**Linear Algebra**

**Digital Assignment 1**

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**15BCE0763**

**LU FACTORIZATION**

L U decomposition of a matrix is the factorization of a given square matrix into two triangular matrices, one upper triangular matrix and one lower triangular matrix, such that the product of these two matrices gives the original matrix. It was introduced by Alan Turing in 1948, who also created the turing machine.

This method of factorizing a matrix as a product of two triangular matrices has various applications such as solution of a system of equations, which itself is an integral part of many applications such as finding current in a circuit and solution of discrete dynamical system problems; finding the inverse of a matrix and finding the determinant of the matrix.

Basically, the L U decomposition method comes handy whenever it is possible to model the problem to be solved into matrix form. Conversion to the matrix form and solving with triangular matrices makes it easy to do calculations in the process of finding the solution.

A square matrix A can be decomposed into two square matrices L and U such that A = L U where U is an upper triangular matrix formed as a result of applying Gauss Elimination Method on A; and L is a lower triangular matrix with diagonal elements being equal to 1.

**Advantages of LU factorization**

Suppose we want to solve a m×n system AX = b.

If we can find a LU-decomposition for A , then to solve AX =b, it is enough to solve the systems

LY=b

UX=Y

Thus the system LY = b can be solved by the method of forward substitution and the system UX = Y

can be solved by the method of backward substitution.

**Using LU decomposition to solve a system of equations**

Once a matrix A has been decomposed into lower and upper triangular parts it is possible to

obtain the solution to AX = B in a direct way.

The procedure can be summarised as follows

• Given A, find L and U so that A = LU. Hence LUX = B.

• Let Y = UX so that LY = B. Solve this triangular system for Y .

• Finally solve the triangular system UX = Y for X.

The benefit of this approach is that we only ever need to solve triangular systems. The cost is

that we have to solve two of them.

Although not all matrices can be written in the ‘LU’ form

An invertible matrix A has an LU decomposition provided that all its leading submatrices have non-zero determinants. The k-th leading submatrix of A is denoted Ak and is the k × k matrix found by looking only at the top k rows and leftmost k columns.

This can be overcome with the re-order of the rows in the invertible matrix so that all of the sumatrices have non-zero determinents.

**Gauss Elimination Method**

According to the Gauss Elimination method:

1. Any zero row should be at the bottom of the matrix.

2. The first non zero entry of each row should be on the right hand side of the first non zero entry of the preceding row.

This method reduces the matrix to row echelon form.

**Steps for L U Decomposition**

Given a set of linear equations, first convert them into matrix form A X = C where A is the coefficient matrix, X is the variable matrix and C is the matrix of numbers on the right hand side of the equations.

Now, reduce the coefficient matrix A , i.e., matrix obtained from the coefficients of variables in all the given equations such that for ‘n’ variables we have an nXn matrix, to row echelon form using Gauss Elimination Method. The matrix so obtained is U.

To find L, we have two methods. The first one is to assume the remaining elements as some artificial variables, make equations using A = L U and solve them to find those artificial variables.

The other method is that the remaining elements are the multiplier coefficients because of which the respective positions became zero in the U matrix. (This method is a little tricky to understand by words but would get clear in example below)

Now, we have A (the nXn coefficient matrix), L (the nXn lower triangular matrix), U (the nXn upper triangular matrix), X (the nX1 matrix of variables) and C (the nX1 matrix of numbers on the right hand side of the equations).

The given system of equations is A X = C. We substitute A = L U. Thus, we have L U X = C.

We put Z = U X, where Z is a matrix or artificial variables and solve for L Z = C first and then solve for U X = Z to find X or the values of the variables, which was required.

**LU factorization with Partial Pivoting**

It turns out that a proper permutation in rows (or columns) is sufficient for the LU factorization. The LU factorization with Partial Pivoting (LUP) refers often to the LU factorization with row permutations only,

PA = LU

where L and U are again lower and upper triangular matrices, and P is a permutation matrix which, when left-multiplied to A, reorders the rows of A. It turns out that all square matrices can be factorized in this form,[2] and the factorization is numerically stable in practice.

**LU factorization with full pivoting**

An LU factorization with full pivoting involves both row and column permutations,

PAQ = LU

where L, U and P are defined as before, and Q is a permutation matrix that reorders the columns of A.

**LDU decomposition**

An LDU decomposition is a decomposition of the form

A = LDU

where D is a diagonal matrix and L and U are unit triangular matrices, meaning that all the entries on the diagonals of L and U are one.

Above we required that A be a square matrix, but these decompositions can all be generalized to rectangular matrices as well. In that case, L and D are square matrices both of which have the same number of rows as A, and U has exactly the same dimensions as A. Upper triangular should be interpreted as having only zero entries below the main diagonal, which starts at the upper left corner.

**Applications of LU :**

1)Solving linear equations

Given a system of linear equations in matrix form

A x = b

we want to solve the equation for x given A and b. Suppose we have already obtained the LUP decomposition of A such that PA = LU, so L U x = P b.

In this case the solution is done in two logical steps:

First, we solve the equation L y = P b for y;

Second, we solve the equation U x = y for x.

**2)Inverting a matrix**

When solving systems of equations, b is usually treated as a vector with a length equal to the height of matrix A. Instead of vector b, we have matrix B, where B is an n-by-p matrix, so that we are trying to find a matrix X (also a n-by-p matrix):

A X = L U X = B.

3) Computing the determinant