (3.13)$$

$$= \int_{-\infty}^{\infty} \delta(t)\, dt = 1 \qquad (3.14)$$

3.6 $e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow}?$

Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then,

$$g(t)e^{j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(t)e^{-j2\pi(f-f_0)t}\, dt \qquad (3.15)$$

$$= F(f - f_0) \qquad (3.16)$$

Using (3.9) in (3.14), $1 \overset{\mathcal{F}}{\longleftrightarrow} \delta(-f)$. Hence, applying (3.16),

$$e^{-j2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} \delta(-(f + f_0)) = \delta(f + f_0) \qquad (3.17)$$

3.7 $\cos(2\pi f_0 t) \overset{\mathcal{F}}{\longleftrightarrow}?$

Solution: Using the linearity of the Fourier Transform and (**??**),

$$\cos(2\pi f_0 t) = \frac{1}{2}\left(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}\right) \qquad (3.18)$$

$$\overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\left(\delta(f + f_0) + \delta(f - f_0)\right) \qquad (3.19)$$

3.8 Find the Fourier Transform of $x(t)$ and plot it. Verify using python.

Solution: Substituting (2.12) in (2.1),

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} c_k \delta(f + kf_0) \qquad (3.20)$$

$$= \frac{2A_0}{\pi} \sum_{k=-\infty}^{\infty} \frac{\delta(f + 2kf_0)}{1 - 4k^2} \qquad (3.21)$$

The python code `codes/3_8.py` verifies (3.21).

3.9 Show that

$$\text{rect}\, t \overset{\mathcal{F}}{\longleftrightarrow} \text{sinc}\, t \qquad (3.22)$$

Verify using python.
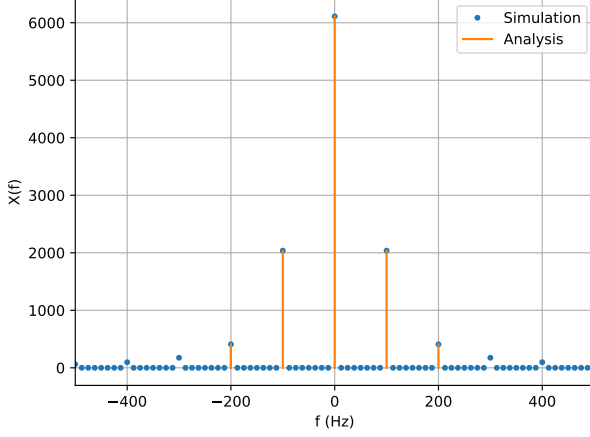
Fig. 3.8: Fourier Transform of $x(t)$.

Solution: We write

$$\operatorname{rect} t \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} \operatorname{rect} t\, e^{-\jmath 2\pi ft}\, dt \qquad (3.23)$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\jmath 2\pi ft}\, dt \qquad (3.24)$$

$$= \frac{e^{\jmath \pi f} - e^{-\jmath \pi f}}{\jmath 2\pi f} = \frac{\sin \pi f}{\pi f} = \operatorname{sinc} f \quad (3.25)$$

The python code `codes/3_9.py` verifies (3.25).
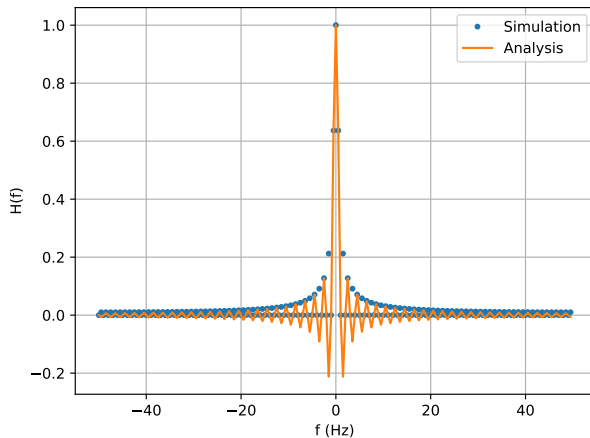
Fig. 3.9: Fourier Transform of rect(t).

3.10 $\operatorname{sinc} t \overset{\mathcal{F}}{\longleftrightarrow}$? Verify using python.
Solution: From (3.9), we have

$$\operatorname{sinc} t \overset{\mathcal{F}}{\longleftrightarrow} \operatorname{rect}(-f) = \operatorname{rect} f \qquad (3.26)$$

Since rect f is an even function. The python code `codes/3_10.py` verifies (3.26).
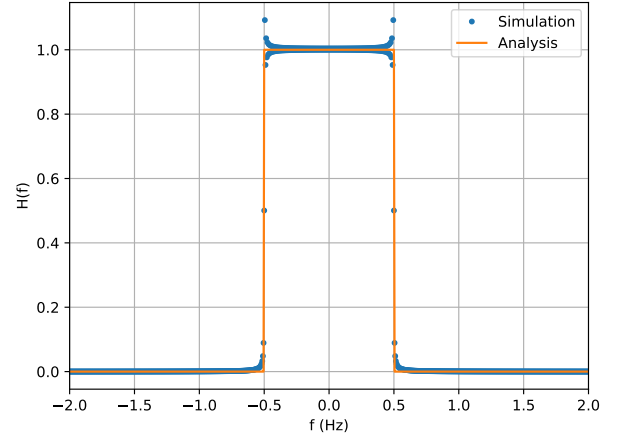

Fig. 3.10: Fourier Transform of sinc(t).

4 FILTER

4.1 Find $H(f)$ which transforms $x(t)$ to DC 5V.
Solution: The function $H(f)$ is a low pass filter which filters out even harmonics and leaves the zero frequency component behind. The rectangular function represents an ideal low pass filter. Suppose the cutoff frequency is $f_c = 50$ Hz, then

$$H(f) = \operatorname{rect} \frac{f}{2f_c} = \begin{cases} 1 & |f| < f_c \\ 0 & \text{otherwise} \end{cases} \qquad (4.1)$$

Multiplying by a scaling factor to get DC 5V,

$$H(f) = \frac{\pi V_0}{2A_0} \operatorname{rect}\left(\frac{f}{2f_c}\right) \qquad (4.2)$$

where $V_0 = 5$ V.
4.2 Find $h(t)$.
Solution: Suppose $g(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f)$. Then, for some nonzero $a \in \mathbb{R}$

$$g(at) \overset{\mathcal{F}}{\longleftrightarrow} \int_{-\infty}^{\infty} g(at) e^{-\jmath 2\pi ft}\, dt \qquad (4.3)$$

$$= \frac{1}{a} \int_{-\infty}^{\infty} g(u) e^{\left(-\jmath 2\pi \frac{f}{a} t\right)}\, dt \qquad (4.4)$$

$$= \frac{1}{a} G\left(\frac{f}{a}\right) \qquad (4.5)$$

where we have substituted $u := at$. Using (4.5) of the Fourier Transform in (4.1),

$$h(t) = 2 * f_c \operatorname{sinc}(2 * f_c t) \qquad (4.6)$$

4.3 Verify your result using convolution.
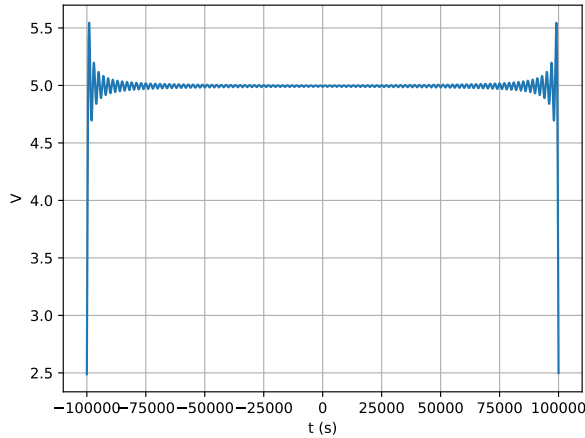**Solution:** The Python code `codes/4_3.py` verifies the result by plotting the graph below.

Fig. 4.3: Convolution of the two signals.

5 FILTER DESIGN

5.1 Design a Butterworth filter for $H(f)$.
Solution: The Butterworth filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \left(\frac{f}{f_c}\right)^{2n}\right)} \qquad (5.1)$$

where n is the order of the filter and f_c is the cutoff frequency. The loss at frequency f is given by

$$\text{Loss} = -10 \log_{10} |H(f)|^2 \qquad (5.2)$$
$$= -20 \log_{10} |H(f)| \qquad (5.3)$$

Assuming the cutoff frequency $f_c = 10$ Hz, the Python code `codes/5_1.py` plots the magnitude response for different values of n as shown below. At $f = 10 f_c = 2 f_0$, we obtain a 40 dB loss for $n = 2$, which is ideal. Hence, we choose $n = 2$. Further, note that the DC gain is $\frac{5}{12}$.

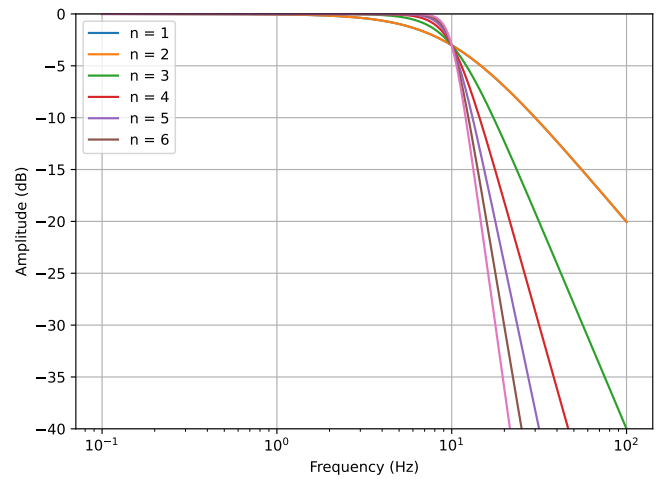5.2 Design a Chebyshev filter for $H(f)$.

Fig. 5.1: Butterworth response for various n.

Solution: The Chebyshev filter has an amplitude response given by

$$|H(f)|^2 = \frac{1}{\left(1 + \epsilon^2 C_n^2\left(\frac{f}{f_c}\right)^2\right)} \qquad (5.4)$$

where n is the order of the filter, ϵ is the ripple, f_c is the cutoff frequency and C_n denotes the nth order Chebyshev polynomial. Assuming the cutoff frequency to be at f_0, the Python codes `codes/5_2.py` and `codes/5_3.py` plot the magnitude response for different values of n.
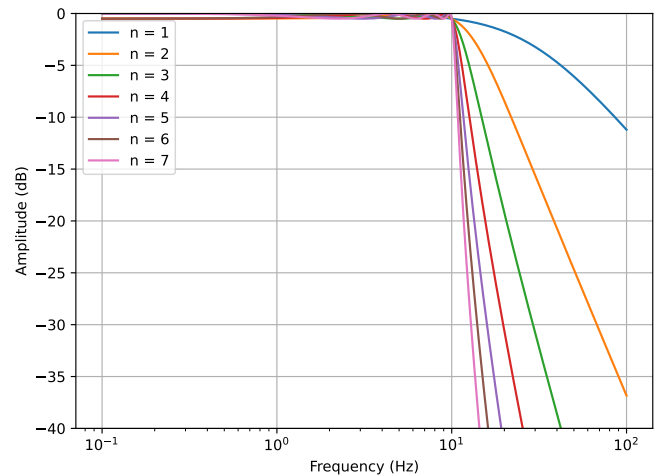

Fig. 5.2: Chebyshev response for various n.

At $f = 10 f_c$, we obtain close to 40 dB loss for $n = 2$, which is ideal. Hence, we choose $n = 2$. Further, note that the DC gain is $\frac{5}{12}$.

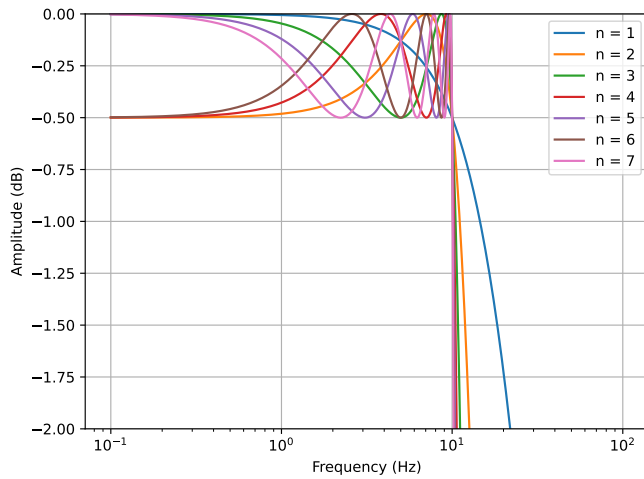5.3 Design a circuit for your Butterworth filter.

Fig. 5.2: Chebyshev response for various n, in detail.

The L-C network is shown below

Fig. 5.4: L-C Chebyshev Filter

Solution: Looking at the table of normalized g values of the Butterworth filter for order 2, we see that

$$g_1 = C_1 = 1.5774 \qquad g_2 = L_1 = 0.4226 \quad (5.5)$$

De-normalizing these values,

$$C_1' = \frac{C_1}{\omega_c} = 2.51 \times 10^{-2} \text{ F} \qquad (5.6)$$

$$L_1' = \frac{L_1}{\omega_c} = 6.72 \text{ mH} \qquad (5.7)$$
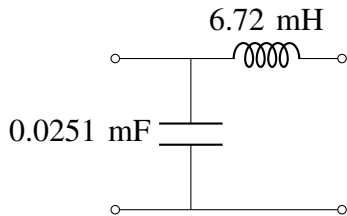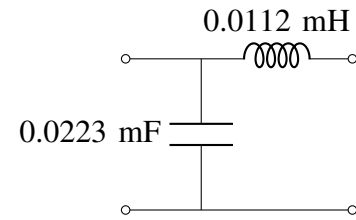
The L-C network is shown below

Fig. 5.3: L-C Butterworth Filter

5.4 Design a circuit for your Chebyshev filter.
Solution: Looking at the table of normalized g values of the Chebyshev filter for order 2 and 0.5 dB ripple, we see that

$$g_1 = C_1 = 1.4029 \qquad g_2 = L_1 = 0.7071 \quad (5.8)$$

De-normalizing these values,

$$C_1' = \frac{C_1}{\omega_c} = 2.23 \times 10^{-2} \text{ F} \qquad (5.9)$$

$$L_1' = \frac{L_1}{\omega_c} = 1.12 \times 10^{-2} \text{ H} \qquad (5.10)$$