1.1 Propositions and logical operations

Conditional proposition from English sentences.
Logic is the study of formal reasoning. A statement in a spoken language, such as in English, is often ambiguous in its meaning. By contrast, a statement in logic always has a well defined meaning. Logic is important in mathematics for proving theorems. Logic is also used in computer science in areas such as artificial intelligence for automated reasoning and in designing digital circuits. Logic is useful in any field in which it is important to make precise statements. In law, logic can be used to define the implications of a particular law. In medicine, logic can be used to specify precisely the conditions under which a particular diagnosis would apply.

The most basic element in logic is a proposition. A ***proposition*** is a statement that is either true or false.

Table 1.1.1: Examples of propositions: Statements that are either true or false.

Proposition	Truth value
There are an infinite number of prime numbers.	True
The Declaration of Independence was signed on July 4, 1812.	False

Propositions are typically declarative sentences. For example, the following are *not* propositions.

Table 1.1.2: English sentences that are not propositions.

Sentence	Comment
What time is it?	A question, not a proposition. A question is neither true nor false.
Have a nice day.	A command, not a proposition. A command is neither true nor false.

A proposition's ***truth value*** is a value indicating whether the proposition is actually true or false. A proposition is still a proposition whether its truth value is known to be true, known to be false, unknown, or a matter of opinion. The following are all propositions.

Table 1.1.3: Examples of propositions and their truth values.

Proposition	Comment
Two plus two is four.	Truth value is true.
Two plus two is five.	Truth value is false.
Monday will be cloudy.	Truth value is unknown.
The movie was funny.	Truth value is a matter of opinion.
The extinction of the dinosaurs was caused by a meteor.	Truth value is unknown.

PARTICIPATION ACTIVITY 1.1.1: Propositions.

Indicate which statements are propositions.

1) 10 is a prime number.

- Proposition
- Not a proposition

2) Shut the door.

- Proposition
- Not a proposition

3) All politicians are dishonest.

- Proposition
- Not a proposition

4) Would you like some cake?

- Proposition
- Not a proposition

5) Interest rates will rise this year.

- Proposition
- Not a proposition

The conjunction operation

Propositional variables such as p, q, and r can be used to denote arbitrary propositions, as in:

p: January has 31 days.
q: February has 33 days.

A ***compound proposition*** is created by connecting individual propositions with logical operations. A ***logical operation*** combines propositions using a particular composition rule. For example, the conjunction operation is denoted by $\land$. The proposition $p \land q$ is read "p and q" and is called the ***conjunction*** of p and q. $p \land q$ is true if both p is true and q is true. $p \land q$ is false if p is false, q is false, or both are false.

Using the definitions for $p \land q$ given above, the proposition $p \land q$ is expressed in English as:

$p \land q$: January has 31 days and February has 33 days.

Proposition p's truth value is true — January does have 31 days. Proposition q's truth value is false — February does not have 33 days. The compound proposition $p \land q$ is therefore false, because it is not the case that both propositions are true.

A ***truth table*** shows the truth value of a compound proposition for every possible combination of truth values for the variables contained in the compound proposition. Every row in the truth table shows a particular truth value for each variable, along with the compound proposition's corresponding truth value. Below is the truth table for $p \land q$, where ***T*** represents true and ***F*** represents false.

PARTICIPATION ACTIVITY 1.1.2: Truth table for the conjunction operation.

Animation captions:

1. $p \land q$ is true only when both p and q are true.
2. $p \land q$ is false for all other combinations.

Different ways to express a conjunction in English

Define the propositional variables p and h as:

p: Sam is poor.
h: Sam is happy.

There are many ways to express the proposition $p \land h$ in English. The sentences below have slightly different meanings in English but correspond to the same logical meaning.

Table 1.1.4: Examples of different ways to express a conjunction in English.

p and h	Sam is poor and he is happy.
p, but h	Sam is poor, but he is happy.
Despite the fact that p, h	Despite the fact that Sam is poor, he is happy.
Although p, h	Although Sam is poor, he is happy.

The disjunction operation

The disjunction operation is denoted by $\lor$. The proposition $p \lor q$ is read "p or q", and is called the ***disjunction*** of p and q. $p \lor q$ is true if either one of p or q is true, or if both are true. The proposition $p \lor q$ is false if neither p nor q is true. Using the same p and q from the example above, $p \lor q$ is the statement:

p ∨ q: January has 31 days or February has 33 days.

The proposition p ∨ q is true because January does have 31 days. The truth table for the ∨ operation is given below.

PARTICIPATION ACTIVITY 1.1.3: Truth table for the disjunction operation.

Animation captions:

1. p ∨ q is true when either of p or q is true.
2. p ∨ q is false only when p and q are both false.

Ambiguity of "or" in English

The meaning of the word "or" in common English depends on context. Often when the word "or" is used in English, the intended meaning is that one or the other of two things is true, but not both. One would normally understand the sentence "Lucy is going to the park or the movie" to mean that Lucy is either going to the park, or is going to the movie, but not both. Such an either/or meaning corresponds to the "exclusive or" operation in logic. The ***exclusive or*** of p and q evaluates to true when p is true and q is false or when q is true and p is false. The ***inclusive or*** operation is the same as the disjunction (∨) operation and evaluates to true when one or both of the propositions are true. For example, "Lucy opens the windows or doors when warm" means she opens windows, doors, or possibly both. Since the inclusive or is most common in logic, it is just called "or" for short.

PARTICIPATION ACTIVITY 1.1.4: Truth table for the exclusive or.

The exclusive or operation is usually denoted with the symbol ⊕. The proposition p ⊕ q is true if exactly one of the propositions p and q is true but not both. This question asks you to fill in the truth table for p ⊕ q.

p	q	p ⊕ q
T	T	1?
T	F	2?
F	T	3?

F	F	4?

1) What is the truth value for the square labeled 1?
   - True
   - False

2) What is the truth value for the square labeled 2?
   - True
   - False

3) What is the truth value for the square labeled 3?
   - True
   - False

4) What is the truth value for the square labeled 4?
   - True
   - False

The ***negation*** operation acts on just one proposition and has the effect of reversing the truth value of the proposition. The negation of proposition p is denoted ¬p and is read as "not p". Since the negation operation only acts on a single proposition, its truth table only has two rows for the proposition's two possible truth values.

PARTICIPATION ACTIVITY 1.1.5: Truth table for the negation operation.

Animation captions:

1. The truth value of ¬p is the opposite of the truth value of p.

PARTICIPATION ACTIVITY 1.1.6: Applying logical operations.

Assume propositions p, q, and r have the following truth values:

p is true
q is true
r is false

What are the truth values for the following compound propositions?

1) $p \wedge q$

- True
- False

2) $\neg r$

- True
- False

3) $p \wedge r$

- True
- False

4) $p \vee r$

- True
- False

5) $p \vee q$

- True
- False

Example 1.1.1: Searching the web.

The language of logic is useful in database searches, such as searching the web. Suppose one is interested in finding web pages related to higher education. A search on the term "college" could potentially miss many pages related to universities. A search on "college OR university" would yiel results on both topics. A search on "dogs AND fleas" would yield pages that pertain to both dogs and fleas. A typical web search engine, though, implicitly uses an AND operation for multiple worc in queries like "dogs fleas".

Additional exercises

Exercise 1.1.1: Identifying propositions.

About

Determine whether each of the following sentences is a proposition. If the sentence is a proposition, then write its negation.

(a) Have a nice day.

Solution

(b) The soup is cold.

Solution

(c) The patient has diabetes.

Solution

(d) The light is on.

Solution

(e) It's a beautiful day.

Solution

(f) Do you like my new shoes?

Solution

(g) The sky is purple.

Solution

(h) $2 + 3 = 6$

Solution

(i) Every prime number is even.

Solution

(j) There is a number that is larger than 17.

Solution

Exercise 1.1.2: Expressing English sentences using logical notation.

About

Express each English statement using logical operations ∨, ∧, ¬ and the propositional variables t and m defined below. The use of the word "or" means inclusive or.

t: The patient took the medication.
n: The patient had nausea.
m: The patient had migraines.

(a)
The patient had nausea and migraines.

(b)
The patient took the medication, but still had migraines.

(c)
The patient had nausea or migraines.

(d)
The patient did not have migraines.

(e)
Despite the fact that the patient took the medication, the patient had nausea.

(f)
There is no way that the patient took the medication.

Exercise 1.1.3: Applying logical operations.

About

Assume the propositions p, q, r, and s have the following truth values:

p is false
q is true
r is false
s is true

What are the truth values for the following compound propositions?

(a)
$\neg p$

(b)
$p \lor r$

(c)
$q \land s$

(d)
$q \lor s$

(e)
$q \oplus s$

(f)
$q \oplus r$

Exercise 1.1.4: Truth values for statements with inclusive and exclusive or.

About

Indicate whether each statement is true or false, assuming that the "or" in the sentence means t inclusive or. Then indicate whether the statement is true or false if the "or" means the exclusive c

(a) February has 31 days or the number 5 is an integer.

(b) The number π is an integer or the sun revolves around the earth.

(c) 20 nickels are worth one dollar or whales are mammals.

(d) There are eight days in a week or there are seven days in a week.

(e) January has exactly 31 days or April has exactly 30 days.

How was this section? **Provide feedback**

1.2 Evaluating compound propositions

A compound proposition can be created by using more than one operation. For example, the proposition $p \lor \neg q$ evaluates to true if p is true or the negation of q is true.

The order in which the operations are applied in a compound proposition such as $p \lor \neg q \land r$ may affect the truth value of the proposition. In the absence of parentheses, the rule is that negation is applied first, then conjunction, then disjunction:

Figure 1.2.1: Order of operations in absence of parentheses.

1. ¬ (not)
2. ∧ (and)
3. ∨ (or)

For example, the proposition p ∨ q ∧ r should be read as p ∨ (q ∧ r), instead of (p ∨ q) ∧ r. However, good practice is to use parentheses to specify the order in which the operations are to be performed, as in p ∨ (q ∧ r). One exception to using parentheses in compound propositions is with the negation operation. Parentheses around ¬p are usually omitted to make compound propositions more readable. Because the negation operation is always applied first, the proposition ¬p ∨ q is evaluated as (¬p) ∨ q instead of ¬(p ∨ q). Also, when there are multiple ∨ operations or multiple ∧ operations, such as in the compound proposition p ∨ q ∨ r or the compound proposition p ∧ q ∧ r, parentheses are usually omitted because the order in which the operations are applied does not affect the final truth value.

PARTICIPATION ACTIVITY 1.2.1: Evaluating compound propositions.

Animation captions:

1. Compound proposition $p \land \neg(q \lor r)$ is evaluated by filling in the given truth values for variables p, q, and r.
2. and then evaluating the operations in the required order.
3. The compound proposition $p \land \neg(q \lor r)$ evaluates to false.

PARTICIPATION ACTIVITY 1.2.2: Evaluating complex compound propositions.

Assume the propositions p, q, r have the following truth values:

p is true
q is true

r is false

What are the truth values for the following compound propositions?

1) $p \vee \neg q$

- True
- False

2) $\neg r \wedge (p \vee \neg q)$

- True
- False

3) $\neg(p \wedge \neg r)$

- True
- False

4) $(p \vee r) \wedge \neg p$

- True
- False

CHALLENGE ACTIVITY 1.2.1: Write proposition using symbols.

Start

Define the proposition in symbols using:

- p: The weather is bad.
- q: The trip is cancelled.
- r: The trip is delayed.

Proposition in words: The trip is not delayed.

Proposition in symbols:

1	2	3	4	5

Check Next

Example 1.2.1: Searching the web -- continued.

Compound propositions can be created with logical operations to conduct refined web searches. Suppose one is interested in studying jaguars (the animal from the cat family). Try searching (e.g at google.com) for the term "jaguar" — the results may include numerous hits related to the car "Jaguar". To avoid results involving cars, try a second search using the query "jaguar AND -car" — the "-" symbol indicates negation. Notice that results are then mostly about the animal.

Filling in the rows of a truth table

A truth table for a compound statement will have a row for every possible combination of truth assignments for the statement's variables. If there are n variables, there are 2^n rows. In the truth table for the compound proposition (p ∨ r) ∧ ¬q, there are three variables and 2^3 = 8 rows.

Table 1.2.1: Truth table with three variables.

p	q	r	$(p \vee r) \wedge \neg q$
T	T	T	F
T	T	F	F
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	F
F	F	T	T
F	F	F	F

Note that the T and F values for each row are unique, and that they are chosen methodically to create all possible combinations. (The column for variable r alternates T F T F..., the column for q alternates T T F F ..., etc.)

PARTICIPATION ACTIVITY 1.2.3: Number of entries in a truth table.

1) How many rows are in a truth table for a compound proposition with propositional variables p, q, and r?

Check Show answer

2) How many rows are in a truth table for the proposition (p ∧ q) ∨ (¬r ∧ ¬q) ∨ ¬(p ∧ t) ?

Check Show answer

When filling out a truth table for a complicated compound proposition, completing intermediate columns for smaller parts of the full compound proposition can be helpful.

PARTICIPATION ACTIVITY 1.2.4: Truth table with intermediate columns.

Animation captions:

1. The truth table for $\neg q \wedge (p \vee r)$ can be computed by first filling in a column for $\neg q$,
2. then filling in a column for $(p \vee r)$,
3. and finally filling in the column for $\neg q \wedge (p \vee r)$ using the intermediate columns.

PARTICIPATION ACTIVITY 1.2.5: Filling in a truth table.

Indicate how the missing items in the truth table below should be filled in:

p	q	r	p ∧ ¬q	(p ∧ ¬q) ∨ r
T	T	T	F	*(A)*
T	T	F	F	F
T	F	T	T	T
T	F	F	*(B)*	T
F	T	T	F	T
F	T	F	F	*(C)*
F	F	T	F	T

F	F	*(D)*	F	F

1) What is the correct value for A?

- T
- F

2) What is the correct value for B?

- T
- F

3) What is the correct value for C?

- T
- F

4) What is the correct value for D?

- T
- F

CHALLENGE ACTIVITY 1.2.2: Truth tables for compound propositions.

Start

Using the pattern above, fill in all combinations of p and q.

p	q
T	T
T	
	T

1	2	3	4	5	6	7	8

Check Next

Example 1.2.2: Logic in electronic devices.

Most of the devices and appliances used in everyday life are controlled by electronic circuitry that works according to the laws of logic. A designer of an electronically-controlled device must first express the desired behavior in logic, and then test that the device behaves as desired under ever set of circumstances.

An electronic fan automatically turns on and off depending on the humidity in a room. Since the humidity detector is not very accurate, the fan will stay on for 20 minutes once triggered in order t ensure that the room is cleared of moisture. There is also a manual off switch that can be used to override the automatic functioning of the control.

Define the following propositions:

M: The fan has been on for twenty minutes.
H: The humidity level in the room is low.
O: The manual "off" button has been pushed.

The fan will turn off if the following proposition evaluates to true:

$(M \land H) \lor O$

A truth table can be useful in testing the device to make sure it works as intended under every set circumstances. The following table might be used by a technician testing the electronic fan.

M	H	O	Should be off? (T: yes)	Your observation
T	T	T	T	
T	T	F	T	
T	F	T	T	
T	F	F	F	
F	T	T	T	
F	T	F	F	
F	F	T	T	
F	F	F	F	

Additional exercises

Exercise 1.2.1: Truth values for compound English sentences.

About

Determine whether the following propositions are true or false:

(a) 5 is an odd number and 3 is a negative number.

(b) 5 is an odd number or 3 is a negative number.

(c) 8 is an odd number or 4 is not an odd number.

(d) 6 is an even number and 7 is odd or negative.

(e) It is not true that 7 is an odd number or 8 is an even number.

Exercise 1.2.2: Translating English statements into logic.

About

Express each statement in logic using the variables:

p: It is windy.
q: It is cold.
r: It is raining.

(a)
It is windy and cold.

(b)
It is windy but not cold.

(c)
It is not true that it is windy or cold.

(d)
It is raining and it is windy or cold.

(e)
It is raining and windy or it is cold.

(f)
It is raining and windy but it is not cold.

Exercise 1.2.3: Truth values for compound propositions.

About

The propositional variables, p, q, and s have the following truth assignments: p = T, q = T, s = F. G
the truth value for each proposition.

(a)
$p \lor \neg q$

(b)
$(p \land q) \lor s$

(c)
$p \land (q \lor s)$

(d)
$p \land \neg(q \lor s)$

(e)
$\neg(q \land p \land \neg s)$

(f)
$\neg(p \land \neg(q \land s))$

Exercise 1.2.4: Writing truth tables.

About

Write a truth table for each expression.

(a)
$\neg p \oplus q$

(b)
$\neg(p \vee q)$

(c)
$r \vee (p \wedge \neg q)$

(d)
$(r \vee p) \wedge (\neg r \vee \neg q)$

Exercise 1.2.5: Translating compound propositions into English sentences.

About

Express the following compound propositions in English using the following definitions:

p: I am going to a movie tonight.
q: I am going to the party tonight.

(a)
$\neg p$

(b)
$p \land q$

(c)
$p \land \neg q$

(d)
$\neg p \lor \neg q$

(e)
$\neg(p \land q)$

Exercise 1.2.6: Multiple disjunction or conjunction operations.

About

Suppose that p, q, r, s, and t are all propositional variables.

(a)
Describe in words when the expression $p \lor q \lor r \lor s \lor t$ is true and when it is false.

(b)
Describe in words when the expression $p \land q \land r \land s \land t$ is true and when it is false.

Exercise 1.2.7: Expressing a set of conditions using logical operations.

About

Consider the following pieces of identification a person might have in order to apply for a credit card:

B: Applicant presents a birth certificate.
D: Applicant presents a driver's license.
M: Applicant presents a marriage license.

Write a logical expression for the requirements under the following conditions:

(a) The applicant must present either a birth certificate, a driver's license or a marriage license.

(b) The applicant must present at least two of the following forms of identification: birth certificate, driver's license, marriage license.

(c) Applicant must present either a birth certificate or both a driver's license and a marriage license.

Exercise 1.2.8: Finding truth values to make two logical expressions evaluate to different values.

About

Give truth values for the propositional variables that cause the two expressions to have different truth values.

For example, given p ∨ q and p ⊕ q, the correct answer would be p = q = T, because when p and q are both true, p ∨ q is true but p ⊕ q is false. Note that there may be more than one correct answ

(a) r ∧ (p ∨ q)
(r ∧ p) ∨ q

(b) ¬p ∧ q
¬(p ∧ q)

(c) p ∨ q
(¬p ∧ q) ∨ (p ∧ ¬q)

Exercise 1.2.9: Boolean expression to express a condition on the input variables.

About

(a) Give a logical expression with variables p, q, and r that is true if p and q are false and r is tru and is otherwise false.

How was this section? **Provide feedback**

1.3 Conditional statements

The ***conditional operation*** is denoted with the symbol →. The proposition p → q is read "if p then q". The proposition $p \to q$ is false if p is true and q is false; otherwise, $p \to q$ is true.

A compound proposition that uses a conditional operation is called a ***conditional proposition***. A conditional proposition expressed in English is sometimes referred to as a ***conditional statement***, as in "If there is a traffic jam today, then I will be late for work."

In $p \to q$, the proposition p is called the ***hypothesis***, and the proposition q is called the ***conclusion***. The truth table for $p \to q$ is given below.

Table 1.3.1: Truth table for the conditional operation.

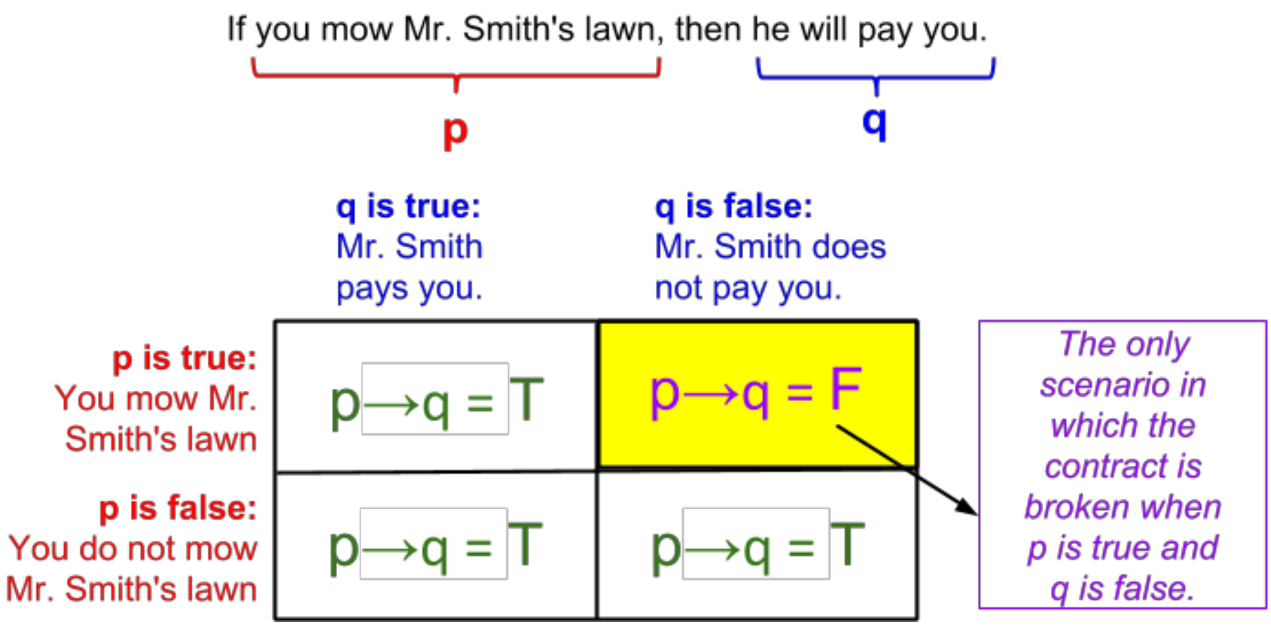
p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A conditional proposition can be thought of like a contract between two parties, as in:

If you mow Mr. Smith's lawn, then he will pay you.

The only way for the contract between you and Mr. Smith to be broken, is for you to mow Mr. Smith's lawn and for him not to pay you. If you do not mow his lawn, then he can either pay you or not, and the contract is not broken. In the words of logic, the only way for a conditional statement to be false is if the hypothesis is true and the conclusion is false. If the hypothesis is false, then the conditional statement is true regardless of the truth value of the conclusion.

Figure 1.3.1: A conditional statement illustrated.



PARTICIPATION ACTIVITY 1.3.1: Understanding conditional statements.

Each question has a proposition p that is a conditional statement. Truth values are also given for the individual propositions contained in that conditional statement. Indicate whether the conditional statement p is true or false.

1) p: If it rains today, I will have my umbrella.
   It is raining today.
   I do not have my umbrella.
   - True
   - False

2) p: If Sally took too long getting ready, she missed the bus.
   Sally did not take too long getting ready.
   Sally missed the bus.
   - True
   - False

3) p: If it is sunny out, I ride my bike.
It is not sunny out.
I am not riding my bike.

- True
- False

There are many ways to express the conditional statement p → q in English:

Table 1.3.2: English expressions of the conditional operation.

Consider the propositions:

p: You mow Mr. Smith's lawn.
q: Mr. Smith will pay you.

If p, then q.	If you mow Mr. Smith's lawn, then he will pay you.
If p, q.	If you mow Mr. Smith's lawn, he will pay you.
q if p	Mr. Smith will pay you if you mow his lawn.
p implies q.	Mowing Mr. Smith's lawn implies that he will pay you.
p only if q.	You will mow Mr. Smith's lawn only if he pays you.
p is sufficient for q.	Mowing Mr. Smith's lawn is sufficient for him to pay you.
q is necessary for p.	Mr. Smith's paying you is necessary for you to mow his lawn.

There is sometimes some confusion about the fact that the statement "p only if q" is the same as the proposition p → q. Both statements mean that the only way for p to be true is if q is also true.

PARTICIPATION ACTIVITY 1.3.2: Conditional proposition from English sentences.

This question uses the following propositions:

p: I will share my cookie with you.
q: You will share your soda with me.

Select the conditional statement that has the same logical meaning as the English sentence given.

1) If you share your soda with me, then I will share my cookie with you.
   - ○ $q \rightarrow p$
   - ○ $p \rightarrow q$

2) Me sharing my cookie with you is sufficient for you to share your soda with me.
   - ○ $q \rightarrow p$
   - ○ $p \rightarrow q$

3) I will share my cookie with you only if you share your soda with me.
   - ○ $q \rightarrow p$
   - ○ $p \rightarrow q$

CHALLENGE ACTIVITY 1.3.1: Convert proposition from words to symbols.

Start

Define the proposition in symbols using:

- p: The weather is bad.
- q: The trip is cancelled.
- r: The trip is delayed.

Proposition in words: If the weather is good, then the trip will not be cancelled.

Proposition in symbols:

1	2	3	4	5

Check Next

The converse, contrapositive, and inverse

Three conditional statements related to proposition p → q are so common that they have special names. The ***converse*** of p → q is q → p. The ***contrapositive*** of p → q is ¬q → ¬p.The ***inverse*** of p → q is ¬p → ¬q.

Table 1.3.3: The converse, contrapositive, and inverse.

Proposition:	p → q	Ex: If it is raining today, the game will be cancelled.
Converse:	q → p	If the game is cancelled, it is raining today.
Contrapositive:	¬q → ¬p	If the game is not cancelled, then it is not raining today.
Inverse:	¬p → ¬q	If it is not raining today, the game will not be cancelled.

PARTICIPATION ACTIVITY 1.3.3: Converse, contrapositive, and inverse of a conditional proposition.

Consider the conditional statement below:

If he studied for the test, then he passed the test.

Match each statement below to the term describing how it is related to the statement above.

Contrapositive **Inverse** **Converse**

If he did not pass the test, then he did not study for the test.

If he passed the test, then he studied for the test.

If he did not study for the test, then he did not pass the test.

Reset

The biconditional operation

If p and q are propositions, the proposition "p if and only if q" is expressed with the ***biconditional operation*** and is denoted $p \leftrightarrow q$. The proposition $p \leftrightarrow q$ is true when p and q have the same truth value and is false when p and q have different truth values.

Alternative ways of expressing $p \leftrightarrow q$ in English include "p is necessary and sufficient for q" or "if p then q, and conversely". The term ***iff*** is an abbreviation of the expression "if and only if", as in "p iff q". The truth table for $p \leftrightarrow q$ is given below:

Table 1.3.4: Truth table for the biconditional operation.

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Compound propositions with conditional and biconditional operations

The conditional and biconditional operations can be combined with other logical operations, as in (p → q) ∧ r. If parentheses are not used to explicitly indicate the order in which the operations should be applied, then ∧, ∨, and ¬ should be applied before → or ↔. Thus, the proposition p → q ∧ r should be evaluated as p → (q ∧ r). Good practice, however, is to use parentheses so that the order of operations is clear.

PARTICIPATION ACTIVITY 1.3.4: Example of evaluating a compound proposition with a biconditional.

Animation captions:

1. The compound proposition, $p \vee (q \leftrightarrow r)$ is evaluated by first filling in the given truth values p, q and r.
2. The biconditional operation is evaluated first,
3. then the negation and then the disjunction, yielding a truth value for the entire compound proposition.

PARTICIPATION ACTIVITY 1.3.5: Evaluating compound propositions with conditional and biconditional operations.

Assume the propositions p, q, r, and s have the following truth values:

p is true
q is true
r is false
s is false

What are the truth values for the following compound propositions?

1) $s \rightarrow q$

- True
- False

2) $(r \leftrightarrow s) \wedge q$

- True
- False

3) $q \rightarrow \neg r$

- True
- False

4) $(q \wedge s) \rightarrow p$

- True
- False

5) $(p \leftrightarrow r) \wedge (\neg r \wedge \neg s)$

- True
- False

6) $q \rightarrow \neg(r \vee q)$

- True
- False

CHALLENGE ACTIVITY 1.3.2: Truth tables for conditional propositions.

Start

Fill in p→q.

p	q	p→q
T	T	
T	F	
F	T	
F	F	

1 2 3 4 5

Check Next

Example 1.3.1: Automatic degree requirements check.

Large universities with thousands of students usually have an automated system for checking whether a student has satisfied the requirements for a particular degree before graduation. Degre requirements can be expressed in the language of logic so that they can be checked by a comput program. For example, let X be the proposition that the student has taken course X. For a degree Computer Science, a student must take one of three project courses, P1, P2, or P3. The student must also take one of two theory courses, T1 or T2. Furthermore, if the student is an honors student, he or she must take the honors seminar S. Let H be the proposition indicating whether th student is an honors student. We can express these requirements with the following proposition:

$(P1 \lor P2 \lor P3) \land (T1 \lor T2) \land (H \to S)$

Additional exercises

Exercise 1.3.1: Truth values for conditional statements in English.

About

Which of the following conditional statements are true and why?

(a) If February has 30 days, then 7 is an odd number.

(b) If January has 31 days, then 7 is an even number.

(c) If 7 is an odd number, then February does not have 30 days.

(d) If 7 is an even number, then January has exactly 28 days.

Exercise 1.3.2: The inverse, converse, and contrapositive of conditional sentences in English.

About

Give the inverse, converse and contrapositive for each of the following statements:

(a)
If she finished her homework, then she went to the party.

(b)
If he trained for the race, then he finished the race.

(c)
If the patient took the medicine, then she had side effects.

(d)
If it was sunny, then the game was held.

(e)
If it snowed last night, then school will be cancelled

Exercise 1.3.3: Truth values for the inverse, contrapositive, and converse of a conditional statement.

About

State the inverse, contrapositive, and converse of each conditional statement. Then indicate whether the inverse, contrapositive, and converse are true.

(a)
If 3 is a prime number then 5 is an even number.

(b)
If $7 < 5$, then $5 < 3$.

(c)
If 5 is a negative number, then 3 is a positive number.

Exercise 1.3.4: Truth tables for logical expressions with conditional operations.

About

Give a truth table for each expression.

(a)
$(\neg p \land q) \to p$

(b)
$(p \to q) \to (q \to p)$

(c)
$(p \lor q) \leftrightarrow (q \to \neg p)$

(d)
$(p \leftrightarrow q) \oplus (p \leftrightarrow \neg q)$

(e)
$(p \lor q) \leftrightarrow (q \land p)$

Exercise 1.3.5: Expressing conditional statements in English using logic.

About

Define the following propositions:

c: I will return to college.
j: I will get a job.

Translate the following English sentences into logical expressions using the definitions above:

(a) Not getting a job is a sufficient condition for me to return to college.

(b) If I return to college, then I won't get a job.

(c) I am not getting a job, but I am still not returning to college.

(d) I will return to college only if I won't get a job.

(e) There's no way I am returning to college.

(f) I will get a job and return to college.

Exercise 1.3.6: Expressing English sentences in if-then form.

About

Give an English sentence in the form "If...then...." that is equivalent to each sentence.

(a) Maintaining a B average is sufficient for Joe to be eligible for the honors program.

(b) Maintaining a B average is necessary for Joe to be eligible for the honors program.

(c) Rajiv can go on the roller coaster only if he is at least four feet tall.

(d) Rajiv can go on the roller coaster if he is at least four feet tall.

Exercise 1.3.7: Expressing conditional statements in English using logic.

About

Define the following propositions:

s: a person is a senior
y: a person is at least 17 years of age
p: a person is allowed to park in the school parking lot

Express each of the following English sentences with a logical expression:

(a) A person is allowed to park in the school parking lot only if they are a senior and at least seventeen years of age.

(b) A person can park in the school parking lot if they are a senior or at least seventeen years o age.

(c) Being 17 years of age is a necessary condition for being able to park in the school parking l

(d) A person can park in the school parking lot if and only if the person is a senior and at least years of age.

(e) Being able to park in the school parking lot implies that the person is either a senior or at lea 17 years old.

Exercise 1.3.8: Translating logical expressions into English.

About

Define the following propositions:

w: the roads were wet
a: there was an accident
h: traffic was heavy

Express each of the logical expressions as an Engilsh sentence:

(a)
$w \rightarrow h$

(b)
$w \land a$

(c)
$\neg(a \land h)$

(d)
$h \rightarrow (a \lor w)$

(e)
$w \land \neg h$

Exercise 1.3.9: Translating English propositions into logical expressions.

About

Use the definitions of the variables below to translate each English statement into an equivalent logical expression.

- y: the applicant is at least eighteen years old
- p: the applicant has parental permission
- c: the applicant can enroll in the course

(a)
The applicant is not eighteen years old but does have parental permission.

(b)
If the applicant is at least eighteen years old or has parental permission, then the applicant can enroll in the course.

(c)
The applicant can enroll in the course only if the applicant has parental permission.

(d)
Having parental permission is a necessary condition for enrolling in the course.

Exercise 1.3.10: Determining if a truth value of a compound expression is known given a partial truth assignment.

About

The variable p is true, q is false, and the truth value for variable r is unknown. Indicate whether th truth value of each logical expression is true, false, or unknown.

(a)

$p \to (q \land r)$

(b)

$(p \lor r) \to r$

(c)

$(p \lor r) \leftrightarrow (q \land r)$

(d)

$(p \land r) \leftrightarrow (q \land r)$

(e)

$p \to (r \lor q)$

(f)

$(p \land q) \to r$

Exercise 1.3.11: Finding logical expressions to match a truth table.

About

For each table, give a logical expression whose truth table is the same as the one given.

(a)

p	q	?
T	T	F
T	F	T
F	T	F
F	F	F

(b)

p	q	?
T	T	F
T	F	T
F	T	T
F	F	F

How was this section? Provide feedback

1.4 Logical equivalence

A compound proposition is a ***tautology*** if the proposition is always true, regardless of the truth value of the individual propositions that occur in it. A compound proposition is a ***contradiction*** if the proposition is always false, regardless of the truth value of the individual propositions that occur in it. $p \lor \neg p$ is a simple example of a tautology since the proposition is always true whether p is true or false. The fact that $p \lor \neg p$ is a tautology can be verified in a truth table, which shows that every truth value in the rightmost column is true.

Table 1.4.1: Truth table for tautology $p \vee \neg p$.

p	$\neg p$	$p \vee \neg p$
T	F	T
F	T	T

Similarly, the proposition $p \wedge \neg p$ is an example of a simple contradiction, because the proposition is false regardless of whether p is true or false. The truth table below shows that $p \wedge \neg p$ is a contradiction because every truth value in the rightmost column is false.

Table 1.4.2: Truth table for contradiction $p \wedge \neg p$.

p	$\neg p$	$p \wedge \neg p$
T	F	F
F	T	F

Showing that a compound proposition is not a tautology only requires showing a particular set of truth values for its individual propositions that cause the compound proposition to evaluate to false. For example, the proposition $(p \wedge q) \rightarrow r$ is not a tautology because when $p = q = T$ and $r = F$, then $(p \wedge q) \rightarrow r$ is false. Showing that a compound proposition is not a contradiction only requires showing a particular set of truth values for its individual propositions that cause the compound proposition to evaluate to true. For example, the proposition $\neg(p \vee q)$ is not a contradiction because when $p = q = F$, then $\neg(p \vee q)$ is true.

PARTICIPATION ACTIVITY 1.4.1: Identifying tautologies and contradictions.

Determine whether the following compound propositions are tautologies, contradictions, or neither.

1) $p \leftrightarrow \neg p$

- Tautology
- Contradiction
- Neither a tautology or a contradiction

2) $p \rightarrow \neg p$

- Tautology
- Contradiction
- Neither a tautology or a contradiction

3) $(p \wedge q) \rightarrow p$

- Tautology
- Contradiction.
- Neither a tautology or a contradiction.

Showing logical equivalence using truth tables

Two compound propositions are said to be ***logically equivalent*** if they have the same truth value regardless of the truth values of their individual propositions. If s and r are two compound propositions, the notation $s \equiv r$ is used to indicate that r and s are logically equivalent. For example, p and $\neg\neg p$ have the same truth value regardless of whether p is true or false, so $p \equiv \neg\neg p$. Propositions s and r are logically equivalent if and only if the proposition $s \leftrightarrow r$ is a tautology.

A truth table can be used to show that two compound propositions are logically equivalent.

PARTICIPATION ACTIVITY 1.4.2: Showing $p \rightarrow \neg p \equiv \neg p$ with a truth table.

Animation captions:

1. $p \rightarrow \neg p$ can be shown to be equivalent to $\neg p$ by filling in a column for $\neg p$,
2. then filling in a column for $p \rightarrow \neg p$, and verifying that the two columns are the same.

Table 1.4.3: Truth table to show: $\neg p \lor \neg q \equiv \neg(p \land q)$.

p	q	¬p	¬q	p ∧ q	¬(p ∧ q)	¬p ∨ ¬q
T	T	F	F	T	**F**	**F**
T	F	F	T	F	**T**	**T**
F	T	T	F	F	**T**	**T**
F	F	T	T	F	**T**	**T**

Showing that two propositions are not logically equivalent only requires showing a particular set of truth values for their individual propositions that cause the two compound proposition to have different truth values. For example, $p \leftrightarrow r$ and $p \rightarrow r$ are not logically equivalent because when p = F and r = T, then $p \leftrightarrow r$ is false but $p \rightarrow r$ is true.

PARTICIPATION ACTIVITY 1.4.3: Logical equivalence by truth table.

The table below shows the truth table for three compound propositions:

- $(p \lor \neg q) \rightarrow r$
- $(p \leftrightarrow q) \rightarrow r$
- $\neg r \rightarrow (\neg p \land q)$

p	q	r	(p ∨ ¬q) → r	(p ↔ q) → r	¬r → (¬p ∧ q)
T	T	T	T	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	F	T	F
F	T	T	T	T	T

F	T	F	T	T	T
F	F	T	T	T	T
F	F	F	F	F	F

1) Use the truth table to determine which logical equivalence is true.

- $(p \lor \neg q) \to r \equiv (p \leftrightarrow q) \to r$
- $(p \leftrightarrow q) \to r \equiv \neg r \to (\neg p \land q)$
- $(p \lor \neg q) \to r \equiv \neg r \to (\neg p \land q)$

De Morgan's laws

De Morgan's laws are logical equivalences that show how to correctly distribute a negation operation inside a parenthesized expression. Both versions of De Morgan's laws are particularly useful in logical reasoning. The first De Morgan's law is:

$$\neg(p \lor q) \equiv (\neg p \land \neg q)$$

When the negation operation is distributed inside the parentheses, the disjunction operation changes to a conjunction operation. Consider an English example with the following propositions for p and q.

p: The patient has migraines
q: The patient has high blood pressure

The use of the English word "or" throughout the example is assumed to be disjunction (i.e., the inclusive or). De Morgan's law says that the following two English statements are logically equivalent:

It is not true that the patient has migraines or high blood pressure.
The patient does not have migraines and does not have high blood pressure.

The logical equivalence $\neg(p \lor q) \equiv (\neg p \land \neg q)$ can be verified using a truth table. Alternatively, reasoning about which truth assignments cause the expressions $\neg(p \lor q)$ and $(\neg p \land \neg q)$ to evaluate to true provides intuition about why the two expressions are logically equivalent.

PARTICIPATION ACTIVITY 1.4.4: Reasoning about De Morgan's laws.

1) There is only one truth assignment for p and q that makes the expression $\neg(p \lor q)$ evaluate to true. Which one is it?

- ○ p = q = T
- ○ p = T and q = F
- ○ p = q = F

2) There is only one truth assignment for p and q that makes the expression $\neg p \land \neg q$ evaluate to true. Which one is it?

- ○ p = q = T
- ○ p = T and q = F
- ○ p = q = F

The second version of De Morgan's law swaps the role of the disjunction and conjunction:

$$\neg(p \land q) \equiv (\neg p \lor \neg q)$$

Continuing with the same example, the following two statements are logically equivalent:

It is not true that the patient has migraines and high blood pressure.
The patient does not have migraines or does not have high blood pressure.

The logical equivalence $\neg(p \land q) \equiv (\neg p \lor \neg q)$ can be verified using a truth table. Alternatively, reasoning about which truth assignments cause the expressions $\neg(p \land q)$ and $(\neg p \lor \neg q)$ to evaluate to false provides intuition about why the two expressions are logically equivalent.

PARTICIPATION ACTIVITY 1.4.5: Reasoning about De Morgan's laws.

1) There is only one truth assignment for p and q that makes the expression $\neg(p \wedge q)$ evaluate to false. Which one is it?

- p = q = T
- p = T and q = F
- p = q = F

2) There is only one truth assignment for p and q that makes the expression $\neg p \vee \neg q$ evaluate to false. Which one is it?

- p = q = T
- p = T and q = F
- p = q = F

PARTICIPATION ACTIVITY 1.4.6: Matching equivalent English expressions using De Morgan's laws.

Select the English sentence that is logically equivalent to the given sentence.

1) It is not true that the child is at least 8 years old and at least 57 inches tall.

- The child is at least 8 years old and at least 57 inches tall.
- The child is less than 8 years old or shorter than 57 inches.
- The child is less than 8 years old and shorter than 57 inches.

2) It is not true that the child is at least 8 years old or at least 57 inches tall.

- The child is at least 8 years old or at least 57 inches tall.
- The child is less than 8 years old or shorter than 57 inches.
- The child is less than 8 years old and shorter than 57 inches.

Additional exercises

Exercise 1.4.1: Proving tautologies and contradictions.

About

Show whether each logical expression is a tautology, contradiction or neither.

(a)

$(p \lor q) \lor (q \rightarrow p)$

(b)

$(p \rightarrow q) \leftrightarrow (p \land \neg q)$

(c)

$(p \rightarrow q) \leftrightarrow p$

(d)

$(p \rightarrow q) \lor p$

(e)

$(\neg p \lor q) \leftrightarrow (p \land \neg q)$

(f)

$(\neg p \lor q) \leftrightarrow (\neg p \land q)$

Exercise 1.4.2: Truth tables to prove logical equivalence.

About

Use truth tables to show that the following pairs of expressions are logically equivalent.

(a)
$p \leftrightarrow q$ and $(p \to q) \land (q \to p)$

(b)
$\neg(p \leftrightarrow q)$ and $\neg p \leftrightarrow q$

(c)
$\neg p \to q$ and $p \lor q$

Exercise 1.4.3: Proving two logical expressions are not logically equivalent.

About

Prove that the following pairs of expressions are not logically equivalent.

(a)
$p \to q$ and $q \to p$

(b)
$\neg p \to q$ and $\neg p \lor q$

(c)
$(p \to q) \land (r \to q)$ and $(p \land r) \to q$

(d)
$p \land (p \to q)$ and $p \lor q$

Exercise 1.4.4: Proving whether two logical expressions are equivalent.

About

Determine whether the following pairs of expressions are logically equivalent. Prove your answe
the pair is logically equivalent, then use a truth table to prove your answer.

(a)
$\neg(p \lor \neg q)$ and $\neg p \land q$

(b)
$\neg(p \lor \neg q)$ and $\neg p \land \neg q$

(c)
$p \land (p \rightarrow q)$ and $p \rightarrow q$

(d)
$p \land (p \rightarrow q)$ and $p \land q$

Exercise 1.4.5: Logical equivalence of two English statements.

About

Define the following propositions:

- j: Sally got the job.
- l: Sally was late for her interview
- r: Sally updated her resume.

Express each pair of sentences using a logical expression. Then prove whether the two expressions are logically equivalent.

(a) If Sally did not get the job, then she was late for interview or did not update her resume.
If Sally updated her resume and did not get the job, then she was late for her interview.

(b) If Sally did not get the job, then she was late for interview or did not update her resume.
If Sally updated her resume and was not late for her interview, then she got the job.

(c) If Sally got the job then she was not late for her interview.
If Sally did not get the job, then she was late for her interview.

(d) If Sally updated her resume or she was not late for her interview, then she got the job.
If Sally got the job, then she updated her resume and was not late for her interview.

Exercise 1.4.6: Applying De Morgan's laws.

About

Translate each English sentence into a logical expression using the propositional variables defin below. Then negate the entire logical expression using parentheses and the negation operation. Apply De Morgan's law to the resulting expression and translate the final logical expression back into English.

- p: the applicant has written permission from his parents
- e: the applicant is at least 18 years old
- s: the applicant is at least 16 years old

(a)
The applicant has written permission from his parents and is at least 16 years old.

(b)
The applicant has written permission from his parents or is at least 18 years old.

How was this section? **Provide feedback**

1.5 Laws of propositional logic

If two propositions are logically equivalent, then one can be substituted for the other within a more complex proposition. The compound proposition after the substitution is logically equivalent to the compound proposition before the substitution.

For example $p \to q \equiv \neg p \vee q$. Therefore,

$$(p \vee r) \wedge (\neg p \vee q) \equiv (p \vee r) \wedge (p \to q)$$

In the next example, the logical equivalence $p \to q \equiv \neg p \vee q$ is applied where the variables p and q represent compound propositions:

$$(\neg t \wedge r) \to (\neg s \vee t) \equiv \neg(\neg t \wedge r) \vee (\neg s \vee t)$$

PARTICIPATION ACTIVITY 1.5.1: Substituting logically equivalent propositions.

Use the logical equivalence $\neg(p \vee q) \equiv \neg p \wedge \neg q$ to match logically equivalent propositions below:

$(s \wedge t) \vee \neg(t \vee r)$ $(\neg s \wedge \neg t) \vee (t \vee r)$ $\neg((s \wedge t) \vee (t \vee r))$

$(s \wedge t) \vee (\neg t \wedge \neg r)$

$\neg(s \wedge t) \wedge \neg(t \vee r)$

$\neg(s \vee t) \vee (t \vee r)$

Reset

Using the laws of propositional logic to show logical equivalence

Substitution gives an alternate way of showing that two propositions are logically equivalent. If one proposition can be obtained from another by a series of substitutions using equivalent expressions, then the two propositions are logically equivalent. The table below shows several laws of propositional logic that are particularly useful for establishing the logical equivalence of compound propositions:

Table 1.5.1: Laws of propositional logic.

Idempotent laws:	$p \vee p \equiv p$	$p \wedge p \equiv p$
Associative laws:	$(p \vee q) \vee r \equiv p \vee (q \vee r)$	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
Commutative laws:	$p \vee q \equiv q \vee p$	$p \wedge q \equiv q \wedge p$
Distributive laws:	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
Identity laws:	$p \vee F \equiv p$	$p \wedge T \equiv p$
Domination laws:	$p \wedge F \equiv F$	$p \vee T \equiv T$
Double negation law:	$\neg\neg p \equiv p$	
Complement laws:	$p \wedge \neg p \equiv F$ $\neg T \equiv F$	$p \vee \neg p \equiv T$ $\neg F \equiv T$
De Morgan's laws:	$\neg(p \vee q) \equiv \neg p \wedge \neg q$	$\neg(p \wedge q) \equiv \neg p \vee \neg q$
Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
Conditional identities:	$p \rightarrow q \equiv \neg p \vee q$	$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

PARTICIPATION ACTIVITY 1.5.2: The laws of propositional logic can be used to show logical equivalence.

Animation content:

undefined

Animation captions:

1. The proof that p --> q is equivalent to¬q--> ¬p starts with p --> q and applies the Conditional identity to get ¬p ∨ q.
2. The Commutative law is applied to get q ∨ ¬p.
3. The Double negation law is applied to get ¬¬q ∨ ¬p.

4. Finally, the Conditional identity is applied again to get ¬q --> ¬p.

PARTICIPATION ACTIVITY 1.5.3: The laws of propositional logic can be used to simplify compound propositions.

Animation content:

`undefined`

Animation captions:

1. The proof that ¬(p ∨ q) ∨ (¬p ∧ q) is equivalent to ¬p starts with ¬(p ∨ q) ∨ (¬p ∧ q) and appli De Morgan's law to get ¬p ∧ ¬q) ∨ (¬p ∧ q).
2. The Distributive law is applied to get ¬p ∧ (¬q ∨ q) .
3. The Commutative law is applied to get ¬p ∧ (q ∨ ¬q).
4. The Complement law is applied to get ¬p ∧ T.
5. Finally, the Identity law is applied again to get ¬p.

PARTICIPATION ACTIVITY 1.5.4: Using the laws of propositional logic to show logical equivalence.

Put the steps in the correct order to show that ¬(p → q) ≡ p ∧ ¬q. Each step should follow from the previous step using the given law.

¬(p → q) **¬¬p ∧ ¬q De Morgan's Law** **p ∧ ¬q Double negation law**

¬(¬p ∨ q) Conditional identity

1

2

3

4

Reset

CHALLENGE ACTIVITY 1.5.1: Reduce the proposition using laws.

Need help with this tool?

Start

Simplify $\neg\neg m \vee F$ to m

Laws

Distributive

$(a \wedge b) \vee (a \wedge c)$	$\equiv$	$a \wedge (b \vee c)$
$(a \vee b) \wedge (a \vee c)$	$\equiv$	$a \vee (b \wedge c)$

Commutative

$a \vee b$	$\equiv$	$b \vee a$
$a \wedge b$	$\equiv$	$b \wedge a$

Complement

$a \vee \neg a$	$\equiv$	T
$a \wedge \neg a$	$\equiv$	F

Identity

$a \wedge T$	$\equiv$	a
$a \vee F$	$\equiv$	a

Double negation

$\neg\neg a$	$\equiv$	a

1	2	3

Check Next

CHALLENGE ACTIVITY 1.5.2: Reduce the proposition using laws, including de Morgan's and conditional.

Start

Simplify $(m \wedge q) \vee \neg(\neg m \vee q)$ to m

Lav

Distributive

$(a \wedge b) \vee (a \wedge c)$	$\equiv$	$a \wedge (b$
$(a \vee b) \wedge (a \vee c)$	$\equiv$	$a \vee (b$

Commutative

$a \vee b$	$\equiv$	$b \vee$
$a \wedge b$	$\equiv$	$b \wedge$

De Morgan's

$\neg(a \wedge b)$	$\equiv$	$\neg a \vee$
$\neg(a \vee b)$	$\equiv$	$\neg a \wedge$

Conditional

$a \rightarrow b$	$\equiv$	$\neg a \vee$
$a \leftrightarrow b$	$\equiv$	$(a \rightarrow b) \wedge$

1	2	3

Check Next

CHALLENGE ACTIVITY 1.5.3: Expand then reduce the proposition.

Start

Simplify $\neg(p \land (s \lor \neg p))$ to $\neg p \lor \neg s$

Lav

Distributive

$(a \land b) \lor (a \land c)$	$\equiv$	$a \land (b$
$(a \lor b) \land (a \lor c)$	$\equiv$	$a \lor (b$

Commutative

$a \lor b$	$\equiv$	$b \lor$
$a \land b$	$\equiv$	$b \land$

De Morgan's

$\neg(a \land b)$	$\equiv$	$\neg a \lor$
$\neg(a \lor b)$	$\equiv$	$\neg a \land$

Conditional

$a \to b$	$\equiv$	$\neg a \lor$
$a \leftrightarrow b$	$\equiv$	$(a \to b) \land$

1	2

Check Next

Additional exercises

Exercise 1.5.1: Label the steps in a proof of logical equivalence.

About

Below are several proofs showing that two logical expressions are logically equivalent. Label the steps in each proof with the law used to obtain each proposition from the previous proposition. The first line in the proof does not have a label.

(a)

$(p \to q) \land (q \lor p)$
$(\neg p \lor q) \land (q \lor p)$
$(q \lor \neg p) \land (q \lor p)$
$q \lor (\neg p \land p)$
$q \lor (p \land \neg p)$
$q \lor F$
q

(b)

$(\neg p \lor q) \to (p \land q)$
$\neg(\neg p \lor q) \lor (p \land q)$
$(\neg\neg p \land \neg q) \lor (p \land q)$
$(p \land \neg q) \lor (p \land q)$
$p \land (\neg q \lor q)$
$p \land T$
p

(c)

$r \lor (\neg r \to p)$
$r \lor (\neg\neg r \lor p)$
$r \lor (r \lor p)$
$(r \lor r) \lor p$
$r \lor p$

Exercise 1.5.2: Using the laws of logic to prove logical equivalence.

About

Use the laws of propositional logic to prove the following:

(a)

$\neg p \to \neg q \equiv q \to p$

(b)

$p \land (\neg p \to q) \equiv p$

(c)

$(p \to q) \land (p \to r) \equiv p \to (q \land r)$

(d)

$\neg p \to (q \to r) \equiv q \to (p \lor r)$

(e)

$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$

(f)

$\neg(p \lor (\neg p \land q)) \equiv \neg p \land \neg q$

(g)

$(p \land q \land \neg r) \lor (p \land \neg q \land \neg r) \equiv p \land \neg r$

(h)

$p \leftrightarrow (p \land r) \equiv \neg p \lor r$

(i)

$(p \land q) \to r \equiv (p \land \neg r) \to \neg q$

Exercise 1.5.3: Using the laws of logic to prove tautologies.

About

Use the laws of propositional logic to prove that each statement is a tautology.

(a)
$(p \land q) \to (p \lor r)$

(b)
$p \to (r \to p)$

(c)
$\neg r \lor (\neg r \to p)$

(d)
$\neg(p \to q) \to \neg q$

(e)
$\neg p \to (p \to q)$

Exercise 1.5.4: Logical relationships between the inverse, converse, and contrapositive.

About

Use the laws of propositional logic to prove each of the following assertions. Start by defining a generic conditional statement $p \rightarrow q$, and then restate the assertion as the equivalence or non-equivalence of two propositions using p and q. Finally prove that the two propositions are equivalent or non-equivalent.

For example, the statement: "A conditional statement is not logically equivalent to its converse" i proven by showing that that $p \rightarrow q$ is not logically equivalent to $q \rightarrow p$.

(a) A conditional statement is not logically equivalent to its converse.

(b) A conditional statement is not logically equivalent to its inverse.

(c) A conditional statement is logically equivalent to its contrapositive.

(d) The converse and inverse of a conditional statement are logically equivalent.

Exercise 1.5.5: Logical equivalence of two mathematical statements.

About

(a) Show that the two sentences below are logically equivalent. Express each pair of sentences using a logical expression. Then prove whether the two expressions are logically equivalent Note: you can assume that x and y are real numbers, so if x is not irrational, then x is rationa and if x is not rational, then x is an irrational number.
   - If x is a rational number and y is an irrational number then x-y is an irrational number.
   - If x is a rational number and x-y is an rational number then y is a rational number.

How was this section? Provide feedback

1.6 Predicates and quantifiers

Many mathematical statements contain variables. The statement "x is an odd number" is not a proposition because its truth value depends on the value of variable x. If x = 5, the statement is true. If x = 4, the statement is false. The truth value of the statement can be expressed as a function P of the variable x, as in P(x). The expression P(x) is read "P of x". A logical statement whose truth value is a function of one or more variables is called a ***predicate***. If P(x) is defined to be the statement "x is an odd number", then P(5) corresponds to the statement "5 is an odd number". P(5) is a proposition because it has a well defined truth value.

A predicate can depend on more than one variable. Define the predicates Q and R as:

$$Q(x, y) : x^2 = y$$

$$R(x, y, z) : x + y = z$$

The proposition Q(5, 25) is true because $5^2 = 25$. The proposition R(2, 3, 6) is false because $2 + 3 \neq 6$.

The ***domain*** of a variable in a predicate is the set of all possible values for the variable. For example, a natural domain for the variable x in the predicate "x is an odd number" would be the set of all integers. If the domain of a variable in a predicate is not clear from context, the domain should be given as part of the definition of the predicate.

PARTICIPATION ACTIVITY 1.6.1: Truth values for predicates on specific inputs.

The domain for all the variables in the following predicates is the set of positive integers:

P(x): x is a prime number
L(x, y): $x < y$
S(x, y, z): $x^2 + y^2 >= z^2$

1) Is P(7) true or false?

- True
- False

2) Is L(6, 6) true or false?

- True
- False

3) Is S(3, 4, 5) true or false?

- True
- False

Statements outside the realm of mathematics can also be predicates. For example, consider the statement: "The city has a population over 1,000,000." The "city" is the variable and the domain is defined to be the set of all cities in the United States. When the city is New York, the statement becomes: "New York has a population over 1,000,000" and the statement is true. When the city is Toledo, the statement becomes: "Toledo has a population over 1,000,000" and the statement is false.

Note that it may happen that a statement P(x) is true for all values in the domain. However, if the statement contains a variable, the statement is still considered to be a predicate and not a proposition. For example, if P(x) is the statement "x + 1 > 1" and the domain is all positive integers, the statement is true for each value in the domain. However, P(x) is considered to be a predicate and not a proposition because it contains a variable.

PARTICIPATION ACTIVITY 1.6.2: Predicates and propositions.

Which sentences are propositions and which are predicates? The domain is the set of all positive integers.

1) x is odd.

- Proposition
- Predicate

2) 23 is a prime number.

- Proposition
- Predicate

3) $\frac{1}{1+x} < 1$

- Proposition
- Predicate

4) 16 = x^2

- Proposition
- Predicate

Universal quantifier

If all the variables in a predicate are assigned specific values from their domains, then the predicate becomes a proposition with a well defined truth value. Another way to turn a predicate into a proposition is to use a quantifier. The logical statement ∀x P(x) is read "for all x, P(x)" or "for every x, P(x)". The statement ∀x P(x) asserts that P(x) is true for every possible value for x in its domain. The symbol ∀ is a ***universal quantifier*** and the statement ∀x P(x) is called a ***universally quantified statement***. ∀x P(x) is a proposition because it is either true or false. ∀x P(x) is true if and only if P(n) is true for every n in the domain.

If the domain is a finite set of elements {a_1, a_2, ..., a_k}, then:

$$\forall x P(x) \equiv P(a_1) \wedge P(a_2) \wedge ... \wedge P(a_k)$$

The equivalence symbol means that the two expressions always have the same truth value, regardless of the truth values for P(a_1), ...,P(a_n). If the domain is the set of students in a class and the predicate A(x) means that student x completed the assignment, then the proposition ∀x A(x) means: "Every student completed the assignment." Establishing that ∀x A(x) is true requires showing that each and every student in the class did in fact complete the assignment.

Some universally quantified statements can be shown to be true by showing that the predicate holds for an arbitrary element from the domain. An "arbitrary element" means nothing is assumed about the element other than the fact that it is in the domain. In the following example, the domain is the set of all positive integers:

$$\forall x(\frac{1}{x+1} < 1)$$

The statement is true because when x is assigned any arbitrary value from the set of all positive integers, the inequality $\frac{1}{1+x} < 1$ holds.

PARTICIPATION ACTIVITY 1.6.3: Proving ∀x (1/(x+1) < 1) is true for an arbitrary positive real number x.

Animation captions:

1. The proof starts with the fact that $0 < x$ for all positive integers and adds 1 to both sides to $1 < 1 + x$.
2. Dividing both sides by $(x + 1)$ gives that $\frac{1}{(x+1)} \leq 1$. The inequality is true for all positive integers x.

A ***counterexample*** for a universally quantified statement is an element in the domain for which the predicate is false. A single counterexample is sufficient to show that a universally quantified statement is false. For example, consider the statement ∀x ($x^2 > x$), in which the domain is the set of positive integers. When x = 1, then $x^2 = x$ and the statement $x^2 > x$ is not true. Therefore x = 1 is a counterexample that shows the statement "∀x ($x^2 > x$)" is false.

PARTICIPATION ACTIVITY 1.6.4: Truth values for universally quantified statements.

In the following questions, the domain is the set of all positive integers. Indicate whether the universally quantified statement is true or false.

1) ∀x ($x^2 > 0$).

- True
- False

2) ∀x ($x - 1 \geq 0$).

- True
- False

3) ∀x (x - 1 > 0).

- True
- False

Existential quantifier

The logical statement ∃x P(x) is read "There exists an x, such that P(x)". The statement ∃x P(x) asserts that P(x) is true for at least one possible value for x in its domain. The symbol ∃ is an ***existential quantifier*** and the statement ∃x P(x) is called a ***existentially quantified statement***. ∃x P(x) is a proposition because it is either true or false. ∃x P(x) is true if and only if P(n) is true for at least one value n in the domain of variable x.

If the domain is a finite set of elements $\{a_1, a_2, ..., a_k\}$, then:

$$\exists x\, P(x) \equiv P(a_1) \vee P(a_2) \vee \dots \vee P(a_k)$$

If the domain is the set of students in a class and the predicate A(x) means that student x completed the assignment, then ∃x A(x) is the statement: "There is a student who completed the assignment." Establishing that ∃x A(x) is true only requires finding one particular student who completed the assignment. However, showing that ∃x A(x) is false requires showing that every student in the class did not complete the assignment.

Some existentially quantified statements can be shown to be false by showing that the predicate is false for an arbitrary element from the domain. For example, consider the existentially quantified statement in which the domain of x is the set of all positive integers:

$$\exists x\, (x + 1 < x)$$

The statement is false because no positive integer satisfies the expression x + 1 < x.

PARTICIPATION ACTIVITY 1.6.5: Showing ∃x (x + 1 < x) is false.

Animation captions:

1. The proof starts with the inequality $x + 1 < x$ and subtracts x from both sides to get $1 <$
2. Since $1 < 0$ is false, the inequality $x + 1 < x$ is false for every x and therefore $\exists x(x + 1$ is false.

PARTICIPATION ACTIVITY 1.6.6: Truth values for existentially quantified statements.

In the following questions, the domain is the set of all positive integers. Indicate whether the existentially quantified statement is true or false.

1) $\exists x(x^2 < 0)$.

- True
- False

2) $\exists x(x - 1 > 0)$.

- True
- False

3) $\exists x(x^2 = x)$.

- True
- False

PARTICIPATION ACTIVITY 1.6.7: Proving universal and existentially quantified statements.

Match what needs to be done to show that an existential or universally quantified statement is true or false.

Show the following statement is false: ∃x P(x).

Show the following statement is false: ∀x P(x).

Show the following statement is true: ∀x P(x).

Show the following statement is true: ∃x P(x).

Give a particular element n in the domain for which P(n) is true.

Show that for every element n in the domain, P(n) is false.

Show that for every element n in the domain, P(n) is true.

Give a counterexample: a particular element n in the domain for which P(n) is false.

Reset

Additional exercises

Exercise 1.6.1: Which expressions with predicates are propositions?

About

Predicates P, T and E are defined below. The domain of discourse is the set of all positive integers.

$$P(x) : x \text{ is even}$$
$$T(x, y) : 2^x = y$$
$$E(x, y, z) : x^y = z$$

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a)
P(3)

(b)
¬P(3)

(c)
$T(5, 32)$

(d)
$T(5, x)$

(e)
$E(6, 2, 36)$

(f)
$E(2, y, 7)$

(g)
$P(3) \lor T(5, 32)$

(h)
$T(5, 16) \rightarrow E(6, 3, 36)$

Exercise 1.6.2: Truth values for quantified statements about integers.

About

In this problem, the domain of discourse is the set of all integers. Which statements are true? If a existential statement is true, give an example. If a universal statement is false, give a counterexample.

(a)
$\exists x\ (x + x = 1)$

(b)
$\exists x\ (x + 2 = 1)$

(c)
$\forall x\ (x^2 - x \neq 1)$

(d)
$\forall x\ (x^2 - x \neq 0)$

(e)
$\forall x\ (x^2 > 0)$

(f)
$\exists x\ (x^2 > 0)$

Exercise 1.6.3: Translating mathematical statements in English into logical expressions.

About

Consider the following statements in English. Write a logical expression with the same meaning
The domain of discourse is the set of all real numbers.

(a)
There is a number whose cube is equal to 2.

(b)
The square of every number is at least 0.

(c)
There is a number that is equal to its square.

(d)
Every number is less than or equal to its square.

Exercise 1.6.4: Truth values for quantified statements for a given set of predicates.

About

The domain for this problem is a set a, b, c, d. The table below shows the value of three predicates for each of the elements in the domain. For example, $Q(b)$ is false because the truth value in row b, column Q is F.

	P	Q	R
a	T	T	F
b	T	F	F
c	T	F	F
d	T	F	F

Which statements are true? Justify your answer.

(a) ∀x P(x)

(b) ∃x P(x)

(c) ∀x Q(x)

(d) ∃x Q(x)

(e) ∀x R(x)

(f) ∃x R(x)

Exercise 1.6.5: Converting a quantified expression to an equivalent logical expression.

About

P(x) is a predicate and the domain for the variable x is {1, 2, 3, 4}. For each of the logical expressions given, give an equivalent logical expression that does not use quantifiers.

(a)

$\forall x\, P(x)$

(b)

$\exists x\, P(x)$

How was this section? Provide feedback

1.7 Quantified statements

Universally and existentially quantified statements can also be constructed from logical operations. Consider an example in which the domain is the set of positive integers and define the following predicates:

$$P(x)\text{: x is prime}$$

$$O(x)\text{: x is odd}$$

The proposition $\exists x\, (P(x) \land \neg O(x))$ states that there exists a positive number that is prime and not odd. This proposition is true because of the number $x = 2$.

The proposition $\forall x\, (P(x) \rightarrow O(x))$ says that for every positive integer x, if x is prime then x is odd. This proposition is false, because of the counterexample $x = 2$. Since 2 is prime and not odd, the conditional statement $P(2) \rightarrow O(2)$ is false.

The universal and existential quantifiers are generically called ***quantifiers***. A logical statement that includes a universal or existential quantifier is called a ***quantified statement***. The quantifiers ∀ and ∃ are applied before the logical operations (∧, ∨, →, and ↔) used for propositions. This means that the statement ∀x P(x) ∧ Q(x) is equivalent to (∀x P(x)) ∧ Q(x) as opposed to ∀x (P(x) ∧ Q(x)).

PARTICIPATION ACTIVITY 1.7.1: Evaluating quantified statements.

For the following questions, the domain for the variable x is the set of all positive integers. The first three questions use predicates O and M which are defined as follows:

O(x): x is odd
M(x): x is an integer multiple of 4 (e.g., 4, 8, 12,...)

Indicate whether each quantified statement is true or false:

1) ∃x (O(x) ∧ M(x))
- True
- False

2) ∃x (¬O(x) ∧ ¬M(x))
- True
- False

3) ∀x (M(x) → ¬O(x))
- True
- False

4) ∀x ((x = 1) ∨ (x^2 ≠ x))
- True
- False

A variable x in the predicate P(x) is called a ***free variable*** because the variable is free to take on any value in the domain. The variable x in the statement ∀x P(x) is a ***bound variable*** because the variable is bound to a quantifier. A statement with no free variables is a proposition because the statement's truth value can be determined.

In the statement (∀x P(x)) ∧ Q(x), the variable x in P(x) is bound by the universal quantifier, but the variable x in Q(x) is not bound by the universal quantifier. Therefore the statement (∀x P(x)) ∧ Q(x) is not a proposition. In contrast, the universal quantifier in the statement ∀x (P(x) ∧ Q(x)) binds both occurrences of the variable x. Therefore ∀x (P(x) ∧ Q(x)) is a proposition.

PARTICIPATION ACTIVITY 1.7.2: Free and bound variables in quantified statements.

1) The expression ∃x P(x) is a proposition.

- True
- False

2) The expression (∃x S(x)) ∨ R(x) is a proposition.

- True
- False

3) The expression ∃x (S(x) ∨ R(x)) has a free variable.

- True
- False

4) The expression ∀x P(x) ∨ ∃x Q(x) is a proposition.

- True
- False

Logical equivalence with quantified statements

Two quantified statements (whether they are expressed in English or the language of logic) have the same logical meaning if they have the same truth value regardless of value of the predicates for the elements in the domain. Consider as an example a domain consisting of a set of people invited to a party. Define the predicates:

P(x): x came to the party

S(x): x was sick

The statement "Everyone was not sick" is logically equivalent to "$\forall x\ \neg S(x)$" because the two statements have the same truth value regardless of who was invited to the party and whether they were sick.

The table below gives an example of a set of people who could have been invited to the party and the value of the predicate S(x) and P(x) for each person. For example, Gertrude came to the party (i.e., P(Gertrude) = T) because the truth value in the row labeled Gertrude and column labeled P(x) is true.

Table 1.7.1: Values for predicates S(x) and P(x) for a particular group.

Name	S(x)	P(x)
Joe	F	T
Theodore	T	F
Gertrude	F	T
Samuel	F	F

For the group of people in the domain, the statement "Someone was sick and came to the party" is false because there is no individual for whom S(x) and P(x) are true. However, $\exists x\ (S(x) \vee P(x))$ is true because, for example, S(Joe) ∨ P(Joe) is true. Therefore the two statements "Someone was sick and came to the party" and "$\exists x\ (S(x) \vee P(x))$" are not logically equivalent.

PARTICIPATION ACTIVITY 1.7.3: Quantified statements in logic and English.

For the following questions, the domain for the variable x is a group of employees working on a project. The predicate N(x) says that x is a new employee. The predicate D(x) says that x met his deadline. Consider the group defined in the table below:

Name	N(x)	D(x)
Happy	F	F

Sleepy	F	T
Grumpy	T	T
Bashful	T	T

1) Is the statement "Every new employee met his deadline" true or false for the group defined in the table?

   - True
   - False

2) Is the statement "$\forall x\ (N(x) \land D(x))$" true or false for the group defined in the table?

   - True
   - False

3) Select the logical expression that is equivalent to the statement: "Every new employee met his deadline".

   - $\forall x\ (N(x) \land D(x))$
   - $\forall x\ (N(x) \rightarrow D(x))$

PARTICIPATION ACTIVITY 1.7.4: Quantified statements in logic and English.

For the following questions, the domain for the variable x is a group of employees working on a project. The predicate N(x) says that x is a new employee. The predicate D(x) says that x met his deadline. Consider the group defined in the table below:

Name	N(x)	D(x)
Happy	F	F
Sleepy	F	T
Grumpy	T	F

Bashful	T	F

1) Is the statement "There is a new employee who met his deadline" true or false for the group defined in the table?

- True
- False

2) Is the statement "$\exists x\ (N(x) \rightarrow D(x))$" true or false for the group defined in the table?

- True
- False

3) Select the logical expression that is equivalent to the statement: "There is a new employee who met his deadline".

- $\exists x\ (N(x) \land D(x))$
- $\exists x\ (N(x) \rightarrow D(x))$

Example 1.7.1: Translating quantified statements from English to logic.

The domain in this problem are the members of a board of directors for a company who are considering a proposal for that company. Define the following predicates:

R(x): person x read the proposal.
V(x): person x voted in favor of the proposal.

Translate each of the following sentences into an equivalent logical expression:

1. Everyone who read the proposal voted in favor of it.

   Solution: $\forall x\ (R(x) \rightarrow V(x))$. Video explanation of the solution (2:01)

2. Someone who did not read the proposal, voted in favor of it.

   Solution: $\exists x\ (\neg R(x) \land V(x))$. Video explanation of the solution (1:09)

3. Someone did not read the proposal and someone voted in favor of it.

   Solution: $\exists x\ \neg R(x) \land \exists x\ V(x)$. Video explanation of the solution (2:57)

PARTICIPATION ACTIVITY 1.7.5: Quantified statements expressed in English.

In the following question, the domain is the set of fourth graders at Lee Elementary School. The predicates P and Q are defined as follows:

P(x): x took the math test
Q(x): x is present today

Match the English sentence with the corresponding logical proposition.

$\exists x\ (Q(x) \land \neg P(x))$ $\exists x\ \neg P(x)$ $\forall x\ (Q(x) \land P(x))$ $\forall x\ (Q(x) \rightarrow P(x))$

Every student was present and took the math test.

There is a student who did not take the math test.

Every student who is present took the math test.

There is a student who is present and did not take the math test.

Reset

CHALLENGE ACTIVITY 1.7.1: Give an example or counterexample of quantified statement.

Mark the statement as true or false.

Start

$$\exists x\,(A(x) \wedge D(x))$$

True | False

Names	A(x)	B(x)	C(x)	D(x)
Ann	F	T	T	T
Bob	T	T	F	T
Joe	F	T	F	T

1 2 3 4 5 6 7 8

Check Next

Additional exercises

Exercise 1.7.1: Determining whether a quantified statement about the integers is true.

About

Predicates P and Q are defined below. The domain of discourse is the set of all positive integers

P(x): x is prime
Q(x): x is a perfect square (i.e., $x = y^2$, for some integer y)

Indicate whether each logical expression is a proposition. If the expression is a proposition, then give its truth value.

(a)
$\exists x\, Q(x)$

(b)
$\forall x\, Q(x) \land \neg P(x)$

(c)
$\forall x\, Q(x) \lor P(3)$

(d)
$\exists x\, (Q(x) \land P(x))$

(e)
$\forall x\, (\neg Q(x) \lor P(x))$

Exercise 1.7.2: Translating quantified statements from English to logic, part 1.

About

In the following question, the domain of discourse is a set of students at a university. Define the following predicates:

> E(x): x is enrolled in the class
> T(x): x took the test

Translate the following English statements into a logical expression with the same meaning.

(a)
Someone who is enrolled in the class took the test.

(b)
All students enrolled in the class took the test.

(c)
Everyone who took the test is enrolled in the class.

(d)
At least one student who is enrolled in the class did not take the test.

Exercise 1.7.3: Translating quantified statements in English into logic, part 2.

About

In the following question, the domain of discourse is the set of employees at a company. One of the employees is named Sam. Define the following predicates:

T(x): x is a member of the executive team
B(x): x received a large bonus

Translate the following English statements into a logical expression with the same meaning.

(a) Someone did not get a large bonus.

(b) Everyone got a large bonus.

(c) Sam did not get a large bonus even though he is a member of the executive team.

(d) Someone who is not on the executive team received a large bonus.

(e) Every executive team member got a large bonus.

Exercise 1.7.4: Translating quantified statements from English to logic, part 3.

About

In the following question, the domain of discourse is a set of employees who work at a compan
Ingrid is one of the employees at the company. Define the following predicates:

S(x): x was sick yesterday
W(x): x went to work yesterday
V(x): x was on vacation yesterday

Translate the following English statements into a logical expression with the same meaning.

(a) At least one person was sick yesterday.

(b) Everyone was well and went to work yesterday.

(c) Everyone who was sick yesterday did not go to work.

(d) Yesterday someone was sick and went to work.

(e) Everyone who did not got to work yesterday was sick.

(f) Everyone who missed work was sick or on vacation (or both).

(g) Someone who missed work was neither sick nor on vacation.

(h) Each person missed work only if they were sick or on vacation (or both).

(i) Ingrid was sick yesterday but she went to work anyway.

(j) Someone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression (x ≠ Ingrid).)

(k) Everyone besides Ingrid was sick yesterday. (Note that the statement does not indicate whether or not Ingrid herself was sick yesterday. Also, for this question, you will need the expression ($x \neq$ Ingrid).)

Exercise 1.7.5: Translating quantified statements from English to logic, part 4.

About

A student club holds a meeting. The predicate M(x) denotes whether person x came to the meet on time. The predicate O(x) refers to whether person x is an officer of the club. The predicate D(x indicates whether person x has paid his or her club dues. The domain is the set of all members (the club. Give a logical expression that is equivalent to each English statement.

(a) Someone is not an officer.

(b) All the officers came on time to the meeting.

(c) Everyone was on time for the meeting.

(d) Everyone paid their dues or came on time to the meeting.

(e) At least one person paid their dues and came on time to the meeting.

(f) There is an officer who did not come on time for the meeting.

Exercise 1.7.6: Translating quantified statements from logic to English.

About

In the following question, the domain of discourse is the set of employees of a company. Define the following predicates:

A(x): x is on the board of directors
E(x): x earns more than $100,000
W(x): x works more than 60 hours per week

Translate the following logical expressions into English:

(a)
$\forall x\ (A(x) \rightarrow E(x))$

(b)
$\exists x\ (E(x) \land \neg W(x))$

(c)
$\forall x\ (W(x) \rightarrow E(x))$

(d)
$\exists x\ (\neg A(x) \land E(x))$

(e)
$\forall x\ (E(x) \rightarrow (A(x) \lor W(x)))$

(f)
$\exists x\ (A(x) \land \neg E(x) \land W(x))$

Exercise 1.7.7: Determining whether a quantified logical statement is true and translating into English, part 1.

About

The domain of discourse is a group working on a project at a company. Define the following predicates.

- D(x): x missed the deadline.
- N(x): x is a new employee.

Consider a situation in which there are five people in the group. The following table gives values the predicates D and N for each member of the group. For example, Bert did not miss the deadli because the truth value in the row labeled Bert and the column labeled D(x) is F.

Using these values, determine whether each quantified expression evaluates to true or false. The translate the statement into English.

Name	D(x)	N(x)
Sam	T	F
Beth	T	T
Melanie	F	T
Al	T	T
Bert	F	T

(a)
$\forall x \, (D(x) \lor N(x))$

(b)
$\forall x \, ((x \neq \text{Sam}) \rightarrow N(x))$

(c)
$\exists x \, (\neg D(x) \land N(x))$

(d)
$\forall x \, (\neg D(x) \rightarrow \neg N(x))$

(e)
$N(\text{Bert}) \rightarrow D(\text{Bert})$

(f)
$\forall x\ (\neg N(x) \rightarrow D(x))$

(g)
$\forall x\ ((x \neq \text{Bert}) \rightarrow D(x))$

(h)
$\exists x\ (\neg D(x) \land \neg N(x))$

(i)
$\forall x\ (D(x) \leftrightarrow N(x))$

(j)
$D(\text{Sam}) \land N(\text{Sam})$

Exercise 1.7.8: Determining whether a quantified logical statement is true and translating into English, part 2.

About

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo
D(x): x was given the medication
A(x): x had fainting spells
M(x): x had migraines

Suppose that there are five patients who participated in the study. The table below shows the names of the patients and the truth value for each patient and each predicate:

Name	P(x)	D(x)	A(x)	M(x)
Frodo	T	F	F	T
Gandalf	F	T	F	F
Gimli	F	T	T	F

Aragorn	T	F	T	T
Bilbo	T	T	F	F

For each of the following quantified statements, indicate whether the statement is a proposition the statement is a proposition, give its truth value and translate the expression into English.

(a)
$\exists x\ (M(x) \land D(x))$

(b)
$\exists x\ M(x) \land \exists x\ D(x)$

(c)
$\exists x\ M(x) \land D(x)$

(d)
$\forall x\ (A(x) \lor M(x))$

(e)
$\forall x\ (M(x) \leftrightarrow A(x))$

(f)
$\forall x\ ((M(x) \land A(x)) \rightarrow \neg D(x))$

(g)
$\exists x\ (D(x) \land \neg A(x) \land \neg M(x))$

(h)
$\forall x\ (D(x) \rightarrow (A(x) \lor M(x)))$

Exercise 1.7.9: Determining whether a quantified logical statement is true, part 1.

About

The domain for this question is the set {a, b, c, d, e}. The following table gives the value of

The domain for this question is the set {a, b, c, d, e}. The following table gives the value of predicates P, Q, and R for each element in the domain. For example, Q(c) = T because the truth value in the row labeled c and the column Q is T. Using these values, determine whether each quantified expression evaluates to true or false.

	P(x)	Q(x)	R(x)
a	T	T	F
b	T	F	F
c	F	T	F
d	T	T	F
e	T	T	T

(a)
$\forall x\ (R(x) \lor Q(x) \rightarrow P(x))$

(b)
$\exists x((x \neq b) \land \neg Q(x))$

(c)
$\exists x((x = c) \rightarrow P(x))$

(d)
$\exists x(Q(x) \land R(x))$

(e)
$Q(a) \land P(d)$

(f)
$\forall x\ ((x \neq b) \rightarrow Q(x))$

(g)
$\forall x\ (P(x) \lor R(x))$

(h)
$\forall x\,(R(x) \rightarrow P(x))$

(i)
$\exists x(Q(x) \vee R(x))$

Exercise 1.7.10: Determining whether a quantified logical statement is true, part 2.

About

A student club holds a meeting. The predicate M(x) denotes whether person x came to the meet on time. The predicate O(x) refers to whether person x is an officer of the club. The predicate D(x indicates whether person x has paid his or her club dues. The domain is the set of all members of the club. The names of the members and their truth values for each of the predicates is given in the following table. Indicate whether each expression is true or false. If a universal statement is true, give a counterexample. If an existential statement is true, give an example.

Name	M(x)	O(x)	D(x)
Hillary	T	F	T
Bernie	F	T	F
Donald	F	T	F
Jeb	F	T	T
Carly	F	T	F

(a)
$\forall x\,\neg(O(x) \leftrightarrow D(x))$

(b)
$\forall x\,((x \neq \text{Jeb}) \rightarrow \neg(O(x) \leftrightarrow D(x)))$

(c)
$\forall x\,(\neg O(x) \rightarrow D(x))$

(d)
$\exists x\ (M(x) \wedge D(x))$

(e)
$\forall x\ (M(x) \vee O(x) \vee D(x))$

(f)
$\forall x\ \neg D(x)$

(g)
$M(\text{Jeb}) \wedge D(\text{Hillary})$

(h)
$D(\text{Bernie}) \wedge O(\text{Bernie})$

(i)
$\exists x\ (O(x) \rightarrow M(x))$

(j)
$\exists x\ (M(x) \wedge O(x) \wedge D(x))$

How was this section? **Provide feedback**

1.8 De Morgan's law for quantified statements

De Morgan's law for quantified statements

The negation operation can be applied to a quantified statement, such as $\neg\forall x\ F(x)$. If the domain for the variable x is the set of all birds and the predicate F(x) is "x can fly", then the statement ¬∀x F(x) is equivalent to:

"Not every bird can fly."

which is logically equivalent to the statement:

"There exists a bird that cannot fly."

The equivalence of the previous two statements is an example of De Morgan's law for quantified statements, which is formally stated as ¬∀x F(x) ≡ ∃x ¬F(x). The diagram below illustrates that for a finite domain, De Morgan's law for universally quantified statements is the same as De Morgan's law for propositions:

Figure 1.8.1: De Morgan's law for universally quantified statements.

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\begin{array}{ccc} \neg \forall x\ P(x) & \equiv & \exists x\ \neg P(x) \\ \equiv & & \equiv \\ \neg(P(a_1) \land P(a_2) \land \ldots \land P(a_n)) & \equiv & \neg P(a_1) \lor \neg P(a_2) \lor \ldots \lor \neg P(a_n) \end{array}$$

Similarly, consider the statement ¬∃x A(x) in which the domain is the set of children enrolled in a class and A(x) is the predicate "x is absent today". The statement is expressed in English as:

"It is not true that there is a child in the class who is absent today."

which is logically equivalent to:

"Every child in the class is not absent today."

The logical equivalence of the last two statements is an example of the second of De Morgan's laws for quantified statements: ¬∃x P(x) ≡ ∀x ¬P(x). The diagram below illustrates that for a finite domain, De Morgan's law for existentially quantified statements is the same as De Morgan's law for propositions:

Figure 1.8.2: De Morgan's law for existentially quantified statements.

Domain of discourse = $\{a_1, a_2, \ldots, a_n\}$

$$\neg \exists x \, P(x) \equiv \forall x \, \neg P(x)$$

$$\equiv \qquad\qquad \equiv$$

$$\neg(P(a_1) \lor P(a_2) \lor \ldots \lor P(a_n)) \equiv \neg P(a_1) \land \neg P(a_2) \land \ldots \land \neg P(a_n)$$

Table 1.8.1: Summary of De Morgan's laws for quantified statements.

$\neg\forall x \, P(x)$	$\equiv$	$\exists x \, \neg P(x)$
$\neg\exists x \, P(x)$	$\equiv$	$\forall x \, \neg P(x)$

PARTICIPATION ACTIVITY 1.8.1: De Morgan's law can be used to simplify an existentially quantified statement.

Animation captions:

1. Start with $\neg\exists x(P(x) \rightarrow \neg Q(x))$ and apply De Morgan's law to get $\forall x \neg(P(x) \rightarrow \neg Q(x))$.
2. Then apply the Conditional Identity to get $\forall x \neg(\neg P(x) \lor \neg Q(x))$.
3. Then apply De Morgan's law to get $\forall x(\neg\neg P(x) \land \neg\neg Q(x))$.
4. Finally, apply the Double Negation law to get $\forall x(P(x) \land Q(x))$.

PARTICIPATION ACTIVITY 1.8.2: De Morgan's law can be used to simplify a universally quantified statement.

Animation captions:

1. Start with $\neg\forall x(P(x) \land \neg Q(x))$ and apply De Morgan's law to get $\exists x\neg(P(x) \land \neg Q(x))$.
2. Then apply De Morgan's law to get $\exists x(\neg P(x) \lor \neg\neg Q(x))$.
3. Finally, apply the Double Negation law to get $\exists x(\neg P(x) \lor Q(x))$.

PARTICIPATION ACTIVITY 1.8.3: Applying De Morgan's laws for quantified statements.

Match the logically equivalent propositions.

¬∃x (¬P(x) ∧ Q(x))   ¬∀x ¬P(x)   ¬∀x (P(x) ∧ ¬Q(x))   ¬∃x (P(x) ∨ Q(x))

∃x P(x)

∃x (¬P(x) ∨ Q(x))

∀x (¬P(x) ∧ ¬Q(x))

∀x (P(x) ∨ ¬Q(x))

Reset

Additional exercises

Exercise 1.8.1: Applying De Morgan's law for quantified statements to logical expressions.

About

Apply De Morgan's law to each expression to obtain an equivalent expression in which each negation sign applies directly to a predicate. (i.e., $\exists x\,(\neg P(x) \lor \neg Q(x))$ is an acceptable final answe but not $\neg\exists x\, P(x)$ or $\exists x\, \neg(P(x) \land Q(x))$).

(a)
$\neg\exists x\, P(x)$

(b)
$\neg\exists x\, (P(x) \lor Q(x))$

(c)
$\neg\forall x\, (P(x) \land Q(x))$

(d)
$\neg\forall x\, (P(x) \land (Q(x) \lor R(x)))$

Exercise 1.8.2: Applying De Morgan's law for quantified statements to English statements.

About

In the following question, the domain of discourse is a set of male patients in a clinical study. Define the following predicates:

P(x): x was given the placebo
D(x): x was given the medication
M(x): x had migraines

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negatio operation applies directly to a predicate and then translate the logical expression back into Engli

Sample question: Some patient was given the placebo and the medication.

- $\exists x\, (P(x) \land D(x))$
- Negation: $\neg\exists x\, (P(x) \land D(x))$
- Applying De Morgan's law: $\forall x\, (\neg P(x) \lor \neg D(x))$
- English: Every patient was either not given the placebo or not given the medication (or botl

(a) Every patient was given the medication.

(b) Every patient was given the medication or the placebo or both.

(c) There is a patient who took the medication and had migraines.

(d) Every patient who took the placebo had migraines. (Hint: you will need to apply the conditio identity, $p \to q \equiv \neg p \lor q$.)

(e) There is a patient who had migraines and was given the placebo.

Exercise 1.8.3: Applying De Morgan's law for quantified statements to English statements.

About

In the following question, the domain of discourse is a set of students who show up for a test. Define the following predicates:

> P(x): x showed up with a pencil
> C(x): x showed up with a calculator

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until each negatio operation applies directly to a predicate and then translate the logical expression back into Engli

Sample question: Every student showed up with a calculator.

- $\forall x\ C(x)$
- Negation: $\neg\forall x\ C(x)$
- Applying De Morgan's law: $\exists x\ \neg C(x)$
- English: Some student showed up without a calculator.

(a)
At least one of the students showed up with a pencil.

(b)
Every student showed up with a pencil or a calculator (or both).

(c)
Every student who showed up with a calculator also had a pencil.

(d)
There is a student who showed up with both a pencil and a calculator.

(e)
Some student showed up with a pencil or a calculator.

(f)
Every student showed up with a pencil and a calculator.

Exercise 1.8.4: Using De Morgan's law for quantified statements to prove logical equivalence.

About

Use De Morgan's law for quantified statements and the laws of propositional logic to show the following equivalences:

(a)
$\neg\forall x\,(P(x) \wedge \neg Q(x)) \equiv \exists x\,(\neg P(x) \vee Q(x))$

(b)
$\neg\forall x\,(\neg P(x) \rightarrow Q(x)) \equiv \exists x\,(\neg P(x) \wedge \neg Q(x))$

(c)
$\neg\exists x\,(\neg P(x) \vee (Q(x) \wedge \neg R(x))) \equiv \forall x\,(P(x) \wedge (\neg Q(x) \vee R(x)))$

How was this section? Provide feedback

1.9 Nested quantifiers

If a predicate has more than one variable, each variable must be bound by a separate quantifier. A logical expression with more than one quantifier that bind different variables in the same predicate is said to have ***nested quantifiers***. The examples below show several logical expressions and which variables are bound in each. The logical expression is a proposition if all the variables are bound.

Figure 1.9.1: Nested quantifiers and bound variables.

$\forall x \exists y\ P(x,y)$	x and y are both bound.
$\forall x\ P(x,y)$	x is bound and y is free.
$\exists y \exists z\ T(x,y,z)$	y and z are bound. x is free.

PARTICIPATION ACTIVITY 1.9.1: Free and bound variables with nested quantifiers.

1) Is the variable y bound in the expression $\forall x\ Q(x, y)$?
- Yes
- No

2) Is the following logical expression a proposition: $\forall z \exists y\ Q(x, y, z)$?
- Yes
- No

Nested quantifiers of the same type

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate M to be:

$$M(x, y)\text{: x sent an email to y}$$

and consider the proposition: $\forall x \forall y\ M(x, y)$. The proposition can be expressed in English as:

$$\forall x \forall y\ M(x, y) \quad \leftrightarrow \quad \text{"Everyone sent an email to everyone."}$$

The statement $\forall x \forall y\ M(x, y)$ is true if for every pair, x and y, M(x, y) is true. The universal quantifiers include the case that x = y, so if $\forall x \forall y\ M(x, y)$ is true, then everyone sent an email to everyone else and everyone sent an email to himself or herself. The statement $\forall x \forall y\ M(x, y)$ is false if there is any pair, x and y, that causes M(x, y) to be false. In particular,

$\forall x \forall y\ M(x, y)$ is false even if there is a single individual who did not send himself or herself an email.

Now consider the proposition: $\exists x \exists y\ M(x, y)$. The proposition can be expressed in English as:

$\exists x \exists y\ M(x, y)$ $\leftrightarrow$ "There is a person who sent an email to someone."

The statement $\exists x \exists y\ M(x, y)$ is true if there is a pair, x and y, in the domain that causes M(x, y) to evaluate to true. In particular, $\exists x \exists y\ M(x, y)$ is true even in the situation that there is a single individual who sent an email to himself or herself. The statement $\exists x \exists y\ M(x, y)$ is false if all pairs, x and y, cause M(x, y) to evaluate to false.

PARTICIPATION ACTIVITY 1.9.2: Nested quantifiers of the same type.

In the following question, the domain is the set of all non-negative integers. xy means x times y.

1) $\forall x \forall y\ (xy = 1)$
- True
- False

2) $\exists x \exists y\ (xy = 1)$
- True
- False

3) $\exists x \exists y\ ((x+y = x) \land (y \neq 0))$.
- True
- False

4) $\forall x \forall y\ ((x+y \neq x) \lor (y = 0))$.
- True
- False

Alternating nested quantifiers

A quantified expression can contain both types of quantifiers as in: $\exists x \forall y\ M(x, y)$. The quantifiers are applied from left to right, so the statement $\exists x \forall y\ M(x, y)$ translates into English as:

$$\exists x\ \forall y\ M(x, y) \quad \leftrightarrow \quad \text{"There is a person who sent an email to everyone."}$$

Switching the quantifiers changes the meaning of the proposition:

$$\forall x\ \exists y\ M(x, y) \quad \leftrightarrow \quad \text{"Every person sent an email to someone."}$$

In reasoning whether a quantified statement is true or false, it is useful to think of the statement as a ***two player game*** in which two players compete to set the statement's truth value. One of the players is the "existential player" and the other player is the "universal player". The variables are set from left to right in the expression. The table below summarizes which variables are set by which player and the goal of each player:

Table 1.9.1: Nested quantifiers as a two-person game.

Player	Action	Goal
Existential player	Selects values for existentially bound variables	Tries to make the expression true
Universal player	Selects values for universally bound variables	Tries to make the expression false

If the predicate is true after all the variables are set, then the quantified statement is true. If the predicate is false after all the variables are set, then the quantified statement is false. Consider as an example the following quantified statement in which the domain is the set of all integers:

$$\forall x\ \exists y\ (x + y = 0)$$

The universal player first selects the value of x. Regardless of which value the universal player selects for x, the existential player can select y to be -x, which will cause the sum x + y to be 0. Because the existential player can always succeed in causing the predicate to be true, the statement $\forall x\ \exists y\ (x + y = 0)$ is true.

Switching the order of the quantifiers gives the following statement:

$$\exists x\ \forall y\ (x + y = 0)$$

Now, the existential player goes first and selects a value for x. Regardless of the value chosen for x, the universal player can select some value for y that causes the predicate to be false.

For example, if x is an integer, then y = -x + 1 is also an integer and x + y = 1 ≠ 0. Thus, the universal player can always win and the statement ∃x ∀y (x + y = 0) is false.

PARTICIPATION ACTIVITY 1.9.3: Examples showing reasoning about nested quantifiers using two player games.

Animation captions:

1. Is $\forall x \exists y (y^2 = x)$ true? The domain is the set of integers. If the universal player selects $x =$ then the existential player cannot find an integer y such that $x = y^2$.
2. The universal player wins, so $\forall x \exists y (y^2 = x)$ is false.
3. Is $\exists x \forall y (x + y = y)$ true? If the existential player selects $x = 0$, then for any y that the universal player selects, $0 + y = y$.
4. The existential player wins, so $\exists x \forall y (x + y = y)$ is true.

PARTICIPATION ACTIVITY 1.9.4: Truth values for statements with nested quantifiers.

In the following question, the domain is the set of all real numbers. xy means x times y.

1) ∀x ∃y (xy = 1)

- True
- False

2) ∃x ∀y (xy = 1)

- True
- False

3) ∃x ∀y (xy = y)

- True
- False

4) ∀x ∃y (x^2 = y)

- True
- False

De Morgan's law with nested quantifiers

De Morgan's law can be applied to logical statements with more than one quantifier. Each time the negation sign moves past a quantifier, the quantifier changes type from universal to existential or from existential to universal:

Table 1.9.2: De Morgan's laws for nested quantified statements.

$\neg\forall x\ \forall y\ P(x, y) \equiv \exists x\ \exists y\ \neg P(x, y)$
$\neg\forall x\ \exists y\ P(x, y) \equiv \exists x\ \forall y\ \neg P(x, y)$
$\neg\exists x\ \forall y\ P(x, y) \equiv \forall x\ \exists y\ \neg P(x, y)$
$\neg\exists x\ \exists y\ P(x, y) \equiv \forall x\ \forall y\ \neg P(x, y)$

Consider a scenario in which the domain is the set of all students in a school. The predicate L(x, y) indicates that x likes y. The statement $\exists x\ \forall y\ L(x, y)$ is read as:

$\exists x\ \forall y\ L(x, y) \leftrightarrow$ There is a student who likes everyone in the school.

The negation of the statement is:

$\neg\exists x\ \forall y\ L(x, y) \leftrightarrow$ There is no student who likes everyone in the school.

Applying De Morgan's laws yields: $\neg\exists x\ \forall y\ L(x, y) \equiv \forall x\ \exists y\ \neg L(x, y)$ which is translated into:

$\forall x\ \exists y\ \neg L(x, y) \leftrightarrow$ Every student in the school has someone that they do not like.

PARTICIPATION ACTIVITY 1.9.5: De Morgan's law for nested quantifiers.

Animation content:

`undefined`

Animation captions:

1. If De Morgan's law is applied once to $\neg\forall x\forall yP(x, y)$, the result is $\exists x\neg\forall yP(x, y)$.

2. Applying De Morgan's law again gives $\exists x \exists y \neg P(x, y)$.
3. Applying De Morgan's law once to $\neg \forall x \exists y P(x, y)$ gives $\exists x \neg \exists y P(x, y)$. The second applica gives $\exists x \forall y \neg P(x, y)$.
4. Applying De Morgan's law once to $\neg \exists x \forall y P(x, y)$ gives $\forall x \neg \forall y P(x, y)$. The second applica gives $\forall x \exists y \neg P(x, y)$.
5. Applying De Morgan's law once to $\neg \exists x \exists y P(x, y)$ gives $\forall x \neg \exists y P(x, y)$. The second applica gives $\forall x \forall y \neg P(x, y)$.

PARTICIPATION ACTIVITY 1.9.6: De Morgan's law for nested quantifiers.

Match logically equivalent propositions.

¬∃x ∃y (¬P(x) ∧ Q(y)) ¬∃x ∀y (P(x) ∨ ¬Q(y)) ¬∀x ∃y (P(x) ∨ Q(y))

¬∀x ∀y (P(x) ∧ Q(y))

∃x ∃y (¬P(x) ∨ ¬Q(y))

∃x ∀y (¬P(x) ∧ ¬Q(y))

∀x ∃y (¬P(x) ∧ Q(y))

∀x ∀y (P(x) ∨ ¬Q(y))

Reset

Additional exercises

Exercise 1.9.1: Which logical expressions with nested quantifiers are propositions? About

The table below shows the value of a predicate M(x, y) for every possible combination of values the variables x and y. The domain for x and y is {1, 2, 3}. The row number indicates the value for and the column number indicates the value for y. For example M(1, 2) = F because the value in r 1, column 2, is F.

M		y		
		1	2	3
x	1	T	F	T
	2	T	F	T
	3	T	T	F

Indicate whether each of the logical expressions is a proposition. If so, indicate whether the proposition is true or false.

(a)
M(1, 1)

(b)
∀y M(x, y)

(c)
∃x M(x, 3)

(d)
∃x ∃y M(x, y)

(e)
∃x ∀y M(x, y)

(f)
M(x, 2)

(g)
$\exists y \, \forall x \, M(x, y)$

Exercise 1.9.2: Truth values for statements with nested quantifiers - small finite domain.

About

The tables below show the values of predicates P(x, y), Q(x, y), and S(x, y) for every possible combination of values of the variables x and y. The row number indicates the value for x and the column number indicates the value for y. The domain for x and y is {1, 2, 3}.

P	1	2	3
1	T	F	T
2	T	F	T
3	T	T	F

Q	1	2	3
1	F	F	F
2	T	T	T
3	T	F	F

S	1	2	3
1	F	F	F
2	F	F	F
3	F	F	F

Indicate whether each of the quantified statements is true or false.

(a)
$\exists x \, \forall y \, P(x, y)$

(b)
$\exists x \, \forall y \, Q(x, y)$

(c)
$\exists x \, \forall y \, P(y, x)$

(d)
$\exists x \, \exists y \, S(x, y)$

(e)
$\forall x \, \exists y \, Q(x, y)$

(f) $\forall x\ \exists y\ P(x, y)$

(g) $\forall x\ \forall y\ P(x, y)$

(h) $\exists x\ \exists y\ Q(x, y)$

(i) $\forall x\ \forall y\ \neg S(x, y)$

Exercise 1.9.3: Truth values for mathematical expressions with nested quantifiers.

About

Determine the truth value of each expression below. The domain is the set of all real numbers.

(a) $\forall x \exists y\ (xy > 0)$

(b) $\exists x \forall y\ (xy = 0)$

(c) $\forall x \forall y \exists z\ (z = (x - y)/3)$

(d) $\forall x \exists y \forall z\ (z = (x - y)/3)$

(e) $\forall x \forall y\ (xy = yx)$

(f)
$\exists x \exists y \exists z \, (x^2 + y^2 = z^2)$

(g)
$\forall x \exists y \; y^2 = x$

(h)
$\forall x \exists y \, (x < 0 \lor y^2 = x)$

(i)
$\exists x \; \exists y \, (x^2 = y^2 \land x \neq y)$

(j)
$\exists x \; \exists y \, (x^2 = y^2 \land |x| \neq |y|)$

(k)
$\forall x \; \forall y \, (x^2 \neq y^2 \lor |x| = |y|)$

Exercise 1.9.4: De Morgan's law and nested quantifiers.

About

Write the negation of each of the following logical expressions so that all negations immediately precede predicates. In some cases, it may be necessary to apply one or more laws of propositio logic.

(a)

$\forall x \, \exists y \, \exists z \, P(y, x, z)$

(b)

$\forall x \, \exists y \, (P(x, y) \land Q(x, y))$

(c)

$\exists x \, \forall y \, (P(x, y) \rightarrow Q(x, y))$

(d)

$\exists x \, \forall y \, (P(x, y) \leftrightarrow P(y, x))$

(e)

$\exists x \, \exists y \, P(x, y) \land \forall x \, \forall y \, Q(x, y)$

Exercise 1.9.5: Applying De Morgan's law to English statements with nested quantifiers.

About

The domain for variables x and y is a group of people. The predicate F(x, y) is true if and only if x a friend of y. For the purposes of this problem, assume that for any person x and person y, either is a friend of y or x is an enemy of y. Therefore, ¬F(x, y) means that x is an enemy of y.

Translate each statement into a logical expression. Then negate the expression by adding a negation operation to the beginning of the expression. Apply De Morgan's law until the negation operation applies directly to the predicate and then translate the logical expression back into English.

(a)
Everyone is a friend of everyone.

(b)
Someone is a friend of someone.

(c)
Someone is a friend of everyone.

(d)
Everyone is a friend of someone.

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How was this section? **Provide feedback**

1.10 More nested quantified statements

This section has been set as optional by your instructor.

Using logic to express "everyone else"

Consider a scenario where the domain is a group of people who are all working on a joint project. Define the predicate M(x, y) that indicates whether x sent an email to y. The statement ∀x ∀y M(x, y) asserts that every person sent an email to every other person and every person sent an email to himself or herself. How could we use logic to express that everyone sent an email to everyone else without including the case that everyone sent an email to himself or herself? The idea is to use the conditional operation: (x ≠ y) → M(x, y).

The table below shows a group of four people and the truth value of M(x, y) for each pair. For example, Agnes sent an email to Fred (i.e., M(Agnes, Fred) = T) because the truth value in the row labeled Agnes and the column labeled Fred is T.

Figure 1.10.1: Email predicate truth values.

x \ y	Agnes	Fred	Sue	Marge
Agnes	T	T	T	T
Fred	T	F	T	T
Sue	T	T	T	T
Marge	T	T	T	F

The statement ∀x ∀y M(x, y) is false because M(Fred, Fred) and M(Marge, Marge) are both false. However, the statement

$$\forall x\ \forall y\ ((x \neq y) \rightarrow M(x, y))$$

is true. The statement says that for every pair, x and y, if x and y are different people then x sent an email to y. That is, everyone sent an email to everyone else. The statement is true for the table above because for every pair that is not on the diagonal of the table (i.e., for every pair such that x ≠ y), M(x, y) is true.

PARTICIPATION ACTIVITY 1.10.1: Expressing 'someone else' in logic.

The diagrams below shows a scenario for the predicate "x sent an email to y" for a particular group of people.

x \ y	Agnes	Fred	Sue	Marge
Agnes	F	F	T	F
Fred	F	T	F	F
Sue	F	F	F	T
Marge	F	T	F	F

1) Indicate whether the following statement is true for the group in the table:
"Everyone sent an email to someone else"

- True
- False

2) Is the statement "∀x ∃y M(x, y)" true for the group in the table?

- True.
- False

3) Consider the statement $\forall x\ \exists y\ ((x \neq y) \land M(x, y))$. In the two player game for the group in the table, if the universal player selects x = Sue, who will the existential player select for y?

- Fred
- Sue
- Marge

4) Is the statement "$\forall x\ \exists y\ ((x \neq y) \land M(x, y))$" true for the group in the table?

- True
- False

PARTICIPATION ACTIVITY 1.10.2: Expressing 'someone else' in logic, cont.

The diagrams below shows a scenario for the predicate "x sent an email to y" for a particular group of people.

		y			
		Sandy	Jen	Frank	Gary
x	Sandy	F	F	T	F
	Jen	F	F	T	F
	Frank	F	F	F	T
	Gary	F	T	F	F

1) Indicate whether the following statement is true or false for the group in the table:
"Everyone sent an email to someone else"

- True
- False

2) Is the statement "$\forall x \; \exists y \; ((x \neq y) \wedge M(x, y))$" true or false for the group in the table?

- True
- False

3) Are the two statements below logically equivalent?
"Everyone sent an email to someone else"
and
$\forall x \; \exists y \; ((x \neq y) \wedge M(x, y))$

- Yes
- No

Expressing uniqueness in quantified statements

An existentially quantified statement evaluates to true even if there is more than one element in the domain that causes the predicate to evaluate to true. If the domain is a set of people who attend a meeting and the predicate L(x) indicates whether or not x came late to the meeting, then the statement $\exists x \; L(x)$ is true if there are one, two or more people who came late.

PARTICIPATION ACTIVITY 1.10.3: Using logic to express that exactly one person came late to the meeting.

Animation captions:

1. How to express: "Exactly one person was late to the meeting." L(x) means x was late to the meeting. ∃x L(x) means that someone was late to the meeting.
2. A way is needed to express that x is the only person who came late to the meeting.
3. Add that for every y, if y ≠ x, then y was not late to the meeting:
   $\exists x(L(x) \wedge \forall y((x \neq y) \rightarrow \neg L(y))$.

PARTICIPATION ACTIVITY 1.10.4: Expressing uniqueness in quantified statements.

Consider a domain consisting of a set of people attending a meeting. The predicate L(x) indicates whether the person was late to the meeting. The table below shows a sample domain and the value of the predicate L for each person attending the meeting.

Name	L(x)
Shirley	F
Rob	T
Ted	F
Mindy	T

1) Indicate whether the following statement is true for the group in the table:
"Exactly one person was late for the meeting."

- True
- False

2) If x = Shirley is the following statement true:
L(Shirley) ∧ ∀y ((Shirley ≠ y) → ¬L(y))

- True
- False

3) If x = Rob, is the following statement true: $L(Rob) \wedge \forall y\, ((Rob \neq y) \rightarrow \neg L(y))$

- True
- False

PARTICIPATION ACTIVITY 1.10.5: Expressing uniqueness in quantified statements, cont.

Consider a domain consisting of a set of people attending a meeting. The predicate L(x) indicates whether the person was late to the meeting. The table below shows a sample domain and the value of the predicate L for each person attending the meeting.

Name	L(x)
Shirley	F
Rob	T
Ted	F
Mindy	F

1) Indicate whether the following statement is true for the group in the table: "Exactly one person was late for the meeting."

- True
- False

2) If x = Rob, is the following statement true: $L(Rob) \wedge \forall y\, ((Rob \neq y) \rightarrow \neg L(y))$

- True
- False

Moving quantifiers in logical statements

Now consider a set of people at a party as the domain. We would like to find a logical expression that is equivalent to the statement: "Every adult is married to someone at the party." There are two predicates:

M(x, y): x is married to y.
A(x): x is an adult.

Here is an equivalent statement that is closer in form to a logical expression: "For every person x, if x is an adult, then there is a person y to whom x is married." The logic is expressed as:

$$\forall x\ (A(x) \rightarrow \exists y\ M(x, y))$$

Since y does not appear in the predicate A(x), "∃y" can be moved to the left so that it appears just after the ∀x resulting in the following equivalent expression:

$$\forall x\ \exists y(A(x) \rightarrow M(x, y))$$

PARTICIPATION ACTIVITY 1.10.6: Example of moving quantifiers in logical statements.

Animation captions:

1. How to express: "Every adult is married to exactly one adult." A(x) means x is an adult. M(x, y) means x is married to y.
2. The first step is to express: "Every adult is married to someone." For every x, if x is an adult, t there is a y such that x is married to y: $\forall x(A(x) \rightarrow \exists y M(x, y))$.
3. The next step is to express that y is the only person x is married to: for every z, if z is not y, t x is not married to z: $\forall x(A(x) \rightarrow (\exists y M(x, y) \land \forall z((z \neq y) \rightarrow \neg M(x, z))))$.
4. The quantifiers for y and z can be moved to the front just after ∀x because ∃y does not pas references to y and ∀z does not pass references to z.

PARTICIPATION ACTIVITY 1.10.7: Nested quantifiers expressed in English.

Match each proposition to the corresponding English sentence. The domain is the set of all students in a math class. The two predicates are defined as:

P(x, y): x knows y's phone number.
H(x): x has the homework assignment.

$\exists x \forall y P(x, y)$ $\forall x \forall y P(x, y)$ $\exists x \forall y (H(x) \wedge P(x, y))$ $\forall x \exists y (P(x, y) \wedge H(y))$

Every student knows every student's phone number.

Some student knows every student's phone number.

Every student knows the phone number of another student who has the homework assignment.

There is a student who has the homework assignment and knows every student's phone number.

Reset

Additional exercises

Exercise 1.10.1: Truth values for expressions with nested quantifiers.

About

The domain of discourse for this problem is a group of three people who are working on a projec To make notation easier, the people are numbered 1, 2, 3. The predicate M(x, y) indicates whethe has sent an email to y, so M(2, 3) is read "Person 2 has sent an email to person 3." The table belc shows the value of the predicate M(x,y) for each (x,y) pair. The truth value in row x and column y gives the truth value for M(x,y).

M	1	2	3
1	T	T	T
2	T	F	T

3	T	T	F

Indicate whether the quantified statement is true or false. Justify your answer.

(a)
$\forall x \, \forall y \; M(x,y)$

(b)
$\forall x \, \forall y \; ((x \neq y) \rightarrow M(x,y))$

(c)
$\exists x \, \exists y \; \neg M(x,y)$

(d)
$\exists x \, \exists y \; ((x \neq y) \wedge \neg M(x,y))$

(e)
$\forall x \, \exists y \; \neg M(x,y)$

(f)
$\exists x \, \forall y \; M(x,y)$

Exercise 1.10.2: Truth values for mathematical statements with nested quantifiers.

About

The domain for all variables in the expressions below is the set of real numbers. Determine whether each statement is true or false. Justify your answer.

(a)

$\forall x\ \exists y\ (x + y = 0)$

(b)

$\exists x\ \forall y\ (x + y = 0)$

(c)

$\exists x\ \forall y\ (xy = y)$

(d)

$\exists x\ \exists y\ ((x^2 = y^2) \land (x \neq y))$

(e)

$\forall x\ \forall y\ \exists z\ (z = (x + y)/2)$

(f)

$\forall x\ \exists y\ \forall z\ (z = (x + y)/2)$

Exercise 1.10.3: Showing non-equivalence for expressions with nested quantifiers.

About

Show that the two quantified statements in each problem are not logically equivalent by filling in table so that, for the domain of discourse {a, b, c}, the values of the predicate P you select for the table causes one of the statements to be true and the other to be false. For example, the table below shows that $\forall x\ \forall y\ P(x, y)$ and $\exists x\ \exists y\ P(x, y)$ are not logically equivalent because for the given values of the predicate P, $\forall x\ \forall y\ P(x, y)$ is false and $\exists x\ \exists y\ P(x, y)$ is true.

P	a	b	c
a	T	T	T
b	T	F	T
c	T	T	F

(a)

$\forall x\ \exists y\ P(x, y)$ and $\exists x\ \forall y\ P(x, y)$

(b)

$\forall x\ \exists y\ ((x \neq y) \land P(x, y))$ and $\forall x\ \exists y\ P(x, y)$

(c)

$\exists x\ \exists y\ (P(x, y) \land P(y, x))$ and $\exists x\ \exists y\ P(x, y)$

Exercise 1.10.4: Mathematical statements into logical statements with nested quantifiers.

About

Translate each of the following English statements into logical expressions. The domain of discourse is the set of all real numbers.

(a) There are two numbers whose ratio is less than 1.

(b) The reciprocal of every positive number is also positive.

(c) There are two numbers whose sum is equal to their product.

(d) The ratio of every two positive numbers is also positive.

(e) The reciprocal of every positive number less than one is greater than one.

(f) There is no smallest number.

(g) Every number besides 0 has a multiplicative inverse.

(h) Every number besides 0 has a unique multiplicative inverse.

Exercise 1.10.5: Statements with nested quantifiers: English to logic, part 1.

About

The domain of discourse is the members of a chess club. The predicate B(x, y) means that pers
x has beaten person y at some point in time. Give a logical expression equivalent to the followin
English statements. You can assume that it is possible for a person to beat himself or herself.

(a)
Sam has been beaten by someone.

(b)
Everyone has been beaten before.

(c)
No one has ever beaten Nancy.

(d)
Everyone has won at least one game.

(e)
No one has beaten both Ingrid and Dominic.

(f)
Josephine has beaten everyone else.

(g)
Nancy has beaten exactly one person.

(h)
There are at least two members who have never been beaten.

Exercise 1.10.6: Statements with nested quantifiers: English to logic, part 2.

About

The domain for the variables x and y are the set of musicians in an orchestra. The predicates S, B, and P are defined as:

S(x): x plays a string instrument
B(x): x plays a brass instrument
P(x, y): x practices more than y

Give a quantified expression that is equivalent to the following English statements:

(a) There are no brass players in the orchestra.

(b) Someone in the orchestra plays a string instrument and a brass instrument.

(c) There is a brass player who practices more than all the string players.

(d) All the string players practice more than all the brass players.

(e) Exactly one person practices more than Sam.

(f) Sam practices more than anyone else in the orchestra.

Exercise 1.10.7: Statements with nested quantifiers: English to logic, part 3.

About

The domain of discourse is a group working on a project at a company. One of the members of
group is named Sam. Define the following predicates.

- P(x, y): x knows y's phone number. (A person may or may not know their own phone numb
- D(x): x missed the deadline.
- N(x): x is a new employee.

Give a logical expression for each of the following sentences.

(a) Someone knows everyone's phone number.

(b) Everyone knows someone's phone number.

(c) There is at least one new employee who missed the deadline.

(d) Sam knows the phone number of everyone who missed the deadline.

(e) There is a new employee who knows everyone's phone number.

(f) Exactly one new employee missed the deadline.

Exercise 1.10.8: Statements with nested quantifiers: English to logic, part 4.

About

A student club holds an election for officers. Before the voting, members can nominate each oth
It is also possible for a member to nominate himself or herself. Some of the members are new members. Some of the members are currently officers. The domain is the set of members of the club. One of the members of the club is named Sam. Define the following predicates.

- N(x, y): person x nominated person y for a position.
- W(x): person x is a new member.
- O(x): person x is currently an officer.

Give a quantified expression that is logically equivalent to each of the following statements.

(a)
All the new members nominated all the officers.

(b)
One of the current officers did not nominate anyone.

(c)
Everyone nominated someone.

(d)
Someone nominated everyone.

(e)
Everyone nominated someone besides themselves. (This statement does not specify wheth anyone nominated themselves or not.)

(f)
Exactly one person nominated Sam.

Exercise 1.10.9: Statements with nested quantifiers: English to logic, part 5.

About

The domain of discourse is a set of animals on a farm. One of the bunnies on the farm is named Fluffy. Use the definitions of the predicates below to translate each English statement into an equivalent logical expression.

- B(x): x is a bunny
- H(x): x is a horse
- F(x): x has been fed
- W(x, y): x weighs more than y

(a)
There is a horse who has not been fed.

(b)
Every bunny has been fed.

(c)
Exactly one horse has not been fed.

(d)
Every horse weighs more than every bunny.

(e)
Fluffy weighs more than every other bunny.

Exercise 1.10.10: Statements with nested quantifiers: variables with different domains.

About

The domain for the first input variable to predicate T is a set of students at a university. The domain for the second input variable to predicate T is the set of Math classes offered at that university. The predicate T(x, y) indicates that student x has taken class y. Sam is a student at th university and Math 101 is one of the courses offered at the university. Give a logical expression each sentence.

(a) Sam has taken Math 101.

(b) Every student has taken at least one math class.

(c) Every student has taken at least one class besides Math 101.

(d) There is a student who has taken every math class besides Math 101.

(e) Everyone besides Sam has taken at least two different math classes.

(f) Sam has taken exactly two math classes.

How was this section? **Provide feedback**