DS 5220 Homework 1

1. Given $\vec{u} = [2,5,1]^T$, $\vec{v} = [3,0,4]^T$, $\vec{w} = [1,2,5]^T$ for finding

a. $3\vec{u} + 4\vec{u}$

$$\Rightarrow 3\begin{bmatrix}2\\5\\1\end{bmatrix} + 4\begin{bmatrix}3\\0\\4\end{bmatrix} =$$

$\therefore$ Transpose - rows will be changed to columns and vice versa

$$\Rightarrow \begin{bmatrix}6\\15\\3\end{bmatrix} + \begin{bmatrix}12\\0\\16\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}18\\15\\19\end{bmatrix} = 3\vec{u} + 4\vec{v}$$

b. $2\vec{u} + 5\vec{v} - 4\vec{w}$

$$\Rightarrow 2\begin{bmatrix}2\\5\\1\end{bmatrix} + 5\begin{bmatrix}3\\0\\4\end{bmatrix} + -4\begin{bmatrix}1\\2\\5\end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix}4\\10\\2\end{bmatrix} + \begin{bmatrix}15\\0\\20\end{bmatrix} - \begin{bmatrix}4\\8\\20\end{bmatrix} \Rightarrow$$

$$\Rightarrow \begin{bmatrix}19\\10\\22\end{bmatrix} - \begin{bmatrix}4\\8\\20\end{bmatrix} = \begin{bmatrix}15\\2\\2\end{bmatrix} = 2\vec{u} + 5\vec{v} - 4\vec{w}$$

2. Given c_1, c_2, c_3 are the coefficients and $u_1 = [1,-1,1]^T$
$u_2 = [1,3,2]^T$, $u_3 = [2,0,1]^T$ and

$$\sum_{i=1}^{3} c_i u_i = [0,3,5]^T$$

∵ transpose of matrix is to change rows into columns and vice versa.

$$\Rightarrow c_1 u_1 + c_2 u_2 + c_3 u_3 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow c_1 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} + c_3 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ -c_1 \\ c_1 \end{bmatrix} + \begin{bmatrix} c_2 \\ 3c_2 \\ 2c_2 \end{bmatrix} + \begin{bmatrix} 2c_3 \\ 0 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 + c_2 + 2c_3 \\ -c_1 + 3c_2 + 0 \\ c_1 + 2c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

Comparing both sides. we get three equations

$c_1 + c_2 + 2c_3 = 0$ —①

$-c_1 + 3c_2 = 3$ ② $c_1 + 2c_2 + c_3 = 5$ —③

from equation 2

$\Rightarrow 3c_2 - c_1 = 3$

$\Rightarrow 3c_2 - 3 = c_1$ — ④

Substituting equation 4 in ① and ③ equations.

$c_1 + c_2 + 2c_3 = 0$

$3c_2 - 3 + c_2 + 2c_3 = 0$

$4c_2 + 2c_3 = 3$ — ⑤

$c_1 + 2c_2 + c_3 = 5$

$3c_2 - 3 + 2c_2 + c_3 = 5$

$\Rightarrow 5c_2 + c_3 = 8$ — ⑥

Solving ⑤ and ⑥ equations.

$4c_2 + 2c_3 = 3$ multiplying equation ⑥ ×2

$5c_2 + c_3 = 8 \times 2$

$\Rightarrow$

$4c_2 + 2c_3 = 3$

$\underline{10c_2 + 2c_3 = 16}$

$+6c_2 = +13$

$\Rightarrow \boxed{c_2 = 13/6}$

from equation 4

$3c_2 - 3 = c_1$

$\Rightarrow \cancel{3}\left(\frac{13}{\cancel{6}\,2}\right) - 3 = c_1$

$\Rightarrow \frac{13-6}{2} \Rightarrow \frac{7}{2}$

hence $\boxed{c_1 = 7/2}$

now from equat 5

$4c_2 + 2c_3 = 3$

$\frac{26}{3} + 2c_3 = 3$

$2c_3 = -\frac{26}{3} + 3. \Rightarrow -17/3 \quad -17/3$

$\boxed{c_3 = -17/6}$, $\boxed{c_3 = -17/6}$,

hence $c_1 = 7/2$, $c_2 = 13/6$, $c_3 = -17/6$.

3. let $\vec{u} = [5, 4, 1]^T$, $\vec{v} = [3, -4, 1]^T$, $\vec{w} = [1, -2, 3]^T$

by collinear condition - if the vectors to be collinear their cross product should be 0. ie $\vec{u} \times \vec{v} = 0$.

hence.

$$\vec{u} \times \vec{v} = \left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & 4 & 1 \\ 3 & -4 & 1 \end{bmatrix}\right| = \hat{i}(4+4) - \hat{j}(5-3) + \hat{k}(-20-12)$$

$$\Rightarrow 8\hat{i} - 2\hat{j} + -32\hat{k} \quad -①$$

from above we can say that $\vec{u} \times \vec{v} \neq 0$ and hence $\vec{u}$ and $\vec{v}$ are not collinear.

$$\vec{v} \times \vec{w} = \left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & 1 \\ 1 & -2 & 3 \end{bmatrix}\right| = \hat{i}(-12+2) - \hat{j}[9-1] + \hat{k}(-6+4)$$

$$\Rightarrow -10\hat{i} - 8\hat{j} + -2\hat{k} \quad -②$$

from above we can say that $\vec{v} \times \vec{w} \neq 0$ and hence $\vec{v}$ and $\vec{w}$ are not collinear.

$$\vec{w} \times \vec{u} = \left|\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 5 & 4 & 1 \end{bmatrix}\right| = \hat{i}(-2-12) - \hat{j}(1-15) + \hat{k}(4+10)$$

$$\Rightarrow -14\hat{i} + 14\hat{j} + 14\hat{k}. - ①$$

from above we can say that $\vec{w} \times \vec{u} \neq 0$ and hence $\vec{u}$ and $\vec{w}$ are not collinear.

6. Orthogonality condition is. If two vectors to be orthogonal their dot product should be '0'.

i) $\vec{u} \cdot \vec{v}$

$$\Rightarrow \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} = 5.3 + 4.(-4) + 1.1$$

$$\Rightarrow 15 - 16 + 1 = 0$$

hence $\vec{u} \cdot \vec{v} = 0$ so $\vec{u}$ and $\vec{v}$ are orthogonal.

ii) $\vec{v}, \vec{w}$

$$\begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = 3 + (-4)(-2) + 3$$

$$\Rightarrow 3 + 8 + 3 \Rightarrow 14 \quad \vec{v} \cdot \vec{w} \neq 0$$

As $\vec{v}$ and $\vec{w}$ dot product is not '0' hence these vectors are not othogonal

(iii) $\vec{w}$ and $\vec{u}$

$$\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} = \left[5 - 8 + 3 \right] = 0$$

as $\vec{w} \cdot \vec{u} = 0$ hence $\vec{w}$ and $\vec{u}$ vectors are orthogonal.

(c) Are $\vec{u}, \vec{v}$ and $\vec{w}$ linearly independent.

Three vectors are independent only if $c_1 \vec{v_1} + c_2 \vec{v_2} + c_3 \vec{v} = 0$

(i.e) $c_1 = c_2 = c_3 = 0$.

$$c_1 \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$$

(or)

$$\begin{bmatrix} 5 & 3 & 1 \\ 4 & -4 & -2 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \rightarrow R_2 + R_3$$

$$\begin{bmatrix} 5 & 3 & 1 \\ 5 & -3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_2 \rightarrow R_1 + R_2$$

$$\begin{bmatrix} 5 & 3 & 1 \\ 10 & -6 & 0 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 - 5R_3 - R_1$$

$$\begin{bmatrix} 5 & 3 & 1 \\ 0 & -6 & 0 \\ 0 & 2 & 14 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow 3R_3 + R_2$$

c

$$\begin{bmatrix} 5 & 3 & 1 \\ 0 & -6 & 0 \\ 0 & 0 & 42 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and hence c_1 c_2 c_3

$$\left[\begin{array}{ccc|c} 5 & 3 & 1 & 0 \\ 0 & -6 & 0 & 0 \\ 0 & 0 & 42 & 0 \end{array}\right]$$

$5c_1 + 3c_2 + c_3 = 0$ — ①

$0 + 6c_2 + = 0$ — ②

$0 + 0\ 42c_3 = 0$ — ③

from 3 we get $c_3 = 0$ → back substituting

→ from 2 we get $c_2 = 0$ and by substituting we get $c_1 = 0$ and hence the only case we get is $c_1 = c_2 = c_3 = 0$ a

$\vec{c} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ → $\vec{u}, \vec{v}, \vec{w}$ are linearly independent.

4. $\vec{u} = [1,2,1]^T$ and $\vec{v} = [3,2,4]^T$.

L_2 distance $\Rightarrow$ euclidean distance.

$$\Rightarrow L_2(\vec{u},\vec{v}) = \sqrt{\sum_{i=1}^{3} [\vec{u}_i - \vec{v}_i]^2}$$

$$\Rightarrow \vec{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$L_2 \Rightarrow \sqrt{(1-3)^2 + (2-2)^2 + (1-4)^2}$$

$$\Rightarrow \sqrt{(-2)^2 + 0^2 + 3^2}$$

$$\Rightarrow \sqrt{4+9} = \sqrt{13}$$

hence L_2 distance between $\vec{u}$ and $\vec{v}$ is $\sqrt{13} \Rightarrow 3.6$

L_1 distance $\Rightarrow L_1(\vec{u},\vec{v}) = \sum_{i=1}^{3} |\vec{u}_i - v_i|$

$$u_1 \Rightarrow \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \vec{v} \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

$$\boxed{\begin{array}{l} L_2 \text{ distance} = \sqrt{13} \\ L_1 \text{ distance} = 5 \end{array}}$$

$$\Rightarrow |1-3| + |2-2| + |1-4|$$

$$\Rightarrow |-2| + |0| + |-3|$$

$$\Rightarrow 2+3 = 5$$

hence L_1 distance between $\vec{u}$ and $\vec{v}$ = 5

5. $\vec{z} = [1, 0, 5]^T$ should be written as linear combination of $\vec{P_1} = [1, 0, 1]^T$, $\vec{P_2} = [1, 1, 3]^T$, $\vec{P_3} = [2, 0, -1]^T$

to be written in linear combination general form is $\sum_{i=1}^{n} c_i u_i$. hence writing that way.

$$\Rightarrow \quad c_1\vec{P_1} + c_2\vec{P_2} + c_3\vec{P_3} = \vec{z}$$

$$\Rightarrow \quad c_1\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}1\\1\\3\end{bmatrix} + c_3\begin{bmatrix}2\\0\\-1\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}c_1\\0\\c_1\end{bmatrix} + \begin{bmatrix}c_2\\c_2\\3c_2\end{bmatrix} + \begin{bmatrix}2c_3\\0\\-c_3\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}c_1 + c_2 + 2c_3\\0 + c_2 + 0\\c_1 + 3c_2 - c_3\end{bmatrix} = \begin{bmatrix}1\\0\\5\end{bmatrix}$$

Comparing both sides.

$\Rightarrow \quad c_1 + c_2 + 2c_3 = 1$ — ①

$c_2 = 0$ — ②

$c_1 + 3c_2 + -c_3 = 5$ — ③.

from ② we get $c_2 = 0$. substituting it in ① and ③ hence

$c_1 + 2c_3 = 1$ — 4

$c_1 + 3(0) - c_3 = 5$ — 5

$\Rightarrow$ Solving 4 and 5

$c_1 + 2c_3 = 1$

$-\,c_1 - c_3 = 5$

$3c_3 = -4$

$\boxed{c_3 = -4/3}$ — 6

Substituting 6 in 4

$c_1 + 2c_3 = 1$

$c_1 - 8/3 = 1$

$c_1 = 1 + 8/3$

$\boxed{c_1 = 11/3}$ — 7

hence $c_1 = 11/3$, $c_2 = 0$, $c_3 = -4/3$.

6. Find all t, k so that $\vec{u} = [t, k]^T$ and $\vec{v} = [3, -4]^T$ are orthogonal and $\vec{u}$ has unit length.

For two vectors to be orthogonal their dot product is 0. (or) should be 0. hence

$$\begin{bmatrix} t \\ k \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -4 \end{bmatrix} = 0$$

$\Rightarrow \quad 3t - 4k = 0 \quad -①$

from the info given its given $\vec{u}$ has unit length hence distance of $\vec{u}$ is $\sqrt{\sum_{i=1}^{2} u_i^2}$

$\Rightarrow$

$\sqrt{t^2 + k^2} = 1$ S.O.B.S. squaring on both sides.

$\Rightarrow \quad t^2 + k^2 = 1 \quad -②$

$\Rightarrow$ from ① $\quad 3t - 4k = 0$

$\Rightarrow \quad t = 4/3\, k \quad -③$

Substituting ③ in ②

$\Rightarrow \quad ((4/3)k)^2 + k^2 = 1$

$\Rightarrow \quad \frac{16k^2}{9} + k^2 = 1$

$\rightarrow \quad \frac{25k^2}{9} = 1 \quad \Rightarrow \quad k^2 = 9/25$

$\boxed{k = \pm 3/5}$.

Scanned with OKEN Scanner

and hence substituting k in ①

$t = 4/\cancel{3} \cdot \cancel{3}/5$

$t = 4/5$ —④

$t = 4/\cancel{3}(-(\cancel{3}/5))$

$\Rightarrow -4/5$.

hence we get two pairs of $(t, k) \Rightarrow$

~~(3/5, 4/5) and t=3/~~ $(4/5, 3/5)$ and $(-4/5, -3/5)$

7. Given X and Y ~~b~~ random variables

$$f_{(x,y)}(x,y) = \begin{cases} k/16, & x \in (-4,4) \text{ and } y \in (2,4) \\ 0 & \text{other.} \end{cases}$$

a. find constant k

b. Proove (or) Disapprove X and Y are independent

c. Find $P[Y \leq 3 \mid X \geq 0]$:

$$1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\, dy.$$

$$\Rightarrow \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} k/16 \; dx\, dy$$

$$\Rightarrow \int_{2}^{4} \int_{-4}^{4} k/16 \; dx\, dy$$

$$\Rightarrow \int_{2}^{4} \int_{-4}^{4} \frac{kx}{16}\, dy$$

$$\Rightarrow \int_{2}^{4} \left[\frac{k(4)}{16} - \frac{k(-4)}{16} \right] dy = \int_{2}^{4} \frac{8k}{16_2}\, dy$$

$$\Rightarrow \left\{ \frac{ky}{2} \right\}_2^4 \Rightarrow k\left\{ \frac{4}{2} - \frac{2}{2} \right\}$$

$$k[2-1] = 1$$

$$\boxed{k=1} \;\; // \; -①$$

from equation ① $k = 1$ →

$\boxed{k=1}$ //

$$f(x,y) = \begin{cases} 1/16 & x \in (-4,4) \text{ and } y \in (2,4) \\ 0 & \text{other.} \end{cases}$$

b. Prove (or) disapprove X and Y are independent

$$f_{XY}(x,y) = f_x(x) \cdot f_y(y).$$

$$f_x(x) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dy$$

$$\Rightarrow \int_{-\infty}^{\infty} k/16\,dy \Rightarrow \int_{2}^{4} 1/16\,dy \Rightarrow \int_{2}^{4} 1/16\,y$$

$$\Rightarrow \quad \Rightarrow \frac{1}{16}[4-2) \Rightarrow \boxed{1/8}\,//.$$

$$f_y(y) = \int_{-\infty}^{\infty} f_{xy}(x,y)\,dx$$

$$\Rightarrow \int_{-4}^{4} 1/16\,dx \Rightarrow 1/16\,(4+4) \Rightarrow 8/16 = \boxed{1/2}\,//$$

we know $f_{xy}(x,y) = f_x(x) \cdot f_y(y)$. so x and y are

⇒ independent.

$\cancel{\boxed{1/7\ 1/6}}$. hence.

$$f_{xy}(x,y) = 1/16 \text{ if } x \in (-4,4) \text{ and } y \in (2,4) = f_x(x) \cdot f_y(y).$$

$P(Y \leq 3 \mid X \geq 0)$ & $f_{xy}(x, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{xy}(x, y)\,dx\,dy$

$$\Rightarrow \int_2^3\int_0^4 \frac{1}{16}\,dx\,dy$$

$$\Rightarrow \int_2^3 \left[x/16 \right]_0^4 dy$$

$$\Rightarrow \int_2^3 \frac{1}{4}\,dy \Rightarrow \left[\frac{1}{4}y\right]_2^3 \Rightarrow \frac{1}{4}[3-2]$$

$$= 1/4 \cdot 4$$

hence $P(Y \leq 3 \mid X \geq 0) = 1/4 \rightarrow$.

$$P(X \geq 0) = \int_2^4\int_0^4 \frac{1}{16}\,dx\,dy$$

$$\Rightarrow \int_2^4\int_0^4 \frac{1}{16} x\,dy$$

$$\Rightarrow \int_2^4 \frac{1}{4}\,dy$$

$$\Rightarrow \left[\frac{1}{4}\right]_2^4 \Rightarrow \frac{2}{\cancel{4}2} = 1/2$$

$$P(Y \leq 3 / X \geq 0) \Rightarrow \frac{P(Y \leq 3 \mid X \geq 0)}{P(X \geq 0)} = \frac{1/4}{1/2} \Rightarrow 1/2$$

8. The probability $\overset{\text{selecting}}{\text{coin}}$ C_1 is $1/3$. $-P(C_1)$

as C_1 and C_2 are complimentary events

$$P(C_1)+P(C_2)=1$$

$$1/3 + P(C_2) = 1$$

$$\boxed{P(C_2) = 2/3}$$ // — ①

Total that C_1 produces head with a probability $1/7$ this can be related $P(H/C_1)$.

Same for C_2 probability of getting a head is $7/8$. $P(H/C_2)$.

The probality that head occurs any ways

→ is getting probabity of head when coin C_1 is selected + getting head when coin C_2 is selected.

⇒ $\frac{1}{3} \cdot \frac{1}{7} + \frac{2}{3} \cdot \frac{7}{8}$ is total probability of getting head

⇒ $\frac{1}{21} + \frac{7}{12} = \frac{12 + 7 \times 21}{12 \times 21} \Rightarrow 4 + 49/84 = 53/84$.

Now that asked that getting a head when C_1 is used

By total pro

⇒ $P(H/C_1) =$ P

$$\Rightarrow \frac{2/3 \cdot 7/8}{53/84}$$

$$\Rightarrow \frac{\cancel{14} \times \cancel{84}}{53 \times \cancel{12} \times \cancel{8}} = 49/53$$

hence probality that a head produces and C_2 is selected is 49/53.

(b) Total probability from the above we could get is

getting coin 1 and getting head $= 1/3 \cdot 1/7$

getting coin 2 and getting head $= 2/3 \cdot 7/8$

$1/21 + 7/12 \Rightarrow 53/84$.

$\Rightarrow 53/84$

and it is not equal to $1/2$ and

hence by this we disprove that probability of getting head is $1/2$. and it is

P(getting head). $= 53/84$.