Chaos and Strange Attractors - The Lorenz Model

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Abstract

Most real world phenomenon like weather, fluid flow and economics exhibit random states of disorder and are chaotic in nature. In this project, we present the Lorenz system to model the chaotic behaviour of dynamic systems and also show that small changes in initial conditions produce large changes in long-term outcome, leading to chaos. We studied various characteristics of the model such the steady state conditions, period doubling window etc. Additionally, we also simulated the lorenz model to study the effects of varying different parameters. The Malkus Waterwheel (Lorenz Mill) was also shown as an application of the lorenz system.

Keywords Chaos · Lorenz Model · Butterfly Effect · Strange Attractors · Malkus Waterwheel

1 The Accidental Discovery - History of Lorenz system

In 1961, Edward Norton Lorenz was using a simple digital computer to simulate weather patterns by modeling 12 variables. The idea behind the Lorenz Equation came when Lorenz ran his program with data rounded off from a previous experiment. He recalls the moment he realised the **'chaos' present in weather systems**:

"I typed in some of the intermediate conditions which the computer had printed out as new initial conditions to start another computation and then went out for a while. Afterwards, I found that the solution was not the same as the one I had before. I soon found that the reason was that the numbers I had typed in were not the same, but were rounded off numbers. The small difference between something retained to six decimal places and rounded off to three had amplified in the course of two months of simulated weather until the difference was as big as the signal itself. And to me this implied that if the real atmosphere behaved in this method then we simply couldn't make forecasts two months ahead. The small errors in observation would amplify until they became large."

To his surprise, the weather that the machine began to predict was completely different from the previous calculation. The culprit: a rounded decimal number on the computer printout. The computer worked with 6-digit precision, but the printout rounded variables off to a 3-digit number, so a value like 0.506127 printed as 0.506. This difference is tiny, and the consensus at the time would have been that it should have no practical effect. However, **Lorenz discovered that small changes in initial conditions produced large changes in long-term outcome.**

2 General Definitions

2.1 Non-Linear System of Differential Equations

Scalar first order differential equations are of the form $u_t = f(u,t)$, but non-linear system of different equations are represented as $\mathbf{u}_t = \mathbf{f}(\mathbf{u},t)$ where,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_d \end{bmatrix} \quad and \quad \mathbf{f}(\mathbf{u},t) = \begin{bmatrix} f_1(\mathbf{u},t) \\ f_2(\mathbf{u},t) \\ \vdots \\ f_d(\mathbf{u},t) \end{bmatrix}$$

are now vectors with d components. We denote by $\mathbf{u}_t$ the component-wise time-derivative; that is, $\mathbf{u}_t = \mathbf{f}(\mathbf{u}, t)$ can be written out explicitly

$$\mathbf{u}_t = \begin{bmatrix} du_1/dt \\ du_2/dt \\ \vdots \\ du_d/dt \end{bmatrix} = \begin{bmatrix} f_1(\mathbf{u}, t) \\ f_2(\mathbf{u}, t) \\ \vdots \\ f_d(\mathbf{u}, t) \end{bmatrix} = \mathbf{f}(\mathbf{u}, t)$$

2.2 Autonomous Systems

An autonomous differential equation is a system of ordinary differential equations which does not depend on the independent variable. It is of the form

$$\frac{d}{dt}X(t) = F(X(t))$$

where X takes values in n-dimensional Euclidean space and t is usually time. It is distinguished from systems of differential equations of the form

$$\frac{d}{dt}X(t) = G(X(t), t)$$

in which the law governing the rate of motion of a particle depends not only on the particle's location, but also on time; such systems are not autonomous. Autonomous systems are closely related to dynamical systems. Any autonomous system can be transformed into a dynamical system and, using very weak assumptions, a dynamical system can be transformed into an autonomous systems.

2.3 Critical Points

We have talked about autonomous systems above, and considering an autonomous system which is a non-linear set of equations, $\mathbf{u}_t = \mathbf{f}(\mathbf{u}, t)$, we say that if $\mathbf{f}(\mathbf{u}, t) = \mathbf{0}$, then $\mathbf{u}$ is called a critical point. At these points, we have equilibrium.

2.4 Stability of Critical Points

A critical point $\mathbf{u}^0$ of $\mathbf{u}' = \mathbf{f}(\mathbf{u})$ is said to be stable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that every solution $\mathbf{u} = \varphi(t)$ which at $t = t_0$ satisfies $\|\varphi(t_0) - \mathbf{u}'\| < \delta$ exists for all $t \geq t_0$ and satisfies $\|\varphi(t) = \mathbf{u}'\| < \varepsilon$ for all $t \geq t_0$.
A critical point $\mathbf{u}^0$ is said to be asymptotically stable if it is stable and there exists $\delta_0 > 0$ such that if $\left\|\varphi(t_0) = \vec{x}^0\right\| < \delta_0$ then $\lim_{t\to\infty} \varphi(t) = \mathbf{u}^0$.

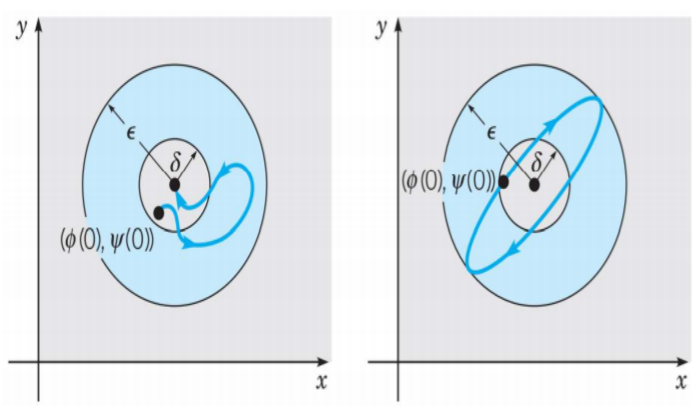

Figure 1: ϵ vs δ Relationship

The idea of stability is that if a solution "starts" near a critical point, then the solution stays near the critical point (but need not approach the critical point).

2.5 Jacobian Matrix

Consider the function $F : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, where

$$F(x_1, x_2, \ldots, x_n) = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}$$

The partial derivatives of $f_1(x_1, \ldots, x_n), \ldots, f_m(x_1, \ldots, x_n)$ (if they exist) can be organized in an $m \times n$ matrix. The Jacobian matrix of $F(x_1, x_2, \ldots, x_n)$ denoted by J_F is as follows:

$$J_F(x_1, \ldots, x_n) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_\infty} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Its importance lies in the fact that it represents the best linear approximation to a differentiable function near a given point.

3 Chaos Theory

Chaos theory is a branch of mathematics focusing on the study of chaos — dynamical systems whose apparently random states of disorder and irregularities are actually governed by underlying patterns and deterministic laws that are highly sensitive to initial conditions. Chaos is sometimes viewed as extremely complicated information, rather than as an absence of order. Chaotic systems remain deterministic, though their long-term behavior can be difficult to predict with any accuracy. This theory states that the within the apparent randomness of chaotic complex systems, there are underlying patters, interconnectedness, self-similarity, repetition and self-organization. Chaos was defined by Edward Lorenz as: "Where the present determines the future, but the approximate present does not approximately determine the future,"

3.1 The Butterfly Effect and Deterministic Chaos

This phenomenon explains the sensitive dependence of non-linear equations on initial conditions.The butterfly effect is called so because, the nonlinear equations that govern the weather have such an incredible sensitivity to initial conditions, that a butterfly flapping its wings in Brazil could set off a tornado in Texas. In popular media the 'butterfly effect' stems from the real-world implications that in any physical system, in the absence of perfect knowledge of the initial conditions (even the minuscule disturbance of the air due to a butterfly flapping its wings), our ability to predict its future course will always fail. Small differences in initial conditions, such as those due to errors in measurements or due to rounding errors in numerical computation, can yield widely diverging outcomes for such dynamical systems, rendering long-term prediction of their behavior impossible in general. A deterministic system is a system in which no randomness is involved in the development of future states of the system. Even if systems are deterministic, where the future is predictable, there is still a possibility that the future cannot be predicted due to minor changes in the initial conditions, leading to chaotic behavior. This behaviour is known as **deterministic chaos**.

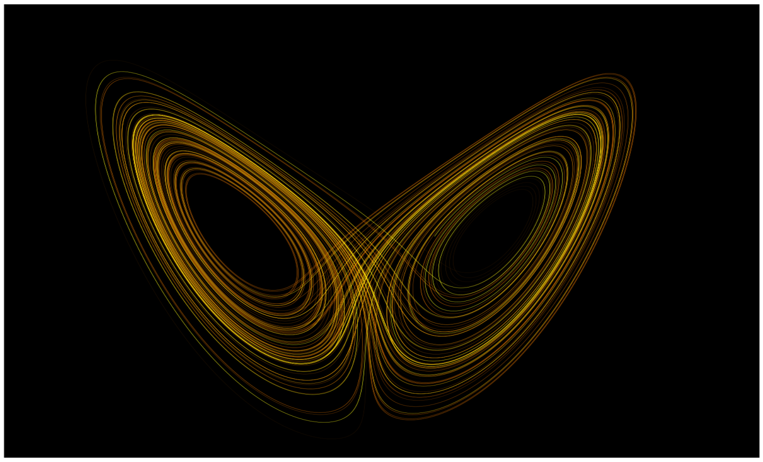

Figure 2: A plot of the Lorenz Attractor depicting the Butterfly Effect

3.2 Chaotic Dynamics

Chose in general means a state of disorder, but in Chaos Theory, it is defined more precisely, where a a dynamical system is defined as chaotic if it has the following properties:

- **Non-periodic behaviour**
  - A chaotic system may have sequences of values for the evolving variable that exactly repeat themselves, giving periodic behavior starting from any point in that sequence. However, such periodic sequences are repelling rather than attracting, meaning that if the evolving variable is outside the sequence, however close, it will not enter the sequence and in fact, will diverge from it. Thus for almost all initial conditions, the variable evolves chaotically with non-periodic behavior.
- **It must be sensitive to initial conditions**
  - Sensitivity to initial conditions means that each point in a chaotic system is arbitrarily closely approximated by other points that have significantly different future paths or trajectories. Thus, an arbitrarily small change or perturbation of the current trajectory may lead to significantly different future behavior.
- **It must be topologically transitive**
  - A map $f : X \to X$ is said to be topologically transitive if for any pair of non-empty open sets $U, V \subset X$, there exists $k > 0$ such that $f^k(U) \cap V \neq \emptyset$. Intuitively, if a map is topologically transitive then given a point x and a region V, there exists a point y near x whose orbit passes through V. This implies that is impossible to decompose the system into two open sets.

3.2.1 Example of Chaos Theory - Double Pendulum Model

The Double Pendulum is a physical extension of the simple single pendulum which changes the system of equations describing it into a complex system with a chaotic regime. Simply put, a double pendulum is a pendulum with another pendulum attached at the end of the first pendulum. This change causes the system to exhibit vastly chaotic and complex motion which is highly sensitive to the initial conditions of the system.

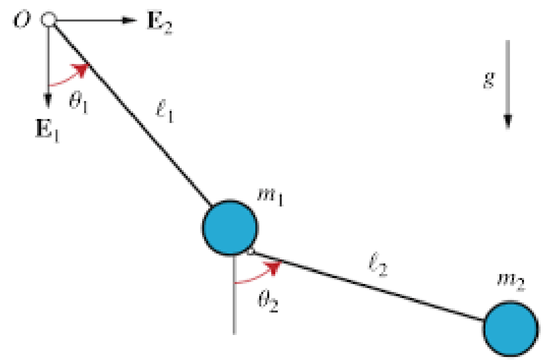

Figure 3: Analytical figure of a double pendulum system

For large motions and energy states, the system is chaotic and changes method of oscillation and motion erratically. However for small initial energy states, it is non-chaotic and is similar to the motion of the linear double spring setup.

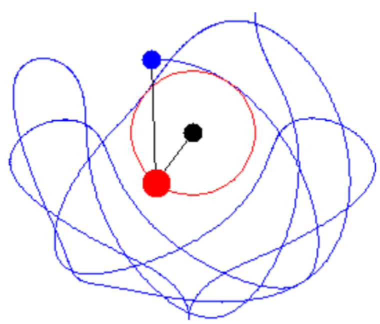

Figure 4: Erratic trajectory of a double pendulum

4 An Overview of the Lorenz Model

The Lorenz system is a system of ordinary differential equations first studied by Edward Lorenz. In 1963, Lorenz published his paper, *Deterministic Nonperiodic Flow*[1], in which he showed that **tiny differences in the initial conditions actually amount to dramatic differences in the systems behaviour over time.**

He developed a simplified mathematical model for **Rayleigh-Bénard convection**, which is an atmospheric convection model simulating a thermally driven fluid convection between two parallel plates. If the vertical temperature difference is small, then there is a linear variation of temperature with altitude but no significant motion of the fluid layer. However, if is large enough, then the warmer air rises, displacing the cooler air above it, and a steady convective motion results. If the temperature difference increases further, then eventually the steady convective flow breaks up and a more complex and turbulent motion ensues.

What drove Lorenz to find the set of three dimensional ordinary differential equations was the search for an equation that would “model some of the unpredictable behavior which we normally associate with the weather”

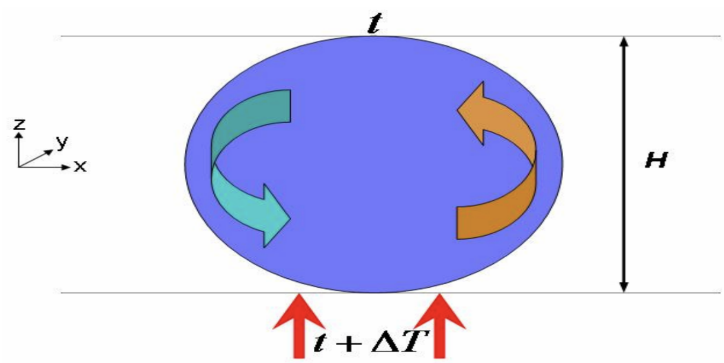

Figure 5: Atmospheric temperature convection model

In March 1963, Lorenz introduced, "ordinary differential equations whose solutions afford the simplest example of deterministic non periodic flow and finite amplitude convection". Lorenz found that when applying the Fourier Series to one of Rayleigh's convection equations that, "all except three variables tended to zero, and that these three variables underwent irregular, apparently non periodic functions". He then used these variables to construct a simple model based on the 2-dimensional representation of the earth's atmosphere. The following three coupled ordinary differential equation is known as the **Lorenz Equation**:

$$\frac{dx}{dt} = \dot{X} = \sigma(y - x)$$
$$\frac{dy}{dt} = \dot{Y} = \rho x - xz - y$$
$$\frac{dz}{dt} = \dot{Z} = xy - \beta z$$

where the three parameters σ, ρ, β are positive and are called the Prandtl number, the Rayleigh number, and a physical proportion, respectively. It is important to note that the x, y, z are not spacial coordinates. The "x is proportional to the intensity of the convective motion, while y is proportional to the temperature difference between the ascending and descending currents, similar signs of x and y denoting that warm fluid is rising and cold fluid is descending. The variable z is proportional to the distortion of vertical temperature profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries."

4.1 Derivation of the Lorenz Equations

The Lorenz equations were derived from the convection equations of Saltzman which came from the investigation of a fluid of uniform depth H, with a temperature difference between upper and lower layer of ΔT , in particular with linear temperature variation. In the special case where there is no variation with respects to the y-axis, Saltzman provided the governing equations:

$$\frac{\partial}{\partial t}\nabla^2\psi = -\frac{\partial\left(\psi, \nabla^2\psi\right)}{\partial(x,z)} + v\nabla^4\psi + g\alpha\frac{\partial\theta}{\partial x}$$
$$\frac{\partial}{\partial t}\theta = -\frac{\partial(\psi,\theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial\psi}{\partial x} + \kappa\nabla^2\theta$$

where ψ is a stream function for the two-dimensional motion, θ is the temperature deviation from the steady state. ψ , $\nabla^2\psi$ vanish at the upper and lower boundaries, and g , α , v , κ are, respectively, constants of gravitational acceleration, coefficient of thermal expansion, kinematic viscosity and thermal conductivity. Rayleigh discovered that at a critical point (known now as the Rayleigh number) these equations show a convective motion. Lorenz then defined three time dependent variables:

- X proportional to the intensity of the convection motion
- Y proportional to the temperature difference between the ascending and descending currents.

- Z proportional to the difference of the vertical temperature profile from linearity.

Substituting the above into the Saltzman equations, and further derivation, we get Lorenz equations:

$$\frac{dX}{dt} = \dot{X} = \sigma(Y - X)$$
$$\frac{dY}{dt} = \dot{Y} = \rho X - XZ - Y$$
$$\frac{dZ}{dt} = \dot{Z} = XY - \beta Z$$

5 Characteristics of the Model

The Lorenz system is **nonlinear, non-periodic, three-dimensional and deterministic**. The first equation is linear, but the **second and third equations involve quadratic non-linearities**. It is notable for having chaotic solutions for certain parameter values and initial conditions. It shows that physical systems can be completely deterministic and yet still be inherently unpredictable even in the absence of quantum effects.

- **Symmetry**
  The Lorenz equation has the following symmetry of ordinary differential equation. They are invariant under the following transformation:
  $$(X, Y, Z) \mapsto (-X, -Y, Z)$$
  Proof : By substitution of the transformation into the Lorenz equations,
  $$-\frac{dX}{dt} = \sigma(-(-X) + (-Y)) \Rightarrow \frac{dX}{dt} = \sigma(Y - X)$$
  $$-\frac{dY}{dt} = \rho(-X) - (-Y) - (-X)Z \Rightarrow \frac{dY}{dt} = X(\rho - Z) - Y$$
  $$\frac{dZ}{dt} = -\beta(Z) + (-X)(-Y) \Rightarrow \frac{dZ}{dt} = XY - \beta Z$$

- **Dissipative**
  The Lorenz system is dissipative i.e. volumes in phase-space contract under the flow. (As t -> ∞, V -> 0)

- **Invariance of z-axis**
  The invariance of the Z-axis implies that a solution that starts on the z-axis (i.e. x=y=0) will remain on the z-axis. Therefore, all the trajectories on the Z-axis will remain on the Z -axis, and approach the origin. Furthermore, since
  $$X = 0, Y > 0 \Rightarrow \frac{dX}{dt} > 0 and X = 0, Y < 0 \Rightarrow \frac{dX}{dt} < 0$$
  All trajectories that rotate about the Z-axis must move clockwise with increasing time (looking from above onto the XY plane).

- **Equilibrium Points**
  By solving the lorenz equations, there are three equilibrium points - (0,0,0), K_1=(-$\sqrt{\beta(\rho-1)}$, $-\sqrt{\beta(\rho-1)}$,ρ - 1), K_2 = ($\sqrt{\beta(\rho-1)}$, $\sqrt{\beta(\rho-1)}$,ρ - 1)

5.1 Fixed points of the Lorenz Model

A fixed point is a point that does not change upon application of a map, system of differential equations, etc. In particular, a fixed point of a function $f(x)$ is a point x_0 such that

$$f(x_0) = x_0$$

Fixed points are also called critical points or equilibrium points. Points of an autonomous system of ordinary differential equations at which:

$$\begin{cases} \frac{dx_1}{dt} = f_1(x_1, \ldots, x_n) = 0 \\ \vdots \\ \frac{dx_n}{dt} = f_n(x_1, \ldots, x_n) = 0 \end{cases}$$

are known as fixed points.

If a variable is slightly displaced from a fixed point, it may
(1) move back to the fixed point ("asymptotically stable" or "superstable"),
(2) move away ("unstable"), or
(3) move in a neighborhood of the fixed point but not approach it ("stable" but not "asymptotically stable").

While Lorenz found chaos to be a large factor in meteorology, the equation he created does not exhibit chaos for all parameters. In fact, there are many parameter values where the function is stable and contain fixed points. We will now explore how Lorenz came to realize fixed points of his system, as well as for what values the equation exhibits chaos.

To find fixed points of the Lorenz Equation, we will first solve for its equilibria. To find these equilibrium points, we will set $\dot{X}$, $\dot{Y}$ and $\dot{Z}$ to 0.

$$\sigma(Y - X) = 0$$
$$\rho X - XZ - Y = 0$$
$$XY - \beta Z = 0$$

From the first equation we have Y = X. Then, eliminating Y from the second and third equations, we obtain

$$Y = \rho X - XZ$$
$$X = X(\rho - Z) => Z = \rho - 1$$
$$X^2 - (\rho - 1)\beta = 0$$
$$X = Y = \pm\sqrt{\beta(\rho - 1)}$$

From above, the three equilibrium points are - (0,0,0), K_1 = (-$\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}$,$\rho$ - 1), K_2 = ($\sqrt{\beta(\rho-1)}$, $\sqrt{\beta(\rho-1)}$,ρ - 1)

The behaviour of the Lorenz Equation is complex, so we consider σ = 10 and β = 8/3, and study the effects with change of ρ.

- **Case 1:** 0 < ρ < 1
  (0, 0, 0) yields the only real fixed point of the equilibrium points. There is no chaos when 0 < ρ < 1
- **Case 2:** ρ = 1
  We identify ρ = 1 as a bifurcation point, as the other two equilibrium points will appear when ρ > 1. There is no chaos when ρ = 1
- **Case 3:** ρ > 1
  We consider two fixed points K_1 = (-$\sqrt{\beta(\rho-1)}, -\sqrt{\beta(\rho-1)}$,$\rho$ - 1), K_2 = ($\sqrt{\beta(\rho-1)}$, $\sqrt{\beta(\rho-1)}$,ρ - 1) We check the stability of these points by linearizing the Lorenz Equation and finding its eigenvalues. A point is stable when its eigenvalues are all negative. We will linearize the system near an already established equilibrium point from above, call it $(\bar{X}, \bar{Y}, \bar{Z})$ using the Jacobian Matrix. This gives

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} -\sigma & \sigma & 0 \\ \rho - \bar{Z} & -1 & \bar{X} \\ \bar{Y} & \bar{X} & -\beta \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

  To get the eigenvalues of the above 3×3 matrix (call it A), we solve $det(A - \lambda I) = 0$ This yields

$$det \begin{bmatrix} -\sigma - \lambda & \sigma & 0 \\ \rho - \bar{Z} & -1 - \lambda & \bar{X} \\ \bar{Y} & \bar{X} & -\beta - \lambda \end{bmatrix} = 0$$

$$\lambda^3 + (\beta + \sigma + 1)\lambda^2 + \left(\beta + \beta\sigma + \sigma - \rho\sigma + \sigma\bar{Z} + \bar{X}^2\right)\lambda + \beta\sigma(1-\rho) + \sigma\left(\overline{XY} + \bar{X}^2 + \beta\bar{Z}\right) = 0 \qquad (1)$$

If we take $(\bar{X}, \bar{Y}, \bar{Z})$ to be the equilibrium point $(0,0,0)$, we get

$$\lambda^3 + (\beta + \sigma + 1)\lambda^2 + (\beta + \beta\sigma + \sigma - \rho\sigma)\lambda + \beta\sigma(1-\rho) = 0$$

$-\beta$ is a solution, so we can factor to get

$$(\lambda + \beta)\left(\lambda^2 + (\sigma+1)\lambda + \sigma(1-\rho)\right) = 0$$

Thus the eigenvalues are

$$\lambda_1, \lambda_2 = \frac{-\sigma - 1 \pm \sqrt{(\sigma+1)^2 + 4\sigma(\rho-1)}}{2}, \lambda_3 = -\beta$$

When $\rho > 1$ however, $\lambda_1 > 0\lambda_2, \lambda_3 < 0$ and $(0,0,0)$ is not stable and thus not a fixed point for $\rho > 1$ If we take $(\bar{X}, \bar{Y}, \bar{Z})$ to be either K_1 or K_2 and plug them into (1), we end up with eigenvalues

$$\mu^3 + (\beta + \sigma + 1)\mu^2 + (\sigma + \rho)\beta\mu + (1-\rho)2\sigma\beta = 0$$

All three eigenvalues μ_1, μ_2, μ_3 will be negative when

$$\rho < \frac{\sigma(\sigma + \beta + 3)}{\sigma - \beta - 1} = \rho_c$$

Substituting $\sigma = 10, \beta = 8/3$, we get $\rho < \frac{470}{19} \approx 24.74 = \rho_c$ Thus, $\mathrm{K}_1,\ \mathrm{K}_2$ are stable and fixed points when $1 < \rho < 24.74$. When $\rho \geq 24.74$, not all of μ_1, μ_2, μ_3 will be negative, and $\mathrm{K}_1,\ \mathrm{K}_2$ will not be stable and thus not fixed points.

At $\rho > \rho_c$, the three equilibrium points $((0,0,0), \mathrm{K}_1,\ \mathrm{K}_2\ )$ are unstable and are not fixed points. They do not approach infinity, but rather enter a region around the origin. This is where we begin to see chaotic behaviour of the Lorenz Equation.

Value of ρ	**Fixed Points**
$[0-1]$	$(0,0,0)$
$(1, 24.74)$	$\mathrm{K}_1,\ \mathrm{K}_2$
$[24.74 - 30.1)$	None, chaos occurs
$[30.1 - \infty)$	intermittency

5.2 Sensitivity to Initial Conditions (Butterfly Effect)

The solutions of the Lorenz equations are extremely sensitive to perturbations in the initial conditions. Usually, if one were to start any system from two extremely close points then the trajectories should remain extremely close for all time. However, this is not the case with the Lorenz equations, no matter how close you put two starting points, within a relative time scale the trajectories will part and take completely independent paths

Beyond a certain time, the long term predictions of physical systems are impossible due to this property, eg. weather. Given two starting trajectories in the phase space that are infinitesimally close, with initial separation δZ_0, the two trajectories end up diverging at a rate given by

$$|\delta \mathbf{Z}(t)| \approx e^{\lambda t} |\delta \mathbf{Z}_0|$$

where t is the time and λ is the Lyapunov exponent. The Lyapunov exponent measures the sensitivity to initial conditions, in the form of rate of exponential divergence from the perturbed initial conditions.

Simulating the Lorenz equations from $X_1 = X_2 = 10$,$Y_1 = Y_2 = 0$, $Z_1 = 10$ and $Z_2 = 10.00000000001$, then for the first 25 time units the trajectories appear to be identical but beyond 30 time units they are totally unrelated to each other.

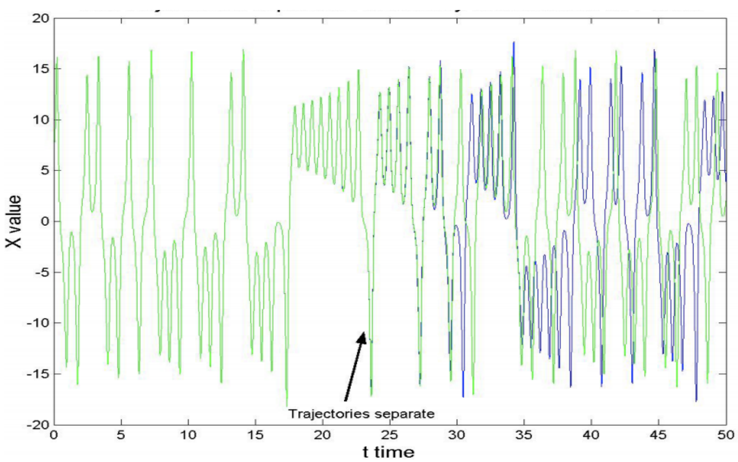

Figure 6: Separation of two trajectories when initial points differ by 10^{-10} units

5.3 Period Doubling Windows

The period doubling window property occurs when the trajectories enter a stable orbit around the two steady state solutions, even though the steady state solutions are unstable. This property holds only for values of ρ above ρ_c(=24.74).

Taking examples of some different period doubling windows:

With period doubling window for values 99.53 < ρ < 100.79, σ =10 and b = 8/3

- **[1-2-2] -** In this the trajectories move around one steady state solution and then twice around the second and back to the first once more (denoted [1-2-2]).

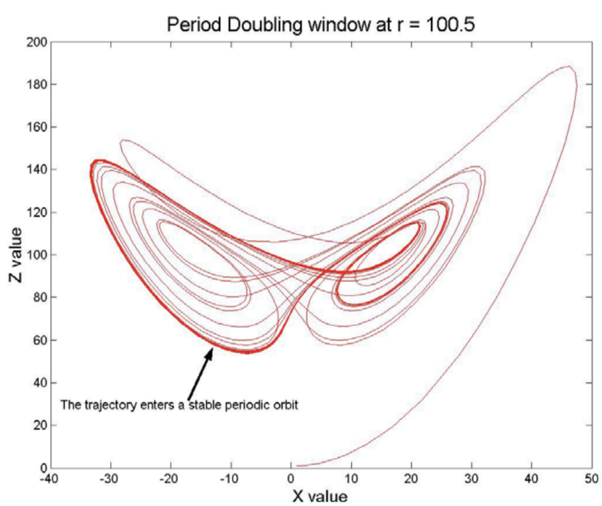

Figure 7: [1-2-2]

- $[1-2-2]^2$ **-** As the value of r decreases the stable orbit widens or doubles, creating the orbit to do as above but then repeat it once more before returning to the original orbit (denoted [1-2-2-1-2-2] or **[1-2-2]**^{2}).

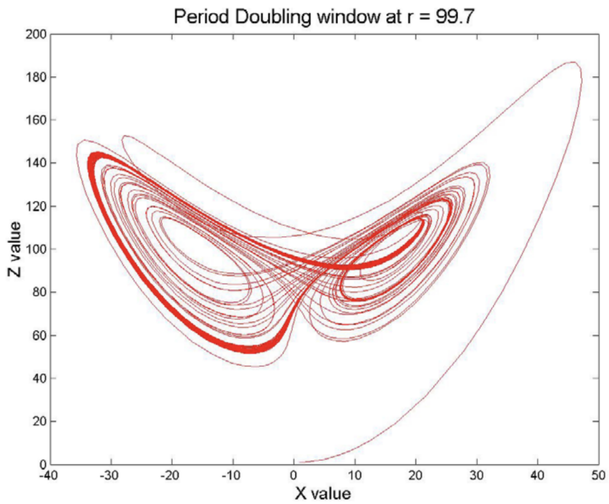

Figure 8: $[1-2-2]^2$

- **[1-2-2] to [2-1-2] -** If we continue to decrease the value of ρ further, the cascade effect still occurs but the orbits reflect and therefore the trajectories orbit the other steady state solution twice whereas before it would have been just one (i.e. [1-2-2] to [2-1-2]).

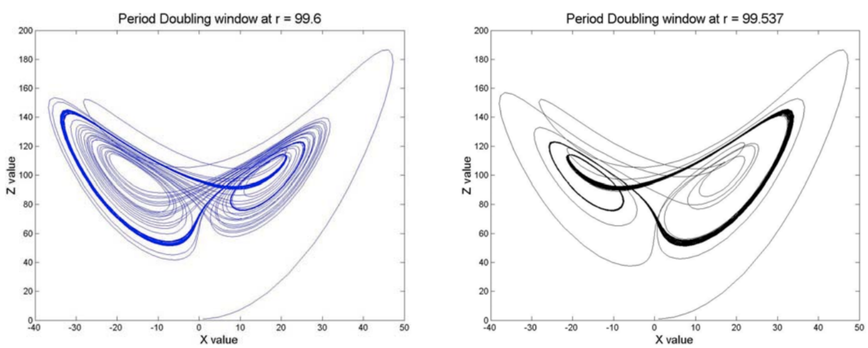

Figure 9: Orbit flipping from [1-2-2] to [2-1-2]

With period doubling window for values 145 < ρ < 166, σ =10 and b = 8/3 This period doubling window is symmetric about the two steady state solutions (denoted [2-2-2])

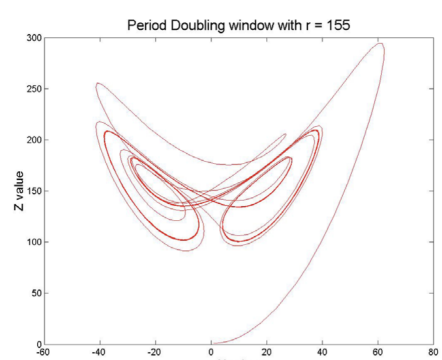

Figure 10: [2-2-2]

5.4 Intermittent Chaos

In dynamical systems, intermittency is the irregular alternation of phases of apparently periodic and chaotic dynamics. When the trajectories are close to a period doubling window (below or above the window), the trajectories exhibit intermittent chaos. This is where the trajectory can be stable oscillatory and then suddenly switches to chaos and then revert back to the oscillatory trajectory. Moreover, as one moves further and further from the windows, the intermittent chaos seems to become more and more frequent until it becomes dominant and then pure chaos returns.

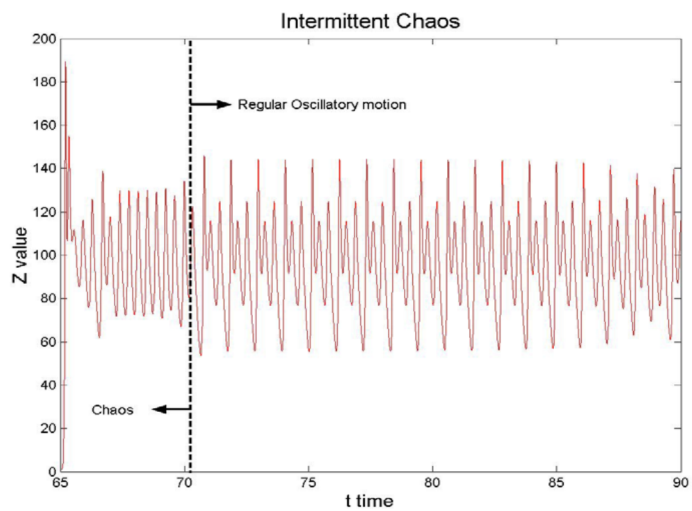

Figure 11: Separation of two trajectories when initial points differ by 10^{-10} units.

Chaos exists and as one moves towards the periodic window intermittent chaos appears until chaos ceases leaving periodic orbits in the window and then as one moves out of the window the intermittent chaos returns once more and finally as one increases still further the chaos totally takes over once again.

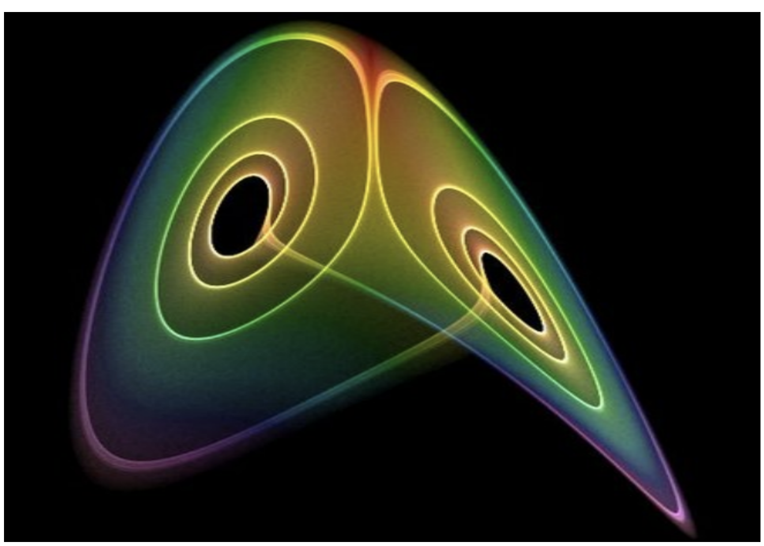

Figure 12: Lorenz attractor showing intermittency. The system spends long periods close to the bright periodic orbit, occasionally moving away for phases of chaotic dynamics that cover the rest of the attractor.

5.5 Attraction towards Steady State solutions

To analyse the lorenz system with values close to ρ_c, consider the values σ =10, b = 8/3 and ρ = 22.2, caculating the steady state solution we get,

$$\left(\pm\sqrt{\frac{848}{15}}, \pm\sqrt{\frac{848}{15}}, \frac{106}{5}\right) \approx (\pm 7.52, \pm 7.52, 21.2)$$

With ρ set, the theory implies that the system should be stable (that is any trajectory should in fact tend to one of the given steady state solutions). The graph below shows the X vs Z representation with the above initialised values, tending to steady state.

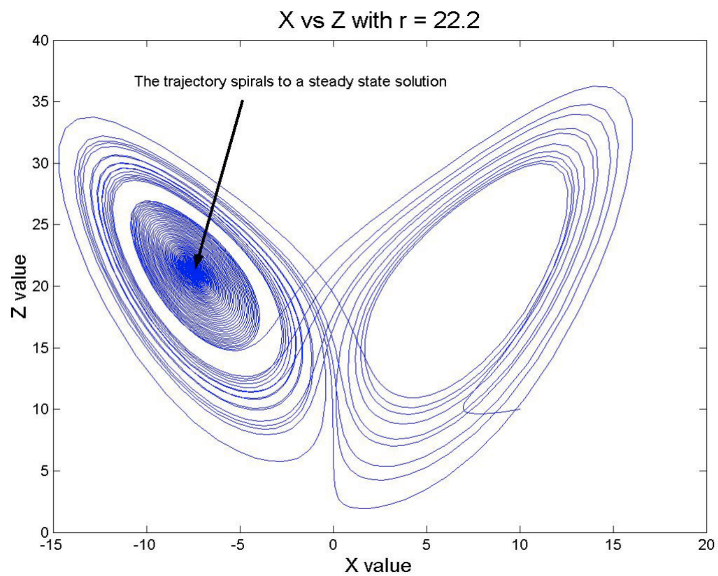

Figure 13: The trajectory enters a region where it is attracted to one of the steady solution

X versus time graph shows the pre-chaotic state where the solution seems to be chaotic but then settles down to oscillatory and finally tends to the steady state solution.

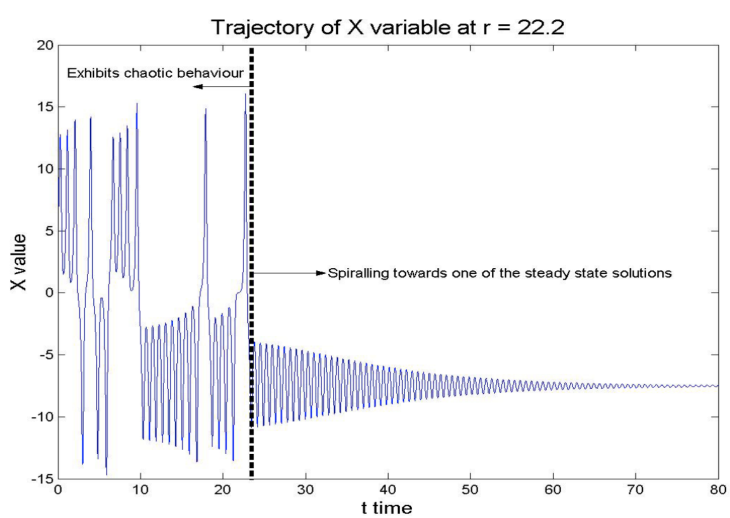

Figure 14: Chaotic behaviour later spiralling towards a steady state solution

6 Lorenz Attractor - a Paradigm for Chaos

The Lorenz attractor is a set of chaotic solutions of the Lorenz system.

6.1 What is an Attractor?

In the field of dynamical systems, an attractor is a set of numerical values toward which a system tends to evolve, for a wide variety of starting conditions of the system. System values that get close enough to the attractor values remain close even if slightly disturbed.

An attractor is a set of points to which all neighbouring trajectories converge. Stable fixed points and stable limit cycles are examples. **An attractor is a closed set A with the following properties:**

1. **A is an invariant set:** any trajectory x(t) that starts in A stays in A for all time ($t \to \infty$).
2. **A attracts an open set of initial conditions:** there is an open set U containing A such that if $x(0) \in U$, then the distance from x(t) to A tends to zero as $t \to \infty$. Hence A attracts all trajectories that start sufficiently close to it. The largest such U is called the basin of attraction of A.
3. **A is minimal:** there is no proper subset of A that satisfies conditions (1) and (2).

An attractor is defined as the smallest unit which cannot be itself decomposed into two or more attractors with distinct basins of attraction. This restriction is necessary since a dynamical system may have multiple attractors, each with its own basin of attraction.

Conservative systems do not have attractors, since the motion is periodic. For dissipative dynamical systems, however, volumes shrink exponentially so attractors have 0 volume in n-dimensional phase space.

6.2 Strange Attractor

A strange attractor is said to be an attractor that exhibits sensitive dependence on initial conditions and it has a fractal geometric structure. It is an attracting set that has zero measure in the embedding phase space and has fractal dimension. Trajectories within a strange attractor appear to skip around randomly. The Lyapunov exponent $\lambda > 0$ for a strange attractor.

- **FRACTAL Geometric structure:** A fractal is a never-ending pattern. Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over in an ongoing feedback loop. Driven by recursion, fractals are images of dynamic systems – the pictures of Chaos.

Figure 15: Fractal

The trajectories of the attractor cannot cross or merge, hence the two surfaces of the strange attractor can only appear to merge. **Lorenz** concluded that **"there is an infinite complex of surfaces"** where they appear to merge. Today this "infinite complex of surfaces" would be called a **FRACTAL**.

A fractal is a set of points with zero volume but infinite surface area.

Usually strange attractors are chaotic, but strange nonchaotic attractors also exist. If a strange attractor is chaotic, exhibiting sensitive dependence on initial conditions, then any two arbitrarily close alternative initial points on the attractor, after any of various numbers of iterations, will lead to points that are arbitrarily far apart (subject to the confines of the attractor), and **after any of various other numbers of iterations will lead to points that are arbitrarily close together.**

Thus a dynamic system with a chaotic attractor is locally unstable yet globally stable: once some sequences have entered the attractor, nearby points diverge from one another but never depart from the attractor. The dynamics inside the chaotic attractors is peculiar it manages to combine attracting flow with exponential divergence of the same flow.

6.3 Theory of Lorenz Attractor - A Strange Attractor

The Lorenz attractor is an example of a strange attractor. Strange attractors are unique from other phase-space attractors in that one does not know exactly where on the attractor the system will be. Two points on the attractor that are near each other at one time will be arbitrarily far apart at later times. The only restriction is that **the state of system remain on the attractor.**

Strange attractors are also unique in that they never close on themselves — the motion of the system never repeats (non-periodic). **The Lorenz attractor was the first strange attractor**, but there are many systems of equations that give rise to chaotic dynamics. Examples of other strange attractors include the Rössler and Hénon attractors.

6.4 Simulations and Results for Lorenz attractor

The Lorenz System demonstrates chaos at certain parameter values and its attractor is fractal.

- The initial condition values set are
$$x = 0.1, y = 1, z = 1$$
- The system parameters are set to as follows:
$$\sigma = 10, \rho = 28, \beta = \frac{8}{3}$$

With these parameter values, the system exhibits deterministic chaos. It has a strange attractor with a fractal structure.

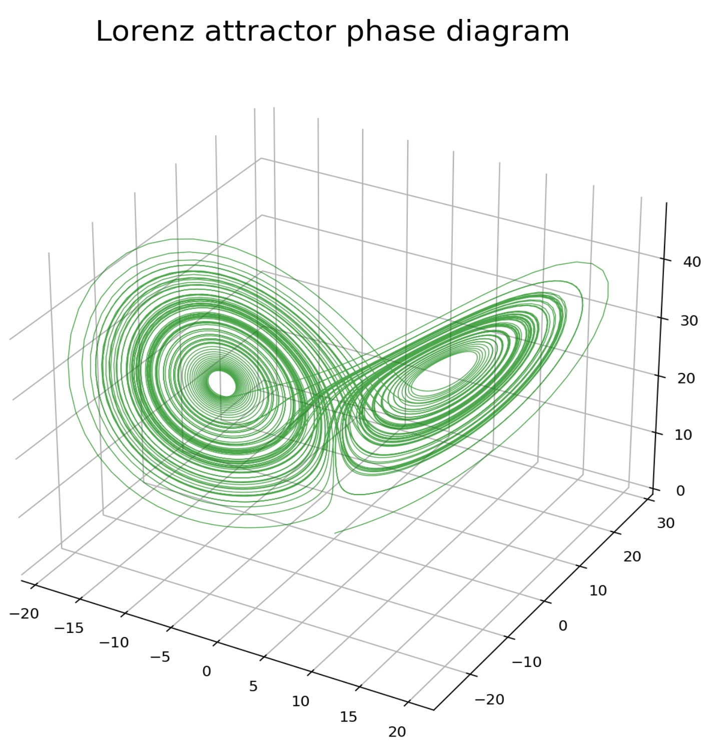

Figure 16: Lorenz Attractor Phase Diagram

In three dimensions, these trajectories never overlap and the system never lands on the same point twice, due to its fractal geometry. We visualize the same trajectory phase planes.

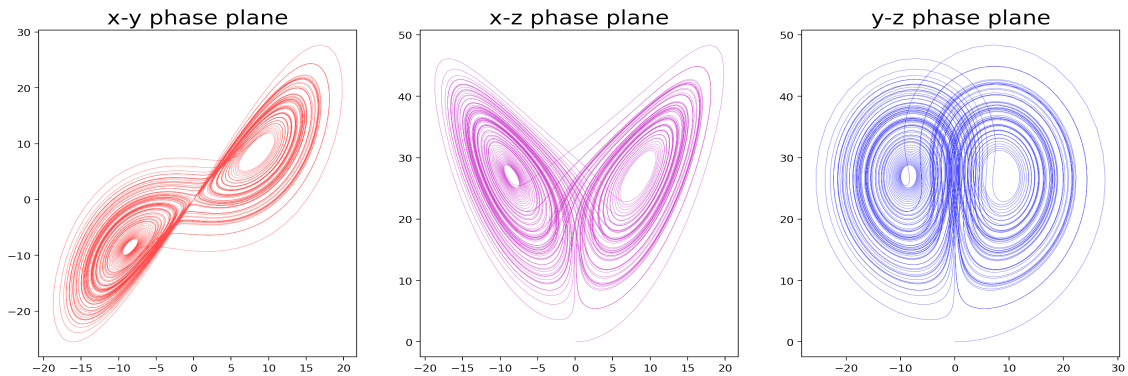

Figure 17: Lorenz Attractor Phase Diagram

6.5 Bifurcation Analysis of Lorenz Attractor

- **Pitchfork Bifurcation**
  In bifurcation theory, a pitchfork bifurcation is a type of local bifurcation where the system transitions from one fixed point to three fixed points. The standard form of the Pitchfork bifurcation is:

$$\frac{dx}{dt} = x\left(a - bx^2\right)$$

  From the standard form it is obvious that the steady state solutions are at $x = 0, \pm\sqrt{\frac{a}{b}}$, and so we know there are three solutions.
  There are always three solutions, but they are not all stable at the same time.

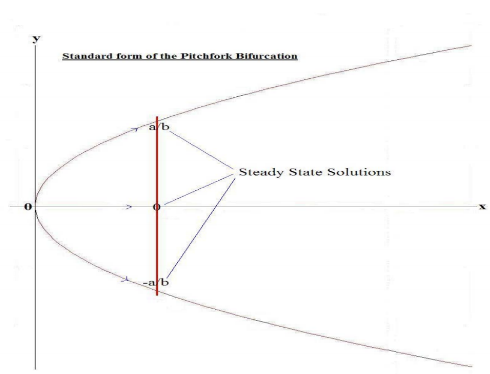

Figure 18: Steady state solutions of Pitchfork Bifurcation

- **Hopf Bifurcation**
  A Hopf bifurcation is a critical point where a system's stability switches and a periodic solution arises. It is a local bifurcation in which a fixed point of a dynamical system loses stability, as a pair of complex conjugate eigenvalues — of the linearization around the fixed point—crosses the complex plane imaginary axis. Pitchfork and Hopf bifurcations have two types – supercritical and subcritical.
  The normal form of a Hopf bifurcation is:

$$\frac{dz}{dt} = z\left((\lambda + i) + b|z|^2\right)$$

  where z, b are both complex and λ is a parameter. $b = \alpha + i\beta$. The number a is called the first Lyapunov coefficient.

– If a is negative then there is a stable limit cycle for $\lambda > 0$:

$$z(t) = re^{i\omega t}$$

where

$$r = \sqrt{-\lambda/\alpha} and \omega = 1 + \beta r^2$$

The bifurcation is then called **supercritical**.

– If a is positive then there is an unstable limit cycle for $\lambda < 0$. The bifurcation is called **subcritical**.

For $r < 1, (\sigma = 10, \beta = 8/3)$ there is only one stable fixed point located at the origin. At this point, all orbits converge to the origin - a global attractor. (r = ρ here)

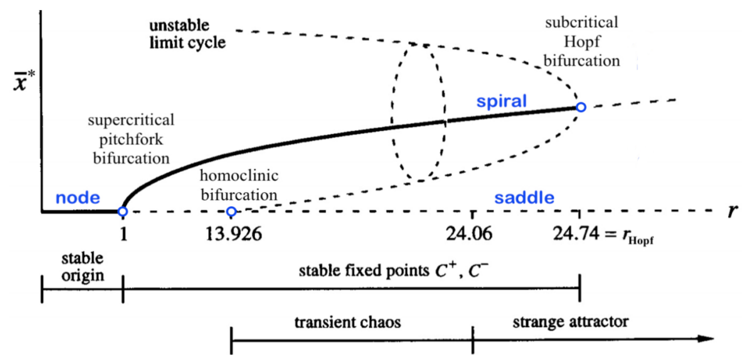

Figure 19: Behaviour for small values of r while keeping other parameters constant where x is the distance from the origin. The origin is globally stable node for r < 1. At r = 1, the origin loses stability by a super-critical pitchfork bifurcation, and a symmetric pair of attracting fixed points (stable spirals) is born, in the above schematic, only one of the pair is shown. At r_{Hopf} = 24.74 the fixed points lose stability.

A supercritical pitchfork bifurcation occurs at $r = 1$, and for $r > 1$ two additional fixed points appear at

$$(x^*, y^*, z^*) = C^{\pm} = (\pm\sqrt{\beta(r-1)}, \pm\sqrt{\beta(r-1)}, r-1)$$

These correspond to steady convection. Fixed points are stable only if

$$r < r_{Hopf}, \quad r_{Hopf} = \sigma\frac{\sigma + \beta + 3}{\sigma - \beta - 1} = 24.74,$$

which can hold only for positive r and $d > b + 1$. At a critical value $r = r_{Hopf}$, both stable fixed points lose stability through a **subcritical Hopf bifurcation**.

As we decrease r from $r_{Hopf,}$ the unstable limit cycles expand and pass precariously close to the saddle point at the origin. At $r = 13.926$, the cycles touch the saddle point and become homoclinic orbits; hence we have a homoclinic bifurcation which is referred to as the *first homoclinic explosion*. Below $r = 13.926$ there are no limit cycles.

The region $13.926 < r < 24.06$ is referred to as transient chaos region. Here, the chaotic trajectories eventually settle at C^+ or C^-.

For $r > 24.06$ and $r > r_{Hopf} = 28$ (immediate vicinity): no stable limit-cycles exist; trajectories do not escape to infinity (dissipation). Almost all initial conditions (I.C.s) will tend to an invariant set – the Lorenz attractor – a strange attractor and a fractal.

For $r \gg r_{Hopf}$ different types of chaotic dynamics exist e.g. noisy periodicity, transient and intermittent chaos. One can even find transient chaos settling to periodic orbits.

6.6 The Lorenz Map - Order from Chaos

6.6.1 Creation of 1-D Maps using Coweb diagram

The general form of 1-D map is the following

$$x_{n+1} = f\left(x_n\right),$$

where f is the given function, $n \in \mathbb{Z}^+$ is the number of iterates applied to an initial condition $x_0 \in \mathbb{R}$ or $x_0 \in \mathbb{C}$.

A graphical way of generating the iterates x_n is to construct the cobweb diagram.

To construct a cobweb diagram one ideally selects an initial condition x_0 in the basin of the map. The basin of a map includes values x_n that return other points on the same map and not the infinity. Move vertically to the function f(x) and register the result, move horizontally to the diagonal and repeat. The resulting map iterates x_n

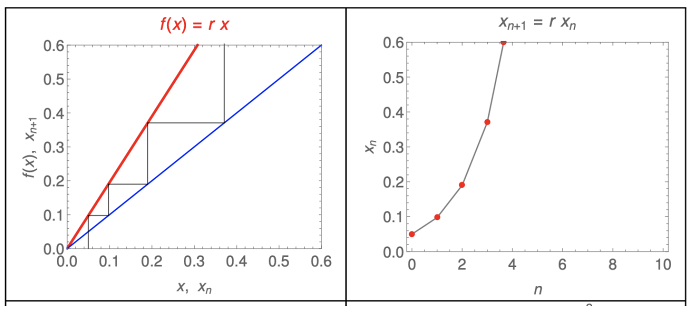

Figure 20: An example of 1-D map

6.6.2 Plotting the Lorenz Map

Lorenz stated that *"the trajectory apparently leaves one spiral only after exceeding some critical distance from the center. Moreover, the extent to which this distance is exceeded appears to determine the point at which the next spiral is entered; this in turn seems to determine the number of circuits to be executed before changing spirals again. It therefore seems that* ***some single feature of a given circuit should predict the same feature of the following circuit****."*

Considering the Z-Y view of the Lorenz strange attractor, the following $z(t)$ graph was created.

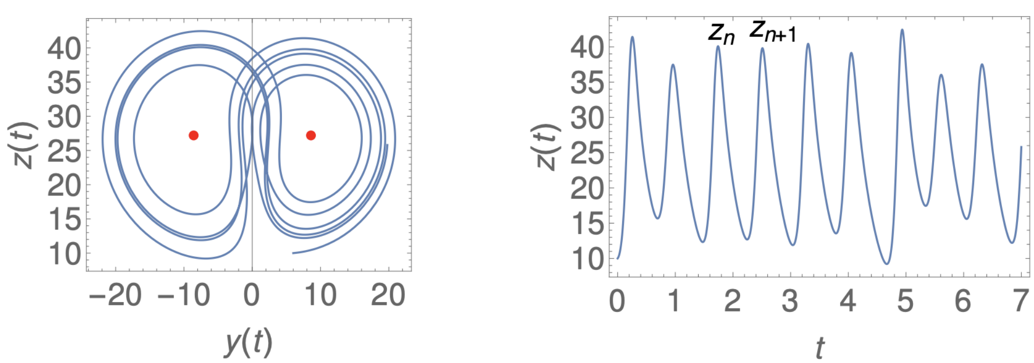

Figure 21: Z-Y view of Lorenz attractor and Z vs time graph

The **"single feature"** he focused on was z_n, the nth local maximum of $z(t)$.

Lorenz's idea is that z_n should predict z_{n+1}. To check this, he numerically integrated the equations for $t \gg 1$, then found the local maxima of $z(t)$, and finally plotted z_{n+1} vs z_n. The data from the chaotic time series appear to fall neatly on a curve.

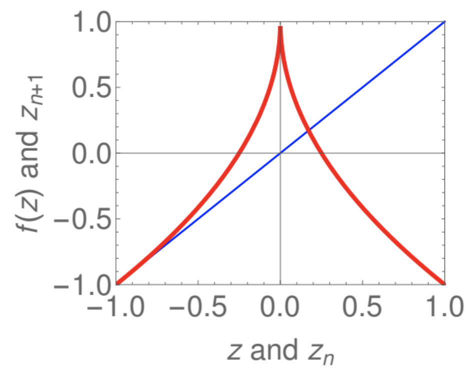

Figure 22: Lorenz Map

By this ingenious trick, Lorenz was able to extract order from chaos. The normalised map $z_{n+1} = f(z_n)$ shown above is called the Lorenz map.

7 Applications

The Lorenz equations are used in simplified models for lasers, dynamos, thermosyphons, brushless DC motors, electric circuits, chemical reactions and forward osmosis. The chaotic Lorenz model is significantly important to show that **long term weather prediction and economic forecast is not possible**.

The Lorenz model is also used to show the chaotic motion in Malkus waterwheel (Lorenz mill).

7.1 Lorenz Mill

The Lorenz Mill is a physical experiment designed to shed light on and demonstrate the application of the Lorenz system of equations. The design consisted of a water wheel with buckets at the ends of the spokes of the wheel. It is similar in construction to a regular water wheel except that the buckets have a leak from which water can flow out. Water is poured in from the top through a tap constantly at a fixed rate, providing energy to the system at a steady rate while the leaking water depletes energy from the system at a steady rate. The spinning wheel displays chaotic motion, speeding up, slowing down, changing directions, stopping and oscillating erratically between these motions. Like most chaotic systems, at a low rate of flow of water from the tap above the waterwheel, the motion is non-chaotic and the wheel rotates in the same circular direction but after a critical rate of influx of water and energy, the system enters into a chaotic state with almost unpredictable motions.

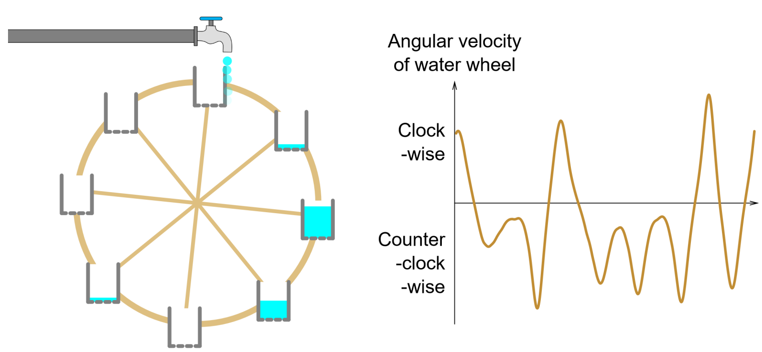

Figure 23: The design of the Lorenz water mill

If the values of the x and y coordinates of the center of mass of the water wheel are logged along with the angular velocity of the wheel, they form a system of Lorenz equations. Upon graphing them we get an output in the shape of the Lorenz strange attractor.

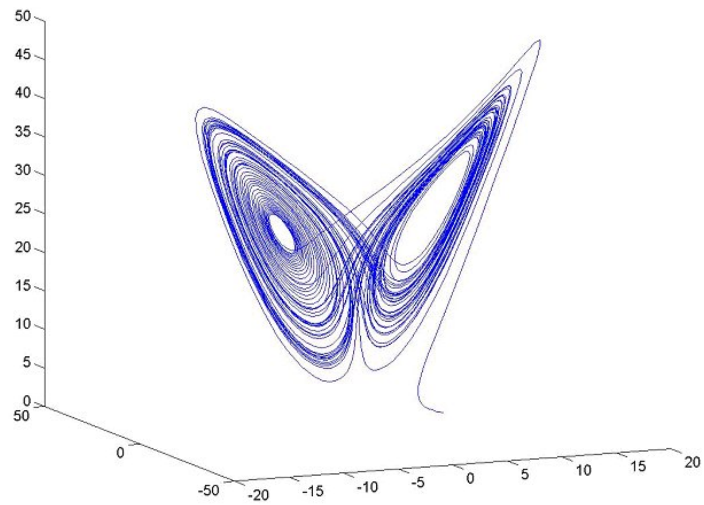

Figure 24: Plot of the angular velocity and x and y coordinates of the center of mass of the Lorenz mill

Since its proposal this model has been used in other applications such as oceanic flow and electro-rotation dynamics.

8 Conclusion

Though discovered by accident, the Lorenz Equation has had a significant contribution to mathematics and many other disciplines. In particular, the Lorenz Equation helped pioneer the study of chaos and sensitive dependence on initial conditions. The Lorenz Equation has made quantifying chaos possible which has inspired many mathematicians to research and study chaos.

9 Acknowledgements

The completion of this project has only been possible because of the guidance and assistance we received from various resources and we are grateful for this opportunity that we got to further our learning.

We would like to take this opportunity to thank our professor , Lakshmi Burra for motivating us to work on this project as a part of our coursework in “Differential Equations”.

We are extremely thankful to ma’am for the constant support and guidance, and widening our knowledge on the topic.

References

[1] Lorenz, E. N. “Deterministic Nonperiodic Flow,” In *Journal of the Atmospheric Sciences 20 (1963)*, pp. 130–141.

[2] Sparrow, C. The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors in *(New York/Berlin: Springer-Verlag, 1982).*

[3] J. Yorke and E. Yorke Metastable chaos: The transition to sustained chaotic behavior in the lorenz model in *Journal of Statistical Physics*

[4] P. Grassberger and I. Procaccia. Measuring the strangeness of strange attractors *Physica D: Nonlinear Phenomena*

[5] Christopher T. M. Clack DYNAMICS OF THE LORENZ EQUATIONS in *School of Mathematics*

[6] Chaos Theory and the Lorenz Equation: History, Analysis, and Application - *University of Waterloo*